EXPLORING THE NUMERICAL RANGE OF BLOCK TOEPLITZ OPERATORS

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ABSTRACT

Exploring the Numerical Range of Block Toeplitz Operators

Brooke Randell

We will explore the numerical range of the block Toeplitz operator with symbol function $\phi(z) = A_0 + zA_1$, where $A_0, A_1 \in M_2(\mathbb{C})$. A full characterization of the numerical range of this operator proves to be quite difficult and so we will focus on characterizing the boundary of the related set, $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$, in a specific case. We will use the theory of envelopes to explore what the boundary looks like and we will use geometric arguments to explore the number of flat portions on the boundary. We will then make a conjecture as to the number of flat portions on the boundary of the numerical range for any 2×2 matrices A_0 and A_1. We finish by providing examples of flat portions on the boundary of the numerical range when $A_0, A_1 \in M_n(\mathbb{C})$, for $3 \leq n \leq 5$.

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1 Background

1.1 The Numerical Range

The main focus of this paper is to characterize the numerical range of a specific type of operator acting on a Hilbert space. A Hilbert space $\mathcal{H}$ is defined to be a complete inner product space. This means that $\mathcal{H}$ is an inner product space with the property that every Cauchy sequence converges in $\mathcal{H}$. Note that the inner product induces a norm on $\mathcal{H}$. To learn more about Hilbert spaces, see [1] Chapter 8.

Given a linear operator A on a Hilbert space $\mathcal{H}$, the numerical range of A is the set

$$W(A) = \{\langle Av, v\rangle : v \in \mathcal{H}, ||v|| = 1\}.$$

The Toeplitz-Hausdorff Theorem [7,16] states that for any bounded linear operator A on $\mathcal{H}$, $W(A) \subseteq \mathbb{C}$ is convex, indicating that $W(A)$ contains the line segment between every pair of points in $W(A)$.

Often when a mathematician discusses the numerical range of an operator, they also discuss the spectrum of that operator: Given a bounded linear operator A on a Hilbert space $\mathcal{H}$, the spectrum of A is the set

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}.$$

It turns out that for any bounded linear operator A on $\mathcal{H}$, the spectrum of A is contained in the closure of the numerical range of A. To read a proof of this fact, see [12] pg.24.

Unsurprisingly, we can also calculate the numerical range and spectrum of a matrix M using the above definitions. The Toeplitz-Hausdorff Theorem and the theorem detailing the relationship between the numerical range and the spectrum, also hold in the matrix case. In the finite matrix case, the spectrum of the matrix M consists of the eigenvalues of the matrix. To learn more, see [8] pg.1-2 and pg.5-6. Given a 2×2 matrix M with complex entries, we can say even more about the numerical range of M. From here on, we let $M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices with complex coefficients. Recall that the trace of a matrix $M \in M_n(\mathbb{C})$, denoted $\mathrm{tr}M$, is the sum of the eigenvalues of M. Equivalently, $\mathrm{tr}M$ is the sum of the diagonal entries of M.

Elliptical Range Theorem

Let $M \in M_2(\mathbb{C})$ and let $M_0 = M - (\frac{1}{2}\mathrm{tr}M)I$. The numerical range of M is a potentially degenerate elliptical disk satisfying the following conditions:

- The center of the elliptical disk is $\frac{1}{2}\mathrm{tr}M$;
- The length of the major axis of the elliptical disk is $\sqrt{\mathrm{tr}M_0^*M_0 + 2|\det M_0|}$;
- The length of the minor axis of the elliptical disk is $\sqrt{\mathrm{tr}M_0^*M_0 - 2|\det M_0|}$;
- The foci of the elliptical disk are the eigenvalues of M.

To read a more in-depth discussion of the Elliptical Range Theorem, see [8] pg.20-24. As an illustration of the three theorems discussed above, consider the following plot of the numerical range and spectrum of the 2×2 matrix, $B = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}$:

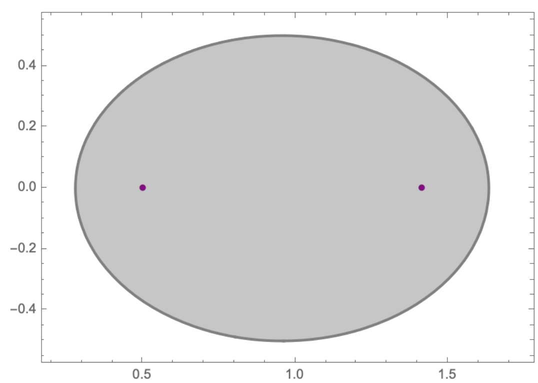

Figure 1: $W(B)$ **is the gray elliptical disk and** $\sigma(B)$ **is the two purple points.**

Note that $W(B)$ is convex, $\sigma(B) \subseteq W(B)$, and $W(B)$ is an elliptical disk as expected.

1.2 Toeplitz Operators

Before defining our operator, we need to understand the underlying spaces that we will be working with. From here on, we use $\mathbb{D}$ to denote the open unit disk and $\partial\mathbb{D}$ to denote the boundary of the unit disk. We also use $||\cdot||_{\mathbb{C}^n}$ to denote the norm induced by the standard inner product on $\mathbb{C}^n$.

- L^2 is the Hilbert space of equivalence classes of square-integrable Lebesgue measurable functions acting on the unit circle.
- $H^2(\mathbb{D}) = H^2$ is the Hilbert space of analytic functions acting on $\mathbb{D}$ whose power series have square-summable coefficients in $\mathbb{C}$.

Given two elements, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in H^2, the inner product on H^2 is $\langle f, g\rangle = \sum_{n=0}^{\infty} a_n \bar{b}_n$. Although not readily obvious, H^2 is a subspace of L^2 (see [2] pg.1722).

- A function ϕ acting on the unit circle is essentially bounded if there exists $m \in \mathbb{R}$ such that the set $\{e^{i\theta} : |\phi(e^{i\theta})| > m\}$ has Lebesgue measure 0.

Any bounded function is said to be essentially bounded because there exists an $m \in \mathbb{R}$ such that $\{e^{i\theta} : |\phi(e^{i\theta})| > m\} = \emptyset$.

- L^∞ is the space of equivalence classes of essentially bounded Lebesgue measurable functions.
- $L^\infty_{n\times n}$ is the space of $n \times n$ matrices whose matrix entries live in L^∞.

To illustrate what elements in this space look like, consider

$$F(z) = \begin{bmatrix} p_{11}(z) & p_{12}(z) & p_{13}(z) \\ p_{21}(z) & p_{22}(z) & p_{23}(z) \\ p_{31}(z) & p_{32}(z) & p_{33}(z) \end{bmatrix},$$

where each $p_{ij} : \partial\mathbb{D} \to \mathbb{C}$ is a polynomial. Note that each p_{ij} is bounded on $\partial\mathbb{D}$ and so, each p_{ij} is contained in L^∞. This implies that $F \in L^\infty_{3\times 3}$.

- $H_n(\mathbb{D})$ is the space of functions mapping $\mathbb{D}$ to $\mathbb{C}^n$ that are analytic at each $z \in \mathbb{D}$.

- $H^2_n(\mathbb{D})$ is the space of functions in $H_n(\mathbb{D})$ such that each $f \in H_n(\mathbb{D})$ can be represented as the power series $f(z) = \sum_{j=0}^{\infty} a_j z^j$, where $\sum_{j=0}^{\infty} ||a_j||^2_{\mathbb{C}^n} < \infty$.

Given two elements, $f(z) = \sum_{j=0}^{\infty} a_j z^j$ and $g(z) = \sum_{j=1}^{\infty} b_j z^j$ in $H^2_n(\mathbb{D})$, the inner product on $H^2_n(\mathbb{D})$ is $\langle f, g\rangle = \sum_{j=0}^{\infty} \langle a_j, b_j\rangle_{\mathbb{C}^n}$. To illustrate what elements in this space look like, consider

$$g(z) = \sum_{n=0}^{\infty} \begin{bmatrix} \frac{1}{n} \\ \frac{1}{2^n} \end{bmatrix} z^n.$$

Notice that $g \in H^2_2(\mathbb{D})$ since

$$\begin{aligned} \sum_{n=0}^{\infty} \left|\left| \begin{bmatrix} \frac{1}{n} \\ \frac{1}{2^n} \end{bmatrix} \right|\right|^2_{\mathbb{C}^2} &= \sum_{n=0}^{\infty} \Big(\frac{1}{n^2} + \frac{1}{4^n}\Big) \\ &< \infty. \end{aligned}$$

- H^∞ is the space of functions that are both analytic and bounded on $\mathbb{D}$.

We will be studying the numerical range of Toeplitz operators. A Toeplitz operator with symbol function $\phi \in L^\infty$ is the map $T_\phi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ defined by $T_\phi f(z) = P\phi(z)f(z),$ for all $f \in H^2(\mathbb{D})$, where P is the orthogonal projection from L^2 onto H^2. The corresponding Toeplitz matrix is constant along its diagonals with respect to some basis. In this paper, we will focus specifically on block Toeplitz operators. These are Toeplitz operators whose symbol functions are matrix-valued and whose corresponding matrices are constant along the block diagonals. We will investigate the numerical range of the block Toeplitz operator with symbol function $\phi(z) = A_0 + zA_1$, for $A_0, A_1 \in M_n(\mathbb{C})$. Since our symbol function is analytic and matrix-valued, we will be using a modified version of the definition discussed above. We define $T_{A_0+zA_1} : H^2_2(\mathbb{D}) \to H^2_2(\mathbb{D})$ by $T_{A_0+zA_1} f(z) = (A_0 + zA_1)f(z),$ for all $f \in H^2_2(\mathbb{D})$. We make the following claim about the matrix of $T_{A_0+zA_1}$:

Claim 1

The block Toeplitz matrix corresponding to the Toeplitz operator $T_{A_0+zA_1}$, for $A_0, A_1 \in M_2(\mathbb{C})$, is

$$\begin{bmatrix} A_0 & 0 & 0 & 0 & \cdots \\ A_1 & A_0 & 0 & 0 & \cdots \\ 0 & A_1 & A_0 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Proof. Consider the Toeplitz operator $T_{A_0+zA_1}$, for $A_0, A_1 \in M_2(\mathbb{C})$. Note that we can write,

$$A_0 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

where $a_{ij}, b_{ij} \in \mathbb{C}$ for all $1 \leq i, j \leq 2$. Therefore,

$$A_0 + zA_1 = \begin{bmatrix} a_{11} + zb_{11} & a_{12} + zb_{12} \\ a_{21} + zb_{21} & a_{22} + zb_{22} \end{bmatrix}.$$

The standard basis for $H_2^2(\mathbb{D})$ is given by,

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} z \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ z \end{bmatrix}, \begin{bmatrix} z^2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ z^2 \end{bmatrix}, \cdots \right\}.$$

To find the first four columns of the matrix for $T_{A_0+zA_1}$ with respect to this standard basis, we compute the following:

$$T_{A_0+zA_1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a_{11} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_{21} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + b_{11} \begin{bmatrix} z \\ 0 \end{bmatrix} + b_{21} \begin{bmatrix} 0 \\ z \end{bmatrix},$$

$$T_{A_0+zA_1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = a_{12} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + b_{12} \begin{bmatrix} z \\ 0 \end{bmatrix} + b_{22} \begin{bmatrix} 0 \\ z \end{bmatrix},$$

$$T_{A_0+zA_1} \begin{bmatrix} z \\ 0 \end{bmatrix} = a_{11} \begin{bmatrix} z \\ 0 \end{bmatrix} + a_{21} \begin{bmatrix} 0 \\ z \end{bmatrix} + b_{11} \begin{bmatrix} z^2 \\ 0 \end{bmatrix} + b_{21} \begin{bmatrix} 0 \\ z^2 \end{bmatrix},$$

$$T_{A_0+zA_1} \begin{bmatrix} 0 \\ z \end{bmatrix} = a_{12} \begin{bmatrix} z \\ 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 \\ z \end{bmatrix} + b_{12} \begin{bmatrix} z^2 \\ 0 \end{bmatrix} + b_{22} \begin{bmatrix} 0 \\ z^2 \end{bmatrix}.$$

So the first four columns of our matrix are,

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & \cdots \\ a_{21} & a_{22} & 0 & 0 & \cdots \\ b_{11} & b_{12} & a_{11} & a_{12} & \cdots \\ b_{21} & b_{22} & a_{21} & a_{22} & \cdots \\ 0 & 0 & b_{11} & b_{12} & \cdots \\ 0 & 0 & b_{21} & b_{22} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} A_0 & 0 & \cdots \\ A_1 & A_0 & \cdots \\ 0 & A_1 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

If we continued computing the rows in this manner we would end up with the matrix,

$$\begin{bmatrix} A_0 & 0 & 0 & 0 & \cdots \\ A_1 & A_0 & 0 & 0 & \cdots \\ 0 & A_1 & A_0 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Thus, the block Toeplitz matrix corresponding to $T_{A_0+zA_1}$ has matrix representation

$$\begin{bmatrix} A_0 & 0 & 0 & 0 & \cdots \\ A_1 & A_0 & 0 & 0 & \cdots \\ 0 & A_1 & A_0 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

with respect to the standard basis of $H_2^2(\mathbb{D})$. $\square$

In the case where the symbol function $A \in L^\infty$ is defined to be $A(z) = \sum_{j=-\infty}^{j=\infty} A_j z^j$, the corresponding block Toeplitz matrix is

$$\begin{bmatrix} A_0 & A_{-1} & A_{-2} & A_{-3} & \cdots \\ A_1 & A_0 & A_{-1} & A_{-2} & \cdots \\ A_2 & A_1 & A_0 & A_{-1} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

In this paper, we will only be studying symbol functions of the form $A(z) = A_0 + zA_1$.

Many facts are known about the numerical range of a Toeplitz operator. Klein [9] completely characterized the numerical range of Toeplitz operators with scalar-valued symbol functions. We will be interested in the following special case:

Theorem [Klein (1972)]

Assume $f : \mathbb{D} \to \mathbb{C}$. If $f \in H^\infty$, then $W(T_f)$ is the convex hull of the range of f on $\mathbb{D}$.

The convex hull of a set S is the intersection of every convex set containing S. To read the proof of this theorem or to learn more about Klein's work with the numerical range of Toeplitz operators, see [9]. In this paper, we will be making use of the following generalization of Klein's theorem, proved by Bebiano and Spitkovsky [2]. Bebiano and Spitkovsky expanded upon Klein's results by characterizing the numerical range of Toeplitz operators with matrix-valued symbol functions. Before, stating the theorem, however, we must define an additional term:

- Let $\phi \in L^\infty$ and let μ denote the Lebesgue measure. The essential range of ϕ, is the set $\mathcal{R}(\phi) = \big\{\lambda : \forall \epsilon > 0, \mu\big(\{e^{i\theta} : |\phi(e^{i\theta}) - \lambda| < \epsilon\}\big) > 0\big\}$.

If ϕ is continuous on $\partial\mathbb{D}$, $\mathcal{R}(\phi)$ is the range of ϕ. To read a proof of this fact, see [11] pg.6. Note that the definition of the essential range of ϕ also holds when $\phi \in L^\infty_{n\times n}$. In this case, however, the absolute value in our definition is replaced with a matrix norm.

Theorem A [Bebiano and Spitkovsky (2011)]

The closure of $W(T_A)$ is the convex hull of $\{W(M) : M \in \mathcal{R}(A)\}$, where A is the matrix-valued symbol of the Toeplitz operator T_A and $\mathcal{R}(A)$ is the essential range of A.

In the case where the matrix-valued symbol function A is analytic on the closure of $\mathbb{D}$, which occurs when this holds for each matrix entry, this theorem tells us that the closure of $W(T_A)$ is the convex hull of the set $\{W(A(z)) : z \in \partial\mathbb{D}\}$. To read the proof of Theorem A, see [2] pg.1722-1723. In our paper, we will use this result to conclude that the closure of $W(T_{A_0+zA_1})$ is equal to the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$. In our proof of Theorem 1, we will apply the Elliptical Range Theorem to $W(A_0 + zA_1)$ for fixed $z \in \partial\mathbb{D}$, analyze the shape of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$, and use

Bebiano and Spitkovsky's result to make a conclusion about the shape of the closure of $W(T_{A_0+zA_1})$.

One way to classify the numerical range of an individual matrix is to determine the number of flat portions on its boundary. Over the course of this paper, we will ask a similar question about the closure of $W(T_{A_0+zA_1})$, which includes an infinite number of matrix numerical ranges. Various facts about the number of flat portions on the boundary of the numerical range of an individual $n \times n$ matrix are known. For example, it is known that an $n \times n$ matrix has at most $\frac{n(n-1)}{2}$ flat portions on the boundary of its numerical range. To learn more about this theorem, see [6] pg.132. In our case, we want to answer this flat portion question for the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$, where $A_0, A_1 \in M_2(\mathbb{C})$. This turns out to be a difficult question because this set consists of infinitely many elliptical disks and so there is a huge variety of shapes that can be created from this infinite collection. Consider the numerical range plot corresponding to $A_0 = \begin{bmatrix} 1 & 1 \\ \frac{10}{4}e^{i\frac{\pi}{4}} & -1 \end{bmatrix}$ and $A_1 = \begin{bmatrix} i & e^{i\frac{7\pi}{10}} \\ \frac{10}{4} & -i \end{bmatrix}$:

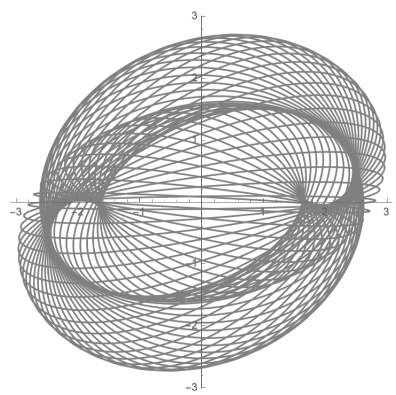

Figure 2: A finite number of boundary curves of the $W(A_0 + zA_1)$ in $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$. For each $z \in \partial\mathbb{D}$, $W(A_0 + zA_1)$ is the corresponding elliptical disk.

Here, the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ appears to have parallel flat portions on its boundary. We will soon see examples in which the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ is a circular disk as well as examples in which the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ has a single flat portion on its boundary. Our Theorem 2 will deal with the latter case.

1.3 Theory of Envelopes

In order to describe the shape of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$, we will apply the theory of envelopes. Given a family of curves, an envelope of this family can be thought of as a curve E that contains the family. The envelope also helps to form the boundary of the family of curves, which will become more evident in the proof of Theorem 1. From here, we define the envelope more rigorously:

- Given a family of curves, a curve E is called an envelope of the family if for every point in E, E intersects one of the curves in the family and E is tangent to that curve at the intersection point.

There are three different ways in which these two conditions can be satisfied, however, we will only focus on one of them:

- Let $\mathcal{F} = \{\Gamma_\theta\}_\theta$ be a family of curves defined in terms of a continuously differentiable function $f(x, y, \theta)$ which is defined so that for each value of θ,

$$\Gamma_\theta = \{x + iy \in \mathbb{C} : f(x, y, \theta) = 0\}.$$

  - The discriminant envelope of the family $\mathcal{F}$ is the curve $\mathcal{E}(\theta)$ consisting of the points $\big(x(\theta), y(\theta)\big)$ for which there exists some θ such that $f(x, y, \theta) = 0$ and $\frac{\partial f}{\partial \theta}(x, y, \theta) = 0$. We define θ be the parameter of $\mathcal{E}$ so that for each value of θ, $\mathcal{E}$ intersects Γ_θ at the point $\big(x(\theta), y(\theta)\big)$ and $\mathcal{E}$ is tangent to Γ_θ at the point $\big(x(\theta), y(\theta)\big)$.

In Theorem 1, we will consider the family of boundary curves of the individual $W(A_0 + zA_1)$ and then calculate their discriminant envelope. The discriminant envelope will then be used to describe the shape of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$. A recent article by Bickel and Gorkin [3] uses the envelope algorithm to compute the boundary of the numerical range of a particular operator and we will do the same for $T_{A_0+zA_1}$. To see how Bickel and Gorkin apply the theory of envelopes, see [3] pg.254-263.

From here, we will prove the following claim about the discriminant envelope:

Claim 2

Let f be a continuously differentiable function that satisfies $(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 \neq 0$ for each fixed $\theta \in (a, b)$. Let E be the curve parametrized by $\big(x(\theta), y(\theta)\big)$, where x and y are continuously differentiable on (a, b). Further assume that $\big(\frac{dx}{d\theta}\big)^2 + \big(\frac{dy}{d\theta}\big)^2 \neq 0$ on the interval (a, b). For each fixed $\theta \in (a, b)$, let $\mathcal{C}_\theta = \big\{(x, y) : f(x, y, \theta) = 0\big\}$. We say that the curve E is the discriminant envelope of the family $\{\mathcal{C}_\theta\}_{\theta \in (a,b)}$ if and only if for all $\theta \in (a, b)$, $f(x, y, \theta) = 0$ and $\frac{\partial f}{\partial \theta}(x, y, \theta) = 0$.

Proof. (modified from [5]) Assume the hypotheses.

($\Rightarrow$) Suppose the curve E is the discriminant envelope of the family $\{\mathcal{C}_\theta\}_{\theta \in (a,b)}$. Let $\theta \in (a, b)$. Thus, E intersects $\mathcal{C}_\theta$ at the point $\big(x(\theta), y(\theta)\big)$. By the definition of $\mathcal{C}_\theta$, $f(x, y, \theta) = 0$ at the point $\big(x(\theta), y(\theta)\big)$. This allows us to substitute $x(\theta)$ for x and $y(\theta)$ for y in $f(x, y, \theta) = 0$. Taking the derivative of $f\big(x(\theta), y(\theta), \theta\big) = 0$ with respect to θ gives, $\frac{\partial f}{\partial \theta}(x(\theta), y(\theta), \theta) = 0$, which by the chain rule can be rewritten as

$$\frac{\partial f}{\partial x}\frac{dx}{d\theta} + \frac{\partial f}{\partial y}\frac{dy}{d\theta} + \frac{\partial f}{\partial \theta}\frac{d\theta}{d\theta} = 0.$$

Since $\frac{d\theta}{d\theta} = 1$, this can be further rewritten as

$$\frac{\partial f}{\partial x}\frac{dx}{d\theta} + \frac{\partial f}{\partial y}\frac{dy}{d\theta} + \frac{\partial f}{\partial \theta} = 0. \tag{0.1}$$

From here, notice that in order for E and $\mathcal{C}_\theta$ to be tangent at the point $\big(x(\theta), y(\theta)\big)$, we need the gradient of $f(x, y, \theta) = 0$ to be orthogonal to the tangent vector to E at this point. The gradient of $f(x, y, \theta) = 0$ at this point is the vector $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})\Big|_\theta$ and the tangent vector to E at this point is $(\frac{dx}{d\theta}, \frac{dy}{d\theta})\Big|_\theta$. So, we require that

$$\Big(\frac{\partial f}{\partial x}\frac{dx}{d\theta} + \frac{\partial f}{\partial y}\frac{dy}{d\theta}\Big)\Big|_\theta = 0.$$

By (0.1), this implies that $\frac{\partial f}{\partial \theta}(x, y, \theta) = 0$. Since $\theta \in (a, b)$ was arbitrary, we conclude that for all $\theta \in (a, b)$, $f(x, y, \theta) = 0$ and $\frac{\partial f}{\partial \theta}(x, y, \theta) = 0$.

($\Leftarrow$) Suppose the curve E satisfies the property that for all $\theta \in (a, b)$, $f(x, y, \theta) = 0$ and $\frac{\partial f}{\partial \theta}(x, y, \theta) = 0$. Let $\theta \in (a, b)$. Once again, we substitute $x(\theta)$ for x and $y(\theta)$ for y in $f(x, y, \theta) = 0$ and then take the derivative with respect to θ. This gives us (0.1). Since $\frac{\partial f}{\partial \theta}(x, y, \theta) = 0$, we plug this into (0.1) to get

$$\frac{\partial f}{\partial x}\frac{dx}{d\theta} + \frac{\partial f}{\partial y}\frac{dy}{d\theta} = 0.$$

Since $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 \neq 0$ and $(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 \neq 0$ by assumption, we conclude that both E and $\mathcal{C}_\theta$ have well-defined tangents at $\big(x(\theta), y(\theta)\big)$ and that E and $\mathcal{C}_\theta$ intersect at $\big(x(\theta), y(\theta)\big)$. Since $\theta \in (a, b)$ was arbitrary, we conclude that this holds for all $\theta \in (a, b)$. Therefore, E is the discriminant envelope of the family $\{\mathcal{C}_\theta\}_{\theta \in (a,b)}$. $\square$

The proof of this claim is a modified version of Courant's proof found in [5] on pg.171-173. This claim still holds if we were to replace the open interval (a, b) with the closed interval $[a, b]$ or if we were to replace the open interval (a, b) with either of the half-open intervals, $[a, b)$ or $(a, b]$.

2 Outline of Paper

The paper is outlined as follows: We start by illustrating how to use the envelope algorithm with a concrete example, so as to expose readers to the algorithm before reading the proof of Theorem 1. A large portion of the paper will then be spent proving Theorem 1, which characterizes the boundary of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ for a specific class of matrices. We can split up the proof of Theorem 1 into three main sections: The first section will apply the Elliptical Range Theorem to determine what the individual $W(A_0 + zA_1)$ look like, the second section will apply the envelope algorithm, and the third section will focus on computing the boundary of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$.

Since the envelope algorithm is quite detailed, we will now spend some time outlining the exact steps. We are interested in finding the envelope of a family of circles and so, the following steps will be written with this in mind. From here on out, we will use the term "pre-envelope curve" to denote the curve being built by the envelope algorithm that may not satisfy all of the conditions of the algorithm and we will use the term "envelope" to denote the curve created via the envelope algorithm that satisfies all of the conditions of the algorithm. The algorithm is as follows:

1. Consider a family of circles $\{\mathcal{C}_\theta\}_{\theta\in[-\pi,\pi)}$ such that for each fixed $\theta \in [-\pi, \pi)$, there is a circle

$$\begin{aligned}\mathcal{C}_\theta &= \big\{(x, y) : f(x, y, \theta) = 0\big\} \\ &= \Big\{\Big(x_c(\theta) + r(\theta)\cos(s), y_c(\theta) + r(\theta)\sin(s)\Big) : s \in [-\pi, \pi)\Big\},\end{aligned}$$

   where $\big(x_c(\theta), y_c(\theta)\big)$ is the center of the circle and $r(\theta)$ is the radius of the circle. From here we let $x(s, \theta) = x_c(\theta) + r(\theta)\cos(s)$ and $y(s, \theta) = y_c(\theta) + r(\theta)\sin(s)$, where $s \in [-\pi, \pi)$. Here, we define s to be the parameter of the circles.

2. Show that $(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$ is nonzero. This condition prevents a curve with fixed θ from having a singularity and so, we will use it to exclude various values of θ from the domain of our pre-envelope curves. See [5] pp.173 for more details. This implies that some of the circles in $\{C_\theta\}_{\theta\in[-\pi,\pi)}$ do not contribute a point to our envelope curves. In our proof of Theorem 1, we will see that this happens for the following two reasons:

(a) $\mathcal{C}_{-\pi}$ will not contribute to our envelope since the pre-envelope curves are undefined at $\theta = -\pi$. However, to get the boundary of $\{\mathcal{D}_\theta\}_{\theta\in[-\pi,\pi)}$, where $\mathcal{D}_\theta$ denotes the circular disk with boundary curve $\mathcal{C}_\theta$, we need to include this circle.

(b) For other values of θ, the circles may be nested. Thus, the outermost circle is the only circle that contributes a point to the envelope curves.

3. Find $\frac{\partial f}{\partial \theta}(x, y, \theta)$. By assumption, $\frac{\partial f}{\partial \theta}(x, y, \theta) = 0$. We will then use $f(x, y, \theta) = 0$ and $\frac{\partial f}{\partial \theta}(x, y, \theta) = 0$ to find the values of s for which our pre-envelope curve is defined. We will specifically be substituting $x(s, \theta)$ for x and $y(s, \theta)$ for y in $\frac{\partial f}{\partial \theta} f(x, y, \theta) = 0$ to get these values of s. This is a valid substitution because $f(x, y, \theta) = 0$ and $\frac{\partial f}{\partial \theta} f(x, y, \theta) = 0$ are satisfied at the point of intersection of the envelope and $\mathcal{C}_\theta$. See [5] pp.172-173 for more details. We will then be solving for s as a function of θ. In the proof of Theorem 1, we will get two solutions and will denote them by $s_1(\theta)$ and $s_2(\theta)$. This indicates that we have two potential envelope curves.

4. Analyze $s_1(\theta)$ and $s_2(\theta)$ to determine the values of θ for which these two functions are defined. In the proof of Theorem 1, we will see that this depends on the value of parameter r_2 and will result in two separate cases. This will give us further information as to which of the circles in $\{C_\theta\}_{\theta\in[-\pi,\pi)}$ contribute a point to our envelope curves.

5. Plug $s_1(\theta)$ into $x(s, \theta)$ and $y(s, \theta)$ to find the parametric equation of the first pre-envelope curve. We denote this curve $E_1(\theta)$ and write $E_1(\theta) = (x_1(\theta), y_1(\theta))$. Similarly, plug $s_2(\theta)$ into $x(s, \theta)$ and $y(s, \theta)$ to find the parametric equation of the second pre-envelope curve. We denote this curve $E_2(\theta)$ and write $E_2(\theta) = (x_2(\theta), y_2(\theta))$. We will then check that $x_1(\theta)$, $x_2(\theta)$, $y_1(\theta)$, and $y_2(\theta)$ are continuously differentiable.

6. Show that our two pre-envelope curves are envelope curves by proving that $(\frac{dx_1}{d\theta})^2 + (\frac{dy_1}{d\theta})^2$ and $(\frac{dx_2}{d\theta})^2 + (\frac{dy_2}{d\theta})^2$ are nonzero. We will see in our proof of Theorem 1 that $E_2(\theta)$ does not always satisfy this condition and so, $E_2(\theta)$ is not always an envelope curve.

7. Construct the discriminant envelope using our envelope curve(s).

This concludes the envelope algorithm.

After applying the envelope algorithm in the proof of Theorem 1, we will use our discriminant envelope to prove that the boundary of $\{\mathcal{D}_\theta\}_{\theta\in[-\pi,\pi)}$ is contained in this envelope and either a finite number of circles or a finite number of points. In a recent article, Bickel, Gorkin, and Tran [4] use envelopes to construct the boundary of several different families of circles. They prove the following theorem:

Theorem B [Bickel, Gorkin, and Tran (2020)]

Let $x_c(t), y_c(t), r(t)$ be functions with $t \in [q_1, q_2]$ and define

$$F(x, y, t) = \big(x - x_c(t)\big)^2 + \big(y - y_c(t)\big)^2 - r(t)^2.$$

For each value of $t \in [q_1, q_2]$, let $\mathcal{C}_t$ be the circle defined by $F(x, y, t) = 0$ and let $\mathcal{D}_t$ be the open disk whose boundary is $\mathcal{C}_t$. Let $\mathcal{F}$ be the family of circles $\mathcal{C}_t$ and let $\Omega = \bigcup_{t \in [q_1, q_2]} \mathcal{D}_t$. Assume that $x_c(t), x_c'(t), x_c''(t), y_c(t), y_c'(t), y_c''(t), r(t), r'(t),$ and $r''(t)$ are continuous on $[q_1, q_2]$. Also assume that for $t \in (q_1, q_2)$,

$$r(t) > 0 \text{ and } r'(t)^2 < x_c'(t)^2 + y_c'(t)^2.$$

Then $\partial\Omega \subseteq \mathcal{E}_2 \cup \mathcal{C}_{q_1} \cup \mathcal{C}_{q_2} \subseteq \mathcal{E}_3 \cup \mathcal{C}_{q_1} \cup \mathcal{C}_{q_2}$.

To read the proof of this theorem, see [4] pg.8-9. Recall that $\mathcal{E}_2$ is the limiting-position envelope of $\mathcal{F}$ and $\mathcal{E}_3$ is the discriminant envelope of $\mathcal{F}$. We will only be computing the discriminant envelope and thus will only use the last containment: $\partial\Omega \subseteq \mathcal{E}_3 \cup \mathcal{C}_{q_1} \cup \mathcal{C}_{q_2}$. This will allow us to make a conclusion about the shape of the boundary of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$.

After finishing the proof of Theorem 1, we will use Bebiano and Spitkovsky's result to extend this result to the operator $T_{A_0+zA_1}$. We will then prove that in this special case, there is at least one flat portion on the boundary of the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$. Then using numerical evidence, we will conjecture as to the number of flat portions on the boundary of the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$, when $A_0, A_1 \in M_2(\mathbb{C})$. We will finish with various examples of flat portions on the boundary of the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ for $A_0, A_1 \in M_n(\mathbb{C})$ and $n > 2$.

3 Application of the Envelope Algorithm

We will now apply the envelope algorithm to a concrete example that is simpler than our ultimate application. Let $A_0 = 0$ and $A_1 = \begin{bmatrix} 2e^{i\frac{\pi}{4}} & 1 \\ 0 & 2e^{i\frac{\pi}{4}} \end{bmatrix}$:

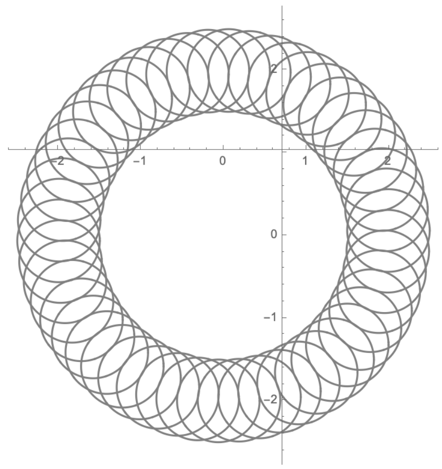

Figure 3: A finite number of boundary curves of the $W(A_0 + zA_1)$ in $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$. For each $z \in \partial\mathbb{D}$, $W(A_0 + zA_1)$ is the corresponding circular disk.

Before applying the envelope algorithm, we need to determine the numerical range of $A_0 + e^{i\theta}A_1$ for each $\theta \in [-\pi, \pi)$. We start by writing,

$$A = A_0 + e^{i\theta}A_1 = \begin{bmatrix} 2e^{i(\theta+\frac{\pi}{4})} & e^{i\theta} \\ 0 & 2e^{i(\theta+\frac{\pi}{4})} \end{bmatrix}$$

From here, we apply the Elliptical Range Theorem: Notice that $\frac{1}{2}\text{tr}A = 2e^{i(\theta+\frac{\pi}{4})}$ and so, for each value of θ, $W(A_0 + e^{i\theta}A_1)$ is an elliptical disk centered at $2e^{i(\theta+\frac{\pi}{4})}$. Next, define

$$A - \Big(\frac{1}{2}\text{tr}A\Big)I = B = \begin{bmatrix} 0 & e^{i\theta} \\ 0 & 0 \end{bmatrix}$$

Notice that $\text{tr}(B^*B) = 1$ and $\det(B) = 0$. So for each value of θ, $W(A_0 + e^{i\theta}A_1)$ is a circular disk with radius $\frac{1}{2}$.

Envelope algorithm:

Step 1: Finding our family of curves and setting up our equations.

For each value of $\theta \in [-\pi, \pi)$, $W(A_0 + e^{i\theta} A_1)$ is a circular disk centered at $2e^{i(\theta+\frac{\pi}{4})}$ with radius $\frac{1}{2}$. So for each $\theta \in [-\pi, \pi)$, let $\mathcal{C}_\theta$ denote the boundary circle of $W(A_0 + e^{i\theta} A_1)$ and let $\mathcal{D}_\theta$ denote the open disk with boundary $\mathcal{C}_\theta$. When $\mathcal{C}_\theta$ is a single point, we define $\mathcal{D}_\theta = \emptyset$. Fix $\theta \in [-\pi, \pi)$. Note that since $\mathcal{C}_\theta = \{(x, y) : f(x, y, \theta) = 0\}$, we can write,

$$f(x,y,\theta) = \Big(x - 2\cos\big(\theta + \frac{\pi}{4}\big)\Big)^2 + \Big(y - 2\sin\big(\theta + \frac{\pi}{4}\big)\Big)^2 - \frac{1}{4}$$

Recall that the envelope algorithm requires that $f(x, y, \theta) = 0$.

Parameterizing $\mathcal{C}_\theta$ with respect to s and θ gives,

$$\begin{aligned} x(s,\theta) &= 2\cos\big(\theta + \frac{\pi}{4}\big) + \frac{1}{2}\cos(s), \text{ and} \\ y(s,\theta) &= 2\sin(\theta + \frac{\pi}{4}) + \frac{1}{2}\sin(s), \text{ for } s \in [-\pi, \pi). \end{aligned} \tag{0.2}$$

Step 2: Checking that $(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$ is nonzero.

Observe that

$$\begin{aligned} \Big(\frac{\partial f}{\partial x}\Big)^2 + \Big(\frac{\partial f}{\partial y}\Big)^2 &= 4\Big[\Big(x - 2\cos\big(\theta + \frac{\pi}{4}\big)\Big)^2 + \big(y - 2\sin(\theta + \frac{\pi}{4})\big)^2\Big] \\ &= 1, \end{aligned}$$

where the second line comes from substituting $x(s, \theta)$ for x and $y(s, \theta)$ for y. Recall that $x(s, \theta)$ and $y(s, \theta)$ come from (0.2). Therefore, $(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 > 0$.

Step 3: Solving for $s(\theta)$ using $\frac{\partial f}{\partial \theta}(x, y, \theta) = 0$.

Taking the partial derivative of f with respect to θ gives,

$$\frac{\partial f}{\partial \theta}(x,y,\theta) = 4x\sin\big(\theta + \frac{\pi}{4}\big) - 4y\cos\big(\theta + \frac{\pi}{4}\big). \tag{0.3}$$

Recall that the envelope algorithm requires that we set (0.3) equal 0. After doing so, we can solve for s in terms of θ to determine the specific points that $\mathcal{C}_\theta$ contributes to the pre-envelope curves: First, we substitute $x(s, \theta)$ for x and $y(s, \theta)$ for y in (0.3), where $x(s, \theta)$ and $y(s, \theta)$ come from (0.2).

Then we solve for $s(\theta)$ to get the following solutions:

$$s_1(\theta) = \theta + \frac{\pi}{4} \text{ and } s_2(\theta) = \theta - \frac{3\pi}{4}.$$

Not only does this tell us that there are two pre-envelope curves, but it also tells us the specific points that $\mathcal{C}_\theta$ contributes to these two curves. We conclude that $\mathcal{C}_\theta$ contributes the point

$$\Big(2\cos\big(\theta + \frac{\pi}{4}\big) + \frac{1}{2}\cos\big(\theta + \frac{\pi}{4}\big), 2\sin\big(\theta + \frac{\pi}{4}\big) + \frac{1}{2}\sin\big(\theta + \frac{\pi}{4}\big)\Big),$$

to the first curve and contributes the point

$$\Big(2\cos\big(\theta + \frac{\pi}{4}\big) + \frac{1}{2}\cos\big(\theta - \frac{\pi}{4}\big), 2\sin\big(\theta + \frac{\pi}{4}\big) + \frac{1}{2}\sin\big(\theta - \frac{\pi}{4}\big)\Big),$$

to the second curve. To illustrate this in more detail, consider Figure 4.

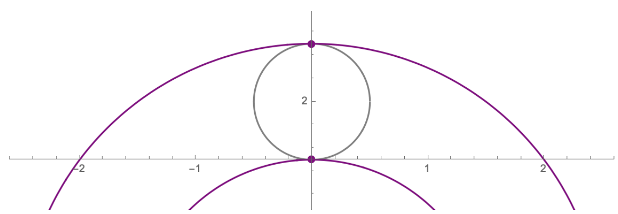

Figure 4: The two purple curves are the pre-envelope curves and the gray circle is $\mathcal{C}_{\frac{\pi}{4}}$. $\mathcal{C}_{\frac{\pi}{4}}$ contributes the point $(0, \frac{5}{2})$ to the top curve and contributes the point $(0, \frac{3}{2})$ to the bottom curve.

Step 4: Determine the values of θ for which $s(\theta)$ is defined.

Note that both $s_1(\theta)$ and $s_2(\theta)$ are defined on $[-\pi, \pi)$.

Step 5: Find the parametric equations for our pre-envelope curves.

Next, we need to obtain the parametric equation for our pre-envelope curves. Plugging $x(s, \theta)$ and $y(s, \theta)$ from (0.2) into $\big(x(s, \theta), y(s, \theta)\big)$ gives us the following curve,

$$E_1(\theta) = \Big(\frac{5}{2}\cos\big(\theta + \frac{\pi}{4}\big), \frac{5}{2}\sin\big(\theta + \frac{\pi}{4}\big)\Big), \text{ for } \theta \in [-\pi, \pi).$$

Similarly, plugging $x(s, \theta)$ and $y(s, \theta)$ from (0.2) into $\big(x(s, \theta), y(s, \theta)\big)$ gives us the following curve,

$$E_2(\theta) = \Big(\frac{3}{2}\cos\big(\theta + \frac{\pi}{4}\big), \frac{3}{2}\sin\big(\theta + \frac{\pi}{4}\big)\Big), \text{ for } \theta \in [-\pi, \pi).$$

From here we let $E_1 = \big(x_1(\theta), y_1(\theta)\big)$ and $E_2 = \big(x_2(\theta), y_2(\theta)\big)$. Thus,

$$\begin{aligned}
x_1(\theta) &= \frac{5}{2}\cos\big(\theta + \frac{\pi}{4}\big),\\
y_1(\theta) &= \frac{5}{2}\sin\big(\theta + \frac{\pi}{4}\big),\\
x_2(\theta) &= \frac{3}{2}\cos\big(\theta + \frac{\pi}{4}\big), \text{ and}\\
y_2(\theta) &= \frac{3}{2}\sin\big(\theta + \frac{\pi}{4}\big).
\end{aligned}$$

Notice that $x_1(\theta)$, $x_2(\theta)$, $y_1(\theta)$, and $y_2(\theta)$ are continuously differentiable.

Step 6: Verify that $(\frac{dx_1}{d\theta})^2 + (\frac{dy_1}{d\theta})^2$ and $(\frac{dx_1}{d\theta})^2 + (\frac{dy_1}{d\theta})^2$ are nonzero.

Notice that $\frac{dx_1}{d\theta} = -\frac{5}{2}\sin\big(\theta + \frac{\pi}{4}\big)$ and $\frac{dy_1}{d\theta} = \frac{5}{2}\cos\big(\theta + \frac{\pi}{4}\big)$. Thus,

$$\Big(\frac{dx_1}{d\theta}\Big)^2 + \Big(\frac{dy_1}{d\theta}\Big)^2 = \frac{25}{4}.$$

Similarly, notice that $\frac{dx_2}{d\theta} = -\frac{3}{2}\sin(\theta + \frac{\pi}{4})$ and $\frac{dy_2}{d\theta} = \frac{3}{2}\cos(\theta + \frac{\pi}{4})$. Thus,

$$\Big(\frac{dx_2}{d\theta}\Big)^2 + \Big(\frac{dy_2}{d\theta}\Big)^2 = \frac{9}{4}.$$

Step 7: Construct the discriminant envelope using our envelope curves.

We conclude that $E_1(\theta)$ and $E_2(\theta)$ form the discriminant envelope of our family.

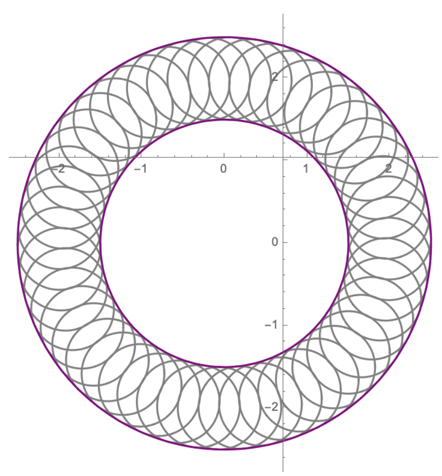

Figure 5: Figure 3 with the discriminant envelope added on.

4 Theorem 1

The goal of the following theorem is to characterize the boundary of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in a special case. The following theorem provides an equation for the discriminant envelope of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ when $A_0 = \begin{bmatrix} r_1 e^{i\theta_1} & ne^{i\gamma} \\ 0 & r_1 e^{i\theta_1} \end{bmatrix}$ and $A_1 = \begin{bmatrix} r_2 e^{i\theta_2} & ne^{i\gamma} \\ 0 & r_2 e^{i\theta_2} \end{bmatrix}$, for $n > 0$, $r_2 > \frac{n}{4}$, and $r_1 \geq 0$. In our proof, we will see that these matrices result in three different cases depending on the value of r_2.

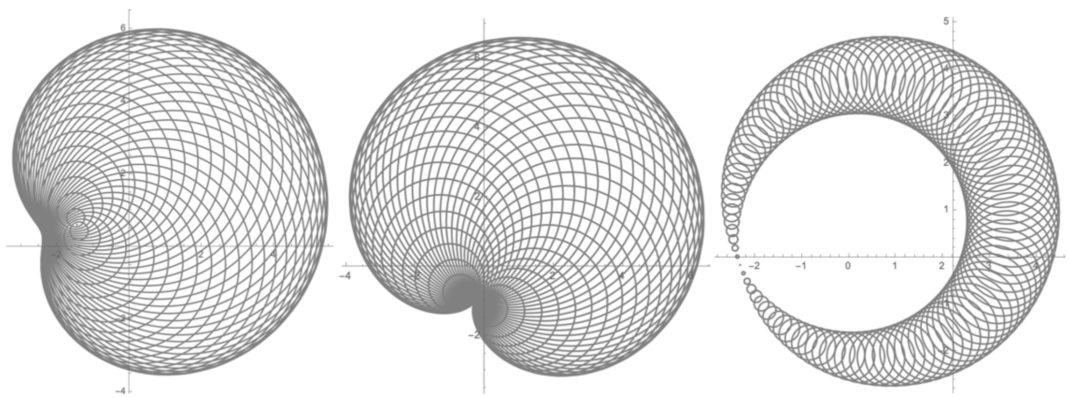

Figure 6: Left to right: $\frac{n}{4} < r_2 < \frac{n}{2}$, $r_2 = \frac{n}{2}$, and $r_2 > \frac{n}{2}$.

The leftmost plot comes from

$$A_0 = \begin{bmatrix} e^{i\frac{\pi}{2}} & 4 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} \frac{3}{2}e^{i\frac{\pi}{12}} & 4 \\ 0 & \frac{3}{2}e^{i\frac{\pi}{12}} \end{bmatrix}.$$

The middle plot comes from

$$A_0 = \begin{bmatrix} e^{i\frac{\pi}{4}} & 4 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} 2e^{i\frac{\pi}{3}} & 4 \\ 0 & 2e^{i\frac{\pi}{3}} \end{bmatrix}.$$

The rightmost plot comes from

$$A_0 = \begin{bmatrix} e^{i\frac{\pi}{3}} & 1 \\ 0 & e^{i\frac{\pi}{3}} \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} 3e^{i\frac{\pi}{9}} & 1 \\ 0 & 3e^{i\frac{\pi}{9}} \end{bmatrix}.$$

The conditions on A_0 and A_1 found in Theorem 1 were chosen so as to prove as general a case as possible while still maintaining this particular limaçon-like shape. Note as well that these conditions produce what looks to be a flat portion on the convex hull of $\{W(A_0+zA_1) : z \in \partial\mathbb{D}\}$. We will deal with the latter observation in Theorem 2. It is also important to note that Mathematica was used as an aid to assist with many of the computations found in this paper and was also used to create a majority of the plots found in this paper.

Theorem 1

Let $A_0 = \begin{bmatrix} r_1 e^{i\theta_1} & n e^{i\gamma} \\ 0 & r_1 e^{i\theta_1} \end{bmatrix}$ and $A_1 = \begin{bmatrix} r_2 e^{i\theta_2} & n e^{i\gamma} \\ 0 & r_2 e^{i\theta_2} \end{bmatrix}$ where $n > 0$, $r_2 > \frac{n}{4}$, and $r_1 \geq 0$ and let $E_1(\theta) = \big(x_1(\theta), y_1(\theta)\big)$ and $E_2(\theta) = \big(x_2(\theta), y_2(\theta)\big)$ where,

$$x_1(\theta) = r_1\cos\theta_1 + r_2\cos(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\cos\Big[\theta_2+\theta-\alpha(\theta)\Big];$$
$$y_1(\theta) = r_1\sin\theta_1 + r_2\sin(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\sin\Big[\theta_2+\theta-\alpha(\theta)\Big];$$
$$x_2(\theta) = r_1\cos\theta_1 + r_2\cos(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\cos\Big[\theta_2+\theta-\pi+\alpha(\theta)\Big];$$
$$y_2(\theta) = r_1\sin\theta_1 + r_2\sin(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\sin\Big[\theta_2+\theta-\pi+\alpha(\theta)\Big],$$

and $\alpha(\theta) = \sin^{-1}(\frac{-n\sin\theta}{2\sqrt{2}r_2\sqrt{1+\cos\theta}})$. Let $s_1 = \cos^{-1}\left(\frac{n^2-8r_2^2}{n^2}\right)$.

1. If $\frac{n}{4} < r_2 < \frac{n}{2}$, then the boundary of $\{W(A_0+zA_1) : z \in \partial\mathbb{D}\}$ is contained in

$$E_1(\theta) \cup E_2(\theta) \cup \mathcal{C}_{s_1} \cup \mathcal{C}_{-s_1}, \text{ for } \theta \in (-s_1, s_1),$$

where $\mathcal{C}_{s_1}$ is the circle centered at $r_1 e^{i\theta_1} + r_2 e^{i(\theta_2+s_1)}$ with radius $\sqrt{n^2-4r_2^2}$ and $\mathcal{C}_{-s_1}$ is the circle centered at $r_1 e^{i\theta_1} + r_2 e^{i(\theta_2-s_1)}$ with radius $\sqrt{n^2-4r_2^2}$;

2. If $r_2 = \frac{n}{2}$, then the boundary of $\{W(A_0+zA_1) : z \in \partial\mathbb{D}\}$ is contained in

$$E_1(\theta) \cup \left\{\left(r_1\cos\theta_1 - \frac{n}{2}\cos\theta_2, r_1\sin\theta_1 - \frac{n}{2}\sin\theta_2\right)\right\}, \text{ for } \theta \in (-\pi, \pi).$$

3. If $r_2 > \frac{n}{2}$, then the boundary of $\{W(A_0+zA_1) : z \in \partial\mathbb{D}\}$ is contained in

$$E_1(\theta) \cup E_2(\theta) \cup \Big\{\Big(r_1\cos\theta_1 - r_2\cos\theta_2, r_1\sin\theta_1 - r_2\sin\theta_2\Big)\Big\}, \text{ for } \theta \in (-\pi, \pi).$$

Proof. **Part 1: Finding Individual Numerical Ranges**

Assume the hypotheses. We will start by determining the numerical range of $A_0 + e^{i\theta}A_1$ for each $\theta \in [-\pi, \pi)$. To do this, we will apply the Elliptical Range Theorem: We can write

$$A_0 + e^{i\theta}A_1 = \begin{bmatrix} r_1e^{i\theta_1} + r_2e^{i(\theta_2+\theta)} & ne^{i\gamma} + ne^{i(\gamma+\theta)} \\ 0 & r_1e^{i\theta_1} + r_2e^{i(\theta_2+\theta)} \end{bmatrix}$$

Now notice that $\frac{1}{2}\text{tr}(A_0 + e^{i\theta}A_1) = r_1e^{i\theta_1} + r_2e^{i(\theta_2+\theta)}$ and so, for each value of θ, $W(A_0 + e^{i\theta}A_1)$ is an elliptical disk centered at $r_1e^{i\theta_1} + r_2e^{i(\theta_2+\theta)}$. Next, let $B = A_0 + e^{i\theta}A_1 - \Big(\frac{1}{2}\text{tr}(A_0 + e^{i\theta}A_1)\Big)I$. Then,

$$B = \begin{bmatrix} 0 & ne^{i\gamma} + ne^{i(\gamma+\theta)} \\ 0 & 0 \end{bmatrix}$$

A calculation shows that $\text{tr}(B^*B) = 2n^2 + 2n^2\cos\theta$ and $\det(B) = 0$. So for each value of θ, $W(A_0 + e^{i\theta}A_1)$ is a circular disk with radius $\frac{n\sqrt{2}}{2}\sqrt{1 + \cos\theta}$.

Part 2: The Envelope Algorithm

Step 1: Finding our family of curves and setting up our equations.

We showed that for each value of $\theta \in [-\pi, \pi)$, $W(A_0 + e^{i\theta}A_1)$ is a circular disk centered at $r_1e^{i\theta_1} + r_2e^{i(\theta_2+\theta)}$ with radius $\frac{n\sqrt{2}}{2}\sqrt{1 + \cos\theta}$. For each value of $\theta \in [-\pi, \pi)$, let $\mathcal{C}_\theta$ denote the boundary circle of $W(A_0 + e^{i\theta}A_1)$ and let $\mathcal{D}_\theta$ denote the open disk with boundary $\mathcal{C}_\theta$. When $\mathcal{C}_\theta$ is a single point, we define $\mathcal{D}_\theta = \emptyset$. Note then that since $\mathcal{C}_{-\pi}$ is a single point, $\mathcal{D}_{-\pi} = \emptyset$. Also note that $\mathcal{C}_{-\pi}$ is the only boundary circle that is a single point. Our goal is to find the discriminant envelope of the family $\{\mathcal{C}_\theta\}_{\theta\in[-\pi,\pi)}$:

Fix $\theta \in [-\pi, \pi)$. Note that $\mathcal{C}_\theta = \{(x, y) : f(x, y, \theta) = 0\}$ for

$$\begin{aligned} f(x,y,\theta) &= \Big(x - r_1\cos\theta_1 - r_2\cos(\theta_2+\theta)\Big)^2 + \Big(y - r_1\sin\theta_1 - r_2\sin(\theta_2+\theta)\Big)^2 - \frac{n^2}{2}(1+\cos\theta) \\ &= x^2 - 2x\Big(r_1\cos\theta_1 + r_2\cos(\theta_2+\theta)\Big) + 2r_1r_2\cos\theta_1\cos(\theta_2+\theta) + r_1^2 + r_2^2 + y^2 \\ &\quad - 2y\Big(r_1\sin\theta_1 + r_2\sin(\theta_2+\theta)\Big) + 2r_1r_2\sin\theta_1\sin(\theta_2+\theta) - \frac{n^2}{2}(1+\cos\theta) \end{aligned}$$

Parametrizing $\mathcal{C}_\theta$ with respect to s and θ gives,

$$\begin{aligned} x(s,\theta) &= r_1\cos\theta_1 + r_2\cos(\theta_2+\theta) + \frac{n\sqrt{2}\cos(s)}{2}\sqrt{1+\cos\theta}, \text{ and} \\ y(s,\theta) &= r_1\sin\theta_1 + r_2\sin(\theta_2+\theta) + \frac{n\sqrt{2}\sin(s)}{2}\sqrt{1+\cos\theta}, \text{ for } s\in[-\pi,\pi). \end{aligned} \tag{0.4}$$

Step 2: Checking that $(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$ is nonzero.

A calculation shows that

$$\begin{aligned} \Big(\frac{\partial f}{\partial x}\Big)^2 + \Big(\frac{\partial f}{\partial y}\Big)^2 &= 4\Big[\Big(x - r_1\cos\theta_1 - r_2\cos(\theta_2+\theta)\Big)^2 + \Big(y - r_1\sin\theta_1 - r_2\sin(\theta_2+\theta)\Big)^2\Big] \\ &= 2n^2(1+\cos\theta), \end{aligned}$$

where the last equality comes from plugging in $x(s,\theta)$ and $y(s,\theta)$ for x and y, respectively. Recall that $x(s,\theta)$ and $y(s,\theta)$ come from (0.4). Therefore,

$$\begin{aligned} &\Big(\frac{\partial f}{\partial x}\Big)^2 + \Big(\frac{\partial f}{\partial y}\Big)^2 > 0 \text{ when } \theta\in(-\pi,\pi) \text{ and,} \\ &\Big(\frac{\partial f}{\partial x}\Big)^2 + \Big(\frac{\partial f}{\partial y}\Big)^2 = 0 \text{ when } \theta = -\pi. \end{aligned}$$

This indicates that the circle $\mathcal{C}_{-\pi}$ does not contribute a point to our pre-envelope curve. From here, we will only be interested in finding envelope curves defined for $\theta\in(-\pi,\pi)$. Thus, $(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2 > 0$ on our domain.

Step 3: Solving for $s(\theta)$ using $\frac{\partial f}{\partial\theta}(x,y,\theta)=0$.

Taking the partial derivative of f with respect to θ gives

$$\frac{\partial f}{\partial\theta}(x,y,\theta) = 2r_2\sin(\theta_2+\theta)\Big(x - r_1\cos\theta_1\Big) - 2r_2\cos(\theta_2+\theta)\Big(y - r_1\sin\theta_1\Big) + \frac{n^2}{2}\sin\theta \tag{0.5}$$

Recall that the envelope algorithm requires setting (0.5) equal to 0. After doing so, plugging $x(s,\theta)$ for x and $y(s,\theta)$ for y into (0.5) gives

$$\begin{aligned} 0 = 2r_2\sin(\theta_2+\theta)&\Big(r_2\cos(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\cos(s)\Big) \\ - 2r_2\cos(\theta_2+\theta)&\Big(r_2\sin(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\sin(s)\Big) + \frac{n^2}{2}\sin\theta. \end{aligned}$$

This can be rewritten as

$$\sin(s - (\theta_2 + \theta)) = \frac{n \sin\theta}{2\sqrt{2}r_2\sqrt{1+\cos\theta}}.$$

Applying arcsine to both sides gives us

$$s - (\theta_2 + \theta) = \sin^{-1}\left(\frac{n \sin\theta}{2\sqrt{2}r_2\sqrt{1+\cos\theta}}\right) \text{ or } s - (\theta_2 + \theta) = \pi - \sin^{-1}\left(\frac{n \sin\theta}{2\sqrt{2}r_2\sqrt{1+\cos\theta}}\right).$$

Thus, we have the following two solutions:

$$s_1(\theta) = \theta_2 + \theta - \sin^{-1}\left(\frac{-n \sin\theta}{2\sqrt{2}r_2\sqrt{1+\cos\theta}}\right) \text{ and } s_2(\theta) = \theta_2 + \theta - \pi + \sin^{-1}\left(\frac{-n \sin\theta}{2\sqrt{2}r_2\sqrt{1+\cos\theta}}\right).$$

Due to the domain of arcsine, we do not yet know where these two equations are defined.

Since we have two solutions, $s_1(\theta)$ and $s_2(\theta)$, we have two pre-envelope curves. For each value of θ, $s_1(\theta)$ and $s_2(\theta)$ allow us to determine the particular points that $\mathcal{C}_\theta$ contributes to these two pre-envelope curves.

Step 4: Determine the values of θ for which $s_1(\theta)$ and $s_2(\theta)$ are defined.

First note that both $s_1(\theta)$ and $s_2(\theta)$ are undefined when $\theta = \pm\pi$, which have already been excluded from our domain. Recall that arcsine is continuously differentiable on the interior of its domain, $[-1, 1]$. Since the domain of arcsine is $[-1, 1]$, we require that

$$\frac{-n \sin\theta}{2\sqrt{2}r_2\sqrt{1+\cos\theta}} \in (-1, 1). \tag{0.6}$$

From here, we make the following claim:

Claim 1

Let $n > 0$ and $r_2 > \frac{n}{4}$. Let

$$\tau_{n,r_2} = \begin{cases} \cos^{-1}\left(\frac{n^2 - 8r_2^2}{n^2}\right), & \frac{n}{4} < r_2 < \frac{n}{2} \\ \pi, & r_2 \geq \frac{n}{2} \end{cases}.$$

For all $\theta \in (-\tau_{n,r_2}, \tau_{n,r_2})$,

$$\frac{-n \sin\theta}{2\sqrt{2} r_2 \sqrt{1 + \cos\theta}} \in (-1, 1).$$

Proof. Assume the hypotheses. Let $\theta \in (-\tau_{n,r_2}, \tau_{n,r_2})$. Using double angle identities we obtain

$$\begin{aligned} \frac{\sin\theta}{\sqrt{1+\cos\theta}} &= \frac{2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})}{\sqrt{2}\cos(\frac{\theta}{2})} \\ &= \sqrt{2}\sin\left(\frac{\theta}{2}\right), \end{aligned}$$

and so,

$$\frac{-n \sin\theta}{2\sqrt{2} r_2 \sqrt{1 + \cos\theta}} = -\frac{n}{2r_2}\sin\left(\frac{\theta}{2}\right).$$

Note that $-\frac{n}{2r_2}\sin\left(\frac{\theta}{2}\right) \in (-1, 1)$ when $\sin\left(\frac{\theta}{2}\right) \in \left(-\frac{2r_2}{n}, \frac{2r_2}{n}\right)$. Thus, when $\frac{r_2}{n} \geq \frac{1}{2}$, this holds for all $\theta \in (-\pi, \pi)$. When $\frac{r_2}{n} \in (\frac{1}{4}, \frac{1}{2})$,

$$\sin\left(\frac{\theta}{2}\right) \in \left(-\frac{2r_2}{n}, \frac{2r_2}{n}\right) \text{ for } \frac{\theta}{2} \in \left(-\sin^{-1}\left(\frac{2r_2}{n}\right), \sin^{-1}\left(\frac{2r_2}{n}\right)\right).$$

From here, let $t = \sin^{-1}(\frac{2r_2}{n})$. Notice that we can write

$$\begin{aligned} \cos 2t &= \cos^2 t - \sin^2 t \\ &= \frac{n^2 - 8r_2^2}{n^2}. \end{aligned}$$

Therefore, $2t = \cos^{-1}\left(\frac{n^2-8r_2^2}{n^2}\right)$, allowing us to conclude that $2\sin^{-1}(\frac{2r_2}{n}) = \cos^{-1}\left(\frac{n^2-8r_2^2}{n^2}\right)$. Thus when $\frac{r_2}{n} \in (\frac{1}{4}, \frac{1}{2})$,

$$\sin\left(\frac{\theta}{2}\right) \in \left(-\frac{2r_2}{n}, \frac{2r_2}{n}\right) \text{ for } \theta \in \left(-\cos^{-1}\left(\frac{n^2 - 8r_2^2}{n^2}\right), \cos^{-1}\left(\frac{n^2 - 8r_2^2}{n^2}\right)\right).$$

We conclude that for all $\theta \in (-\tau_{n,r_2}, \tau_{n,r_2})$, $\frac{-n\sin\theta}{2\sqrt{2}r_2\sqrt{1+\cos\theta}} \in (-1,1)$. $\square$

By Claim 1, we conclude that

- When $\frac{n}{4} < r_2 < \frac{n}{2}$, $s_1(\theta)$ and $s_2(\theta)$ are defined for $\theta \in \Big(-\cos^{-1}\big(\frac{n^2-8r_2^2}{n^2}\big), \cos^{-1}\big(\frac{n^2-8r_2^2}{n^2}\big)\Big)$.
- When $r_2 \geq \frac{n}{2}$, $s_1(\theta)$ and $s_2(\theta)$ are defined for $\theta \in (-\pi, \pi)$.

Step 5: Find the parametric equations for our pre-envelope curves.

In Step 4, we found the values of θ for which $s_1(\theta)$ and $s_2(\theta)$ are defined. Note that our pre-envelope curves will depend on $s_1(\theta)$ and $s_2(\theta)$. As a result, knowing where $s_1(\theta)$ and $s_2(\theta)$ are defined will help us determine where our pre-envelope curves are defined. We will now obtain the parametric equation for these two curves. Plugging $s_1(\theta)$ into $\big(x(s,\theta), y(s,\theta)\big)$ from (0.4) gives us the following curve,

$$E_1(\theta) = \Bigg(r_1\cos\theta_1 + r_2\cos(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\cos[\theta_2+\theta-\alpha(\theta)],$$
$$r_1\sin\theta_1 + r_2\sin(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\sin[\theta_2+\theta-\alpha(\theta)]\Bigg),$$

where $\alpha(\theta) = \sin^{-1}(\frac{-n\sin\theta}{2\sqrt{2}r_2\sqrt{1+\cos\theta}})$. Plugging $s_2(\theta)$ into $\big(x(s,\theta), y(s,\theta)\big)$ from (0.4) gives us the following curve,

$$E_2(\theta) = \Bigg(r_1\cos\theta_1 + r_2\cos(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\cos[\theta_2+\theta-\pi+\alpha(\theta)],$$
$$r_1\sin\theta_1 + r_2\sin(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\sin[\theta_2+\theta-\pi+\alpha(\theta)]\Bigg),$$

where $\alpha(\theta) = \sin^{-1}(\frac{-n\sin\theta}{2\sqrt{2}r_2\sqrt{1+\cos\theta}})$. Note that the domain of $E_1(\theta)$ depends on the domain of $s_1(\theta)$ and the domain of $E_2(\theta)$ depends on the domain of $s_2(\theta)$. We will calculate the exact domain of these two curves in Step 7. Letting $E_1(\theta) = \big(x_1(\theta), y_1(\theta)\big)$ and $E_2(\theta) = \big(x_2(\theta), y_2(\theta)\big)$ gives

$$x_1\big(\theta\big) = r_1\cos\theta_1 + r_2\cos(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\cos[\theta_2+\theta-\alpha(\theta)], \tag{0.7}$$
$$x_2\big(\theta\big) = r_1\cos\theta_1 + r_2\cos(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\cos[\theta_2+\theta-\pi+\alpha(\theta)], \tag{0.8}$$
$$y_1\big(\theta\big) = r_1\sin\theta_1 + r_2\sin(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\sin[\theta_2+\theta-\alpha(\theta)], \tag{0.9}$$
$$y_2\big(\theta\big) = r_1\sin\theta_1 + r_2\sin(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\sin[\theta_2+\theta-\pi+\alpha(\theta)], \tag{0.10}$$

where $\alpha(\theta) = \sin^{-1}(\frac{-n\sin\theta}{2\sqrt{2}r_2\sqrt{1+\cos\theta}})$.

Furthermore, notice that we can use $s_1(\theta)$ and $s_2(\theta)$ to determine where $x_1(\theta)$, $x_2(\theta)$, $y_1(\theta)$, and $y_2(\theta)$ are continuously differentiable. Note that when $\frac{n}{4} < r_2 < \frac{n}{2}$, $s_1(\theta)$ and $s_2(\theta)$ are continuously differentiable for $\theta \in \Big(-\cos^{-1}\big(\frac{n^2-8r_2^2}{n^2}\big), \cos^{-1}\big(\frac{n^2-8r_2^2}{n^2}\big)\Big)$ by Step 4. When $r_2 \geq \frac{n}{2}$, $s_1(\theta)$ and $s_2(\theta)$ are continuously differentiable for $\theta \in (-\pi, \pi)$ by Step 4. Additionally, the other factors and terms in $x_1(\theta), x_2(\theta), y_1(\theta)$ and $y_2(\theta)$ are continuously differentiable on $(-\pi, \pi)$. As a result, we conclude that

- When $\frac{n}{4} < r_2 < \frac{n}{2}$, $x_1(\theta)$, $x_2(\theta)$, $y_1(\theta)$, and $y_2(\theta)$ are continuously differentiable for $\theta \in \Big(-\cos^{-1}\big(\frac{n^2-8r_2^2}{n^2}\big), \cos^{-1}\big(\frac{n^2-8r_2^2}{n^2}\big)\Big)$.
- When $r_2 \geq \frac{n}{2}$, $x_1(\theta)$, $x_2(\theta)$, $y_1(\theta)$, and $y_2(\theta)$ are continuously differentiable for $\theta \in (-\pi, \pi)$.

Step 6: Show that $(\frac{dx_1}{d\theta})^2 + (\frac{dy_1}{d\theta})^2$ and $(\frac{dx_2}{d\theta})^2 + (\frac{dy_2}{d\theta})^2$ are nonzero.

From here we wish to verify that $\big(\frac{dx_1}{d\theta}\big)^2 + \big(\frac{dy_1}{d\theta}\big)^2 \neq 0$ and $\big(\frac{dx_2}{d\theta}\big)^2 + \big(\frac{dy_2}{d\theta}\big)^2 \neq 0$. These are the conditions that turn our pre-envelope curves into envelope curves. A lengthy calculation heavily aided by Mathematica shows that

$$\begin{aligned}\Big(\frac{dx_1}{d\theta}\Big)^2 + \Big(\frac{dy_1}{d\theta}\Big)^2 &= \frac{n^4\cos(2\theta)}{2n^2(\cos\theta-1)+16r_2^2} + \frac{2r_2n^3\cos\theta\sqrt{1+\cos\theta}\sqrt{8+\frac{n^2(\cos\theta-1)}{r_2^2}}}{2n^2(\cos\theta-1)+16r_2^2} \\ &+ \frac{8r_2^2n^2(1+2\cos\theta)}{2n^2(\cos\theta-1)+16r_2^2} + \frac{8r_2^3n\sqrt{1+\cos\theta}\sqrt{8+\frac{n^2(\cos\theta-1)}{r_2^2}}}{2n^2(\cos\theta-1)+16r_2^2} \\ &+ \frac{16r_2^4}{2n^2(\cos\theta-1)+16r_2^2}, \text{ and,}\end{aligned} \tag{0.11}$$

$$\begin{aligned}\Big(\frac{dx_2}{d\theta}\Big)^2 + \Big(\frac{dy_2}{d\theta}\Big)^2 &= \frac{n^4\cos(2\theta)}{2n^2(\cos\theta-1)+16r_2^2} - \frac{2r_2n^3\cos\theta\sqrt{1+\cos\theta}\sqrt{8+\frac{n^2(\cos\theta-1)}{r_2^2}}}{2n^2(\cos\theta-1)+16r_2^2} \\ &+ \frac{8r_2^2n^2(1+2\cos\theta)}{2n^2(\cos\theta-1)+16r_2^2} - \frac{8r_2^3n\sqrt{1+\cos\theta}\sqrt{8+\frac{n^2(\cos\theta-1)}{r_2^2}}}{2n^2(\cos\theta-1)+16r_2^2} \\ &+ \frac{16r_2^4}{2n^2(\cos\theta-1)+16r_2^2}.\end{aligned} \tag{0.12}$$

From here, we have the following claim:

Claim 2

Let $n > 0$ and $r_2 > \frac{n}{4}$. Let

$$\omega_{n,r_2} = \begin{cases} \cos^{-1}\left(\frac{n^2-8r_2^2}{n^2}\right), & \frac{n}{4} < r_2 < \frac{n}{2} \\ \pi, & r_2 > \frac{n}{2} \end{cases}.$$

For all $\theta \in (-\omega_{n,r_2}, \omega_{n,r_2})$,

$$\left(\frac{dx_1}{d\theta}\right)^2 + \left(\frac{dy_1}{d\theta}\right)^2 > 0 \text{ and } \left(\frac{dx_2}{d\theta}\right)^2 + \left(\frac{dy_2}{d\theta}\right)^2 > 0.$$

Note that τ_{n,r_2} from Claim 1 and ω_{n,r_2} from Claim 2 are not the same function.

Proof. Assume the hypotheses. We can rewrite (0.11) and (0.12) as

$$\begin{aligned}\left(\frac{dx_j}{d\theta}\right)^2 + \left(\frac{dy_j}{d\theta}\right)^2 &= \frac{n^4\cos(2\theta)}{2n^2(\cos\theta-1)+16r_2^2} + (-1)^{j-1}\left(\frac{2r_2n^3\cos\theta\sqrt{1+\cos\theta}\sqrt{8+\frac{n^2(\cos\theta-1)}{r_2^2}}}{2n^2(\cos\theta-1)+16r_2^2}\right) \\ &+ \frac{8r_2^2n^2(1+2\cos\theta)}{2n^2(\cos\theta-1)+16r_2^2} + (-1)^{j-1}\left(\frac{8r_2^3n\sqrt{1+\cos\theta}\sqrt{8+\frac{n^2(\cos\theta-1)}{r_2^2}}}{2n^2(\cos\theta-1)+16r_2^2}\right) \\ &+ \frac{16r_2^4}{2n^2(\cos\theta-1)+16r_2^2},\end{aligned}$$

for $j = 1, 2$. Letting $t = \frac{r_2}{n}$ allows us to rewrite this as

$$\begin{aligned}\left(\frac{dx_j}{d\theta}\right)^2 + \left(\frac{dy_j}{d\theta}\right)^2 &= \frac{n^4\cos(2\theta)}{2n^2(8t^2+\cos\theta-1)} + (-1)^{j-1}\left(\frac{2n^4\cos\theta\sqrt{1+\cos\theta}\sqrt{8t^2+\cos\theta-1}}{2n^2(8t^2+\cos\theta-1)}\right) \\ &+ \frac{8t^2n^4(1+2\cos\theta)}{2n^2(8t^2+\cos\theta-1)} + (-1)^{j-1}\left(\frac{8t^2n^4\sqrt{1+\cos\theta}\sqrt{8t^2+\cos\theta-1}}{2n^2(8t^2+\cos\theta-1)}\right) \\ &+ \frac{16t^4n^4}{2n^2(8t^2+\cos\theta-1)},\end{aligned}$$

for $j = 1, 2$. Grouping similar terms allows us to write

$$\begin{aligned}\left(\frac{dx_j}{d\theta}\right)^2 + \left(\frac{dy_j}{d\theta}\right)^2 &= \frac{n^2\Big(\cos^2\theta + 8t^2\cos\theta + (8t^4+4t^2-\frac{1}{2})\Big)}{8t^2+\cos\theta-1} \\ &+ (-1)^{j-1}\left(\frac{n^2\sqrt{1+\cos\theta}\sqrt{8t^2+\cos\theta-1}\Big(\cos\theta+4t^2\Big)}{8t^2+\cos\theta-1}\right),\end{aligned}$$

for $j = 1, 2$. Factoring the trigonometric polynomial in the numerator of the first term gives

$$\left(\frac{dx_j}{d\theta}\right)^2 + \left(\frac{dy_j}{d\theta}\right)^2 = \frac{n^2\Big(\cos\theta - t^2(2\sqrt{2}-4) + \frac{\sqrt{2}}{2}\Big)\Big(\cos\theta + t^2(2\sqrt{2}+4) - \frac{\sqrt{2}}{2}\Big)}{8t^2 + \cos\theta - 1}$$
$$+ (-1)^{j-1}\left(\frac{n^2\sqrt{1+\cos\theta}\sqrt{8t^2+\cos\theta-1}\Big(\cos\theta + 4t^2\Big)}{8t^2+\cos\theta-1}\right),$$

for $j = 1, 2$. Therefore, (0.11) becomes

$$\begin{aligned}\left(\frac{dx_1}{d\theta}\right)^2 + \left(\frac{dy_1}{d\theta}\right)^2 &= \frac{n^2\Big(\cos\theta - t^2(2\sqrt{2}-4) + \frac{\sqrt{2}}{2}\Big)\Big(\cos\theta + t^2(2\sqrt{2}+4) - \frac{\sqrt{2}}{2}\Big)}{8t^2 + \cos\theta - 1} \\ &\quad + \frac{n^2\sqrt{1+\cos\theta}\sqrt{8t^2+\cos\theta-1}\Big(\cos\theta + 4t^2\Big)}{8t^2+\cos\theta-1},\end{aligned} \tag{0.13}$$

and (0.12) becomes

$$\begin{aligned}\left(\frac{dx_2}{d\theta}\right)^2 + \left(\frac{dy_2}{d\theta}\right)^2 &= \frac{n^2\Big(\cos\theta - t^2(2\sqrt{2}-4) + \frac{\sqrt{2}}{2}\Big)\Big(\cos\theta + t^2(2\sqrt{2}+4) - \frac{\sqrt{2}}{2}\Big)}{8t^2 + \cos\theta - 1} \\ &\quad - \frac{n^2\sqrt{1+\cos\theta}\sqrt{8t^2+\cos\theta-1}\Big(\cos\theta + 4t^2\Big)}{8t^2+\cos\theta-1}.\end{aligned} \tag{0.14}$$

To prove that (0.13) and (0.14) are greater than 0, we will rely on the fact that cosine is an even function. We start by considering the denominator:

Subclaim 1

For $\theta \in (-\omega_{n,r_2}, \omega_{n,r_2})$, $8t^2 + \cos\theta - 1 > 0$.

Proof. Notice that $8t^2 + \cos\theta - 1 > 0$ whenever $\cos\theta > 1 - 8t^2$. When $t > \frac{1}{2}$, we have that $1 - 8t^2 < -1$ and thus,

$$\cos\theta > 1 - 8t^2 \text{ for } \theta \in (-\pi, \pi).$$

When $t \in (\frac{1}{4}, \frac{1}{2})$, we have that $-1 < 1 - 8t^2 < \frac{1}{2}$ and thus,

$$\cos\theta > 1 - 8t^2 \text{ for } \theta \in \Big(-\cos^{-1}(1-8t^2), \cos^{-1}(1-8t^2)\Big).$$

From here, note that $\cos^{-1}(1-8t^2) = \cos^{-1}\left(\frac{n^2-8r_2^2}{n^2}\right)$ for $t \in (\frac{1}{4}, \frac{1}{2})$. Putting this all together allows us to conclude that $8t^2 + \cos\theta - 1 > 0$ for $\theta \in (-\omega_{n,r_2}, \omega_{n,r_2})$. ■

Next, we will consider the second term found in both (0.13) and (0.14):

Subclaim 2

For $\theta \in (-\omega_{n,r_2}, \omega_{n,r_2})$,

$$\frac{n^2\sqrt{1+\cos\theta}\sqrt{8t^2+\cos\theta-1}\big(\cos\theta+4t^2\big)}{8t^2+\cos\theta-1} > 0.$$

Proof. The sign of $n^2\sqrt{1+\cos\theta}\sqrt{8t^2+\cos\theta-1}\big(\cos\theta+4t^2\big)$ depends on the sign of $\cos\theta+4t^2$. When $t > \frac{1}{2}$, $\cos\theta + 4t^2 > 0$ for all $\theta \in (-\pi, \pi)$. Therefore, when $t > \frac{1}{2}$,

$$\frac{n^2\sqrt{1+\cos\theta}\sqrt{8t^2+\cos\theta-1}\big(\cos\theta+4t^2\big)}{8t^2+\cos\theta-1} > 0, \text{ for all } \theta \in (-\pi,\pi).$$

When $t \in (\frac{1}{4}, \frac{1}{2})$, $\frac{1}{4} < 4t^2 < 1$ and so,

$$\cos\theta + 4t^2 > 0 \text{ for } \theta \in \Big(-\cos^{-1}(-4t^2), \cos^{-1}(-4t^2)\Big).$$

From here, we will use the fact that $\cos^{-1}(-4t^2) = \cos^{-1}(\frac{-4r_2^2}{n^2})$. When $t \in (\frac{1}{4}, \frac{1}{2})$, $1 - 8t^2 > -4t^2$ and so,

$$\Big(-\cos^{-1}\Big(\frac{n^2-8r_2^2}{n^2}\Big), \cos^{-1}\Big(\frac{n^2-8r_2^2}{n^2}\Big)\Big) \subseteq \Big(-\cos^{-1}\Big(\frac{-4r_2^2}{n^2}\Big), \cos^{-1}\Big(\frac{-4r_2^2}{n^2}\Big)\Big).$$

Therefore, when $t \in (\frac{1}{4}, \frac{1}{2})$,

$$\frac{n^2\sqrt{1+\cos\theta}\sqrt{8t^2+\cos\theta-1}(\cos\theta+4t^2)}{8t^2+\cos\theta-1} > 0 \text{ for } \theta \in \Big(-\cos^{-1}\Big(\frac{n^2-8r_2^2}{n^2}\Big), \cos^{-1}\Big(\frac{n^2-8r_2^2}{n^2}\Big)\Big).$$

Putting this all together allows us to conclude that for $\theta \in (-\omega_{n,r_2}, \omega_{n,r_2})$,

$$\frac{n^2\sqrt{1+\cos\theta}\sqrt{8t^2+\cos\theta-1}\big(\cos\theta+4t^2\big)}{8t^2+\cos\theta-1} > 0.$$

■

Next, we will consider the first term found in both (0.13) and (0.14):

Subclaim 3

For $\theta \in (-\omega_{n,r_2}, \omega_{n,r_2})$,

$$\frac{n^2(\cos\theta - t^2(2\sqrt{2}-4) + \frac{\sqrt{2}}{2})(\cos\theta + t^2(2\sqrt{2}+4) - \frac{\sqrt{2}}{2})}{8t^2 + \cos\theta - 1} > 0.$$

Proof. To start, we will consider the values of θ for which $\cos\theta - t^2(2\sqrt{2}-4) + \frac{\sqrt{2}}{2} > 0$. Since $2\sqrt{2} - 4 < -1$, $t^2(2\sqrt{2}-4) - \frac{\sqrt{2}}{2} < -\frac{\sqrt{2}}{2}$. Additionally, we have that

$$\begin{aligned} &t^2(2\sqrt{2}-4) - \frac{\sqrt{2}}{2} \in (-1, 0) \text{ for all } t \in \left(\frac{1}{4}, \frac{1}{2}\right) \text{ and,} \\ &t^2(2\sqrt{2}-4) - \frac{\sqrt{2}}{2} < -1 \text{ for all } t > \frac{1}{2}. \end{aligned}$$

Thus, when $t > \frac{1}{2}$,

$$\cos\theta > t^2(2\sqrt{2}-4) - \frac{\sqrt{2}}{2} \text{ for } \theta \in (-\pi, \pi).$$

When $t \in (\frac{1}{4}, \frac{1}{2})$,

$$\cos\theta > t^2(2\sqrt{2}-4) - \frac{\sqrt{2}}{2} \text{ for } \theta \in \Big(-\cos^{-1}\Big(t^2(2\sqrt{2}-4) - \frac{\sqrt{2}}{2}\Big), \cos^{-1}\Big(t^2(2\sqrt{2}-4) - \frac{\sqrt{2}}{2}\Big)\Big).$$

For $t \in (\frac{1}{4}, \frac{1}{2})$, $1 - 8t^2 > t^2(2\sqrt{2}-4) - \frac{\sqrt{2}}{2}$ and so,

$$\begin{aligned} &\Big(-\cos^{-1}\Big(\frac{n^2 - 8r_2^2}{n^2}\Big), \cos^{-1}\Big(\frac{n^2 - 8r_2^2}{n^2}\Big)\Big) \subseteq \\ &\qquad \Big(-\cos^{-1}\Big(\frac{r_2^2(4\sqrt{2}-8) - \sqrt{2}n^2}{2n^2}\Big), \cos^{-1}\Big(\frac{r_2^2(4\sqrt{2}-8) - \sqrt{2}n^2}{2n^2}\Big)\Big). \end{aligned}$$

Therefore, when $t \in (\frac{1}{4}, \frac{1}{2})$,

$$\cos\theta - t^2(2\sqrt{2}-4) + \frac{\sqrt{2}}{2} > 0 \text{ for } \theta \in \Big(-\cos^{-1}\Big(\frac{n^2 - 8r_2^2}{n^2}\Big), \cos^{-1}\Big(\frac{n^2 - 8r_2^2}{n^2}\Big)\Big).$$

Putting this all together allows us to conclude that for $\theta \in (-\omega_{n,r_2}, \omega_{n,r_2})$,

$$\frac{n^2\Big(\cos\theta - t^2(2\sqrt{2}-4) + \frac{\sqrt{2}}{2}\Big)}{8t^2 + \cos\theta - 1} > 0.$$

Our next step is to consider the values of θ for which $\cos\theta + t^2(2\sqrt{2}+4) - \frac{\sqrt{2}}{2} > 0$. Notice that $2\sqrt{2}+4 > 6$. Additionally, we have that

$$\begin{aligned}
&-t^2(2\sqrt{2}+4) + \frac{\sqrt{2}}{2} \in \Big(-1, \frac{1}{2}\Big) \text{ for } t \in \Big(\frac{1}{4}, \frac{1}{2}\Big) \text{ and,}\\
&-t^2(2\sqrt{2}+4) + \frac{\sqrt{2}}{2} < -1 \text{ for all } t > \frac{1}{2}
\end{aligned}$$

Thus, when $t > \frac{1}{2}$,

$$\cos\theta > -t^2(2\sqrt{2}+4) + \frac{\sqrt{2}}{2} \text{ for } \theta \in (-\pi, \pi).$$

When $t \in (\frac{1}{4}, \frac{1}{2})$,

$$\begin{aligned}
&\cos\theta > -t^2(2\sqrt{2}+4) + \frac{\sqrt{2}}{2}\\
&\qquad\qquad \text{for } \theta \in \Big(-\cos^{-1}\Big(\frac{\sqrt{2}}{2} - t^2(2\sqrt{2}+4)\Big), \cos^{-1}\Big(\frac{\sqrt{2}}{2} - t^2(2\sqrt{2}+4)\Big)\Big).
\end{aligned}$$

For $t \in (\frac{1}{4}, \frac{1}{2})$, $1 - 8t^2 > -t^2(2\sqrt{2}+4) + \frac{\sqrt{2}}{2}$ and so,

$$\begin{aligned}
&\Big(-\cos^{-1}\Big(\frac{n^2 - 8r_2^2}{n^2}\Big), \cos^{-1}\Big(\frac{n^2 - 8r_2^2}{n^2}\Big)\Big) \subseteq\\
&\qquad \Big(-\cos^{-1}\Big(\frac{-r_2^2(4\sqrt{2}+8) + \sqrt{2}n^2}{n^2}\Big), \cos^{-1}\Big(\frac{-r_2^2(4\sqrt{2}+8) + \sqrt{2}n^2}{n^2}\Big)\Big).
\end{aligned}$$

Therefore, when $t \in (\frac{1}{4}, \frac{1}{2})$,

$$\cos\theta + t^2(2\sqrt{2}+4) - \frac{\sqrt{2}}{2} > 0 \text{ for all } \theta \in \Big(-\cos^{-1}\Big(\frac{n^2 - 8r_2^2}{n^2}\Big), \cos^{-1}\Big(\frac{n^2 - 8r_2^2}{n^2}\Big)\Big).$$

Putting this all together allows us to conclude that for $\theta \in (-\omega_{n,r_2}, \omega_{n,r_2})$,

$$\frac{n^2(\cos\theta - t^2(2\sqrt{2}-4) + \frac{\sqrt{2}}{2})(\cos\theta + t^2(2\sqrt{2}+4) - \frac{\sqrt{2}}{2})}{8t^2 + \cos\theta - 1} > 0.$$

■

By Subclaims 1, 2, and 3, we conclude that for $\theta \in (-\omega_{n,r_2}, \omega_{n,r_2})$, $\left(\frac{dx_1}{d\theta}\right)^2 + \left(\frac{dy_1}{d\theta}\right)^2 > 0$.

From here we need to show that (0.14) is greater than zero. To do this, we must prove the following:

Subclaim 4

For $\theta \in (-\omega_{n,r_2}, \omega_{n,r_2})$,

$$\frac{n^2\big(\cos\theta - t^2(2\sqrt{2}-4) + \frac{\sqrt{2}}{2}\big)\big(\cos\theta + t^2(2\sqrt{2}+4) - \frac{\sqrt{2}}{2}\big)}{8t^2 + \cos\theta - 1} > \frac{n^2\sqrt{1+\cos\theta}\sqrt{8t^2+\cos\theta-1}\big(\cos\theta + 4t^2\big)}{8t^2+\cos\theta-1}.$$

Proof. Notice that this is equivalent to showing that for $\theta \in (-\omega_{n,r_2}, \omega_{n,r_2})$,

$$\Big(\cos\theta - t^2(2\sqrt{2}-4) + \frac{\sqrt{2}}{2}\Big)\Big(\cos\theta + t^2(2\sqrt{2}+4) - \frac{\sqrt{2}}{2}\Big) > \sqrt{1+\cos\theta}\sqrt{8t^2+\cos\theta-1}\big(\cos\theta+4t^2\big).$$

Since the right-hand side is positive by the proof of Subclaim 2, this is equivalent to showing that for $\theta \in (-\omega_{n,r_2}, \omega_{n,r_2})$,

$$\Big(\cos\theta - t^2(2\sqrt{2}-4) + \frac{\sqrt{2}}{2}\Big)^2\Big(\cos\theta + t^2(2\sqrt{2}+4) - \frac{\sqrt{2}}{2}\Big)^2 > (1+\cos\theta)(8t^2+\cos\theta-1)\big(\cos\theta+4t^2\big)^2.$$

A calculation shows that the cosines cancel out leaving us with

$$64t^8 - 64t^6 + 24t^4 - 4t^2 + \frac{1}{4} > 0,$$

which we can factor to get

$$\frac{(1-4t^2)^4}{4} > 0,$$

which is true as long as $t \neq \frac{1}{2}$.

This allows us to conclude that for all $\theta \in (-\omega_{n,r_2}, \omega_{n,r_2})$, $\big(\frac{dx_2}{d\theta}\big)^2 + \big(\frac{dy_2}{d\theta}\big)^2 > 0$. ∎

We conclude that for all $\theta \in (-\omega_{n,r_2}, \omega_{n,r_2})$,

$$\Big(\frac{dx_1}{d\theta}\Big)^2 + \Big(\frac{dy_1}{d\theta}\Big)^2 > 0 \text{ and } \Big(\frac{dx_2}{d\theta}\Big)^2 + \Big(\frac{dy_2}{d\theta}\Big)^2 > 0.$$

□

Claim 3

Let $n > 0$. For $r_2 = \frac{n}{2}$ and $\theta \in (-\pi, \pi)$,

$$\left(\frac{dx_1}{d\theta}\right)^2 + \left(\frac{dy_1}{d\theta}\right)^2 > 0 \text{ and } \left(\frac{dx_2}{d\theta}\right)^2 + \left(\frac{dy_2}{d\theta}\right)^2 = 0.$$

Proof. Assume the hypotheses. We can rewrite (0.11) and (0.12) as

$$\begin{aligned}\left(\frac{dx_j}{d\theta}\right)^2 + \left(\frac{dy_j}{d\theta}\right)^2 = \frac{n^4(2\cos^2\theta - 1)}{2n^2(1+\cos\theta)} + (-1)^{j-1}\left(\frac{2n^4\cos\theta(1+\cos\theta)}{2n^2(1+\cos\theta)}\right) + \frac{2n^4(1+2\cos\theta)}{2n^2(1+\cos\theta)} \\ + (-1)^{j-1}\left(\frac{2n^4(1+\cos\theta)}{2n^2(1+\cos\theta)}\right) + \frac{n^4}{2n^2(1+\cos\theta)},\end{aligned}$$

for $j = 1, 2$. Simplifying this further allows us to conclude that (0.11) can be rewritten as

$$\left(\frac{dx_1}{d\theta}\right)^2 + \left(\frac{dy_1}{d\theta}\right)^2 = 2n^2(1+\cos\theta),$$

which is positive for $\theta \in (-\pi, \pi)$. Similarly, we conclude that (0.12) can be rewritten as

$$\left(\frac{dx_2}{d\theta}\right)^2 + \left(\frac{dy_2}{d\theta}\right)^2 = 0.$$

We conclude that for $r_2 = \frac{n}{2}$ and $\theta \in (-\pi, \pi)$,

$$\left(\frac{dx_1}{d\theta}\right)^2 + \left(\frac{dy_1}{d\theta}\right)^2 > 0 \text{ and } \left(\frac{dx_2}{d\theta}\right)^2 + \left(\frac{dy_2}{d\theta}\right)^2 = 0.$$

□

Step 7: Construct the discriminant envelope using our envelope curves $E_1(\theta)$ and $E_2(\theta)$.

From here, we will use $E_1(\theta)$ and $E_2(\theta)$ to construct the discriminant envelopes for the family $\{\mathcal{C}_\theta\}_{\theta\in[-\pi,\pi)}$. We will split this up into three different cases based on the value of r_2:

Case 1: Suppose $\frac{n}{4} < r_2 < \frac{n}{2}$. Recall that

- In Step 5, we found the curves $E_1(\theta) = \big(x_1(\theta), y_1(\theta)\big)$ and $E_2(\theta) = \big(x_2(\theta), y_2(\theta)\big)$. We also noted that $x_1(\theta)$, $x_2(\theta)$, $y_1(\theta)$, and $y_2(\theta)$ are continuously differentiable for $\theta \in \Big(-\cos^{-1}\big(\frac{n^2-8r_2^2}{n^2}\big), \cos^{-1}\big(\frac{n^2-8r_2^2}{n^2}\big)\Big)$.

- In Step 6, we proved that $\left(\frac{dx_1}{d\theta}\right)^2+\left(\frac{dy_1}{d\theta}\right)^2>0$ and $\left(\frac{dx_2}{d\theta}\right)^2+\left(\frac{dy_2}{d\theta}\right)^2>0$ for $\theta \in \Big(-\cos^{-1}\left(\frac{n^2-8r_2^2}{n^2}\right), \cos^{-1}\left(\frac{n^2-8r_2^2}{n^2}\right)\Big)$.

Putting this all together allows us to conclude that $E_1(\theta)$ and $E_2(\theta)$ satisfy the conditions of the envelope algorithm for $\theta \in \big(-\cos^{-1}\left(\frac{n^2-8r_2^2}{n^2}\right), \cos^{-1}\left(\frac{n^2-8r_2^2}{n^2}\right)\big)$. As a result, we conclude that the two curves

$$E_1(\theta) = \Bigg(r_1\cos\theta_1 + r_2\cos(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\cos[\theta_2+\theta-\alpha(\theta)],$$
$$r_1\sin\theta_1 + r_2\sin(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\sin[\theta_2+\theta-\alpha(\theta)]\Bigg)$$

and

$$E_2(\theta) = \Bigg(r_1\cos\theta_1 + r_2\cos(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\cos[\theta_2+\theta-\pi+\alpha(\theta)],$$
$$r_1\sin\theta_1 + r_2\sin(\theta_2+\theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\sin[\theta_2+\theta-\pi+\alpha(\theta)]\Bigg),$$

with $\alpha(\theta)=\sin^{-1}(\frac{-n\sin\theta}{2\sqrt{2}r_2\sqrt{1+\cos\theta}})$ and $\theta\in(-\cos^{-1}(\frac{n^2-8r_2^2}{n^2}), \cos^{-1}(\frac{n^2-8r_2^2}{n^2}))$, form the discriminant envelope for the family $\{\mathcal{C}_\theta\}_{\theta\in[-\pi,\pi)}$.

On another note, observe that since $E_1(\theta)$ and $E_2(\theta)$ are defined on $(-\cos^{-1}(\frac{n^2-8r_2^2}{n^2}), \cos^{-1}(\frac{n^2-8r_2^2}{n^2}))$, some of the circles in $\{\mathcal{C}_\theta\}_{\theta\in[-\pi,\pi)}$ did not contribute any points to the envelope. Specifically, the only circles that contributed to the envelope were $\mathcal{C}_t$ for $t\in(-\cos^{-1}(\frac{n^2-8r_2^2}{n^2}), \cos^{-1}(\frac{n^2-8r_2^2}{n^2}))$. We claim that this is due to the following nesting phenomenon:

Claim 4

Let $n>0$, $\frac{n}{4}<r_2<\frac{n}{2}$, and $s_1=\cos^{-1}(\frac{n^2-8r_2^2}{n^2})$.

1. For all $\theta\in(s_1,\pi)$, $\mathcal{C}_\theta\subseteq\mathcal{C}_{s_1}$; and
2. For all $\theta\in[-\pi,-s_1)$, $\mathcal{C}_\theta\subseteq\mathcal{C}_{-s_1}$.

Proof. Assume the hypotheses.

Proof of Part (1): Let $\gamma\in(s_1,\pi)$. We wish to show that $\mathcal{C}_\gamma\subseteq\mathcal{C}_{s_1}$. First notice that the equation for $\mathcal{C}_\gamma$ is given by

$$\Big(x - r_1\cos\theta_1 - r_2\cos(\theta_2+\gamma)\Big)^2 + \Big(y - r_1\sin\theta_1 - r_2\sin(\theta_2+\gamma)\Big)^2 = \frac{n^2}{2}(1+\cos\gamma).$$

Similarly, the equation for $\mathcal{C}_{s_1}$ is given by

$$\Big(x - r_1\cos\theta_1 - r_2\cos(\theta_2+s_1)\Big)^2 + \Big(y - r_1\sin\theta_1 - r_2\sin(\theta_2+s_1)\Big)^2 = \frac{n^2}{2}(1+\cos s_1).$$

To show that $\mathcal{C}_\gamma$ is contained in $\mathcal{C}_{s_1}$, we wish to show that

$$(x_{s_1} - x_\gamma)^2 + (y_{s_1} - y_\gamma)^2 < (r_{s_1} - r_\gamma)^2,$$

where (x_{s_1}, y_{s_1}) is the center of $\mathcal{C}_{s_1}$, (x_γ, y_γ) is the center of $\mathcal{C}_\gamma$, r_{s_1} is the radius of $\mathcal{C}_{s_1}$, and r_γ is the radius of $\mathcal{C}_\gamma$. This is equivalent to showing that

$$6r_2^2 - 2r_2^2\cos\gamma + \frac{16r_2^4}{n^2}\cos\gamma - \frac{8r_2^3}{n^2}\sqrt{n^2-4r_2^2}\sin\gamma - \frac{3n^2}{2} - \frac{n^2}{2}\cos\gamma + \sqrt{2}n\sqrt{n^2-4r_2^2}\sqrt{1+\cos\gamma} < 0.$$

From here, let $t = \frac{r_2}{n}$. So, $t \in (\frac{1}{4}, \frac{1}{2})$. Notice then that our inequality becomes

$$\sqrt{1-4t^2}\Big(-\frac{1}{2}\sqrt{1-4t^2}\big[3 + (1+8t^2)\cos\gamma\big] + 2\cos\big(\frac{\gamma}{2}\big) - 8t^3\sin\gamma\Big) < 0. \qquad (0.15)$$

Thus, we just need to show that (0.15) holds for $\gamma \in \big(\cos^{-1}(1-8t^2), \pi\big)$ and for $t \in (\frac{1}{4}, \frac{1}{2})$. From here, set

$$g(t,\gamma) = -\frac{1}{2}\sqrt{1-4t^2}\big[3 + (1+8t^2)\cos\gamma\big] + 2\cos\big(\frac{\gamma}{2}\big) - 8t^3\sin\gamma.$$

We wish to find the maximum of g on $(\frac{1}{4}, \frac{1}{2}) \times (\cos^{-1}(1-8t^2), \pi)$. If the maximum is less than 0, then we know that (0.15) holds. To do so, we will find the maximum of g on the compact set $[\frac{1}{4}, \frac{1}{2}] \times [\cos^{-1}(1-8t^2), \pi]$: Notice that we can take the partials of g with respect to t and γ to get

$$\frac{\partial g}{\partial t} = -\frac{6t}{\sqrt{1-4t^2}}\Big(-1 + \cos\gamma - 8t^2\cos\gamma + 4t\sqrt{1-4t^2}\sin\gamma\Big) \text{ and}$$
$$\frac{\partial g}{\partial \gamma} = -8t^3\cos\gamma - \sin(\frac{\gamma}{2}) + \frac{1}{2}\sqrt{1-4t^2}(1+8t^2)\sin\gamma$$

To find the critical values of g, we set

$$1 - \cos\gamma + 8t^2\cos\gamma - 4t\sqrt{1-4t^2}\sin\gamma = 0 \tag{0.16}$$

$$8t^3\cos\gamma + \sin(\frac{\gamma}{2}) - \frac{1}{2}\sqrt{1-4t^2}(1+8t^2)\sin\gamma = 0 \tag{0.17}$$

By multiplying (0.16) by $-t$ and then adding $-t$(0.16) and (0.17), we get

$$-t(1-\cos\gamma) + \sin\left(\frac{\gamma}{2}\right) - \frac{1}{2}\sqrt{1-4t^2}\sin\gamma = 0,$$

which we can rewrite as

$$-2t\sin^2\left(\frac{\gamma}{2}\right) + \sin\left(\frac{\gamma}{2}\right) - \sqrt{1-4t^2}\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right) = 0.$$

We can further rewrite this as

$$\sin\left(\frac{\gamma}{2}\right)\Big(1 - 2t\sin\left(\frac{\gamma}{2}\right) - \sqrt{1-4t^2}\cos\left(\frac{\gamma}{2}\right)\Big) = 0. \tag{0.18}$$

Note from (0.18) that either

$$\sin\left(\frac{\gamma}{2}\right) = 0, \text{or} \tag{0.19}$$

$$1 - 2t\sin\left(\frac{\gamma}{2}\right) - \sqrt{1-4t^2}\cos\left(\frac{\gamma}{2}\right) = 0. \tag{0.20}$$

Note that (0.19) has no solutions in $[\cos^{-1}(1-8t^2), \pi]$. So, consider (0.20). First notice that,

$$(2t)^2 + (\sqrt{1-4t^2})^2 = 1,$$

and so, there exists an angle $\zeta = \cos^{-1}(\sqrt{1-4t^2})$ such that

$$\cos\zeta = \sqrt{1-4t^2} \text{ and } \sin\zeta = 2t.$$

We can rewrite (0.20) as $\cos\zeta\cos\left(\frac{\gamma}{2}\right) + \sin\zeta\sin\left(\frac{\gamma}{2}\right) = 1$, which becomes $\cos\left(\zeta - \frac{\gamma}{2}\right) = 1$. Therefore, $\zeta - \frac{\gamma}{2} = 0$, which is equivalent to $\cos\left(\frac{\gamma}{2}\right) = \sqrt{1-4t^2}$. Solving for γ in terms of t gives, $\gamma =$

$2\cos^{-1}(\sqrt{1-4t^2})$. From here, let $\nu = \cos^{-1}(\sqrt{1-4t^2})$. Then,

$$\begin{aligned}\cos 2\nu &= \cos^2\nu - \sin^2\nu \\ &= 1 - 8t^2.\end{aligned}$$

So, $2\cos^{-1}(\sqrt{1-4t^2}) = \cos^{-1}(1-8t^2)$. Therefore, the only critical point of g in $[\frac{1}{4}, \frac{1}{2}] \times [\cos^{-1}(1-8t^2), \pi]$ is $\gamma = \cos^{-1}(1-8t^2)$. From here, we check the values of g at $t = \frac{1}{4}$, $t = \frac{1}{2}$, $\gamma = \cos^{-1}(1-8t^2)$, and $\gamma = \pi$:

$$\begin{aligned} g\Big(\frac{1}{4}, \gamma\Big) &= 2\cos\big(\frac{\gamma}{2}\big) - \frac{3\sqrt{3}}{8}(2+\cos\gamma) - \frac{1}{8}\sin\gamma \in \Big[-\frac{3\sqrt{3}}{8}, 0\Big], \text{ for } \gamma \in \Big[\frac{\pi}{3}, \pi\Big] \\ g\Big(\frac{1}{2}, \gamma\Big) &= 2\cos\big(\frac{\gamma}{2}\big) - \sin\gamma = 0, \text{ for } \gamma = \pi \\ g\Big(t, \cos^{-1}(1-8t^2)\Big) &= 0 \text{ for } t \in \Big[\frac{1}{4}, \frac{1}{2}\Big] \\ g(t,\pi) &= -(1-4t^2)^{3/2} \in \Big[-\frac{3\sqrt{3}}{8}, 0\Big], \text{ for } t \in \Big[\frac{1}{4}, \frac{1}{2}\Big] \end{aligned}$$

This allows us to conclude that the maximum of g on $[\frac{1}{4}, \frac{1}{2}] \times [\cos^{-1}(1-8t^2), \pi]$ is 0, which is obtained at the points $(\frac{1}{4}, \frac{\pi}{3})$, $(\frac{1}{2}, \pi)$, $\big(t, \cos^{-1}(1-8t^2)\big)$ for any $t \in [\frac{1}{4}, \frac{1}{2}]$, and $(\frac{1}{2}, \pi)$. Since none of these points is contained in $(\frac{1}{4}, \frac{1}{2}) \times (\cos^{-1}(1-8t^2), \pi)$, we conclude that (0.15) holds. Thus, $\mathcal{C}_\gamma \subseteq \mathcal{C}_{s_1}$ for all such γ. This completes the proof of part (1).

Proof of Part (2): The proof of part 2 follows a similar argument.

We conclude that for all $\theta \in (s_1, \pi)$, $\mathcal{C}_\theta \subseteq \mathcal{C}_{s_1}$ and for all $\theta \in [-\pi, -s_1)$, $\mathcal{C}_\theta \subseteq \mathcal{C}_{-s_1}$. $\square$

Case 2: Suppose $r_2 = \frac{n}{2}$. Recall that

- In Step 5, we found the curve $E_1(\theta) = \big(x_1(\theta), y_1(\theta)\big)$. We also noted that $x_1(\theta)$ and $y_1(\theta)$ are continuously differentiable for $\theta \in (-\pi, \pi)$.
- In Step 6, we proved that $\left(\frac{dx_1}{d\theta}\right)^2 + \left(\frac{dy_1}{d\theta}\right)^2 > 0$ holds for $\theta \in (-\pi, \pi)$.

Putting this all together allows us to conclude that $E_1(\theta)$ satisfies the conditions of the envelope algorithm for $\theta \in (-\pi, \pi)$. As a result, we conclude that the curve

$$E_1(\theta) = \Bigg(r_1 \cos\theta_1 + \frac{n}{2}\cos(\theta_2 + \theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\cos[\theta_2 + \theta - \alpha(\theta)],$$
$$r_1 \sin\theta_1 + \frac{n}{2}\sin(\theta_2 + \theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\sin[\theta_2 + \theta - \alpha(\theta)]\Bigg)$$

with $\alpha(\theta) = \sin^{-1}\left(\frac{-\sin\theta}{\sqrt{2}\sqrt{1+\cos\theta}}\right)$ and $\theta \in (-\pi, \pi)$, forms the discriminant envelope for the family $\{\mathcal{C}_\theta\}_{\theta \in [-\pi, \pi)}$.

On another note, observe that since $E_1(\theta)$ is defined on $(-\pi, \pi)$, every circle except $\mathcal{C}_{-\pi}$ contributes a point to the envelope.

Case 3: Suppose $r_2 > \frac{n}{2}$. Recall that

- In Step 5, we found the curves $E_1(\theta) = \big(x_1(\theta), y_1(\theta)\big)$ and $E_2(\theta) = \big(x_2(\theta), y_2(\theta)\big)$. We also noted that $x_1(\theta)$, $x_2(\theta)$, $y_1(\theta)$, and $y_2(\theta)$ are continuously differentiable for $\theta \in (-\pi, \pi)$.
- In Step 6, we proved that $\left(\frac{dx_1}{d\theta}\right)^2 + \left(\frac{dy_1}{d\theta}\right)^2 > 0$ and $\left(\frac{dx_2}{d\theta}\right)^2 + \left(\frac{dy_2}{d\theta}\right)^2 > 0$ hold for $\theta \in (-\pi, \pi)$.

Putting this all together allows us to conclude that $E_1(\theta)$ and $E_2(\theta)$ satisfy the conditions of the envelope algorithm for $\theta \in (-\pi, \pi)$. As a result, we conclude that the curves

$$E_1(\theta) = \Bigg(r_1 \cos\theta_1 + r_2\cos(\theta_2 + \theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\cos[\theta_2 + \theta - \alpha(\theta)],$$
$$r_1 \sin\theta_1 + r_2\sin(\theta_2 + \theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\sin[\theta_2 + \theta - \alpha(\theta)]\Bigg)$$

and

$$E_2(\theta) = \Bigg(r_1 \cos\theta_1 + r_2 \cos(\theta_2 + \theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\cos[\theta_2 + \theta - \pi + \alpha(\theta)],$$

$$r_1 \sin\theta_1 + r_2 \sin(\theta_2 + \theta) + \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}\sin[\theta_2 + \theta - \pi + \alpha(\theta)] \Bigg),$$

with $\alpha(\theta) = \sin^{-1}(\frac{-n\sin\theta}{2\sqrt{2}r_2\sqrt{1+\cos\theta}})$ and $\theta \in (-\pi, \pi)$, form the discriminant envelope for the family $\{\mathcal{C}_\theta\}_{\theta\in[-\pi,\pi)}$.

On another note, observe that since $E_1(\theta)$ is defined on $(-\pi, \pi)$, every circle except $\mathcal{C}_{-\pi}$ contributes to the envelope.

This concludes the envelope algorithm.

Part 3: Constructing the Boundary

Let $\Omega = \bigcup_{\theta\in[-\pi,\pi)} \mathcal{D}_\theta$. The main goal of this section is to apply Theorem B to our family of circles $\{\mathcal{C}_\theta\}_{\theta\in[-\pi,\pi)}$ to make a conclusion about the boundary of Ω. First note that $\Omega = \bigcup_{\theta\in[-\pi,\pi]} \mathcal{D}_\theta$, since $\mathcal{D}_\pi = \mathcal{D}_{-\pi} = \emptyset$. Next note that for each $\theta \in (-\pi, \pi)$, we have

$$r(\theta) = \frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta} \text{ and } r'(\theta)^2 = \frac{n^2\sin^2\theta}{8(1+\cos\theta)},$$

where $r(\theta)$ is the radius of $\mathcal{C}_\theta$. Notice that $r(\theta), r'(\theta),$ and $r''(\theta)$ are all continuous on $(-\pi, \pi)$. Also notice that $r(\theta) > 0$ on $(-\pi, \pi)$. Recall from part 1 of this proof that for each $\theta \in [-\pi, \pi)$, the center of $\mathcal{C}_\theta$ is given by $r_1 e^{i\theta_1} + r_2 e^{i(\theta_2+\theta)}$. We denote the center of $\mathcal{C}_\theta$ by $(x_c(\theta), y_c(\theta))$ for each value of θ. This gives us

$$x_c(\theta) = r_1\cos\theta_1 + r_2\cos(\theta_2+\theta),$$

$$y_c(\theta) = r_1\sin\theta_1 + r_2\sin(\theta_2+\theta), \text{ and}$$

$$x_c'(\theta)^2 + y_c'(\theta)^2 = r_2^2.$$

Notice that $x_c(\theta), x_c'(\theta),$ and $x_c''(\theta)$ are all continuous on $[-\pi, \pi]$. Similarly, $y_c(\theta), y_c'(\theta),$ and $y_c''(\theta)$ are all continuous on $[-\pi, \pi]$. In order to apply Theorem B, we must show that $r'(\theta)^2 < x_c'(\theta)^2 + y_c'(\theta)^2$

holds on the domain of each envelope curve. In other words, we must show that

$$\frac{n^2 \sin^2 \theta}{8(1+\cos\theta)} < r_2^2, \tag{0.21}$$

for θ in the domain of our envelope curves.

Claim 5

Let $n > 0$ and $r_2 > \frac{n}{4}$. Let

$$\tau_{n,r_2} = \begin{cases} \cos^{-1}\left(\frac{n^2-8r_2^2}{n^2}\right), & \frac{n}{4} < r_2 < \frac{n}{2} \\ \pi, & r_2 \geq \frac{n}{2} \end{cases}.$$

For all $\theta \in (-\tau_{n,r_2}, \tau_{n,r_2})$,

$$\frac{n^2 \sin^2 \theta}{8(1+\cos\theta)} < r_2^2.$$

Proof. Assume the hypotheses. We can rewrite (0.21) as $-n^2\cos^2\theta - 8r_2^2\cos\theta + n^2 - 8r_2^2 < 0$, which can be factored to get,

$$-n^2(\cos\theta + 1)\Big(\cos\theta - 1 + \frac{8r_2^2}{n^2}\Big) < 0. \tag{0.22}$$

Notice that $\cos\theta + 1 > 0$ on $(-\pi, \pi)$.

Consider $r_2 \geq \frac{n}{2}$. Notice that $\frac{8r_2^2}{n^2} \geq 2$ when $r_2 \geq \frac{n}{2}$. Therefore, when $r_2 \geq \frac{n}{2}$,

$$\cos\theta - 1 + \frac{8r_2^2}{n^2} > 0 \text{ on } (-\pi, \pi).$$

Putting this together allows us to conclude that when $r_2 \geq \frac{n}{2}$,

$$(\cos\theta + 1)\Big(\cos\theta - 1 + \frac{8r_2^2}{n^2}\Big) > 0 \text{ on } (-\pi, \pi).$$

As a result, we conclude that when $r_2 \geq \frac{n}{2}$, (0.22) holds on $(-\pi, \pi)$.

Now consider $\frac{n}{4} < r_2 < \frac{n}{2}$. Note that $\frac{8r_2^2}{n^2} \in (\frac{1}{2}, 2)$. Consider $\theta = \pm\cos^{-1}(\frac{n^2-8r_2^2}{n^2})$. When $\theta = \pm\cos^{-1}(\frac{n^2-8r_2^2}{n^2})$,

$$\cos\theta - 1 + \frac{8r_2^2}{n^2} = 0.$$

Then since cosine is an even function, we conclude that

$$\cos\theta - 1 + \frac{8r_2^2}{n^2} > 0 \text{ on } \Big(-\cos^{-1}\Big(\frac{n^2-8r_2^2}{n^2}\Big), \cos^{-1}\Big(\frac{n^2-8r_2^2}{n^2}\Big)\Big).$$

Putting this together allows us to conclude that when $\frac{n}{4} < r_2 < \frac{n}{2}$,

$$(\cos\theta + 1)\Big(\cos\theta - 1 + \frac{8r_2^2}{n^2}\Big) > 0 \text{ on}\Big(-\cos^{-1}\Big(\frac{n^2-8r_2^2}{n^2}\Big), \cos^{-1}\Big(\frac{n^2-8r_2^2}{n^2}\Big)\Big).$$

As a result, we conclude that when $\frac{n}{4} < r_2 < \frac{n}{2}$, (0.22) holds on $(-\cos^{-1}\Big(\frac{n^2-8r_2^2}{n^2}\Big), \cos^{-1}\Big(\frac{n^2-8r_2^2}{n^2}\Big)\Big)$.

Therefore, we conclude that for $\theta \in (-\tau_{n,r_2}, \tau_{n,r_2})$, $\frac{n^2\sin^2\theta}{8(1+\cos\theta)} < r_2^2$. $\square$

From here, we will split into cases:

Case 1: Suppose $\frac{n}{4} < r_2 < \frac{n}{2}$. Let $s_1 = \cos^{-1}(\frac{n^2-8r_2^2}{n^2})$. Recall from part 2 of this proof that the only circles that contribute to our envelope are the $\mathcal{C}_t$ such that $t \in (-s_1, s_1)$. As a result, $\Omega = \bigcup_{t\in[-s_1,s_1]} \mathcal{D}_t$. So by Claim 5 and Theorem B, we conclude that $\partial\Omega \subseteq E_1(\theta) \cup E_2(\theta) \cup \mathcal{C}_{s_1} \cup \mathcal{C}_{-s_1}$. We conclude that for $\frac{n}{4} < r_2 < \frac{n}{2}$,

$$\partial\Omega \subseteq E_1(\theta) \cup E_2(\theta) \cup \mathcal{C}_{s_1} \cup \mathcal{C}_{-s_1},$$

where $\mathcal{C}_{s_1}$ is the circle centered at $r_1e^{i\theta_1} + r_2e^{i(\theta_2+s_1)}$ with radius $\sqrt{n^2-4r_2^2}$ and $\mathcal{C}_{-s_1}$ is the circle centered at $r_1e^{i\theta_1} + r_2e^{i(\theta_2-s_1)}$ with radius $\sqrt{n^2-4r_2^2}$.

Additionally, we have numerical evidence that suggests that the circles $\mathcal{C}_{s_1}$ and $\mathcal{C}_{-s_1}$ both contribute a single point to our boundary.

Case 2: Suppose $r_2 = \frac{n}{2}$. Recall from part 2 of this proof that the only circle that did not contribute to our envelope was $\mathcal{C}_{-\pi}$. As a result, $\Omega = \bigcup_{t\in[-\pi,\pi]} \mathcal{D}_t$. So by Claim 5 and Theorem B, we conclude that $\partial\Omega \subseteq E_1(\theta) \cup \mathcal{C}_{-\pi} \cup \mathcal{C}_{\pi} = E_1(\theta) \cup \mathcal{C}_{-\pi}$. We conclude that for $r_2 = \frac{n}{2}$,

$$\partial\Omega \subseteq E_1(\theta) \cup \mathcal{C}_{-\pi},$$

where $\mathcal{C}_{-\pi} = \big(r_1 \cos\theta_1 - \frac{n}{2}\cos\theta_2, r_1 \sin\theta_1 - \frac{n}{2}\sin\theta_2\big)$.

Case 3: Suppose $r_2 > \frac{n}{2}$. Recall from part 2 of this proof that the only circle that did not contribute to our envelope was $\mathcal{C}_{-\pi}$. As a result, $\Omega = \bigcup_{t\in[-\pi,\pi]} \mathcal{D}_t$. So by Claim 5 and Theorem B, we conclude that $\partial\Omega \subseteq E_1(\theta) \cup E_2(\theta) \cup \mathcal{C}_{-\pi} \cup \mathcal{C}_{\pi} = E_1(\theta) \cup E_2(\theta) \cup \mathcal{C}_{-\pi}$. We conclude that for $r_2 > \frac{n}{2}$,

$$\partial\Omega \subseteq E_1(\theta) \cup E_2(\theta) \cup \mathcal{C}_{-\pi},$$

where $\mathcal{C}_{-\pi} = \big(r_1 \cos\theta_1 - r_2\cos\theta_2, r_1 \sin\theta_1 - r_2\sin\theta_2\big)$.

This completes the proof of Theorem 1.

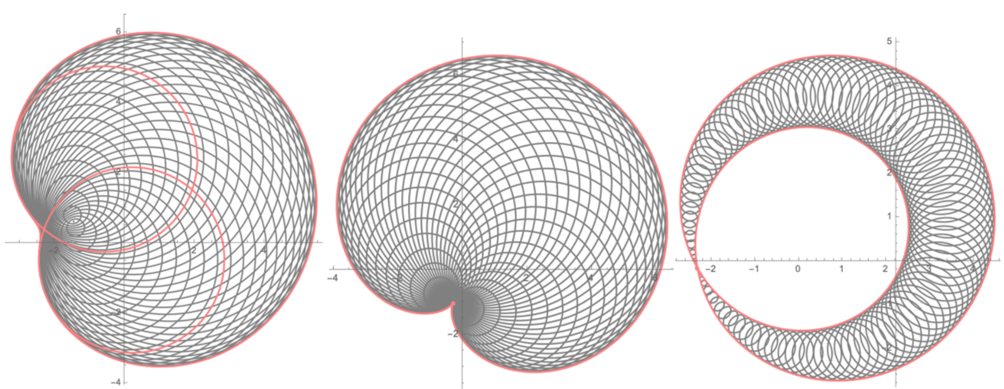

Figure 7: Figure 6 with the proposed boundary obtained from Theorem 1. The boundary of the leftmost plot appears to only need one point each from the two pink circles.

Consider Figure 7. It appears that we have found the boundary in the case where $r_2 \geq \frac{n}{2}$, which corresponds to the middle and rightmost plots. We should be able to prove the reverse containment with a more geometric argument. The leftmost plot, corresponding to the case where $\frac{n}{4} < r_2 < \frac{n}{2}$, has two additional circles that do not seem to contribute much to the boundary. It seems likely that each of these circles contributes one point to the boundary, though it has been difficult to prove.

Although Theorem 1 does not give us the exact boundary of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in this special case, we can still use it to make a conclusion about the closure of the numerical range of $T_{A_0+zA_1}$:

Corollary 1

Let $A_0 = \begin{bmatrix} r_1e^{i\theta_1} & ne^{i\gamma} \\ 0 & r_1e^{i\theta_1} \end{bmatrix}$ and $A_1 = \begin{bmatrix} r_2e^{i\theta_2} & ne^{i\gamma} \\ 0 & r_2e^{i\theta_2} \end{bmatrix}$ where $n > 0$, $r_2 > \frac{n}{4}$, and $r_1 \geq 0$ and let $\phi(z) = A_0 + zA_1$. Let $E_1(\theta) = \big(x_1(\theta), y_1(\theta)\big)$ and $E_2(\theta) = \big(x_2(\theta), y_2(\theta)\big)$, where

$$x_1(\theta) = r_1 \cos\theta_1 + r_2 \cos(\theta_2 + \theta) + \frac{n\sqrt{2}}{2}\sqrt{1 + \cos\theta}\cos[\theta_2 + \theta - \alpha(\theta))];$$
$$y_1(\theta) = r_1 \sin\theta_1 + r_2 \sin(\theta_2 + \theta) + \frac{n\sqrt{2}}{2}\sqrt{1 + \cos\theta}\sin[\theta_2 + \theta - \alpha(\theta))];$$
$$x_2(\theta) = r_1 \cos\theta_1 + r_2 \cos(\theta_2 + \theta) + \frac{n\sqrt{2}}{2}\sqrt{1 + \cos\theta}\cos[\theta_2 + \theta - \pi + \alpha(\theta))];$$
$$y_2(\theta) = r_1 \sin\theta_1 + r_2 \sin(\theta_2 + \theta) + \frac{n\sqrt{2}}{2}\sqrt{1 + \cos\theta}\sin[\theta_2 + \theta - \pi + \alpha(\theta))],$$

and $\alpha(\theta) = \sin^{-1}(\frac{-n\sin\theta}{2\sqrt{2}r_2\sqrt{1+\cos\theta}})$. Let $s_1 = \cos^{-1}\left(\frac{n^2-8r_2^2}{n^2}\right)$.

1. If $\frac{n}{4} < r_2 < \frac{n}{2}$, then the closure of $W(T_\phi)$ is contained in the convex hull of

$$E_1(\theta) \cup E_2(\theta) \cup \mathcal{C}_{s_1} \cup \mathcal{C}_{-s_1}, \text{ for } \theta \in (-s_1, s_1),$$

where $\mathcal{C}_{s_1}$ is the circle centered at $r_1e^{i\theta_1} + r_2e^{i(\theta_2+s_1)}$ with radius $\sqrt{n^2 - 4r_2^2}$ and $\mathcal{C}_{-s_1}$ is the circle centered at $r_1e^{i\theta_1} + r_2e^{i(\theta_2-s_1)}$ with radius $\sqrt{n^2 - 4r_2^2}$.

2. If $r_2 = \frac{n}{2}$, then the closure of $W(T_\phi)$ is contained in the convex hull of

$$E_1(\theta) \cup \left\{\left(r_1 \cos\theta_1 - \frac{n}{2}\cos\theta_2, r_1 \sin\theta_1 - \frac{n}{2}\sin\theta_2\right)\right\}, \text{ for } \theta \in (-\pi, \pi).$$

3. If $r_2 > \frac{n}{2}$, then the closure of $W(T_\phi)$ is contained in the convex hull of

$$E_1(\theta) \cup E_2(\theta) \cup \left\{\left(r_1 \cos\theta_1 - r_2 \cos\theta_2, r_1 \sin\theta_1 - r_2 \sin\theta_2\right)\right\}, \text{ for } \theta \in (-\pi, \pi).$$

Proof. Assume the hypotheses. By Theorem A, we know that the closure of $W(T_\phi)$ for $\phi(z) = A_0 + zA_1$ is the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$. This result follows immediately from Theorem A and Theorem 1.

5 Flat Portions

Next we will analyze the number of flat portions on the boundary of the convex hull of $\{W(A_0+zA_1):z\in\partial\mathbb{D}\}$ in our special case:

Theorem 2

Let $A_0=\begin{bmatrix} r_1e^{i\theta_1} & ne^{i\gamma} \\ 0 & r_1e^{i\theta_1}\end{bmatrix}$ and $A_1=\begin{bmatrix} r_2e^{i\theta_2} & ne^{i\gamma} \\ 0 & r_2e^{i\theta_2}\end{bmatrix}$ where $n>0$, $r_2>\frac{n}{4}$, and $r_1\geq 0$. The convex hull of $\{W(A_0+zA_1):z\in\partial\mathbb{D}\}$ has at least one flat portion on its boundary.

Proof. Assume the hypotheses. Recall from part 1 of the proof of Theorem 1 that for each value of $\theta\in[0,2\pi)$, $W(A_0+e^{i\theta}A_1)$ is a circular disk centered at $r_1e^{i\theta_1}+r_2e^{i(\theta_2+\theta)}$ with radius $\frac{n\sqrt{2}}{2}\sqrt{1+\cos\theta}$. From here, notice that the maximum possible radius for one of these disks is n and the minimum possible radius for one of these disks is 0. Now consider $r_1e^{i\theta_1}+r_2e^{i(\theta_2+\theta)}$ for $\theta\in[0,2\pi)$. We can rewrite this as

$$\Big(r_1\cos\theta_1+r_2\cos(\theta_2+\theta), r_1\sin\theta_1+r_2\sin(\theta_2+\theta)\Big),\ \text{for } \theta\in[0,2\pi).$$

Thus, we have a circle centered at $\big(r_1\cos\theta_1, r_1\sin\theta_1\big)$ with radius r_2. Call this circle C. So, for each $z\in\partial\mathbb{D}$, the center of $W(A_0+zA_1)$ is a distinct point on the circle C.

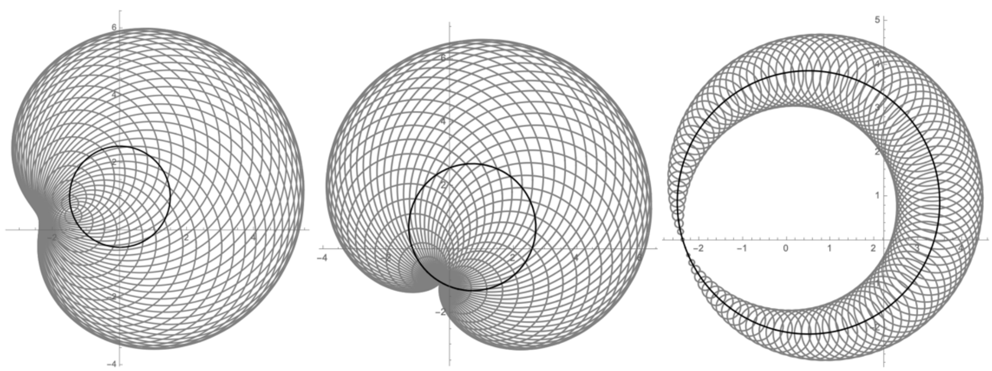

Figure 8: Figure 6 with the circle of centers C added on.

Consider $z=e^{i\pi}$. Note that $W(A_0+zA_1)$ is the circular disk with radius 0, located at $\big(r_1\cos\theta_1-r_2\cos\theta_2, r_1\sin\theta_1-r_2\sin\theta_2\big)$. Consider the line that passes through this point and the center point of C. A calculation shows that the slope of this line is $\tan\theta_2$ and so the equation of this line is

$$y = (\tan\theta_2)x + r_1\big(\sin\theta_1 - \tan\theta_2\cos\theta_1\big). \tag{0.23}$$

Now consider

$$e^{-i\theta_2}(A_0 + e^{i\theta}A_1) = \begin{bmatrix} r_1 e^{i(\theta_1-\theta_2)} + r_2 e^{i\theta} & n e^{i(\gamma-\theta_2)} + n e^{i(\gamma-\theta_2+\theta)} \\ 0 & r_1 e^{i(\theta_1-\theta_2)} + r_2 e^{i\theta} \end{bmatrix}.$$

Note that $\{W(e^{-i\theta_2}(A_0+zA_1)) : z \in \partial\mathbb{D}\}$ is a rotation of the set $\{W(A_0+zA_1) : z \in \partial\mathbb{D}\}$ by an angle of $-\theta_2$. Thus, $e^{-i\theta_2}C$ is centered at $\big(r_1\cos(\theta_1-\theta_2), r_1\sin(\theta_1-\theta_2)\big)$ with radius r_2. By rotating (0.23) by an angle of $-\theta_2$, we get the line $y = r_1\sin(\theta_1-\theta_2)$, which passes through $W(e^{-i\theta_2}(A_0 + e^{i\pi}A_1))$ and the center of $e^{-i\theta_2}C$.

Next, consider the complex conjugate pair $e^{i\alpha}, e^{-i\alpha} \in \partial\mathbb{D}$. Notice that $W(e^{-i\theta_2}(A_0 + e^{i\alpha}A_1))$ is the circular disk centered at $\big(r_1\cos(\theta_1-\theta_2)+r_2\cos\alpha, r_1\sin(\theta_1-\theta_2)+r_2\sin\alpha\big)$ with radius $\frac{n\sqrt{2}}{2}\sqrt{1+\cos\alpha}$. Similarly, $W(e^{-i\theta_2}(A_0+e^{i-\alpha}A_1))$ is the circular disk centered at $\big(r_1\cos(\theta_1-\theta_2)+r_2\cos\alpha, r_1\sin(\theta_1-\theta_2)-r_2\sin\alpha\big)$ with radius $\frac{n\sqrt{2}}{2}\sqrt{1+\cos\alpha}$. We now wish to show that $\{W(e^{-i\theta_2}(A_0+zA_1)) : z \in \partial\mathbb{D}\}$ is symmetric about the line $y = r_1\sin(\theta_1-\theta_2)$:

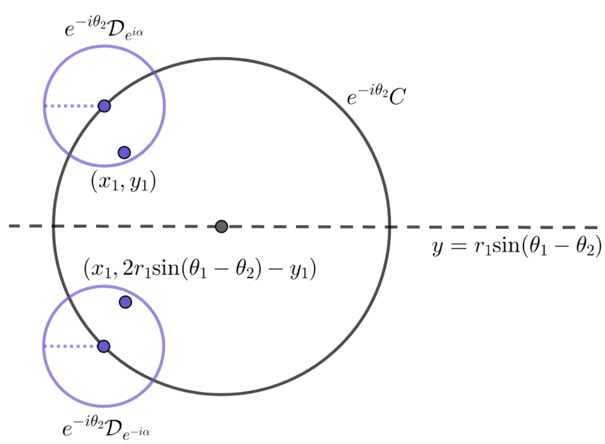

Figure 9: Symmetry of $\{W(e^{-i\theta_2}(A_0+zA_1)) : z \in \partial\mathbb{D}\}$ about the dashed line $y = r_1\sin(\theta_1-\theta_2)$.

Let $z_1 = x_1 + iy_1$ be a point contained in $\{W(e^{-i\theta_2}(A_0 + zA_1)) : z \in \partial\mathbb{D}\}$. We wish to show that this set is symmetric about the line $y = r_1 \sin(\theta_1 - \theta_2)$ and therefore wish to show that $\big(x_1, 2r_1\sin(\theta_1 - \theta_2) - y_1\big)$ is contained in this set. Since z_1 is contained in $\{W(e^{-i\theta_2}(A_0 + zA_1)) : z \in \partial\mathbb{D}\}$, z_1 must be contained in a closed circular disk that corresponds to some $z = e^{i\alpha}$. Call this disk $e^{-i\theta_2}\mathcal{D}_{e^{i\alpha}}$. We know that $e^{-i\theta_2}\mathcal{D}_{e^{i\alpha}}$ is centered at $\big(r_1\cos(\theta_1 - \theta_2) + r_2\cos\alpha, r_1\sin(\theta_1 - \theta_2) + r_2\sin\alpha\big)$ with radius $\frac{n\sqrt{2}}{2}\sqrt{1 + \cos\alpha}$. Since z_1 is contained in $e^{-i\theta_2}\mathcal{D}_{e^{i\alpha}}$, we know that

$$\sqrt{\big(x_1 - r_1\cos(\theta_1 - \theta_2) - r_2\cos\alpha\big)^2 + \big(y_1 - r_1\sin(\theta_1 - \theta_2) - r_2\sin\alpha\big)^2} \leq \frac{n\sqrt{2}}{2}\sqrt{1 + \cos\alpha} \quad (0.24)$$

Now consider the closed circular disk corresponding to $\bar{z} = e^{-i\alpha}$. Call this disk $e^{-i\theta_2}\mathcal{D}_{e^{-i\alpha}}$. We know that $e^{-i\theta_2}\mathcal{D}_{e^{-i\alpha}}$ is centered at $\big(r_1\cos(\theta_1 - \theta_2) + r_2\cos\alpha, r_1\sin(\theta_1 - \theta_2) - r_2\sin\alpha\big)$ with radius $\frac{n\sqrt{2}}{2}\sqrt{1 + \cos\alpha}$. Notice that the distance between $\Big(x_1, 2r_1\sin(\theta_1 - \theta_2) - y_1\Big)$ and $\big(r_1\cos(\theta_1 - \theta_2) + r_2\cos\alpha, r_1\sin(\theta_1 - \theta_2) - r_2\sin\alpha\big)$ is

$$\begin{aligned} d &= \sqrt{\big(x_1 - r_1\cos(\theta_1 - \theta_2) - r_2\cos\alpha\big)^2 + \big(2r_1\sin(\theta_1 - \theta_2) - y_1 - r_1\sin(\theta_1 - \theta_2) + r_2\sin\alpha\big)^2} \\ &= \sqrt{\big(x_1 - r_1\cos(\theta_1 - \theta_2) - r_2\cos\alpha\big)^2 + \big(y_1 - r_1\sin(\theta_1 - \theta_2) - r_2\sin\alpha\big)^2} \\ &\leq \frac{n\sqrt{2}}{2}\sqrt{1 + \cos\alpha}, \end{aligned}$$

by (0.24). Therefore, $\big(x_1, 2r_1\sin(\theta_1 - \theta_2) - y_1\big)$ is contained in $e^{-i\theta_2}\mathcal{D}_{e^{-i\alpha}}$, indicating that $\big(x_1, 2r_1\sin(\theta_1 - \theta_2) - y_1\big)$ is contained in $\{W(e^{-i\theta_2}(A_0 + zA_1)) : z \in \partial\mathbb{D}\}$. We conclude that $\{W(e^{-i\theta_2}(A_0 + zA_1)) : z \in \partial\mathbb{D}\}$ is symmetric about the line $y = r_1\sin(\theta_1 - \theta_2)$ and thus that the convex hull of $\{W(e^{-i\theta_2}(A_0 + zA_1)) : z \in \partial\mathbb{D}\}$ is symmetric about the line $y = r_1\sin(\theta_1 - \theta_2)$.

Next, notice that for each circular disk contained in $\{W(e^{-i\theta_2}(A_0 + zA_1)) : z \in \partial\mathbb{D}\}$, we can find the minimum real part. This will allow us to conclude that there is a flat portion on the left side of the boundary of the convex hull of $\{W(e^{-i\theta_2}(A_0 + zA_1)) : z \in \partial\mathbb{D}\}$. So for each circular disk in $\{W(e^{-i\theta_2}(A_0 + zA_1)) : z \in \partial\mathbb{D}\}$, we are interested in the x-value of $e^{-i\theta_2}C$ minus the radius of the circular disk. The smallest such value will give the minimum real part. So, we want to find the value of θ for which

$$r_1\cos(\theta_1 - \theta_2) + r_2\cos\theta - \frac{n\sqrt{2}}{2}\sqrt{1 + \cos\theta} \quad (0.25)$$

is minimized. Taking the derivative of (0.25) with respect to θ and setting this equal to zero gives

$$-r_2\sin\theta + \frac{n\sqrt{2}}{4}\Big(\frac{\sin\theta}{\sqrt{1+\cos\theta}}\Big) = 0.$$

Notice that since $r_2 > \frac{n}{4}$, the distinct critical points contained in $[0, 2\pi)$ are $\theta = 0, \pi, \pm\cos^{-1}(\frac{n^2}{8r_2^2} - 1)$. The two absolute minimums occur when $\theta = \pm\cos^{-1}(\frac{n^2}{8r_2^2} - 1)$. A calculation shows that

$$r_1\cos(\theta_1 - \theta_2) - \frac{n^2}{8r_2} - r_2$$

is the minimum real part of all of the circular disks contained in $\{W(e^{-i\theta_2}(A_0 + zA_1)) : z \in \partial\mathbb{D}\}$. This minimum real part occurs on the circular disks $e^{-i\theta_2}\mathcal{D}_{\cos^{-1}(\frac{n^2}{8r_2^2}-1)}$ and $e^{-i\theta_2}\mathcal{D}_{-\cos^{-1}(\frac{n^2}{8r_2^2}-1)}$. Then since the convex hull of $\{W(e^{-i\theta_2}(A_0 + zA_1)) : z \in \partial\mathbb{D}\}$ is a convex set, the line segment

$$\begin{aligned}&\vec{r}(t) = \big(r_1\cos(\theta_1 - \theta_2) - \frac{n^2}{8r_2} - r_2, t\big),\\ &\text{for } t \in \Big[r_1\sin(\theta_1 - \theta_2) - \frac{n}{8r_2}\sqrt{16r_2^2 - n^2}, r_1\sin(\theta_1 - \theta_2) + \frac{n}{8r_2}\sqrt{16r_2^2 - n^2}\Big],\end{aligned}$$

must be contained in the convex hull of $\{W(e^{-i\theta_2}(A_0+zA_1)) : z \in \partial\mathbb{D}\}$. Since this line segment connects the two points on the boundaries of the circles with the minimum real part, we know that no other points to the left of this line are contained in the convex hull of $\{W(e^{-i\theta_2}(A_0+zA_1)) : z \in \partial\mathbb{D}\}$. Therefore, this line segment forms a flat portion on the boundary of the convex hull. We conclude that the convex hull of $\{W(e^{-i\theta_2}(A_0+zA_1)) : z \in \partial\mathbb{D}\}$ has at least one flat portion on its boundary.

To get the corresponding flat portion on $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$, we need to multiply $\big(r_1\cos(\theta_1 - \theta_2) - \frac{n^2}{8r_2} - r_2, r_1\sin(\theta_1 - \theta_2) - \frac{n}{8r_2}\sqrt{16r_2^2 - n^2}\big)$ by $e^{i\theta_2}$. This gives the point

$$\begin{aligned}\Big(r_1\cos\theta_1 - \big(r_2 + \frac{n^2}{8r_2}\big)\cos\theta_2 + \frac{n}{8r_2}\sqrt{16r_2^2 - n^2}\sin\theta_2,&\\ r_1\sin\theta_1 - \big(r_2 + \frac{n^2}{8r_2}\big)\sin\theta_2 - \frac{n}{8r_2}\sqrt{16r_2^2 - n^2}\cos\theta_2\Big).&\end{aligned}$$

Multiplying $\big(r_1\cos(\theta_1 - \theta_2) - \frac{n^2}{8r_2} - r_2, r_1\sin(\theta_1 - \theta_2) + \frac{n}{2}\sqrt{1 - \frac{n^2}{16r_2^2}}\big)$ by $e^{i\theta_2}$ gives the point

$$\begin{aligned}\Big(r_1\cos\theta_1 - \big(r_2 + \frac{n^2}{8r_2}\big)\cos\theta_2 - \frac{n}{8r_2}\sqrt{16r_2^2 - n^2}\sin\theta_2,&\\ r_1\sin\theta_1 - \big(r_2 + \frac{n^2}{8r_2}\big)\sin\theta_2 + \frac{n}{8r_2}\sqrt{16r_2^2 - n^2}\cos\theta_2\Big).&\end{aligned}$$

Now consider the line segment

$$\vec{r}(t) = \Big(r_1 \cos(\theta_1 - \theta_2)\cos\theta_2 - (r_2 + \frac{n^2}{8r_2})\cos\theta_2 - t\sin\theta_2,$$
$$r_1 \cos(\theta_1 - \theta_2)\sin\theta_2 - (r_2 + \frac{n^2}{8r_2})\sin\theta_2 + t\cos\theta_2\Big),$$

for $t \in \Big[r_1 \sin(\theta_1 - \theta_2) - \frac{n}{8r_2}\sqrt{16r_2^2 - n^2}, r_1 \sin(\theta_1 - \theta_2) + \frac{n}{8r_2}\sqrt{16r_2^2 - n^2}\Big]$. This line segment forms a flat portion on the boundary of the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$. We conclude that the boundary of the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ has at least one flat portion.

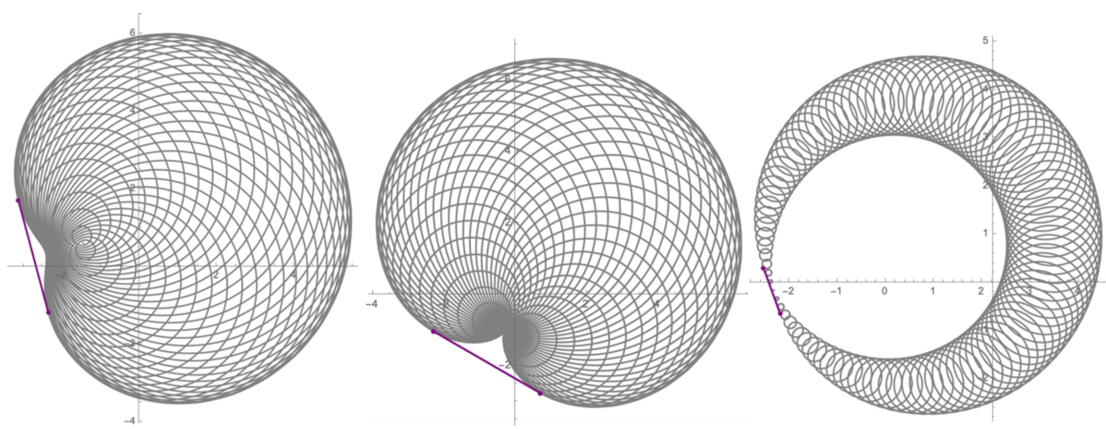

Figure 10: Figure 6 with the flat portions added on.

Now that we know what the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ looks like, we can make a conclusion as to what the closure of the numerical range of the corresponding Toeplitz operator looks like:

Corollary 2

Let $A_0 = \begin{bmatrix} r_1 e^{i\theta_1} & ne^{i\gamma} \\ 0 & r_1 e^{i\theta_1} \end{bmatrix}$ and $A_1 = \begin{bmatrix} r_2 e^{i\theta_2} & ne^{i\gamma} \\ 0 & r_2 e^{i\theta_2} \end{bmatrix}$ where $n > 0$, $r_2 > \frac{n}{4}$, and $r_1 \geq 0$ and let $\phi(z) = A_0 + zA_1$. The closure of $W(T_\phi)$ has at least one flat portion on its boundary.

Proof. By Theorem A, we know that the closure of $W(T_\phi)$ for $\phi(z) = A_0 + zA_1$ is the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$. This result follows immediately from Theorem A and Theorem 2.

There is much numerical evidence to suggest that in our specific case, $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ has exactly 1 flat portion on the boundary of its convex hull, as seen in Figure 0.10. However, it has proved quite difficult to show that this other direction holds.

For the remainder of this paper, we will be analyzing the number of flat portions on the boundary of the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ for any $A_0, A_1 \in M_n(\mathbb{C})$. We start with $n = 2$, our most familiar case. Consider what happens when we calculate the numerical range of $A_0 + zA_1$ for $A_0 = A_1 = \begin{bmatrix} 1 & 1 \\ \frac{10}{4} & -1 \end{bmatrix}$:

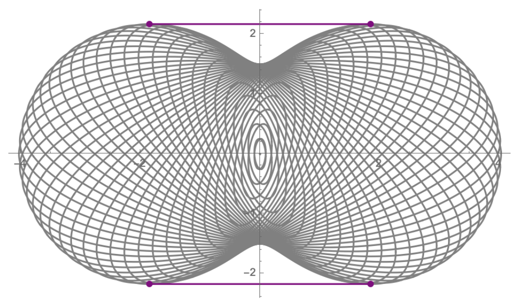

Figure 11: An example of two flat portions on the boundary, denoted by the two purple line segments.

By taking the convex hull of $\{W(A_0+zA_1) : z \in \partial\mathbb{D}\}$ in Figure 11, we find two parallel flat portions. Next, consider the following example where $A_0 = 0$ and $A_1 = \begin{bmatrix} 1 - i\sqrt{2} & 2 \\ 0 & 1 - i\sqrt{2} \end{bmatrix}$:

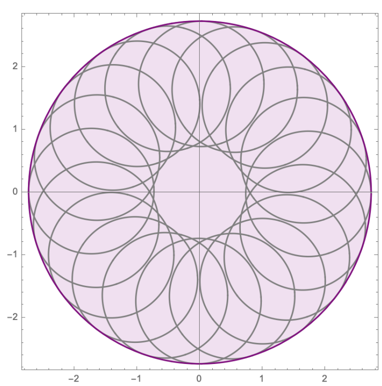

Figure 12: An example of zero flat portions on the boundary.

In this case, the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$, found in Figure 12, is the purple circular disk. Thus, it has no flat portions on its boundary.

The convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ has also been studied by Cal Poly students Sarah Mantell and Mav Lara. In her senior project, Sarah characterized the case in which $A_0, A_1 \in M_2(\mathbb{C})$ are diagonal matrices. To learn more about Sarah's work with Toeplitz operators, see [11]. Over the summer, Mav characterized the case in which $A_0 = 0$ and $A_1 \in M_n(\mathbb{C})$. Mav and I also analyzed conditions on $A_0, A_1 \in M_2(\mathbb{C})$ that produce flat portions. To learn more about the work that Mav and I did this summer, see [14]. Using all of our observations, we make the following conjecture:

Conjecture 1

Let $A_0, A_1 \in M_2(\mathbb{C})$. The boundary of the closure of $W(T_\phi)$, with symbol $\phi(z) = A_0 + zA_1$, has at most two flat portions.

Next, we consider the case where $n = 3$. We have numerical evidence that the boundary of the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ has zero, one, two, or three flat portions. First consider the example with $A_0 = A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{bmatrix}$:

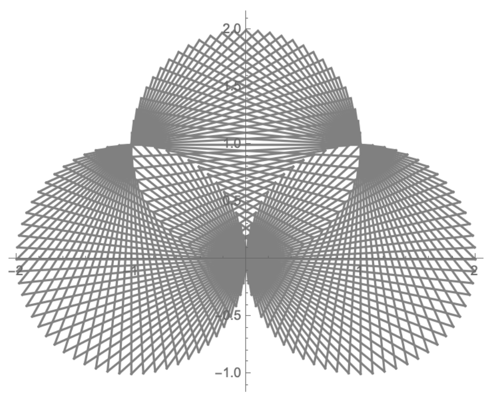

Figure 13: An example of three flat portions when $n = 3$.

Notice that the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 13 will have a triangular shape. Our next example occurs when

$$A_0 = \begin{bmatrix} i & 0 & -2 \\ 0 & 0 & -1 \\ 2 & 1 & \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} i & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \end{bmatrix}:$$

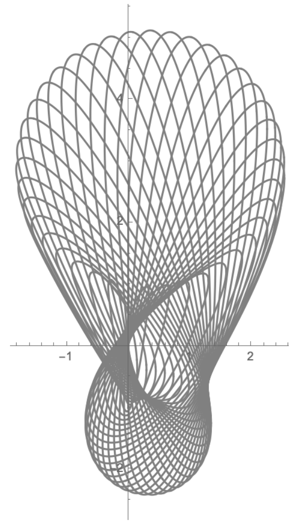

Figure 14: An example of two flat portions when $n = 3$.

The convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 14 will have a more elliptical shape with two flat portions that are not parallel. Next we have the example where $A_0 = A_1 = \begin{bmatrix} 0 & i & i \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix}$:

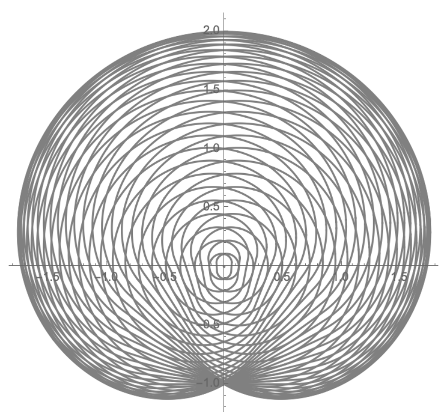

Figure 15: An example of one flat portion when $n = 3$.

The set $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 15 has a shape reminiscent of the shapes in Theorems 1 and 2. Our final example in this case is when $A_0 = 0$ and $A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$:

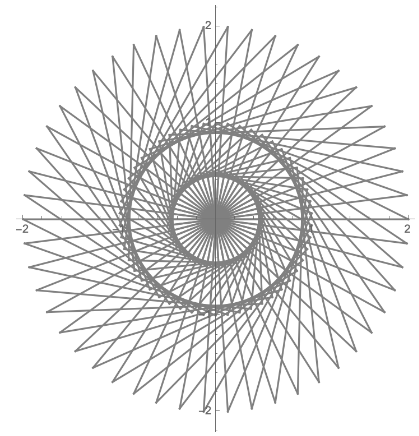

Figure 16: An example of zero flat portions when $n = 3$.

Once again, the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 16 will be a circular disk.

Next, we will consider examples in the case where $n = 4$. We have numerical evidence that the boundary of the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ has zero, one, two, three, or four flat portions. First consider the example with

$$A_0 = A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}:$$

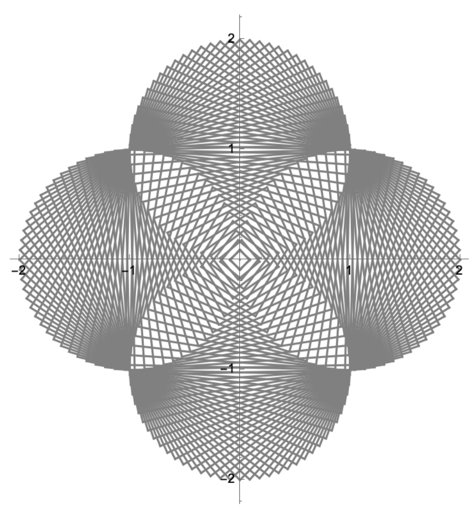

Figure 17: An example of four flat portions when $n = 4$.

We see that the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 17 will have a square shape. Next, consider the example where

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & -\frac{1}{10} & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -i \end{bmatrix}:$$

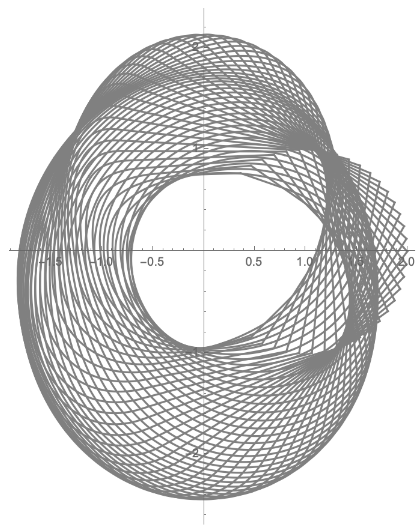

Figure 18: An example of three flat portions when $n = 4$.

Note that the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 18 will have two distinct flat portions on the right side and one flat portion on the left. Next, consider the example where

$$A_0 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & -\frac{1}{10} & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{i}{2} & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -i \end{bmatrix}:$$

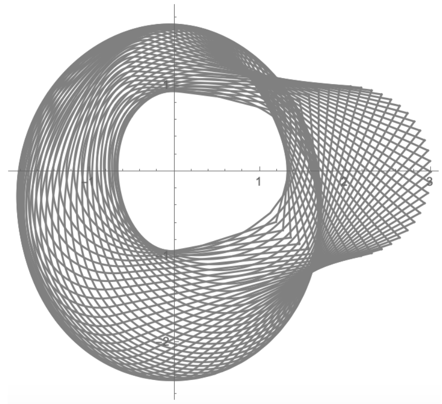

Figure 19: An example of two flat portions when $n = 4$.

Note that the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 19 will have two nonparallel flat portions. Next, consider the example where

$$A_0 = A_1 = \begin{bmatrix} 0 & i & i\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & i\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}:$$

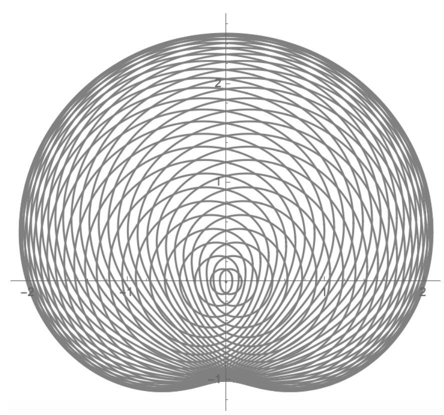

Figure 20: An example of one flat portion when $n = 4$.

Once again, $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 20 is reminiscent of the shapes found in Theorems 1 and 2. Our final example for this case occurs when

$$A_0 = \begin{bmatrix} 0 & 1 & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}:$$

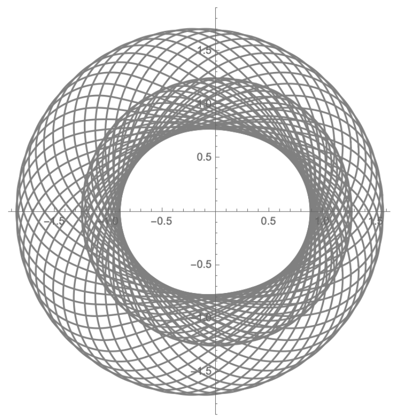

Figure 21: An example of zero flat portions when $n = 4$.

The convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 21 appears to be a rounded shape.

Finally, we will consider examples in the case where $n = 5$. We have numerical evidence that the boundary of the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ has zero, one, two, three, four, five, or six flat portions. First consider the example with

$$A_0 = A_1 = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{6} & 0 & 0 \\ 0 & 0 & 0 & \frac{7}{6}i & 0 \\ 0 & 0 & 0 & 0 & -\frac{7}{6} \end{bmatrix}:$$

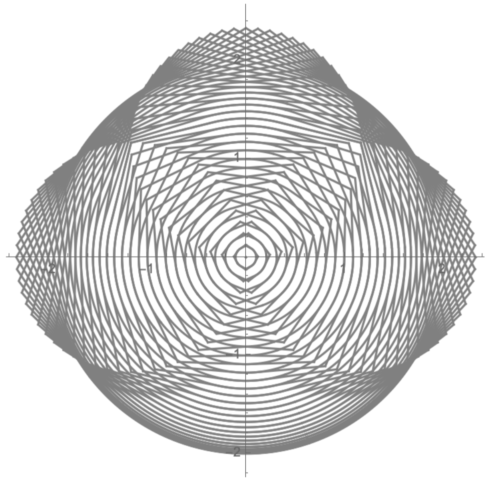

Figure 22: An example of six flat portions when $n = 5$.

We see that the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 22 will have a square shape. Next, consider the plot for

$$A_0 = \begin{bmatrix} e^{i\frac{\pi}{5}} & 0 & 0 & 0 & 0 \\ 0 & e^{i\frac{3\pi}{5}} & 0 & 0 & 0 \\ 0 & 0 & e^{i\pi} & 0 & 0 \\ 0 & 0 & 0 & e^{i\frac{7\pi}{5}} & 0 \\ 0 & 0 & 0 & 0 & e^{i\frac{9\pi}{5}} \end{bmatrix} \text{ and } A_1 = I :$$

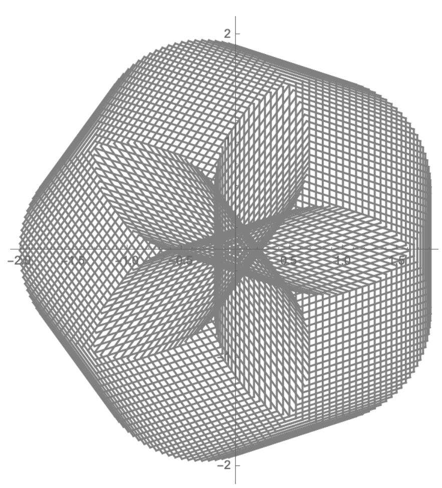

Figure 23: An example of five flat portions when $n = 5$.

Note that the boundary of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 23 appears to have five flat portions prior to taking the convex hull. Next, consider the example with

$$A_0 = A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}:$$

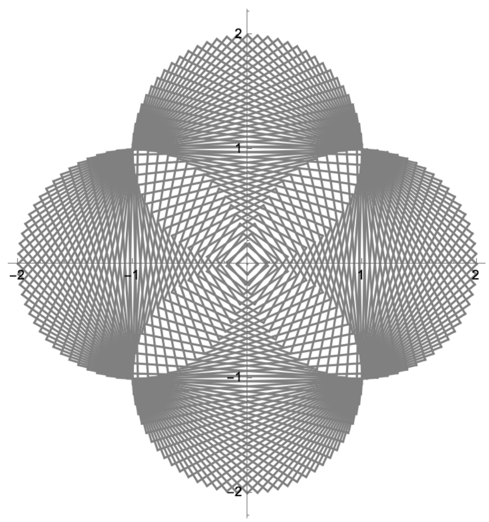

Figure 24: An example of four flat portions when $n = 5$.

Notice that the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 24 will have a square shape. Next, consider the example with

$$A_0 = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{4} \end{bmatrix} \text{ and } A_1 = I:$$

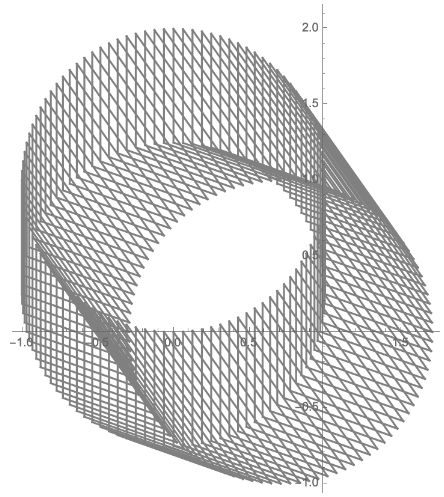

Figure 25: An example of three flat portions when $n = 5$.

Once again, note that the boundary of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 25 appears to have three flat portions prior to taking the convex hull. Next, consider the example with

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 2e^{i\frac{\pi}{8}} & e^{i\frac{\pi}{4}} \\ 0 & 0 & 0 & 0 & 2e^{i\frac{\pi}{8}} \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2}e^{i\frac{\pi}{12}} & e^{i\frac{\pi}{4}} \\ 0 & 0 & 0 & 0 & \frac{1}{2}e^{i\frac{\pi}{12}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}:$$

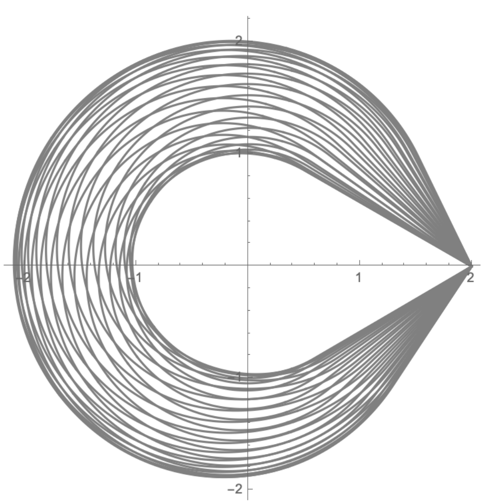

Figure 26: An example of two flat portions when $n = 5$.

Once again, note that the boundary of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 26 appears to have two flat portions prior to taking the convex hull. Next, consider the example with

$$A_0 = A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

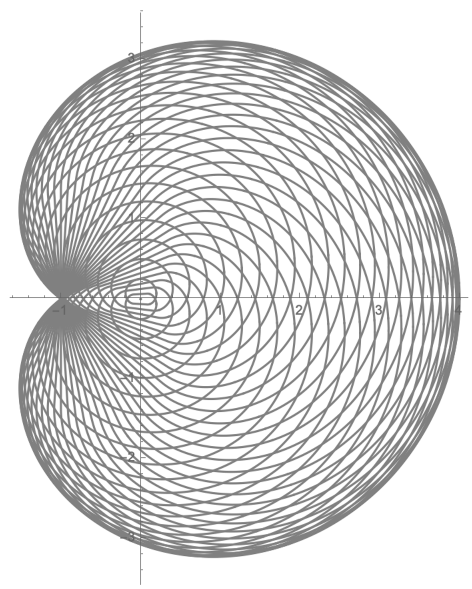

Figure 27: An example of one flat portions when $n = 5$.

Once again, note that $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 27 is reminiscent of the shapes found in Theorems 1 and 2. Finally, consider the example with

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 2e^{i\frac{\pi}{8}} & e^{i\frac{\pi}{4}} \\ 0 & 0 & 0 & 0 & 2e^{i\frac{\pi}{8}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2}e^{i\frac{\pi}{12}} & e^{i\frac{\pi}{4}} \\ 0 & 0 & 0 & 0 & \frac{1}{2}e^{i\frac{\pi}{12}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}:$$

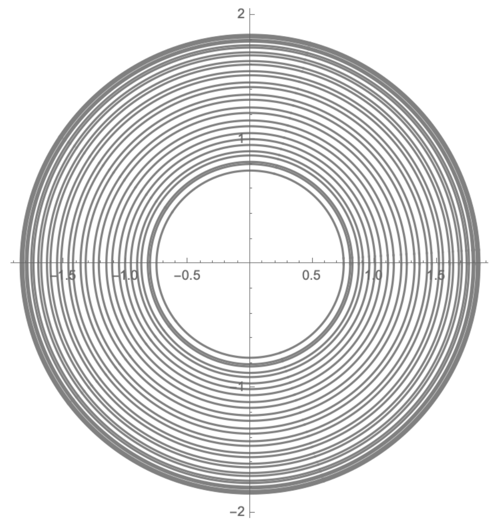

Figure 28: An example of zero flat portions when $n = 5$.

Here the convex hull of $\{W(A_0 + zA_1) : z \in \partial\mathbb{D}\}$ in Figure 28 appears to be a circular disk.

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