# Primer on Nuclear Exchange Models

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**Abstract.** Basic physics is applied to nuclear force exchange models between two nations. Ultimately, this scenario approach can be used to try and answer the age old question of "how much is enough?" This work is based on Chapter 2 of *Physics of Societal Issues: Calculations on National Security, Environment and Energy* (Springer, 2007 and 2014).

## FLAT EARTH

The range of a parabolic missile flight is

$$X = v^2 \sin(2\theta)/g = (10^4 \text{ m/sec})^2 \sin(60^o)/(9.8 \text{ m/sec}^2) = 8800 \text{ km}, \quad (1)$$

with a flight time of

$$t = 2v \sin(30^o) = 10^3 \text{ sec} = 20 \text{ min}, \quad (2)$$

close to the flight time of ICBMs that travel ¼ the Earth's circumference of 10,000 km.

## SPHERICAL EARTH

Elliptical orbits are relatively easy to use, but calculations are more complex when one considers optimum launch angle, gravitational bias error for the non-spherical Earth with concentrated mass (mascons), variable drag forces, and variable rocket thrust. Trajectories can be obtained by Runge Kutta numerical integrations of the basic equations:

$$d^2r/dt^2 - r\,(d\theta/dt)^2 = -GM/r^2, \quad (3)$$

$$d/dt\,(r^2\, d\theta/dt) = 0. \quad (4)$$

The equations in $x$ and $y$ are easier to solve than with those with $r$ and $\theta$ ($R_E$ is the Earth's radius). The equations can be modified for variable thrust, drag forces, the non-spherical Earth, and so forth:

$$d^2x/dt^2 = -x(gR_E^2)/(x^2 + y^2)^{3/2} + F_{other}/m, \quad (5)$$

$$d^2y/dt^2 = -y(gR_E^2)/(x^2 + y^2)^{3/2} + F_{other}/m. \quad (6)$$

## ICBM ACCURACY

The accuracy of a ballistic missile is determined from the following errors:

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(1) terminal velocity $\Delta v = 0.5 \times 10^{-5} v = 0.5 \times 10^{-5}(10^4 \text{ m/sec}) = 0.05$ m/sec, (7)

(2) range, vertical angular error $10^{-5}\theta = (10^{-5})(0.5 \text{ radian}) = 0.5 \times 10^{-5}$ radian, (8)

(3) tracking azimuthal error $\Delta\phi = 10^{-5}$ radian. (9)

The *range error* $\Delta X$ is the product of the fractional range error ($\Delta X/X$) times the range $X$:

$$\Delta X = [2(\Delta v/v) + 2[\Delta\theta/\tan(2\theta)] + (\Delta g/g)]X. \quad (10)$$

Using the above values, *the range error* over the 8800-km range from the velocity error $\Delta v/v$ is

$$\Delta X = 2(\Delta v/v)X = [2(0.05/10^4)](8.8 \times 10^6 \text{ m}) = 88 \text{ m}. \quad (11)$$

The *range error from error in the vertical angle* $\Delta\theta$ is a second order correction when launching at the minimum energy angle of $\theta_{min} = 22^o$ above the horizon for 10,000-km range flights above round Earth. Since we are not at that angle, the range error is

$$\Delta X = 2[\Delta\theta/\tan(2\theta)]X = [2(0.5 \times 10^{-5} \text{ radian})/\tan 60^o](8.8 \times 10^6 \text{ m}) = 51 \text{ m}. \quad (12)$$

If these are random errors, the combined range error is

$$\sigma_x = (88^2 + 51^2)^{1/2} = 102 \text{ m}. \quad (13)$$

If the errors were systematic, the total error could be as large as 139 m. The error in azimuthal angle $\Delta\phi$ gives rise to an error in the tracking direction,

$$\Delta Y = (\Delta\phi)X = (10^{-5})(8.8 \times 10^6 \text{ m}) = 88 \text{ m}. \quad (14)$$

The radial error from the aim point is obtained by combining the range and tracking errors,

$$\sigma_{total} = (101^2 + 88^2)^{1/2} = 130 \text{ m}. \quad (15)$$

## NONSPHERICAL EARTH GRAVITATIONAL BIAS

US and Soviet-Russian ICBMs are intended to travel near the North Pole. Because Earth's polar radius is 21 km (0.3%) smaller than its equatorial radius, guidance computers must take into account the nonspherical Earth. Highly accurate three-dimensional, gravitational multipole-potentials were developed for the Earth by observing variations in satellite orbits. When a satellite approaches a concentrated extra mass, the satellite speeds up slightly and it slows after it passes the mass concentration. Corrections for local gravity at launch sites are important, since slowly rising missiles spend more time near the modified gravitational force.

We consider only the quadrapole term with a simplified approach that uses $\Delta g/g$ to determine the gravitational bias error. To first order, the fractional change in $g$ is proportional to the fractional change in Earth radius, or

$$\Delta g/g \approx \Delta R_E/R_E = 0.003. \quad (16)$$

Because missiles take off and land at about 40$^o$ north latitude, far from the equator, the estimate of the bias error is reduced by a factor of about 3; that is

$$\Delta X = (\Delta g/3g)X = (10^{-3})(8.8 \times 10^6 \text{ m}) \approx 15 \text{ km}, \quad (17)$$

which agrees with accurate estimates. Guidance computers must calculate gravitational bias corrections to better than 1% accuracy because a 15-km error is 100 times larger than 100-m accuracy. The conventional wisdom is that guidance computers can do this calculation.

## KILL PROBABILITY = *f*(*CEP*, *H*, *Y*, *R*, *n*, fratricide)

The "cookie cutter" approximation assumes that a target is destroyed if overpressure exceeds the hardness of the target but it survives if the overpressure is less than the hardness. Reality expects that the step function, cookie cutter probability, which is either 0 or 1, should be smoothed. A two-dimensional Gaussian kill probability density function describes missiles impacting a distance $r$ from a target,

$$p(r) = (1/2\pi\sigma^2)\exp(-r^2/2\sigma^2). \tag{18}$$

In practice the *footprint* of landing missiles is an ellipse, but we will treat it as a circle. If gravitational bias exists, $r$ should be replaced by the vector $(\boldsymbol{r} - \boldsymbol{B})$. By integrating $p(r)$ from $r = 0$ to *CEP* and setting the integrated *single-shot kill probability* (*SSKP*) to 0.5, it is shown that $CEP = 1.18\sigma$. The relation between pressure $p$ and yield $Y$ is obtained by numerically fitting the empirical data as a function of distance $r$ in Glasstone and Dolan[1]

$$p = 14.7(Y/r^3) + 12.8(Y/r^3)^{0.5} \quad \text{(in psi, megaton, nautical miles (1860 m)}, \tag{19}$$

$$p = 6.3(Y/r^3) + 2.20(Y/r^3)^{0.5} \quad \text{(in atmospheres, megaton, km)}. \tag{20}$$

The first term is sufficiently accurate for small distances when attacking silos, but it is not accurate for the greater distances of cities. At close distances the pressure falls with the third power of distance, two powers from geometry and one because the blast pulse width broadens proportional to distance. Using this connection gives the single shot kill probability of warhead destroying a hardened facility.[2]

$$SSKP = 1 - \exp(-Y^{2/3}/0.22H^{2/3}CEP^2), \tag{21}$$

where $Y$ is in Mton, $H$ is in psi and *CEP* is in nautical miles (1860 m). The kill probability for one warhead takes into account reliability of the missile-warhead system,

$$P_{\text{kill-1}} = R \times SSKP. \tag{22}$$

## ACCURACY vs. YIELD

A 1-Mton warhead has an *SSKP* of 90% against a target. By how much can yield be reduced if accuracy is improved by a factor of 2, while retaining the same *SSKP*? Using a fixed *SSKP* argument with $H_1 = H_2$ gives

$$(Y_1/Y_2)^{2/3} = (CEP_1/CEP_2)^2 \quad \text{and} \quad (Y_1/Y_2) = (CEP_1/CEP_2)^3. \tag{23}$$

Thus, a *CEP* reduced by a factor of 2 allows the yield to be reduced by a factor of 8. For our example, this gives a reduced yield of $(1/2)^3$(1 Mton) = 1/8 Mton. US weapon yield was reduced as accuracy was improved by a factor of 4 as Minuteman II (0.2 nmi = 370 m) was replaced with Peacekeeper (0.05 nmi = 90 m). The reduction by a factor of 4 in *CEP* implies that yield could be reduced by $4^3 = 64$, but yield was in fact reduced *only* by a factor of 4 from Minuteman-II to Peacekeeper. The cause of the difference between ratios of 4 and 64 is that the Peacekeeper was designed for harder silos and in an era when higher kill probabilities were sought.

The Soviets always had larger weapons because Soviet accuracy was always surpassed by the US. Even today, the reported accuracy of the SS-18 (0.13 nmi) is about 1/3 that of Peacekeeper's 0.05 nmi. In Senate hearings on SALT, much was made of the large size of Soviet SS-9s as compared to US Minuteman. Senators misled the public by showing large models of Soviet missiles emphasizing launch-weight and yield, but they neglected the two most important parameters, accuracy and reliability.

## ACCURACY vs. HARDNESS

As US accuracy increased, the Soviets moved their ICBMs from launch pads to silos with 300 psi hardness, then to silos with 2000 psi hardness and finally to some silos with a hardness over 5000 psi. During this period US accuracy improved from 1300 m in 1962 to 300 m in 1970 to 90 m in 1986. It is generally accepted that accuracy won the race against hardness. Perhaps super-hardened silos might be able to withstand 20,000 psi, but the cost would become very large. In addition, when crater size becomes similar to *CEP*, the kill mechanism becomes cratering, and not overpressure. US hard-target warheads can produce craters with radii approaching their accuracy.

**Relative Constancy of Hard-Target Yield.** The US and Russia maintain warheads of about one-half Mton for their hard-target weapons. The record was set with the test of the Soviet's 58-Mton weapon in 1962, which was later reported to be but a part of a 100–150 Mton weapon. These parameters are consensus numbers from the International Institute for Strategic Studies.

## RELIABILITY vs. SSKP

Warhead accuracy requires many tasks be carried out reliably. The total reliability of a ballistic missile is the product of the reliabilities for command-control-communication-intelligence ($C^3I$) reliability, missile reliability, and warhead reliability:

$$R_{\text{total}} = R_{\text{C3I}} \times R_{\text{missile}} \times R_{\text{warhead}}. \quad (24)$$

The US Congressional Budget Office quoted a reliability of 85% for US ICBMs. It is generally believed that warheads have a high reliability of greater than 95%, higher than the reliability of the missiles that carry them. The ratio of missile-to-warhead failure rates ($F$) is, perhaps, a factor of 3, from these reliabilities:

$$F_{\text{missile}}/F_{\text{warhead}} = (1 - R_{\text{missile}})/(1 - R_{\text{warhead}}) = (1 - 0.85)/(1 - 0.95) = 3. \quad (25)$$

Consider the case where high-yield, accurate missiles have $SSKP \approx 1$. The survival probability is $1 - P_{\text{kill-1}} = 1 - R$. For hard-target weapons (Peacekeeper, Trident/W88, SS-18, SS-27), the number of surviving targets is essentially determined by the reliability of the attacking system.

## RATE OF CHANGE IN $P_{\text{kill-1}}$

Parameter changes affect kill probabilities. It is useful to take the differential of the single warhead kill probability $P_{\text{kill-1}}$:

$$P_{\text{kill-1}} = R(1 - e^{-\alpha}), \quad (26)$$

where $\alpha = Y^{2/3}/(0.22\ CEP^2\ H^{2/3})$, to obtain

$$\Delta P_{\text{kill-1}}/P_{\text{kill-1}} = \Delta R/R + (2\alpha/3)[\Delta Y/Y - \Delta H/H - 3(\Delta CEP/CEP)]/(e^{\alpha} - 1). \quad (27)$$

**TABLE 1.** Improvements from enhanced *R, Y, H,* and *CEP*. Improvements in one-warhead kill probability, $\Delta P_{\text{kill-1}}/P_{\text{kill-1}}$, from 10% improved reliability, yield, hardness, and accuracy for two situations.

| Attacker | *Y*(Mt) | *H*(psi) | *CEP*(nmi) | *R* | *L* | *P*k1 | *P*k2 | *Δ*P*k1*/P*k1(%):* | *R* | *Y or H* | *CEP* |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A | 0.75 | 2000 | 0.135 | 0.85 | 45 | 62% | 85% | | 10% | 3.3% | 9.8% |
| B | 0.5 | 2000 | 0.05 | 0.9 | 252 | 89.9% | 99% | | 10% | 0.04% | 0.1% |

It takes two A-warheads to accomplish what B can do with one. It follows that A improves its one-warhead kill probability more with 10% improvements than 10% improvements for B. For A, 10% improvements in reliability (D*R*/*R* = 0.1) and accuracy (D*CEP*/*CEP* = 0.1) gives 10% improvements in $P_{\text{kill-1}}$, while a 10% yield increase (D*Y*/*Y* = 0.1) raises $P_{\text{kill-1}}$ by 3.3%. For B, which has much better accuracy, 10% improvement increase $P_{\text{kill-1}}$ by 10% for reliability, 0.04% for yield and 0.1% for *CEP*.

## TWO WARHEADS PER TARGET

The survival probability for one warhead attacking a silo is

$$P_{\text{survive-1}} = 1 - P_{\text{kill-1}} = 1 - R \times SSKP. \quad (28)$$

If two warheads with the same parameters, but coming from different missiles, attack a silo, the survival probability is multiplicative, since the launches are independent actions,

$$P_{\text{survive-2}} = (1 - R \times SSKP)(1 - R \times SSKP) = (1 - R \times SSKP)^2, \quad (29)$$

with a total kill probability of

$$P_{\text{kill-2}} = 1 - (1 - R \times SSKP)^2. \quad (30)$$

Since missile reliability is the most likely failure mode, warheads from different missiles are used to target a silo. A failure of a missile carrying two warheads for one target would cause both the first and second warheads to fail. For the case of $SSKP = 1$ and $R_{\text{missile}} = 0.8$, 80% of the silos would be destroyed and 20% would survive, since second warheads fail with the first failure. If different missiles were used for the two warheads, the kill probability would be raised to 96%:

$$P_{\text{kill-2}} = 1 - (1 - 0.8)^2 = 1 - (0.2)^2 = 96\%, \text{ and } P_{\text{survive-2}} = 4\%. \quad (31)$$

## FRATRICIDE

There are many mechanisms that cause *fratricide*, the killing of one warhead by another: Blast waves and dust can destroy the second warhead; an electromagnetic pulse from the first warhead can destroy the second warhead's electronics; and neutrons from the first warhead can preheat or pre-initiate the second warhead.
For the case of *no fratricide*,

$$P_{\text{kill-2}} = 1 - (1 - R \times SSKP)^2 = (2R \times SSKP) - (R^2 \times SSKP^2). \quad (32)$$

We consider three fratricide situations:

(1) The first warhead destroys the target with a probability of $R \times SSKP$.

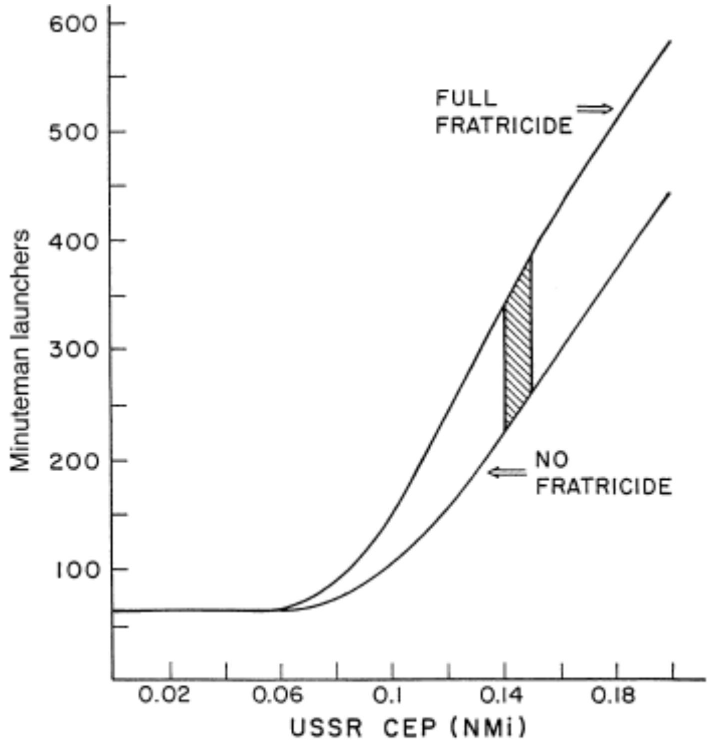
(2) The first warhead misses the target, but destroys the second warhead with reliability $R$.

(3) The first warhead misses the target, but does not destroy the second warhead. For simplicity, we consider *completely effective* fratricide and ignore the third possibility to obtain

$$P_{\text{kill-2-fratricide}} = (2R \times SSKP) - (R^2 \times SSKP). \quad (33)$$

This result slightly differs from $P_{\text{kill-2}}$. For very reliable and lethal weapons ($R = 1$, $SSKP = 1$), fratricide is irrelevant since it takes only one reliable warhead to destroy a silo. However, if reliability is not 1, but the weapons are very lethal with $SSKP = 1$, then two-shot kill probability with fratricide and without fratricide reduces to the same answer, $P_{\text{kill-2-fratricide}} = 2R - R^2 = P_{\text{kill-2}}$. However, when $SSKP$ is not 1 there is a marked difference. In Figure 1 we plot, as an example, the number of surviving silos as a function of accuracy using the above equations for $P_{\text{kill-2}}$ and $P_{\text{kill-2-fratricide}}$. Accuracy is varied while yield and reliability remain fixed. The curves with and without fratricide coincide for accuracy better than 0.06 nautical mile since $SSKP$ approaches 1 at that point, but they separate for larger *CEP*s. The shaded area indicates that, at most, 100 additional silos in 1990 (10% of 1000) could survive because of fratricide.

FIGURE 1. Fratricide. The number of silos that survive calculated as a function of accuracy for two situations: (a) no fratricide and (b) totally effective fratricide when a first warhead misses a target and destroys a second warhead. Totally effective fratricide increases the number of surviving silos by about 10%. These kinds of discussions should include the survivable submarine fleet. [Hafemeister, *Amer. Jour. Physics* 51, 215 (1983)]




## MORE THAN TWO WARHEADS PER TARGET

By simple extension, the *n*-shot kill probabilities is

$$P_{\text{kill-}n} = 1 - (1 - R \times SSKP)^n = 1 - (1 - P_{\text{kill-1}})^n. \tag{34}$$

It does not make great technical sense to use a third warhead on a target when the marginal return is small or when fratricide is increased. For the case of good hard-target weapons, $P_{\text{kill-1}}$ is about 0.9. A second warhead makes an improvement to $P_{\text{kill-2}} = 0.99$, but a third warhead gives only marginal improvement at $P_{\text{kill-3}} = 0.999$. This argument is weakened if $P_{\text{kill-1}}$ is low, say $P_{\text{kill-1}} = 0.5$, giving $P_{\text{kill-2}} = 0.75$ and $P_{\text{kill-3}} = 0.875$. In this case improvements with each additional warhead is larger. Government calculations using conjectured three-warhead targeting by the Soviets were used in Senate testimony by those trying to show vulnerability of US systems.

## HOW MANY NUCLEAR WEAPONS IS ENOUGH?

These estimates are usually based on worst-case-analysis. One usually assumes that these very complicated attacks go as planned. Two–warhead targeting of hardened silos is based on the use of accurate timing to avoid fratricide and other issues. Political debates don't allow arguments that nay-say foreign prowess by saying they are not that competent. But even with worst–case analysis, it easily shown that there is sufficient second-strike prowess remaining after a full-throated attack to deter a sensible attacker. We briefly discuss three types of analysis, from very simple to more complex.

**An equation for the Senate:** The size of the future US arsenal was an issue for the ratification of the Strategic Arms Reduction Treaties. In 1992, the staff of the Foreign Relations Committee estimated the minimum number of surviving warheads after a brutal attack[3] In 1992, it was generally assumed that U.S. nuclear forces under START-1 forces would consist of the following:

- 1400 warheads on ICBMs,
- 3456 warheads on SLBMs and
- 3700 warheads on heavy bombers, for a total of 8500.

The total number of 8500 is gargantuan, much larger than today's number of about 1700. The Senate START-1 report has the following equation for surviving warheads ($WH_{survive}$).

$$WH_{survive} = \text{ICBM } (1400 \times 0.1–0.2) + \text{SLBM } (3456 \times 0.65) = + \text{Heavy Bombers } (3700 \times 0.3)$$

$$= 200 + 2300 + 1100 = 3500 \text{ warheads.} \quad (35)$$

Shortly after START ratification, the US and the Russian Federation agreed on 3500 for the START-2 limit, a 50%reduction from the START-1 limit. On June 26, 1992, *General Colin Powell, Chairman of the Joint Chiefs of Staff, testified the following:*

> We are not dueling with each other, my warhead against your warhead. The question is, does the Untied States force structure give us enough capability to deliver a devastating blow against any nuclear State that may choose to attack us? It if does, then that is a deterrent to that nuclear State ever contemplating such an action.

**More Equations:** Barb Levi and I derived the following equations to describe an attack by red on blue's four types of warheads (WH): silos (l), SLBMs (s), bombers (a) and mobile missiles (m):[4]

$$WH_{used}(\text{red}) = 2L_l(\text{blue}) + 2B_s(\text{blue}) + 16B_a(\text{blue}) + ML_m(\text{blue}), \quad (36)$$

and

$$WH_{destroyed}(\text{blue}) = [1 - (1–R \times SSKP)^2] \times WH_l(\text{blue}) + [1–(1 –R)^2] \times [(1 - f_s) \times WH_s(\text{blue}) +$$

$$(1–f_a) \times W_a(\text{blue})] + R \times (M \times A)/AD \times W_m(\text{blue}). \quad (37)$$

L is the number of missile (s or m) launchers carrying a total of WH warheads, B is the number of bomber bases or SLBM ports, M is the number of warheads dedicated to attacking each mobile missile with devastation area A per warhead, AD is the total mobile dispersal area, f is the fraction of bombers or submarines (a or s) on alert, SSKP is the single-shot-kill probability, R is reliability.

**Exchange Models:** Some models lay down the attacking warheads in a way that maximizes the destruction of warheads by choosing which warhead types to attack the various targets. One such model was called the Price to Attack. Such an application was carried out by Steinbruner, Bing and May.[5]

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