Asymptotic Numbers: II. Order Relation, Infinitesimals and Interval Topology

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It has been shown in [8] that the set of asymptotic numbers $A$ is a system of generalized numbers including isomorphically the set of real numbers $\mathbb{R}$, as well as the field of formal power (asymptotic) series. In the present paper, which is a continuation of [8], an order relation in $A$ is introduced due to $A$ turning out to be a totally-ordered set. The consistency between the order relation and the algebraic operations in $A$ is investigated and in particular, it is shown that the inequalities in $A$ can be added and multiplied as in the set of the real numbers. The notions of infinitesimals (infinitely small numbers), finite and infinitely large numbers are introduced; $A$ turns out to be a non-archimedean set. The usage of infinitesimals as infinitely large numbers along with the real numbers is the reason why the terms and the notions introduced in this paper are very much like those of the non-standard analysis (Robinson's theory of infinitesimals) [7]. In connection with the order relation, an interval topology of $A$ is introduced and some of its properties are established. The theory of asymptotic functions, as well as the applications to the quantum theory, are put off for a future paper.

The notions of the asymptotic numbers [2,4,8] and those of the asymptotic functions [3,8] are introduced as a subsidiary device for investigation of some problems in quantum theory. For further details about the motivation of this work we advise the reader to refer to [2,3,4,5,8]. But the knowledge of [8] is quite sufficient for the understanding of the present paper.

1. Order Relation in the Set of the Asymptotic Numbers

Definition 1 (Order relation). Let $a, b \in A$.

(i) We shall say that $a$ is not larger than $b$ and we write this as $a \leq b$ if for every choice of $a \in A$ there exists $b \in B$ such that $a(s) \leq b(s)$ for all sufficiently small $s \in (0, 1)$ such that $a(s) \leq b(s)$ for all $s \in (0, e)$;

(ii) We shall say that $a$ is smaller than $b$ or that $b$ is larger than $a$ and then we write $a < b$ if $a \leq b$ and $a \neq b$.

Remark: We shall denote by $N, N_0, Z, \mathbb{R}, \mathbb{R}^+$, and $C$ the sets of naturals, integers, real, positive real and complex numbers, respectively. We shall denote by $A$ the set of all real asymptotic numbers (8, Definition 5).

We shall introduce some subsidiary notions with the help of which the above definition could be formulated in a more convenient form.

Definition 2 (Filter $\mathcal{F}$).
(i) We denote by $\mathcal{S}$ the set of all subsets of $(0, 1)$, which contain an interval of the type $(0, \varepsilon)$, $\varepsilon \in (0, 1)$ ($\varepsilon$ is different in general for the different elements of $\mathcal{S}$).

It is easily verified that $\mathcal{S}$ is a filter on $(0, 1)$, i.e. $\mathcal{S}$ possesses the following (filter) properties:

1. $\emptyset \notin \mathcal{S}$,
2. $S, T \in \mathcal{S}$ implies $S \cap T \in \mathcal{S}$,
3. $S \in \mathcal{S}$ and $S \subseteq T \subseteq (0, 1)$ implies $T \in \mathcal{S}$.

With the help of $\mathcal{S}$, Definition 1 could be perphrased as follows (we are going to formulate it as a lemma):

Lemma 1. Let $a, b \in A$. Then $a \leq b$ if and only if for every choice $a \leq a$ there exists $\beta \in b$ such that

$$\{ s: a(s) \leq \beta(s) \} \in \mathcal{S}. \quad (5)$$

Proof. The equivalence between Definition 1 and Lemma 1 follows from the fact that assertion "$\{ s: a(s) \leq \beta(s) \} \in \mathcal{S}$" is equivalent to the assertion "$a(s) \leq \beta(s)$ holds for all sufficiently small $s \in (0, 1)$". Further, we are going to use Lemma 1 together with (1), (2) and (3) first of all.

Remark. Let us note that "$a < b$" is not equivalent to "for every choice of $a \leq a$ there exists $\beta \in b$ such that

$$\{ s: a(s) < \beta(s) \} \in \mathcal{S}. \quad (5)$$

in order to convince ourselves of the above-mentioned remark it would be sufficient to consider the case $a - b$.

Lemma 2. If $a, b \in A$ and $a - b \notin \mathcal{S}$ (8, Definition 5, (v)), then:

i) $a < b$ if and only if for every choice $a \leq a$ there exists $a \leq a$ and $\beta \in b$ ($5$) holds;

ii) $a < b$ if and only if there exists $\alpha \leq a$ and $\beta \in b$ for which ($5$) holds.

Proof. The validity of (5) for all $a \leq a$ and all $\beta \in b$ implies, of course, $a < b$, as well as $a < b$ implies the existence of $a \leq a$ and $\beta \in b$ such that (5) holds, bearing in mind that $\{ s: a(s) = \beta(s) \} \in \mathcal{S}$ is not possible in the case $a - b \notin \mathcal{S}$ ($\{ s: a(s) = \beta(s) \} \in \mathcal{S}$ should mean $a - b \notin \mathcal{S}$).

(i) Let $a \leq b$ and let (5) hold for $a \leq a$ and $\beta \in b$. Let us set (for the sake of convenience) $a - b = c$. Then $c \notin \mathcal{S}$ implies that every representative $\gamma \in c$ can be represented in the form $\gamma(s) = \gamma_0 s^a + \mathcal{A}(s)$, where $\mu \in L$, $\gamma_0 \in R$, $\gamma_0 + 0$ and $\gamma_0$ is the same for all $\gamma \in c$ and $\lim_{s \to 0} \mathcal{A}(s)/s^a = 0$. If $\gamma = a - b$, then we obtain $\gamma_0 < 0$, i.e.

$$\{ s: \gamma(s) < 0 \} \in \mathcal{S} \quad \text{for all } \gamma \in c \quad \text{(but not only for } \gamma = a - b).$$

But the latter means that (5) holds for all $a \leq a$ and all $\beta \in b$;

(ii) Let there exist $a \leq a$ and $\beta \in b$ such that (5) holds. Consequently, there exists $\gamma \in c$ such that $\{ s: \gamma(s) < 0 \} \in \mathcal{S}$, which implies again $\gamma_0 < 0$, i.e. $\{ s: \gamma(s) < 0 \} \in \mathcal{S}$ for all $\gamma \in c$, i.e. (5) holds for all $a \leq a$ and all $\beta \in b$, i.e. $a \leq b$. On the other hand, since $a - b \notin \mathcal{S}$, $a = b$ is impossible, i.e. $a < b$. The proof is finished.

Lemma 3. If $a, b \in A$ and $a - b \notin \mathcal{S}$, then $a \leq b$ if and only if $a \leq b$ (a$\leq b$ means $a \leq b$ or $a = b$).

Proof. $a \leq b$ and even $a < b$ implies, obviously, $a < b$; to believe that we can set $a = \beta \in b$. Let $a \leq b$ and let us assume $a \leq b$. This means, in particular, that for the accuracies $\nu_\alpha$ of $a$ and $\nu_\beta$ of $b$, $\nu_\alpha < \nu_\beta$ holds (and consequently, $\nu_\alpha < \infty$) and every $a \leq a$ can be represented in the form $a = P(a) + \mathcal{A}(s)$, where $P(a)$ is the main part of $a$ (8, 3)), $\lim_{s \to 0} \mathcal{A}(s)/s^a = 0$ and every representative $\beta \in b$ can be represented in the form $\beta(s) - P(a) + \beta_0 s^a + \mathcal{A}(s)$, where $\beta_0$ is a
of \((0,1)\), which contain an in general for the different ele
,1), i. e. \(\delta\) possesses the follow-

\[ T \in \delta, \]

implies \(T \in \delta.\)

erphrased as follows (we are only if for every choice \(a \not\in \delta\).

11 and Lemma 1 follows from equivalent to the assertion \(\sim(s)\)
. Further, we are going to use if all.

equivalent to \(\sim(a \not\in \delta)\).

mentioned remark it would be

\(\sim\) (v), then:

\(a\) and \(b \not\in \delta (5)\) holds;

\(b \not\in \delta\) for which (5) holds.

\(a\) and \(b \not\in \delta\), and \(b \not\in \delta\) such that (5) holds, of possible in the case \(a \not\in \delta\).

\(\beta \not\in \delta\). Let us set (for the sake of every representative \(\gamma \not\in \delta\) can be

\(\gamma \not\in \delta\), if \(\gamma \not\in \delta\) and \(\gamma \not\in \delta\), we obtain \(\gamma \not\in \delta\), i. e.

\(\gamma \not\in \delta\). But the latter means

\(a\) holds. Consequently, there exists again \(\gamma \not\in \delta\), i. e. \(\{s : \gamma(s) < 0\} \not\in \delta\)
11 \(\beta \not\in \delta\), i. e. \(a \not\in \delta\). On the other

\(a \not\in \delta\). The proof is finished.

\(a \not\in \delta\) if and only if \(a \not\in \delta\) (a \not\in \delta)

iously, \(a \not\in \delta\); to believe that we 
\(a \not\in \delta\). This means, in particular,

\(a \not\in \delta\) holds (and consequently, in the form \(a = P(a) + 1, \)
where \((a + 1)^{s/a} \not\in \delta\) and every representative

\((a + 1)^{s/a} \not\in \delta\), where \(\beta_0\) is a
real number and \(\lim_{s \to 0} (a(s)/s^{n+1} = 0. If we choose \(A(s) = s^{n+1}/s, s \in (0,1)\), i. e. \(a(s)\)

\(= P(a) + s^{n+1}/s, s \in (0,1)\), we see that there exists no \(\beta \in \delta\) such that \(\{s : a(s) \leq \beta(s)\} \in \delta\).

The latter contradicts \(a \not\in \delta\). The lemma is proved.

Remark. Let us note that the relation \(a \not\in \delta\) makes sense for the complex
asymptotic numbers too and not only for the real ones (8, Sec. 2).

Theorem 1. The relation \(\leq\) is a relation of non-strictly order in \(A\), i. e.

it is reflexive, anti-symmetric and transitive.

Proof. (i) \(a \leq b\) (reflexive) follows from Lemma 1 by \(a \not\in \delta\) and \(a \not\in \delta\);

(ii) We must prove that \(a \leq b\) and \(b \leq a\) implies \(a \leq b\) (anti-symmetric). Let us

assume \(a \not\in \delta\) and \(b \not\in \delta\), and \(b \not\in \delta\) be chosen arbitrarily. Corresponding to Lemma 2, the sets \(S = \{s : a(s) = \beta(s)\}\) and \(T = \{s : a(s) = \beta(s)\}\) belong to \(\delta, i. e.

\(S, T \in \delta\) and corresponding to (2), \(S \cap T \in \delta\). On the other hand, \(S \cap T = \{s : a(s) = \beta(s)\}\), i. e. \(a \not\in \delta\) which is a contradiction. So we conclude that \(a \not\in \delta\). The

Corresponding to Lemma 3, \(a \not\in \delta\) and \(b \not\in \delta\) are reduced to \(a \not\in \delta\) and \(b \not\in \delta, i. e.

\(a = b = \beta\);

(iii) Let \(a \leq b\) and \(b \leq a\). We must prove \(a \leq c\) (transitive). Let \(a \not\in \delta\) be chosen arbitrarily. Corresponding to Lemma 1, there exist \(a \not\in \delta\) and \(c \not\in \delta\), and \(c \not\in \delta\) such that the sets \(S = \{s : a(s) \leq \beta(s)\}\) and \(T = \{s : a(s) \leq \beta(s)\}\) are elements of \(\delta\). Hence, with the help of (2), we obtain \(S \cap T \in \delta\). Let us set \(U = \{s : a(s) \leq \beta(s)\}\). Obviously, \(S \cap T \subseteq U \subseteq (0,1)\), which is corresponding to (3), leads to \(U \in \delta\), i. e. \(a \leq c\). The proof is completed.

Theorem 2. The relation \(\prec\) is a relation of strict order in \(A\), i. e.

\(\sim\) it is:

(i) \(a \not\in \delta\) for all \(a \in A\) ("\(\sim\) means "the logical contradiction");

(ii) \(a \not\in \delta\) for all \(a \not\in \delta\) implies \(b \not\in \delta\);

(iii) \(a \not\in \delta\) for all \(a \not\in \delta\) and \(b \not\in \delta\) implies \(a \not\in \delta\). Furthermore, \(a \not\in \delta\) if and only if \(a \not\in \delta\) or \(a = b\).

Proof. The theorem is a (standard) consequence of Definition 1 and Theorem 1.

Theorem 3. Every two asymptotic numbers \(a\) and \(b\) are orderable, i. e. for

every \(a, b, \not\in \delta\) one (and only one) of the assertions \(a \not\in \delta\) holds.

(In other words, the order in \(A\) is linear or, which is the same, the order

possesses the property of trichotomy.)

Proof. Let \(a, b \in \delta\) and let us set \(a \not\in \delta - c\). If \(c \not\in \delta\) (i. e. \(a \not\in \delta - b\), then

there exists \(\gamma(s) = \beta(s)\), such that every representative \(\gamma \not\in \delta\) can be

represented in the form \(\gamma(s) = \gamma(s) + 1\), where \(\mu \in \delta\) is the power of \(c\) and \(\lim_{s \to 0} (a(s)/s^{n+1} = 0. If \(\gamma(s) < 0\), then \(\{s : a(s) \leq \beta(s)\} \in \delta\)
and \(\{s : a(s) \leq \beta(s)\} \in \delta\) for all \(a \not\in \delta\) and all \(\beta \not\in \delta\), i. e.

\(a \not\in \delta\). Let \(c \not\in \delta\). Then, corresponding to (8, Theorem 3), \(a \not\in \delta\) and \(b \not\in \delta\) and (one and only one) of the relations \(a \not\in \delta\) holds; these relations are equivalent to \(a \not\in \delta\), \(a \not\in \delta\) and \(b \not\in \delta\), respectively. The theorem is proved.

Definition 3 (Positive and negative numbers). Let \(a \in A\). We say that \(a\) is positive if \(a \not\in \delta\) and we say that \(a\) is negative if \(a \not\in \delta\). The set of all positive

asymptotic numbers is denoted by \(A_+\).

Remark. Corresponding to (8, Theorem 6), \(0 = 0^{n+1}\) is the zero element of

\(A\). The use of the notation \(0^{*}\) instead of \(0^{n+1}\) is based on (8, Definition 12).

Theorem 4. (i) If \(a \in A\), \(a \not\in \delta\), then \(a \not\in \delta\) if and only if \(a = 0\) (respectively), where \(\mu\) is

the power of \(a\) and \(a_\mu\) is the corresponding (for \(k = \mu\) coefficient in the main

part of \(a\) (8, (3));
(ii) The inequalities $0 - 0 < \cdots < 0 < 0 - 1 < \cdots$, $\forall \in \mathbb{Z}$, hold.

**Proof.** The theorem could be proved just like the theorems exposed so far. We notice that the inequalities $a < 0$ have sense in $A$ and $a, 0 \in \mathbb{A}$.

**Corollary 1.** If $a, b \in A$ and $a - b \notin \mathbb{A}$, then $a \leq b$ if and only if $a - b < 0$.

**Corollary 2.**
(i) Every asymptotic zero different from 0 is positive, i.e. $0 > 0$ for all $\forall \in \mathbb{Z}$.
(ii) If $a < 0$, then $a \notin \mathbb{A}$.

**Theorem 3.** Let $a, b, c \in A$. Then: (i) $a < b$ implies $a + c < b + c$; (ii) if $a \neq 0$ and $b \neq 0$, then $a > 0$ implies $a \cdot b > 0$.

**Corollary.** If $a, b \in A$ and $a, b \notin \mathbb{A}$, then $a + b \notin \mathbb{A}$.

**Theorem 5.** Let $a, b \in A$ and $a - b \notin \mathbb{A}$, then $a \leq b$ if and only if $v_a \geq v_b$ (respectively), where $v_a$ and $v_b$ are the accuracies ($8$, Definition 5, (iii)) of $a$ and $b$ respectively.

**Proof.** Consequence of the preceding theorem (or of ($8$, Theorem 3)).

**Corollary.** If $1^a$ and $1^b$, $\lambda^a$, $\lambda^b \in I$ are two asymptotic units, i.e. $1^a$, $1^b$ ($8$, Def. 5, (vi)), then $1^a \leq 1^b$ if and only if $\lambda^a \geq \lambda^b$.

**Theorem 6.** (i) If $a \in A$, then $a \leq 0$ implies $-a > 0$ ($8$, Definition 9);
(ii) if $a \notin A$ and $b \notin \mathbb{A}$, then $a > 0$ implies $a \cdot b > 0$. (We shall recall that $-0 = 0$ ($8$, Theorem 11)).

**Proof.** (i) Let $a \leq 0$.
Corresponding to Corollary 2 of Theorem 4, $a \notin \mathbb{A}$ and consequently, $a \leq 0$ which follows to $-a > 0$, i.e. $-a > 0$, corresponding to Theorem 4;
(ii) is proved analogously to (i).

**Theorem 7.** Let $a \in A$ and $a \notin \mathbb{A}$. The reciprocal number $a^{-1}$ ($8$, Definition 9) is positive (negative) if and only if $a$ is positive (negative).

**Proof.** The theorem follows directly from Theorem 4, bearing in mind, $a \notin \mathbb{A}$.

2. Order Relation and Algebraic Operations

Now we are going to discuss the consistency between the order and the algebraic operations in $A$. We should like to point out in advance that the essential points of this section are Theorem 16, Theorem 17 and Theorem 18. The reader, who is interested only in the final results, could pay attention to these three theorems only (as well as Theorem 21, perhaps).

**Theorem 8.** If $a, b, c \in A$, then: (i) $a \leq b$ implies $a + c \leq b + c$; (ii) $a \geq 0$ and $b \geq 0$ implies $a \cdot b \geq 0$.

**Proof.** (i) Let $a \leq b$ and let $a \in A$ and $c \in \mathbb{C}$ be chosen arbitrarily and for $\beta \in b$ such that $s : \alpha(s) \leq \beta(s) \in \mathbb{C}$ holds. Obviously, $s : \alpha(s) + \gamma(s) \leq \beta(s) + \gamma(s)$, which implies $a + c \leq b + c$;
(ii) Let $a \geq 0$ and $b \geq 0$. If (at least) one of the numbers $a$ and $b$ is asymptotic zero, then $a \cdot b \geq 0$ is obvious. Let $a, b \notin \mathbb{A}$. Corresponding to Lemma 2 (i), for every choice of $a \in A$, $\beta \in b$, $A \in 0 \in \mathbb{A}$ and $A \in 0 \in \mathbb{A}$, $s : \alpha(s) \geq A(s) \in \mathbb{C}$ and $s : \beta(s) \geq A(s) \in \mathbb{C}$ hold. Let $A(s), A(s) \geq 0$ on $(0,1)$ (for example, $A(s) = A(s) = \exp(-1/s)$ on $(0,1)$). We have that

$$\{ s : \alpha(s) \geq A(s) \} \cap \{ s : \beta(s) \geq A(s) \} \subseteq \{ s : \alpha(s) \cdot \beta(s) \geq A(s) \cdot A(s) \}$$

holds, from which, bearing in mind (2) and (3), we obtain $\{ s : \alpha(s) \cdot \beta(s) \geq A(s) \cdot A(s) \} \in \mathbb{C}$. The latter means $a \cdot b \geq 0$, bearing in mind Lemma 2 (ii) and (8, Theorem 18). The proof is completed.
... \forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } |a_n - a| < \epsilon \text{ for all } n \geq N.

Corollary. \( a^2 \geq 0 \) for all \( a \in A \) and \( a^2 = 0 \) if and only if \( a = 0 \).

Theorem 9. If \( a, b, c, d \in A \), then \( a \leq b \) and \( c \leq d \) implies \( a + c \leq b + d \).

Proof. The proof is standard (i.e., the assertion of the theorem follows from Theorem 8 as in any ring or field): \( a \leq b \) implies \( a + c \leq b + c \); \( c \leq d \) implies \( c + b \leq d + b \) from which \( a + c + b \leq b + d \) follows.

Definition 4. Let \( A \times A \times A \) be the set of all triples \( (a, b, c) \), where \( a, b, c \in A \).

We shall separate the following subsets of \( A \times A \times A \):

\[
D_1 = \{(a, b, c) : \mu_a < \mu_b, \mu \in \mathbb{N} \setminus \{0\}\},
\]

\[
D_2 = \{(a, b, c) : a \leq b, c < 0\},
\]

\[
D = D_1 \cup D_2,
\]

where \( \mu_a \) and \( \mu_b \) are the powers \( (8, \text{Definition 5, (iii)}) \) of \( a \) and \( b \) respectively.

Theorem 10. If \( a, b, c \in A \) and \( (a, b, c) \notin D \), then \( a \leq b \) implies \( a \cdot c \leq b \cdot c \) by \( c < 0 \) and \( a \leq b \) implies \( a \cdot c \leq b \cdot c \) by \( c \leq 0 \).

Proof. (i) \( a \cdot c - b \cdot c \notin \mathbb{O} \). Bearing in mind the generalized distributive law \((8, 21)\):

\[
(b-a) \cdot c \leq b \cdot c - a \cdot c \quad (r \text{ is the accuracy of } b \cdot c - a \cdot c),
\]

we get \( (b-a) \cdot c \notin \mathbb{O} \) if \( a \cdot c \geq b \cdot c \), which was required.

(ii) \( a \cdot c - b \cdot c \in \mathbb{O} \). If we assume also that \( c \notin \mathbb{O} \), this implies either \( a \cdot c \leq b \cdot c \) or \( a \cdot c = b \cdot c \). If \( c \in \mathbb{O} \), then \( a \cdot c \leq b \cdot c \) (the case \( a \cdot c = b \cdot c \) is analogous).

Theorem 11. If \( a, c, b \in A \) and \( (a, b, c) \notin D \), then \( a \leq b \) implies \( a \cdot c \leq b \cdot c \) (in spite of \( c < 0 \)).

Proof. In fact, in this case \( a \cdot c - b \cdot c \) is reduced (because of \( \mu_a < \mu_b \) and \( c \notin \mathbb{O} \)) to \( 0^{a^+ r} > 0^{b^+ r} \), where \( r \) is the accuracy of \( c \), i.e. \( 0^{a^+ r} > 0^{b^+ r} \), by \( c \notin \mathbb{O} \).

Theorem 12. If \( (a, b, c) \in D \), then \( a \leq b \) implies \( a \cdot c \leq b \cdot c \) (in spite of \( c < 0 \)).

Proof. In this case, corresponding to \((7)\), \( a \leq b \) is reduced to \( a \in b \) and is satisfied by \( a \in b \) and \( c \notin \mathbb{O} \).

Theorem 13. If \( a, b, c, d \in A \), then \( 0 \leq a \leq b \) and \( 0 \leq c \leq d \) implies \( a \cdot c \leq b \cdot d \).

Proof. First of all, \( 0 \leq a \leq b \) and \( 0 \leq c \leq d \) implies \( (a, b, c) \in D \), \( (a, b, c) \notin D \). Indeed, corresponding to Theorem 4 both \( a \leq b \) and \( \mu_a < \mu_b \) are possible only in the case \( c \cdot a < 0 \), \( c \cdot b < 0 \), where \( c \cdot a \) and \( c \cdot b \) are the first coefficients in the main parts \((8, 3)\) of \( a \) and \( b \), respectively.

However, by Theorem 4 both \( a \leq b \) and \( \mu_a < \mu_b \) are possible only in the case \( a \leq b \) and \( \mu_a < \mu_b \) are possible only in the case \( c \cdot a < 0 \), \( c \cdot b < 0 \), where \( c \cdot a \) and \( c \cdot b \) are the first coefficients in the main parts \((8, 3)\) of \( a \) and \( b \), respectively.

Indeed, corresponding to Theorem 4 both \( a \leq b \) and \( \mu_a < \mu_b \) are possible only in the case \( c \cdot a < 0 \), \( c \cdot b < 0 \), where \( c \cdot a \) and \( c \cdot b \) are the first coefficients in the main parts \((8, 3)\) of \( a \) and \( b \), respectively.

We multiply the first inequality by \( a \) and the second by \( b \) and we obtain \( a \cdot c \leq b \cdot c \) and \( c \cdot b \leq b \cdot d \), i.e. \( a \cdot c \leq b \cdot d \). The proof is completed.

Corollary. If \( a, b \in A \), then \( 0 \leq a \leq b \) implies \( a^n \leq b^n \) for every \( n = 1, 2, \ldots \)

The strict inequality "<" in \( A \) has, with respect to the algebraic operations,
properties quite analogous to those of the non-strict one. Because of the strange algebraic properties of "<" however, we can obtain an equality after adding an asymptotic number to a strict inequality (to both sides of a strict inequality); this property does not have an analogue in the set of real numbers ε. The following lines are devoted just to this special feature of the strict inequalities in A.

Definition 5. We shall denote by $E_+$ the following subset of AXAXA:

$$E_+ = \{(a, b, c) : a - b \not\in \mathbb{N}, \mu_{a - b} > r_c \} \cup \{(a, b, c) : a - b \not\in \mathbb{N}, \mu_{a - b} = r_c \},$$

where $\mu_{a - b}$ is the power of $a - b$ and $r_c$ is the accuracy of $c$.

Theorem 14. If $a, b, c \in A$, then $a < b$ implies $a + c < b + c$ in the case $(a, b, c) \in E_+$ and $a + c = b + c$ in the case $(a, b, c) \in E_-$.

Proof. We showed already that $a < b$ implies $a + c < b + c$. Consequently, we must only specify the cases $(a, b, c) \in E_+$ and $(a, b, c) \in E_-$. To this end it is convenient to put down $a, b$ and $c$ in the normal additive form (8, Definition 13):

$$a = \sum_{n \in \mathbb{N}} a_n s^n + 0^*a, \quad b = \sum_{k \in \mathbb{N}} b_k s^k + 0^*b, \quad c = \sum_{h \in \mathbb{N}} c_h s^h + 0^*c,$$

where $\mu_a, \mu_b$ and $\mu_c$ are the powers of $a, b$ and $c$ respectively, $\mu = \min(\mu_a, \mu_b)$, $a_n > 0, \mu < n < \mu_b$ and $b_n = 0, \mu \leq n < b$ and $r_a, r_b$ and $r_c$ are the accuracies of $a, b$ and $c$ respectively. We obtain $a_n = b_n, \mu < n < a - b$ and in the case $a < b$, $a - b \not\in \mathbb{N}$ (i.e. $a < b$ and $b$ belong also). It is sufficient to use (8, Theorem 3) only. The proof is completed.

Definition 6. We shall denote by $E$ the following subset of AXAXA:

$$E_0 = \{(a, b, 0) : a, b \in A \} \cup \{(a, b, c) : a, b, c \in A, \mu_a = \mu_b, \mu_{a - b} - \mu > \mu_c \},$$

where $\mu_a, \mu_b$ and $\mu_{a - b}$ are the powers of $a, b$ and $a - b$, respectively, and $\mu_c$ is the relative accuracy of $c$.

Theorem 15. If $a, b, c \in A$, then $a + b \implies a + c \leq b + c$ if $a, c \in E_0$ and $a, c \in E_0$. The proof is quite analogous to that of Theorem 14. We shall omit it. In spite of the results of Theorem 14 and Theorem 15 most of the properties of the strict inequalities in $\mathbb{R}$ are valid in $A$ too, e.g.:

Theorem 16. If $a, b \in A$, then $a > 0$ and $b > 0$ imply $a - b > 0$.

Proof. Corresponding to Theorem 8, we have $a, b > 0$. Furthermore, $A$ does not have divisors of zero (8, Theorem 7), i.e. $a, b > 0$, which implies $a - b > 0$.

Theorem 17. If $a, b, c, d \in A$, then $a < b$ and $c < d$ implies $a + c < b + d$.

Proof. Let $a < b$ and $c < d$. We shall show at first that (at least) one of the triples $(a, b, c)$ and $(c, d, b)$ does not belong to $E_+$. Let us assume the opposite, i.e. that $(a, b, c), (c, d, b) \in E_+$.

(i) Let (at least) one of the differences $a - b$ and $c - d$ not be an asymptotic zero — let us assume, for instance, that $a - b \notin \mathbb{N}$, (9) implies $r_b \geq \min(r_a, r_b) = r_{a - b} \geq \mu_{a - b} - r_c = \min(r_c, r_d) = r_{c - d} \geq \mu_{c - d} - r_h$, i.e. $r_b > r_h$ which is a contradiction;

(ii) Let $a - b \notin \mathbb{N}$ and $c - d \notin \mathbb{N}$. We obtain $r_a > r_b$ and $r_a > r_h$ corresponding to Theorem 5 and $\mu_{a - b} = r_c$ and $\mu_{c - d} = r_h$ corresponding to (9). On the other hand, $r_k = \min(r_a, r_b) = r_{a - b} \geq \mu_{a - b} - r_c$ and $r_k = \min(r_c, r_d) = r_{c - d} \geq \mu_{c - d} - r_h$, i.e. $r_k \geq r_h$, which contradicts $r_h \geq r_k$. The subsidiary assumption is proved. Without loss of generality we can assume that $(a, b, c) \notin E_+$. We obtain $a + c < b + c$ and $c + b \leq d + b$, corresponding to Theorem 14 and Theorem 9 respectively, i.e. $a + c < b + d$. The proof is finished.
Theorem 18. If $a, b, c, d \in A$, then $0 < a < b$ and $0 < c < d$ implies $a < c < b$. 
Proof. Let $0 < a < b$ and $0 < c < d$.
(i) We shall show that (at least) one of the triples $(a, b, c)$ and $(c, d, b)$ does not belong to $E_r$. Let us assume that $(a, b, c), (c, d, b) \in E_r$. We obtain $\mu_a = \mu_b = \mu_c = \mu_d = \mu_c = -\mu_d = -\mu_c > \lambda_b$, corresponding to (10). We have also: $\lambda_b = \min (\lambda_a, \lambda_b) = \mu_d = -\mu_c = -\mu_c = \lambda_b$, i.e., $\lambda_b > \lambda_b$, which is a contradiction. Further, let us assume that $(a, b, c) \notin E_r$. On the other hand, as we already showed by proving Theorem 13 $(a, b, c), (c, d, b) \notin D$. Consequently, according to Theorem 10 and Theorem 15, $c < d < b$ and $a < c < d$, i.e., $a < c < b$. The theorem is proved.

Corollary. If $a, b \in A$ and $n \in N (N = \{1, 2, \ldots \})$, then $0 < a < b$ implies $a^n < b^n$.

Theorem 19. Each positive asymptotic number cannot possess more than one square root.
Proof. Let $a > 0$ and $a = b^2$ and $0 < a < b$. From the above Corollary we obtain $b^2 < c^2$, which contradicts $b^2 = c^2$.

Theorem 20. If $a, b \in A$, then $a < b$ implies $a < (a + b)/2 < b$ and $(a + b)/2 < b$ only in the case $a < b \in (2 \in A^\infty (8, Definition 12))$.
Proof.
(i) If $a < b \notin \mathcal{C}$, then, corresponding to the above theorem, $a < c < b$ for $c = (a + b)/2$.
(ii) Let $a < b$ and $a < b \notin (i, e. a < b)$. Corresponding to Theorem 5, $v_a > v_b$, where $v_a$ and $v_b$ are the accuracies of $a$ and $b$ respectively. Let $\epsilon$ be an arbitrary positive asymptotic number different from asymptotic zero, i.e., $\epsilon \notin 0$, with power $\mu_c = v_a$. Then $c = a + \epsilon$ possesses the property (we are looking for) $a < c < b$. Indeed, the positivity of $\epsilon$ implies $a < c$ and $\mu_c = v_a$ implies $a < c$, i.e., $a < c$.

Remark. The condition $\epsilon \notin \mathcal{C}$ was required in the above proof in order to comprise the cases $a < b \notin \mathcal{C}$ and $a < v_a + 1$.

The theorems exposed so far show that the properties of the order in $A$ with respect to the algebraic operations in $A$ are distinguished from the properties of any order ring or field (see, for instance, [1]). As we saw, however, the peculiarities of the order in $A$ (in respect to $\mathfrak{A}$, for instance) refer to rather "narrow" subsets of AXAXA — the sets $D, E_b$, and $E_r$. This gives us a possibility to work in most cases with the inequalities in $A$ by the usual (as in $\mathfrak{A}$) rules. In this respect the non-strict inequality "$\leq"$ is more convenient because the peculiarities mentioned above refer only to the set $D$.

We shall define some notions directly connected with order relation of the asymptotic numbers.

Definition 7 (Magnitude). By the term magnitude $|a|$ of a given asymptotic number $a$ we shall understand the extension of the function $|x|, \forall x \in \mathfrak{A}$ on $A$ at the point $x = a$ (8, Definition 8).
Remark. Corresponding to (8, Definition 8), in order to obtain \(|a|, a \in A\) we must form the set \(a \in A \mid x \in A\). The smallest asymptotic number with respect to the inclusion “\(\subset\)” which covers \(|a|\), i.e. \(a \subset a\), is the magnitude \(|a|\) of \(a\). We shall recall that we called a point “perfect” if \(a^* = a\) and “imperfect” if \(a^* \subset a\) (strict).

**Theorem 22.**

(i) \(a\) exists and \(a \in A\) for every choice of \(a \in A\). The numbers \(a \in A\) are perfect and the numbers \(a \in A\) are imperfect points of the magnitude.

(ii) Moreover, \(a = \max (-a, a)\) holds for every \(a \in A\).

**Proof.**

(i) Let \(a \in A \setminus A\) and let \(u\) and \(u\), be the power and the corresponding coefficient in the main part (8, (3)) of \(a \in A\) implies \(a \notin Z\) and \(a_n \neq 0\). Every representative \(a \in A\) can be represented in the form \(a(s) = a_n s^n + A(s)\), where \(\lim_{s \to 0} (s^n s^n) = 0\). Hence, we draw the conclusion that the point \(s = 0\) is not a non-trivial adherent point of the set of all zeros of \(a\) (i.e. all points \(s\) at which \(a(s) = 0\)). Consequently, for every \(a \in A\) there exists \(E \in A\) such that \(a(s)\) does not change its sign on \(E\). Besides, the sign of \(a(s)\) on \(E\) does not depend on the choice of \(a \in A\), namely, \(a(s) > 0\) if \(a_n > 0\) and \(a(s) < 0\) if \(a_n < 0\). That means \(a(s) = -a(s)\) on \(E\) if \(a_n > 0\) and \(a(s) = -a(s)\) on \(E\) if \(a_n < 0\). The theorem is proved in the considered case;

(ii) Let \(a = 0^* \in A\). We have (obviously) \(0^* \subset 0^*\) (strict), from which follow the existence of \(0^*\), as well as the imperfectness of \(a = 0^*\). Furthermore, we obtain \(0^* = 0^*\), i.e. \(0^* = \max (0^*, -0^*)\) holds too (because \(0^* > 0\) and \(-0^* = 0\)). The proof is completed.

**Theorem 23.** If \(a, b, c \in A\), then:

\[
\begin{align*}
|a| \geq 0 \quad & \text{and} \quad |a| = 0 \quad \text{if and only if} \quad a = 0, \\
|a - b| \leq |a + b| \leq |a| + |b|, \\
|a \cdot b| = |a| \cdot |b|, \\
|a / b| = a / b, \quad b \not\in A.
\end{align*}
\]

The proof is analogous to that of Theorem 22 — we shall not give it.

**Remark.** Definition 7, Theorem 22, (i) and Theorem 23 are valid not only for real asymptotic numbers but also for the complex ones (8, sec. 2).

3. Order Relation in Some Subsets of \(A\)

In (8, Definition 10) we defined the sets of asymptotic numbers \(A^0, A^\infty\) and \(A\) and we proved (8, Theorem 20) that \(A^0\) and \(A^\infty\) are isomorphic (with respect to the algebraic operations) to the field of real numbers \(R\) and \(\tilde{R}\) maps homomorphically on \(R\). The correspondences realizing the isomorphism and homomorphism mentioned above were given by:

\[
\begin{align*}
(11) \quad R^0 \ni a = x + 0^* \leftrightarrow x \in A^0, \quad \nu = 0, \infty, \\
(12) \quad \tilde{R} \ni a = x + 0^* \leftrightarrow x \in A, \quad \nu = 0, 1, \ldots, \infty
\end{align*}
\]

respectively, where the numbers, from \(A^0, A^\infty\) and \(\tilde{R}\) are written in their normal additive form (8, Definition 13). Now we shall consider the properties of the sets \(A^0, A^\infty\) and \(\tilde{R}\) as well as the correspondences (11) and (12) with respect to the order relation in \(A\) and \(R\) respectively.

**Theorem 24.** \(A^0\) and \(A^\infty\) are order fields. The correspondence (11) preserves the order relation (strict and non-strict) in \(A\) and \(R\) respectively, i.e.
in order to obtain \(|a|, a \in \mathbb{A}

\) is the smallest asymptotic number with \(a, \) i.e. \(a \leq a,\) is the magnitude point "perfect" if \(|a|^2 = a\) and \(a \neq \mathbb{0}\).

The numbers \(a \neq \mathbb{0}\) are points of the magnitude; every \(a \in \mathbb{A}\).

Weer and the corresponding coefficients \(\mu \in \mathbb{Z}\) and \(a_\mu \neq 0\). Every \(a = \sum_{\mu} a_\mu s^\mu, a \in \mathbb{A}\) exists \(E \in \mathbb{E}\) such that \(a(s)\) does not depend on the \(s \in \mathbb{E}\) if \(a_\mu < 0\). That \(a < a(s)\) on \(E\) if \(a_\mu < 0\). The theorem:

\((\mu)\) (strict), from which follow the ss distribution of \(a\) at \(a=0\). Furthermore, it holds too (because \(0^+ > 0\) and \(-0^+\)

only if \(a=0,\)

we shall not give it. Theorem 23 are valid not only \(\mathbb{R}\) complex ones (8, sec. 2).

of \(\mathbb{A}\)

to the field of real numbers \(\mathbb{R}\) and \(\mathbb{A}\) are isomorphic (with \(\mathbb{E}\) field of real numbers \(\mathbb{R}\) and \(\mathbb{A}\) makes it possible to realize the isomorphism given by:

\(p = 0, \infty,\)

\(p = 0^+ \cdots, \infty\)

and \(\mathbb{A}\) are written in their normal forms. The correspondence (11) and (12) with respectively, i.e.

\(a, b \in \mathbb{A}, p = 0, \infty\) and \(x, y \in \mathbb{R}\) are their images by (11) resp., then \(a < b\) if and only if \(x < y\).

\(\text{Proof.}\) The first part of the theorem — that \(\mathbb{R}\) and \(\mathbb{A}\) are order fields— follows directly from Theorem 8. Further, let us take note that if \(a, b \in \mathbb{R},\)

\(p = 0, \infty,\) then \(a + b \in \mathbb{A}\) and \(a - b \in \mathbb{A}^\circ\) are equivalent to each other. That proves the theorem.

\(\text{Theorem 25.}\) The mapping (12) of \(\mathbb{R}\) on \(\mathbb{A}\) preserves the non-strict order, i.e. if \(x, y \in \mathbb{R}\) are the images of \(a, b \in \mathbb{A}\) by (12) respectively, then \(a \leq b\) implies \(x \leq y\).

The proof is analogous to that of Theorem 23. We notice that if \(a, b \in \mathbb{A}\)

and \(a + b \in \mathbb{A}\), then \(a < b\) implies \(x = y\) (but not \(x < y\)), i.e. (12) does not preserve (in general) the strict inequality.

In (8, Theorem 23) we showed that the set \(\mathbb{A}\) of all asymptotic numbers with infinite accuracies is a field which is isomorphic to the field of the formal power (asymptotic) series (with real coefficients in the case of real asymptotic numbers). With respect to the order, the following theorem holds:

\(\text{Theorem 26.}\) \(\mathbb{A}\) is an order field. Furthermore, if \(a \in \mathbb{A}\), then \(a = 0\) if and only if \(a_\mu = 0\) respectively, where \(\mu\) is the power and \(a_\mu\) is the first coefficient of the main part of \(a\) respectively.

\(\text{Proof.}\) The theorem is an immediate consequence of Theorem 8. In (8, Sec. 1) we introduced an additive (8, Definition 13):

\(a = q + 0^+, a \in \mathbb{A}, q \in \mathbb{A}^0, 0^+ \in \mathbb{A}\)

and a multiplicative (8, Definition 14):

\(a = r. 1^+, a \in \mathbb{A} \setminus \mathbb{0}, r \in \mathbb{A}^0, 1^+ \in \mathbb{I}\)

form of the \(a\). Further, we developed an algebraic technique (8, Sec. 6) with the symbols (11) and (12), which allowed us to express the algebraic operations and algebraic properties of \(A\) by the algebraic operations and algebraic properties of \(\mathbb{A}\), \(\mathbb{0}\) and \(\mathbb{I}\). The sense of this approach lies in the fact that \(\mathbb{A}\) is a field (8, Theorem 23) and, as we know from Theorem 26, \(\mathbb{A}\) is an order field, and \(\mathbb{0}\) and \(\mathbb{I}\) have comparatively simple algebraic properties, including comparatively simple order (see Theorem 3, and Corollary of Theorem 5). The following theorems allow us to express the order in \(A\) by the order of \(\mathbb{A}\), \(\mathbb{0}\) and \(\mathbb{I}\).

\(\text{Theorem 27.}\) If \(a = q + 0^+\) and \(a' = q + 0^+,\) then \(a < a'\) if and only if:

(i) \(q < q'\) in the case \(a - a' \in \mathbb{0}\);

(ii) \(0^- < 0^-\) in the case \(a - a' \in \mathbb{0}\).

\(\text{Theorem 28.}\) If \(a = r. 1^+\) and \(a' = r. 1^+\), where \(a, a' \in \mathbb{A} \setminus \mathbb{0},\) then \(a < a'\) if and only if:

(i) \(r < r'\) in the case \(a a' \in \mathbb{0}\);

(ii) \(1^+ < 1^+\) in the case \(a a' \in \mathbb{I}\).

\(\text{Theorem 29.}\) Let \(a = p + 0^+\) and \(a = p + 1^+\) be the normal additive (8, Definition 13) and multiplicative (8, Definition 14) forms of \(a\) respectively. Then \(a\) is negative, i.e. \(a < 0,\) if and only if \(p\) is negative, i.e. \(p < 0,\)

Theorems 27, 28 and 29 are an immediate paraprosis of material exhibited so far, therefore we do not rewrite their proofs.
4. Infinitesimals, Finite and Infinitely Large Asymptotic Numbers

Definition 8 (Infinitesimals, finite and infinitely large numbers):
(i) If \( a \in A \), then \( a \) will be called infinitely small or an infinitesimal if \( a < x \) for all \( x \in \mathbb{R} \), \( x > 0 \). The set of all infinitesimals will be denoted by \( \Omega_0 \);
(ii) The number \( a \in A \) will be called finite if there exists \( x \in \mathbb{R} \) such that \( |a| < x \). The set of all finite numbers will be denoted by \( \Omega \);
(iii) The asymptotic number \( a \) will be called infinitely large if \( |x| < a \) for all \( x \in \mathbb{R} \). The set of all infinitely large numbers will be denoted by \( \Omega_\infty \).

Remark 1. As we know from (8, Theorem 20), \( \mathbb{R} \) is isomorphic to \( \mathbb{R}^0 \) and \( \mathbb{R}^\infty \) (8, Definition 10) which, on their part, are subsets of \( A \). So, as we write \( x \in \mathbb{R} \), we have in mind either \( x \in \mathbb{R}^0 \) or \( x \in \mathbb{R}^\infty \).

Remark 2. The asymptotic numbers: \( a = s + 0^1 \), \( b = 2 + s + 0^1 \) and \( c = 1/s + 0^{-1} \) give us examples of an infinitesimal, finite and infinitely large number, respectively. In other words, \( A \) possesses infinitesimals (different from zero), finite and infinitely large numbers. The latter could be formulated in the following way: \( A \) is a non-archimedean order set. (A totally-ordered set \( F \) which contains \( N \) is called archimedean if for every \( a \in F \) there exists \( n \in N \) such that \( |a| < n \), see, for instance, [6].) It is clear also that

\[
\Omega_0 \subset \Omega, \quad \Omega \cap \Omega_\infty = \emptyset, \quad A = \Omega \cup \Omega_\infty.
\]

Remark 3. The above definition makes sense also for the complex asymptotic numbers.

The following theorem establishes a connection between the notions just introduced and properties of the representatives of the asymptotic numbers.

Theorem 30.
(i) The asymptotic number \( a \) is an infinitesimal, i.e. \( a \in \Omega_0 \), if and only if \( \lim_{s \to 0^+} a(s) = 0 \) for all \( a \in A \);
(ii) The asymptotic number \( a \) is finite, i.e. \( a \in \Omega \), if and only if for every \( a \in A \) there exists \( E \in \mathbb{E} \) such that \( a \) is bounded on \( E \). Besides, if \( a \in \Omega \), then there exists \( x \in \mathbb{R} \) such that \( \lim_{s \to 0} a(s) = x \) for all \( a \in A \);
(iii) The asymptotic number \( a \) is infinitely large, i.e. \( a \in \Omega_\infty \), if and only if there exists \( a \in A \) such that \( a \) is unbounded on every \( E \in \mathbb{E} \). Moreover, if \( a \in \Omega_\infty \) and \( a \notin \{0^{-n} : n \in \mathbb{N} \} \), then \( \lim_{s \to 0} a(s) = -\infty \) for all \( a \in A \) in the case \( a < 0 \) and \( \lim_{s \to 0} a(s) = +\infty \) if \( a > 0 \).

Proof. The theorem follows directly from (8, Theorem 24).

Theorem 32. The asymptotic zeros are either infinitesimals or infinitely large numbers. In particular, \( 0^* \in \Omega_0 \) for \( v = 0, 1, \ldots, \infty \) and \( 0^* \in \Omega_\infty \) for \( v = -1, -2, \ldots \). The asymptotic units \( 1^*, \lambda = 0, 1, \ldots, \infty \) are finite numbers and, more strictly, \( 1 \in \Omega \setminus \Omega_0 \) (\( A \) is the set of all asymptotic units).

Proof: Trivial.

Corollary:
\[
\Omega_0 = \{ a \in A : |a| \leq 0^n \},
\]
\[
\Omega = \{ a \in A : |a| < 0^{-1} \},
\]
\[
\Omega_\infty = \{ a \in A : |a| \geq 0^{-1} \}.
\]
Large Asymptotic Numbers

Infinitely large numbers: every small or an infinitesimal if 1 infinitesimals will be denoted

If there exists \( x \in \mathbb{R} \) such that \( x < a \) for a subeet of \( A \). So, we write \( \mathbb{R} \),

\( s+0^1, b=2+s+0^1 \)

and infinitely large numbers, infinitesimal numbers (different from zero), could be formulated in the followin:

(A totally-ordered set \( F \) which \( \forall \ a \in F \) there exists \( n \in N \) such that

Also for the complex asymptotic connection between the notions

infinitesimal, i.e. \( a \in \Omega_n \), if and only if

\( a \in \Omega \), if and only if for every \( a \in A \)

besides, if \( a \in \Omega \), then there

large, i.e. \( a \in \Omega_\infty \), if and only if

in every \( E \in \mathbb{R} \). Moreover, if \( a \in \Omega_\infty \), then all \( a \in a \) in the case \( a<0 \) and

asymptotic)

Theorem 33. If \( a \in A \) and \( a \in \Omega \), then \( a \) is an infinitesimal or infinitely large

if and only if \( a^{-1} \) is infinitely large or an infinitesimal, respectively, and

\( a \in \Omega \setminus \Omega_\infty \) if and only if \( a^{-1} \in \Omega \setminus \Omega_\infty \).

Proof. The theorem follows directly from Definition 8.

Theorem 34.

(i) The set of finite numbers \( \Omega \) is closed with respect to the operations addition, subtraction and multiplication. With respect to the order relation, \( \Omega \) is an archimedean set and 0 is a subset of \( \Omega \).

(ii) The set of finite numbers \( \Omega \) is closed with respect to the order relation, \( \Omega \) is an archimedean set and 0 is a subset of \( \Omega \).

Proof. Let \( a, b \in \Omega \). Corresponding to Definition 8, there exist \( x, y \in \mathbb{R} \) such that

\( a \leq x \) and \( b \leq y \). We obtain

\[ a+b \leq a+b \leq x+y, \]

\( a-b \leq a-b \leq x+y, \)

\( i.e., \) \( a, b \in \Omega \). The assertion of the theorem about the division follows from Theorem 33. The property that \( \Omega \) is an archimedean set follows directly from Definition 8. Let \( 0 \leq a \leq b \in \Omega \). The latter means

\( a=\frac{a}{b} b-b \leq x \) for any \( x \in \mathbb{R}, \) i.e. \( a \in \Omega \). The theorem is proved.

Theorem 35. The set of all infinitesimals \( \Omega_o \) is a prime ideal in the set of the finite numbers \( \Omega \). Furthermore, \( \Omega_o \) possesses the property: For every choice of \( a, b \in \Omega \), 0 \( a \leq b \in \Omega_\infty \) implies \( a \in \Omega_o \).

Proof:

(i) Let \( a, b \in \Omega_o \), i.e. \( a \leq x \) and \( b \leq x \) for every choice of the real num-

ber \( x \). We have \( a+b \leq a+b \leq 2x \), i.e. \( a+b \in \Omega_o \);

(ii) Obviously, \( a \in \Omega_o \) implies \( a \in \Omega_o \);

(iii) Let \( a \in \Omega \) and \( b \in \Omega_o \), which means that there exists a real number \( x_o \) such that

\( a<x_o \). We have \( a, b \geq a, b \leq x_o \) for all real numbers \( x \), i.e. \( a, b \in \Omega_o \). We proved that \( \Omega_o \) is an ideal in \( \Omega \).

(iv) We must prove that \( \Omega_o \) is a prime ideal in \( \Omega \), i.e. that \( a, b \in \Omega_o \) implies

either \( a \in \Omega_o \) or \( b \in \Omega_o \). Let us assume the opposite, i.e. that \( a, b \notin \Omega_o \). We have

\( a \geq x_o \) and \( b \geq y_o \) for some real numbers \( x_o \) and \( y_o \), i.e. \( a, b \)

\( 
\geq x_o, y_o \), i.e. \( a, b \notin \Omega_o \), which is a contradiction;

(v) Let \( 0 \leq a \leq b \in \Omega_o \), i.e. \( a \leq b \leq x \) for all real \( x \). We obtain

\( a \in \Omega_o \), i.e. \( a \in \Omega_o \). The theorem is proved.

Definition 9 (Infinitesimal relation). We shall say that \( a, b \in A \) are infinitely

close and write this as \( a - b \) in an infinitesimal, i.e. if \( a - b \in \Omega_o \).

Theorem 36.

(i) The relation \( \sim \) is an equivalence relation in the set of all finite numbers \( \Omega \);

(ii) If \( a, b, c, d \in \Omega \), then \( a \sim b \) and \( c \sim d \) implies \( a+c \sim b+d \) and \( a \sim b \) and \( c \sim b \).

(iii) Let \( a, b \in \Omega \) and not \( a \sim b \). Then \( a \sim b \) implies \( c \leq d \) for all \( c, d \in \Omega \) such that

\( a \sim c \) and \( b \sim d \);

(iv) The factor-set \( \Omega/\Omega_o \) is isomorphic to the field of the real numbers \( \mathbb{R} \) (or to the field of the complex numbers \( C \) in the case of the complex asymptotic numbers).

Proof. Consequence of Theorem 30 (i).

Remark 1. As \( A \) and \( \Omega \) are not a ring (6, Theorem 6) the properties of the ideals of \( A \), we know from the theory of rings, do not hold (in general) in \( A \) (or in \( \Omega \)). In particular, if \( M \) is an ideal of \( A \) (or \( M \) is an ideal in \( \Omega \)), then the relation \( a \sim b \) if \( a - b \in M \) is not always an equivalence relation in \( A \) (in \( \Omega \)); for example, \( 0 \) does not define such a relation (8, Theorem 18).

Consequently, the assertion of the above theorem is not a standard corollary of Theorem 35. On the other hand, \( \Omega_o \) is not a maximal ideal in \( \Omega \) (in spite of \( \Omega/\Omega_o \) being a field), e.g. \( \Omega_o \cup (\Omega \cap A) \), \( k=0, 1, \ldots, \infty \), are also ideals in \( \Omega \), where \( A \) were defined in (6, Definition 11).
Remark 2. Elements of $\Omega/\Omega_0$ (the equivalence classes) we shall denote by their images in $R$, i.e. as real numbers.

Definition 10 (Real image $r$). The canonical homomorphism of $\Omega$ onto $R$ (the homomorphism of $\Omega$ onto $R$ with kernel $\Omega_0$) will be denoted by $r$. If $a$ is a finite number, i.e. $a \in \Omega$, then its image in $R$ will be denoted by $r(a)$, i.e. $r(a) \in R$. For every $a \in \Omega$. (In the case of complex asymptotic numbers $r(a) \in C$ for all finite complex asymptotic numbers $a$.) $r(a)$ will be called real (or complex, resp.) image of $a$.

Corresponding to Theorem 30, if $a \in \Omega$ and $x \in R$, then $r(a) = x$ if and only if there exists a representative $a \in a$ such that $\lim_{s \to 0} a(s) = x$. In fact, $\lim_{s \to 0} a(s) = x$ for all $a \in a$.

Lemma 4. (i) If $a, b \in \Omega \setminus \Omega_0$, then $r(a) = a_0$, where $a_0$ is the corresponding (the first, i.e. for $k = 0$) coefficient in the main part $(8, (3))$ of $a$; (ii) $r(a) = 0$ for all $a \in \Omega_0$, i.e. for all infinitesimals.

Proof: Trivial.

Theorem 37. The homomorphism $r$ possesses the following properties:
If $a, b \in \Omega$, then:
(i) $a \leq b$ implies $r(a) \leq r(b)$; (ii) $a < b$ implies $r(a) < r(b)$ if and only if $a \sim b$; (iii) $r(a) = r(b)$ if and only if $a = b$.

Proof. We shall prove only the property (i); the remaining ones are proved analogously. Let $a < b$, i.e. (by definition) there exist $a \in a$ and $b \in b$ such that $\{s : a(s) < b(s)\} \in \varepsilon$. The latter means $\lim_{s \to 0} a(s) = \lim_{s \to 0} b(s)$, i.e. $r(a) = r(b)$.

Remark. We proved that a unique real (or complex) number $x = r(a)$ corresponds to every finite asymptotic number $a$. On the other hand, corresponding to $(8, \text{Theorem 20})$, $R$ is isomorphic to two subspaces $R^0$ and $R^\infty$ of $A$ $(8, \text{Definition 10})$, i.e. $R \approx R^0$ and $R \approx R^\infty$. Now, let us restrict ourselves to only one of these isomorphisms: either $R \approx R^0$ or $R \approx R^\infty$. Then the following assertion holds: Every finite real asymptotic number $a$ is infinitely close to a unique real number, namely, $r(a)$, i.e. $a \approx r(a)$ for all $a \in \Omega$.

5. Interval Topology of $A$

Definition 11 (Interval topology):
(i) A non-empty subset $A$ of $A$ will be called interval of $A$ if $a, b \in A, x \in A$ and $a < x < b$ implies $x \in A$; (ii) The subset $s$ of $A$ will be called an open set of $A$ if for every $x \in s$ there exists an interval $A \subseteq s$ such that $x \in A$. (The empty set $\emptyset$ as well as $A$ are open sets of $A$); (iii) The family of all open sets of $A$ will be called the interval topology of $A$; (iv) By the term a neighbourhood of $a \in A$ we shall understand every subset $M$ of $A$ for which there exists an open set $s$ of $A$ so that $a \in s \subseteq M$; (v) Let $a \in A$ and $a_n \in A, n \in N$. Then we shall say that the sequence $\{a_n : n \in N\}$ is convergent (with respect to the interval topology) and $a$ is its limit and we write this as $\lim_{n \to \infty} a_n = a$ (in contrast to "lim", which will be preserved for the ordinary topology of $R$) if for every neighbourhood $M$ of $a$ there exists $n_0 \in N$ such that $a_n \in M$ for all $n \in N, n > n_0$;
(vi) Let \( a_n \in A, \ n \in N \). The sequence \( \{a_n : n \in N\} \) will be called fundamental if for every \( \varepsilon \in A_+ \), \( A_+ \) is the set of all positive asymptotic numbers) there exists \( n_0 \in N \) such that \( a_p - a_q < \varepsilon \) for all \( p, q \in N \) such that \( p > n_0 \) and \( q > n_0 \) (see, for instance, [6]).

We ought to find a more convenient and simpler, if possible, base of neighbourhoods of every asymptotic number. In the case \( \lim a_n = a \in A^\infty \) (8, Definition 11) the family of all intervals of type \( (a - \varepsilon, a + \varepsilon) = \{x \in A : x - a < \varepsilon\} \), where \( \varepsilon \in A_+ \) turns out to be such a base. The following theorem is valid:

Theorem 38 (The case \( \lim a_n = A^\infty \)). Let \( a_n : A, n \in N \) and \( A \in A^\infty \). Then \( \lim a = a \) if and only if for every \( \varepsilon \in A_+ \) there exists \( n_0 \in N \) such that \( a_n - a < \varepsilon \) for all \( n > n_0 \).

Proof. We must prove that the family of intervals \( (a - \varepsilon, a + \varepsilon), \varepsilon \in A_+ \) is a base of neighbourhoods of \( a \). Indeed, \( a', a'' \in A, A \in A^\infty \) and \( a' < a < a'' \) implies \( a'-a, a-a'' \in \emptyset \). Correspoding to Theorem 21, \( a - \varepsilon < a < a + \varepsilon' < a'' \) for \( \varepsilon' = (a'-a)/2 \) and \( \varepsilon'' = (a''-a)/2 \). Let \( \varepsilon = \min (\varepsilon', \varepsilon'') \). We obtain \( a' - a < \varepsilon < a < a'' + \varepsilon < a'' \). The theorem is proved.

The following theorem gives a test for convergence in the case \( \lim a_n = A^\infty \).

Theorem 39 (A test for convergence in the case \( \lim a_n = A^\infty \)). Let \( a_n : A, n \in N \) and \( A \in A^\infty \) be chosen arbitrarily and let

\[ a = \sum_{k=\mu}^{\infty} \alpha_k s^k, \quad a_n = \sum_{k=\mu}^{\infty} \alpha_k s^k \]

be their normal additive forms (8, Definition 13), where \( \mu_n \) and \( r_n \) are the powers and accuracies of \( a_n \) and \( a \) and \( \infty \) are the power and accuracy of \( a \), respectively. Then \( \lim a_n = a \) if and only if the following three conditions are valid:

(i) \( \lim \mu_n = \mu \);

(ii) \( \lim r_n = \infty \);

(iii) For every \( k \in Z, k \geq \mu \) there exists \( n_0 \in N \) such that \( \alpha_{k+n_0} = \alpha_k \) for all \( n > n_0 \).

Remark. As usual, \( \lim \mu_n = \mu \) and \( \lim r_n = \infty \) mean \( \mu \) and \( r \) are the unique adherent points of the sequences \( \{\mu_n : n \in N\} \) and \( \{r_n : n \in N\} \) respectively. In our case \( \mu_n, r_n \in Z \cup \{\infty\} \) and consequently, \( \mu \) and \( r \) are either trivial adherent points of the above sequences, respectively (the sequences are trivial) or these sequences increase unboundedly.

Proof. Let us assume that (i), (ii) and (iii) hold and let \( a \in A_+ \). If \( \mu_n \) are the powers of \( A_+ = a_{n+1} - a_n \) for all \( n \), then \( \lim \mu_n = \infty \) and consequently, \( a^n_{\mu_n} = \mu_n \) holds for every sufficiently large \( n \), where \( \mu_n \) is the power of \( \varepsilon \). Hence, we have \( a_n - a < \varepsilon \) for all sufficiently large \( n \), i.e. \( \lim a_n = a \). Let \( a = \lim a_n = \infty \) hold and let us assume that \( \mu' \in Z \cup \{\infty\} \) is an adherent point of \( \{\mu_n : n \in N\} \) and \( \mu' = \mu \). If we choose \( a \in A_+ \) such that \( \mu_n > \mu' \), then we obtain \( a_n - a > \varepsilon \) for infinitely many \( n \), which contradicts \( \lim a_n = a \). If \( \mu' \in Z \cup \{\infty\} \) is an adherent point of \( \{r_n : n \in N\} \) and \( \mu' = \infty \), then for \( \varepsilon = 0 \), we obtain a contradiction also.

Let us assume that there exists \( \varepsilon \in Z \) such that \( \varepsilon > k \) and \( \alpha_n \geq \alpha_k \) for infinitely many \( n \). The latter means \( \mu_n \leq k \) for infinitely many \( n \) and, consequently, if we choose \( a \in A_+ \) such that \( \mu_n > k \), we obtain \( |a_n - a| > \varepsilon \) for infinitely many \( n \). The proof is completed.
Corollary 1. If \( n \in Z \cup \{\infty\} \), \( n \in N \) and \( \lim_{n \to \infty} n = \infty \), then

\[
\lim_{n \to \infty} \left( \sum_{k=\mu}^{n} a_k s^k + O(n) \right) = \sum_{k=\mu}^{\infty} a_k s^k
\]

for any \( \mu \in Z \) and any \( a_k \in \mathbb{R} \) (or \( a_k \in C \)), \( k = \mu, \mu + 1, \ldots \).

Corollary 2. In the case "\( v_n = \infty \) for all \( n \)" (20) reduces to

\[
\lim_{n \to \infty} \left( \sum_{k=\mu}^{n} a_k s^k \right) = \sum_{k=\mu}^{\infty} a_k s^k
\]

for any \( \mu \in Z \) and any \( a_k \in \mathbb{R} \) (or \( a_k \in C \)).

Corollary 3.

\[
\lim_{n \to \infty} n a_n s^n = 0
\]

holds for arbitrarily chosen \( a_n \in \mathbb{R} \) (or \( a_n \in C \)), \( n \in N \).

Corollary 4. \( \lim_{n \to \infty} n a_n = 0 \) holds. We notice that, corresponding to (8, Definition 12) and (8, Theorem 23), we treat every formal power series as a number of \( A^\infty \); as we did in (21) and (22), where the elements of the sequences on the lefthand sides of these formulas are numbers from \( A^\infty \).

Theorem 40. Let \( a_n \in A_1 \), \( n \in N \). Then the sequence \( \{a_n : n \in N\} \) is fundamental if and only if it is convergent and \( \lim_{n \to \infty} a_n = A^\infty \).

Proof. The proof is quite analogous to that of Theorem 38. We shall not give it.

Definition 12 (Series in \( A \)).

(i) Let \( \mu \in Z \) and \( a_k \in A_1 \), \( k = \mu, \mu + 1, \ldots \) be an arbitrary asymptotic numbers. The sequence \( \{S_n : n = \mu, \mu + 1, \ldots \} \), where

\[
S_n = \sum_{k=\mu}^{n} a_k
\]

will be called an infinite series, and for this sequence the notation

\[
\sum_{n=\mu}^{\infty} a_n
\]

will be used. The elements of the sequence \( S_n, n = \mu, \mu + 1, \ldots \) are called the partial sums of the infinite series;

(ii) An infinite series will be called convergent if the sequence of its partial sums is convergent and the limit of this sequence will be called sum of the series. We shall not use different notations for convergent infinite series and their sum.

Theorem 41. For any \( \mu \in Z \) and any \( a_k \in \mathbb{R} \) (or any \( a_k \in C \)), \( k = \mu, \mu + 1, \ldots \), the series

\[
\sum_{k=\mu}^{\infty} a_k h^k
\]

is convergent on \( D_0 \), i.e. for every infinitesimal \( h \), and (23) is the additive form (8, Definition 13) of its sum.

Proof. In fact, the above theorem is a trivial generalization of Corollary 2 of Theorem 38 (we could obtain the case described in this corollary by \( h = s \)).
which is an infinitesimal). We should still notice that for infinitesimals \( h \) with finite accuracy \( r \) (i.e., for \( h \in \Omega_0 \setminus A^\infty \)) the sum (23) coincides with one of the partial sums of the series (23), namely, with \( S_r \).

Remark. (23) defines a mapping of \( \Omega_0 \) into \( A \), i.e., (23) is an asymptotic function defined on \( \Omega_0 \).

We considered the case \( \lim a_n \in A^\infty \) only. The case \( \lim a_n \notin A^\infty \) is more complicated and not too important for us. That is why we shall briefly expose the results in this case without giving the proofs.

Definition 13 (The set \( E(a) \)). If \( a \in A \), then we shall put

\[
E(a) = \{ x \in A \setminus \emptyset : a = x \} \cup \{ x \in \emptyset : a \in x \},
\]

where \( r \) is the accuracy of \( a \) and \( \mu_r \) is the power of \( r \). (If \( r = \infty \), i.e., \( a \in A^\infty \), then \( E(a) = A_+ \).

Theorem 42 (The case \( \lim a_n \in A \setminus A^\infty \)). If the accuracy \( r \) of the asymptotic number \( a \) is finite, i.e., \( r \in \mathbb{Z} \), then the family of intervals \((a', a' + \epsilon)\) obtained by all \( a' \in A \) such that \( a' < a \) and \( a - a' \in \emptyset \) (i.e., \( a' \subset a \) and all \( \epsilon \in E(a) \)) is a base (which is simple and convenient, in some ways) of the neighbourhoods of \( a \).

Theorem 43. \( A \) is Hausdorff's set (see Theorem 21).

Theorem 44. \( A \) is not Dedekind complete. (An ordered set is called Dedekind complete if every non-empty subset, which is bounded above has a least upper bound.)

Proof. Indeed, the set \( A(\mu) \) of all asymptotic numbers with the same power \( \mu \) is bounded above, namely, every positive asymptotic number \( a \) with power \( \mu_a < \mu \) is an upper bound of \( A(\mu) \). But \( A(\mu) \) has not a least upper bound.

As we know (8, Theorem 20), the set of the real numbers \( \mathbb{R} \) is isomorphic to the sets \( \mathbb{R}^0 \) and \( \mathbb{R}^\infty \) (8, Definition 10). Consequently, the topology of \( A \) induces two topologies on \( \mathbb{R}^0 \) — topology (by means of the isomorphism \( \mathbb{R}^0 \approx \mathbb{R} \)) and \( \mathbb{R}^\infty \) — topology (by means of the isomorphism \( \mathbb{R}^\infty \approx \mathbb{R} \)). The following theorems refer exactly to these two cases.

Theorem 45 (\( \mathbb{R}^0 \) — topology). Let \( a_n \in \mathbb{R}^0 \), \( n \in N \), i.e., \( a_n = x_n + 0^0 \), where \( x_n \in \mathbb{R}, n \in N \). Then the sequence \( \{a_n : n \in N\} \) is convergent (with respect to the topology of \( A \)) if and only if one of the following two conditions is valid:

(i) \( \{x_n : n \in N\} \) is convergent with respect to the topology of \( \mathbb{R} \) and \( x \leq x_n \) for all sufficiently large \( n \in N \) where \( x = \lim x_n \). In this case \( \lim a_n = a \) where \( a = x + 0^0 \), i.e., \( \lim (x_n + 0^0) = \lim x_n + 0^0 \);  

(ii) \( \lim x_n = \infty \). In this case \( \lim a_n = 0^{-1} \).

Theorem 46 (\( \mathbb{R}^\infty \) — topology). Let \( a_n \in \mathbb{R}^\infty \), \( n \in N \), i.e., \( a_n = x_n + 0^0 \) where \( x_n \in \mathbb{R}, n \in N \). Then the sequence \( \{a_n : n \in N\} \) is convergent (with respect to the topology of \( A \)) if and only if one of the following two conditions is valid:

(i) \( \{a_n : n \in N\} \) is trivial, i.e., there exists \( a \in \mathbb{R}^\infty \) such that \( a_n = a \) for all sufficiently large \( n \in N \). In this case \( \lim a_n = a \), of course;

(ii) \( \lim x_n = \infty \). In this case \( \lim a_n = 0^{-1} \).

Theorem 47. Let \( \{a_n : n \in N\} \) and \( \{b_n : n \in N\} \) be two convergent sequences of asymptotic numbers and let \( \lim a_n = a \) and \( \lim b_n = b \). Then the convergence of the sequences \( \{a_n \pm b_n : n \in N\} \), \( \{a_n \cdot b_n : n \in N\} \), or \( \{a_n / b_n : n \in N\} \) implies \( \lim (a_n \pm b_n) = a \pm b \), \( \lim (a_n \cdot b_n) = a \cdot b \), or \( \lim (a_n / b_n) = a / b \) respectively (in the case of division we ought to require still \( b_n \neq 0 \), \( n \in N \)).
We shall stop investigating the properties of the set of asymptotic numbers \(A\). (We have done it perhaps in more details than was necessary for our purposes in physics.) Now, we are ready to define the asymptotic functions and, in particular, Dirac's delta-function, as well as its square, etc., in accordance with our promise made at the beginning of [8]. We shall put this off for a future paper.

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References