Asymptotic Numbers: II. Order Relation, Infinitesimals and Interval Topology

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It has been shown in [8] that the set of asymptotic numbers \( A \) is a system of generalized numbers including isomorphically the set of real numbers \( \mathbb{R} \), as well as the field of formal power (asymptotic) series. In the present paper, which is a continuation of [8], an order relation in \( A \) is introduced due to \( A \) turning out to be a totally-ordered set. The consistency between the order relation and the algebraic operations in \( A \) is investigated and in particular, it is shown that the inequalities in \( A \) can be added and multiplied as in the set of the real numbers. The notions of infinitesimals (infinitely small numbers), finite and infinitely large numbers are introduced; \( A \) turns out to be a non-archimedean set. The usage of infinitesimals as infinitely large numbers along with the real numbers is the reason why the terms and the notions introduced in this paper are very much like those of the non-standard analysis (Robinson's theory of infinitesimals) [7]. In connection with the order relation, an interval topology of \( A \) is introduced and some of its properties are established. The theory of asymptotic functions, as well as the applications to the quantum theory, are put off for a future paper.

The notions of the asymptotic numbers [2, 4, 8] and those of the asymptotic functions [8, 5] are introduced as a subsidiary device for investigation of some problems in quantum theory. For further details about the motivation of this work we advise the reader to refer to [2, 3, 4, 5, 8]. But the knowledge of [8] is quite sufficient for the understanding of the present paper.

1. Order Relation in the Set of the Asymptotic Numbers

Definition 1 (Order relation). Let \( a, b \in A \).

(i) We shall say that \( a \) is not larger than \( b \) and we write this as \( a \leq b \) if for every choice of \( a \in \mathbb{R} \), there exists \( \beta \in \mathbb{R} \) such that \( \alpha(s) \leq \beta(s) \) for all sufficiently small \( s \in (0,1) \) (there exists \( \varepsilon \in (0,1) \) such that \( a(s) \leq \beta(s) \) for all \( s \in (0,\varepsilon) \))

(ii) We shall say that \( a \) is smaller than \( b \) or that \( b \) is larger than \( a \) and then we write \( a < b \) if \( a \leq b \) and \( a \neq b \).

Remark: We shall denote by \( N, N_0, Z, \mathbb{R}, \mathbb{R}_+, \) and \( C \) the sets of naturals, naturals and zero (i. e. 0, 1, 2, \ldots), integers, real, positive real and complex numbers, respectively. We shall denote by \( A \) the set of all real asymptotic numbers (8, Definition 5).

We shall introduce some subsidiary notions with the help of which the above definition could be formulated in a more convenient form.

Definition 2 (Filter \( \mathcal{F} \)).

Some of the results of this paper were reported by the author at the conference "Operatoren-Distributionen und Verwandte Non-Standard Methoden", Oberwolfach, Federal Republic of Germany, 2—8 July, 1978.
(i) We denote by $\mathcal{S}$ the set of all subsets of $\mathbb{R}$, which contain an interval of the type $(0,1)$, i.e., $\mathcal{S}$ is different (in general) for the different elements of $\mathcal{S}$.

It is easily verified that $\mathcal{S}$ is a filter on $\mathbb{R}$, i.e., $\mathcal{S}$ possesses the following (filter) properties:

1. $\emptyset \notin \mathcal{S}$,
2. $S, T \in \mathcal{S}$ implies $S \cap T \in \mathcal{S}$,
3. $S \in \mathcal{S}$ and $S \subseteq T \subseteq (0,1)$ implies $T \in \mathcal{S}$.

With the help of $\mathcal{S}$, Definition 1 could be perphrased as follows (we are going to formulate it as a lemma):

**Lemma 1.** Let $a, b \in A$. Then $a \leq b$ if and only if for every choice $a \notin a$ there exists $\beta \in b$ such that

$$\{ s: a(s) \leq \beta(s) \} \in \mathcal{S}.$$ 

**Proof.** The equivalence between Definition 1 and Lemma 1 follows from the fact that assertion "$\{ s: a(s) \leq \beta(s) \} \in \mathcal{S}$" is equivalent to the assertion "$a(s) \leq \beta(s)$ holds for all sufficiently small $s \in (0,1)$". Further, we are going to use Lemma 1 together with (1), (2) and (3) first of all.

**Remark.** Let us note that "$a < b$" is not equivalent to "for every choice of $a \in a$ there exists $\beta \in b$ such that"

$$\{ s: a(s) < \beta(s) \} \in \mathcal{S}.$$ 

In order to convince ourselves of the above-mentioned remark it would be sufficient to consider the case $a < b$.

**Lemma 2.** If $a, b \in A$ and $a \leq b \in \mathcal{S}$ (Definition 5, (v)), then:

(i) $a \leq b$ if and only if for every choice $a \notin a$ and $\beta \in b$ (5) holds;

(ii) $a < b$ if and only if there exists $a \in a$ and $\beta \in b$ for which (5) holds.

**Proof.** The validity of (5) for all $a \in a$ and all $\beta \in b$, as well as $a < b$ implies the existence of $a \in a$ and $\beta \in b$ such that (5) holds, bearing in mind that $\{ s: a(s) = \beta(s) \} \notin \mathcal{S}$ is not possible in the case $a \leq b \notin \mathcal{S}$.

(i) Let $a < b$ and let (5) hold for $a \in a$ and $\beta \in b$. Let us set (for the sake of convenience) $a - b = c$. Then $c \notin \mathcal{S}$ implies that every representative $\gamma \in c$ can be represented in the form $\gamma(s) = \gamma_0 s^n + \mathcal{A}(s)$, where $\gamma_0 \in \mathbb{R}$, $\gamma_0 + 0$ and $\gamma_0$ is the same for all $\gamma \in c$ and $\lim s_n = 0$. If $\gamma - a - b$, then we obtain $\gamma_0 < 0$, i.e.,

$$\{ s: \gamma(s) < 0 \} \in \mathcal{S}$$

for all $\gamma \in c$ (but not only for $c = a - b$). But the latter means that (5) holds for all $a \in a$ and all $\beta \in b$.

(ii) Let there exist $a \in a$ and $\beta \in b$ such that (5) holds. Consequently, there exists $\gamma \in c$ such that $\{ s: \gamma(s) < 0 \} \in \mathcal{S}$, which implies again $\gamma_0 < 0$, i.e., $\{ s: \gamma(s) < 0 \} \in \mathcal{S}$ for all $\gamma \in c$, i.e., (5) holds for all $a \in a$ and all $\beta \in b$, i.e., $a < b$. On the other hand, since $a - b \notin \mathcal{S}$, $a = b$ is impossible, i.e., $a < b$. The proof is finished.

**Lemma 3.** If $a, b \in A$ and $a - b \notin \mathcal{S}$, then $a \leq b$ if and only if $a \subseteq b$ (i.e., $a \leq b$).

**Proof.** $a \subseteq b$ and even $a \subseteq b$ implies, obviously, $a \subseteq b$; to believe that we can set $a = \beta \in a$. Let $a \leq b$ and let us assume $a \ni b$. This means, in particular, that for the accuracies $a$ of $a$ and $a$ of $b$, $a \ni b$ holds (and consequently, $a \ni \infty$) and every $a \in a$ can be represented in the form $a = P(a) + \mathcal{A}(s)$, where $P(a)$ is the main part of $a$ (8, (3)), $\lim a(s)a = 0$ and every representative $\beta \in b$ can be represented in the form $\beta(s) = P(a) + \beta_0 s^{e_0} + \mathcal{A}(s)$, where $\beta_0$ is a
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Further, we are going to use it all.

equivalent to \( \forall s \in \mathcal{A} \).

mentioned remark it would be

\[ \text{(v)} \]

\[ \begin{align*}
\text{(a)} & \quad \text{if and only if for every choice } a \in \mathcal{A}, \\
\text{(b)} & \quad \text{if and only if for every choice } a \in \mathcal{A}, \\
\text{(c)} & \quad \text{if and only if for every choice } a \in \mathcal{A}, \\
\text{(d)} & \quad \text{if and only if for every choice } a \in \mathcal{A}, \\
\text{(e)} & \quad \text{if and only if for every choice } a \in \mathcal{A}.
\end{align*} \]

\[ \text{Definition 5.} \]

\[ \begin{align*}
\text{(i) anti-reflexive, i.e. } a \leq a & \quad \text{for all } a \in \mathcal{A} \quad \text{("\leq" means "the logical contradiction");} \\
\text{(ii) anti-symmetric, i.e. } a < b & \quad \text{implies } b < a; \\
\text{(iii) transitive, i.e. } a < b & \quad \text{and } b < c \quad \text{implies } a < c. \text{ Furthermore, } a \leq b & \quad \text{if and only if } a < b \quad \text{or } a = b. \\
\end{align*} \]

\[ \text{Proof.} \]

\[ \text{Theorem 1.} \]

\[ \text{Theorem 2.} \]

\[ \text{Theorem 3.} \]

\[ \text{Definition 3. (Positive and negative numbers).} \]

\[ \text{Remark.} \]

\[ \text{Theorem 4.} \]

\[ \text{(i) If } a \in \mathcal{A}, a \notin \mathcal{A}, \text{ then } a \leq 0 \text{ if and only if } a < 0 \text{ (respectively), where } \mu \text{ is the power of } a \text{ and } a_\mu \text{ is the corresponding (for } k = \mu \text{) coefficient in the main part of } a \text{ (8, (3));} \]
(ii) The inequalities \( 0 < 0 < \cdots < 0 < 0^{-1} < \cdots \), \( \forall \in \mathbb{Z} \), hold.

**Proof.** The theorem could be proved just like the theorems exposed so far. We notice that the inequalities \( a < 0 \) have sense in \( A \) and \( a < 0 \) - in \( \mathbb{R} \).

**Corollary 1.** If \( a, b \in A \) and \( a - b \notin \mathbb{R} \), then \( a < b \) if and only if \( a - b < 0 \).

**Corollary 2.** (i) Every asymptotic zero different from 0 is positive, i.e. \( 0 > 0 \) for all \( \forall \in \mathbb{Z} \);
(ii) If \( a < 0 \), then \( a \notin \mathbb{R} \).

**Theorem 5.** If \( a, b \in A \) and \( a - b \notin \mathbb{R} \), then \( a \leq b \) if and only if \( \forall_a \geq \forall_b \) (respectively), where \( \forall_a \) and \( \forall_b \) are the accuracies (8, Definition 5, iii)) of \( a \) and \( b \) respectively.

**Proof.** Consequence of the preceding theorem (or of (8, Theorem 3)).

**Corollary.** If \( 1' \) and \( 1'' \), \( \lambda' \), \( \lambda'' \in I \) are two asymptotic units, i.e. \( 1' \), \( 1'' \in I \) (8, Def. 5, vi)), then \( 1' \leq 1'' \) if and only if \( \lambda' \geq \lambda'' \).

**Theorem 6.** (i) If \( a \in A \), then \( a < 0 \) implies \( -a > 0 \) (8, Definition 9);
(ii) If \( a \in A \) and \( a \notin \mathbb{R} \), then \( a > 0 \) implies \( a < 0 \). (We shall recall that \( -0 = 0 \).

**Proof.** (i) Let \( a < 0 \). Corresponding to Corollary 2 of Theorem 4, \( a \notin \mathbb{R} \) and consequently, \( a < 0 \) which follows to \( -a > 0 \), i.e. \( -a > 0 \), corresponding to Theorem 4;
(ii) is proved analogously to (i).

**Theorem 7.** Let \( a \in A \) and \( a \notin \mathbb{R} \). The reciprocal number \( a^{-1} \) (8, Definition 9) is positive (negative) if and only if \( a \) is positive (negative).

**Proof.** The theorem follows directly from Theorem 4, bearing in mind, \( a \notin \mathbb{R} \).

2. Order Relation and Algebraic Operations

Now we are going to discuss the consistency between the order and the algebraic operations in \( A \). We should like to point out in advance that the essential points of this section are Theorem 16, Theorem 17 and Theorem 18. The reader, who is interested only in the final results, could pay attention to these three theorems only (as well as Theorem 21, perhaps).

**Theorem 8.** If \( a, b, c \in A \), then: (i) \( a \leq b \) implies \( a + c \leq b + c \); (ii) \( a \geq 0 \) and \( \beta \geq 0 \), implies \( a \beta \geq 0 \).

**Proof.**
(i) Let \( a \leq b \) and let \( a, \beta \in A \) and \( \gamma \in A \) be chosen arbitrarily and for \( \beta \in B \) \( \{ s : \alpha(s) \leq \beta(s) \} \in \delta \) holds. Obviously, \( \{ s : \alpha(s) \leq \beta(s) \} = \{ s : \alpha(s) + \gamma(s) \leq \beta(s) + \gamma(s) \} \), which implies \( a + c \leq b + c \);
(ii) Let \( a \geq 0 \) and \( b \geq 0 \). If (at least) one of the numbers \( a \) and \( b \) is asymptotic zero, then \( a = b = 0 \) is obvious. Let \( a, b \notin \mathbb{R} \). Corresponding to Lemma 2 (i), for every choice of \( a \in A \), \( \beta \in B \), \( \lambda \equiv 0 = 0^0 \) and \( \lambda \equiv 0 = 0^0 \), \( \{ s : a(s) \geq A(s) \} \in \delta \) and \( \{ s : \beta(s) \geq \beta(s) \} \in \delta \) hold. Let \( A(s), \beta(s) \geq 0 \) on \( (1,0) \) (for example, \( A(s) = \beta(s) = \exp (-1/s) \) or just \( A = \beta = 0 \) no \( (1,0) \)). We have that \( \{ s : a(s) \geq A(s) \} \cap \{ s : \beta(s) \geq \beta(s) \} \simeq \{ s : a(s) \beta(s) \geq A s \cdot \beta(s) \} \). From which, bearing in mind (2) and (3), we obtain \( \{ s : a(s), \beta(s) \geq A(s), A(s) \} \in \delta \). The latter means \( a, b \geq 0 \), bearing in mind Lemma 2 (ii) and (8, Theorem 18). The proof is completed.
Corollary. \( a^2 = 0 \) for all \( a \in A \) and \( a^2 = 0 \) if and only if \( a = 0 \).

Theorem 9. If \( a, b, c \in A \), then \( a \leq b \) and \( c \leq d \) implies \( a + c \leq b + d \).

Proof. The proof is standard (i.e. the assertion of the theorem follows from Theorem 8 as in any ring or field): \( a \leq b \) implies \( a + c \leq b + c \); \( c \leq d \) implies \( c + d = a + b \) from which \( a + c + b + d \) follows.

Definition 4. Let \( A \times A \times A \) be the set of all triples \((a, b, c)\), where \( a, b, c \in A \).

We shall separate the following subsets of \( A \times A \times A \):

\[
D_1 = \{(a, b, c) : \mu_a < \mu_b, \epsilon \in \mathbb{C} \setminus \{0\}\}
\]

\[
D_2 = \{(a, b, c) : a < b, c < 0\}
\]

\[
D = D_1 \cup D_2,
\]

where \( \mu_a \) and \( \mu_b \) are the powers (8, Definition 5, (iii)) of \( a \) and \( b \) respectively.

Theorem 10. If \( a, b, c \in A \) and \( (a, b, c) \in D \), then \( a \leq b \) implies \( a \leq b \) by \( c \geq 0 \) and \( a \leq b \) implies \( a \leq b \) by \( c \leq 0 \).

Proof.

(i) Let \( a \leq b \in C \), \( \epsilon \in \mathbb{C} \). Bearing in mind the generalized distributive law (8 (21)): \( (b - a) \cdot c + c = b \cdot e - a \cdot c \) (\( \nu \) is the accuracy of \( b \cdot e - a \cdot c \)), we get \( (b - a) \cdot c \in \mathbb{C} \). The accessibility of \( \epsilon \) is a property of \( \mathbb{C} \) (9, Theorem 18), follows to \( b \cdot e - a \cdot c \). By Corollary 1 of Theorem 4, \( a \leq b \) implies \( b = -a \cdot 0 \), which together with \( c = 0 \) follows to \( b \cdot e = a \cdot 0 \). We must again use the generalized distributive law and obtain \( b \cdot e = a \cdot 0 \), which by Corollary 1 of Theorem 4 implies \( a \cdot c = b \cdot c \). If \( c < 0 \), corresponding to Theorem 6, \( c \\leq 0 \) and consequently, \( -c \cdot b < a \cdot 0 \) from which we get \( a \cdot c = b \cdot c \), i.e. \( a \cdot c = b \cdot c; \)

(ii) Let \( a \leq b \in \mathbb{C} \). If we assume also that \( c \in \mathbb{C} \setminus \{0\} \) (this in particular means \( c > 0 \)), then we have \( \mu_a < \mu_b \) and \( a \cdot c = b \cdot c \) will be reduced to \( 0^{\text{ab} - c} < 0^{\text{ab} + c} \) (because of \( \mu_a < \mu_b \)). Let \( c \in \mathbb{C} \). If \( c > 0 \), then \( a \leq b \) implies \( a \cdot c = b \cdot c \) — the case \( a \cdot c \notin \mathbb{C} \) we already considered above and for \( a \cdot c \in \mathbb{C} \), i.e. \( a \cdot c = b \cdot c \). If \( c < 0 \), then \( a \leq b \) (together with (7)) implies either \( a = b \cdot c \), which we considered already, or \( a = b \), which implies \( a = c \cdot b \). The theorem is proved.

Theorem 11. If \( a, c \in A \) and \( (a, b, c) \in D \), then \( a \leq b \) implies \( a \leq b \) (in spite of \( c > 0 \)).

Proof. In fact, in this case \( a \cdot c = b \cdot c \) is reduced (because of \( \mu_a < \mu_b \) and \( c \notin \mathbb{C} \) \( \setminus \{0\} \)) to \( 0^{\text{ac} - c} \geq 0^{\text{ac} + c} \), where \( \nu \) is the accuracy of \( b \cdot c \), i.e. \( 0^{\text{ac} - c} > 0^{\text{ac} + c} \).

Theorem 12. If \( (a, b, c) \in D_2 \), then \( a \leq b \) implies \( a \leq b \) (in spite of \( c > 0 \)).

Proof. In this case, corresponding to (7), \( a \leq b \) is reduced to \( a \leq b \) and corresponding to Theorem 4, \( c < 0 \) means, in fact, \( \nu < 0 \), where \( \mu \) and \( \nu \) are the power and the corresponding coefficient in the main part of \( c(8 (3)) \), respectively. Hence, we get \( a \leq c \cdot b \), i.e. \( a \leq c \cdot b \).

Theorem 13. If \( a, b, c \in A \) and \( 0 \leq a \leq b \) and \( 0 \leq c \leq d \) implies \( a \cdot c = b \cdot d \).

Proof. First of all, \( 0 \leq a \leq b \) and \( 0 \leq c \leq d \) implies \( (a, b, c), (d, b) \in D \). Indeed, corresponding to Theorem 4 both \( a \cdot c = b \cdot d \) and \( \mu_a < \mu_b \) are possible only in the case \( \mu_a < \mu_b \) (\( a \neq b \) and \( b \neq d \) are the first coefficients in the main parts (8 (3)) of \( a \) and \( b \) respectively), which contradicts \( 0 \leq a \), i.e. \( (a, b, c) \in D \). On the other hand, \( 0 \leq c \) shows that \( (a, b, c) \in D_2 \), i.e. \( (a, b, c) \in D \). In the same way we obtain \( (c, d, b) \in D \). Further on the proof is standard (as in any ordered ring or field). We multiply the first inequality by \( a \) and the second by \( b \) and we obtain \( a \cdot c = b \cdot c \) and \( c \cdot b = d \cdot b \), i.e. \( a \cdot c = b \cdot d \). The proof is completed.

Corollary. If \( a, b \in A \), then \( 0 \leq a \leq b \) implies \( a^n \leq b^n \) for every \( n = 1, 2, \ldots \).

The strict inequality "<" in \( A \) has, with respect to the algebraic operations,
properties quite analogous to those of the non-strict one. Because of the strange algebraic properties of "<" however, we can obtain an equality after adding an asymptotic number to a strict inequality (to both sides of a strict inequality); this property does not have an analogue in the set of real numbers $\mathbb{R}$. The following lines are devoted just to this special feature of the strict inequalities in $A$.

**Definition 5.** We shall denote by $E_+$ the following subset of $\mathbb{A}$:

$$E_+ = \{ (a, b, c) : a - b \notin \mathbb{C}, \mu_{a-b} > r_c \} \cup \{ (a, b, c) : a - b \notin \mathbb{C}, \mu_{a-b} = r_c \},$$

where $\mu_{a-b}$ is the power of $a - b$ and $r_c$ is the accuracy of $c$.

**Theorem 14.** If $a, b, c \in A$, then $a < b$ implies $a + c < b + c$ in the case $(a, b, c) \notin E_+$ and $a + c = b + c$ in the case $(a, b, c) \in E_+$.

**Proof.** We showed already that $a < b$ implies $a + c < b + c$. Consequently, we must only specify the cases $(a, b, c) \notin E_+$ and $(a, b, c) \in E_+$. To this end it is convenient to put down $a, b$ and $c$ in the normal additive form (8, Definition 13):

$$a = \sum_{n=1}^{r_a} a_n s^n + 0^n a, \quad b = \sum_{k=1}^{r_b} b_k s^k + 0^n b, \quad c = \sum_{h=1}^{r_c} c_h s^h + 0^n c,$$

where $\mu_a, \mu_b$ and $\mu_c$ are the powers of $a, b$ and $c$ respectively, $\mu = \min(\mu_a, \mu_b)$, $\nu_a = 0, \mu_n = n < a$ and $\nu_b = 0, \mu_n = n < b$ and $r_a, r_b$ and $r_c$ are the accuracies of $a, b$ and $c$ respectively. We obtain $u_a = \beta_{a, n} = n < n \leq \mu_{a-b}$ and in the case $a < b$, $a - b \notin \mathbb{N}$ (i.e. $a < b$) $u_a = \beta_{a, n} = n \leq \mu_{a-b}$ and in the case $a < b, a - b \notin \mathbb{N}$ also. It is sufficient to use (8, Theorem 3) only. The proof is completed.

**Definition 6.** We shall denote by $E$ the following subset of $\mathbb{A}$:

$$E_0 = \{ (a, b, c) : a, b \in A \} \cup \{ (a, b, c) : a, b, c \in A, \mu_a = \mu_b, \mu_{a-b} = \mu > \lambda_c \}$$

where $\mu_a, \mu_b$ and $\mu_{a-b}$ are the powers of $a, b$ and $a - b$, respectively, and $\lambda_c$ is the relative accuracy of $c$.

**Theorem 15.** If $a, b, c \in A$, then $a + b$ implies $a + c = b + c$ in the case $(a, b, c) \notin E_0$ and $a + c = b + c$ in the case $(a, b, c) \in E_0$. The proof is quite analogous to that of Theorem 14. We shall omit it. In spite of the results of Theorem 14 and Theorem 15 most of the properties of the strict inequalities in $\mathbb{R}$ are valid in $A$ too, e.g.:

**Theorem 16.** If $a, b \in A$, then $a > 0$ and $b > 0$ imply $a \cdot b > 0$.

**Proof.** Corresponding to Theorem 8, we have $a > 0$. Furthermore, $A$ does not have divisors of zero (8, Theorem 7), i.e. $a \cdot b \cdot 0 = 0$, which implies $a \cdot b > 0$.

**Theorem 17.** If $a, b, c, d \in A$, then $a < b$ and $c < d$ implies $a + b < c + d$.

**Proof.** Let $a < b$ and $c < d$. We shall show at first that (at least) one of the triples $(a, b, c)$ and $(c, d, b)$ does not belong to $E_+$. Let us assume the opposite, i.e. that $(a, b, c)$ and $(c, d, b) \in E_+$. (i) Let (at least) one of the differences $a - b$ and $c - d$ not be an asymptotic zero — let us assume, for instance, that $a - b \notin \mathbb{C}$, (9) implies $r_b \leq \min(\nu_a, \nu_b) = r_a - b = \mu_{a-b} - r_c \leq \min(r_a, r_b) = r_\lambda = \mu_{a-b} - r_d$, i.e. $r_a > r_b$, which is a contradiction; (ii) Let $a - b \notin \mathbb{C}$ and $c - d \notin \mathbb{C}$. We obtain $r_a > r_b$ and $r_a > r_d$, corresponding to $\mu_{a-b} = \mu_{c-d} = \mu > \lambda_c$, $\nu_a = \nu_b = \nu_d$, and $r_d$ corresponding to (9). On the other hand, $r_b = \min(\nu_a, \nu_b) = r_\lambda = \mu_{a-b} - r_c$. However, we have (9) implies $r_c \leq \min(\nu_a, \nu_b) = r_\lambda = \mu_{a-b} - r_d$, i.e. $r_c \leq r_d$, which contradicts $r_c > r_d$. The subsidiary assumption is proved. Without loss of generality we can assume that $(a, b, c) \notin E_+$. We obtain $a + c < b + c$ and $c + b < d + b$, corresponding to Theorem 14 and Theorem 9 respectively, i.e. $a + c < b + d$. The proof is finished.
Theorem 18. If \( a, b, c, d \in A \), then \( 0 < a < b \) and \( 0 < c < d \) implies \( a < c < b < d \).

Proof. Let \( 0 < a < b \) and \( 0 < c < d \).

(i) We shall show that (at least) one of the triples \((a, b, c)\) and \((c, d, b)\) does not belong to \(E_c\). Let us assume that \((a, b, c)\) and \((c, d, b)\) belong to \(E_c\). We obtain \(u_a = u_b = u_c = u_d\) which is a contradiction. Further, let us assume that \((a, b, c) \in E_c\) and \((c, d, b) \in E_c\). On the other hand, we already showed by proving Theorem 13 \((a, b, c), (c, d, b) \in D\). Consequently, we have \(0 < a < b \) implies \(a < c < b < d\).

(ii) If \(a < b\), then \(a < c < b\). From the above Corollary we obtain \(b < c\), which contradicts \(b = c\).

Theorem 19. Each positive asymptotic number cannot possess more than one square root.

Proof. Let \(a > 0\) and \(a - b^2 = c^2\) and \(0 < c < b\). From the above Corollary we obtain \(b^2 < c^2\), which contradicts \(b^2 = c^2\).

Theorem 20. If \(a, b \in A\), then \(a < b\) implies \(a < (a + b)2 \leq b\) and \((a + b)2 < b\) only in the case \(a = bE\) \((2 \in A^\infty \text{ (8, Definition 12)}\).

Proof.

(i) \(a < b\) implies \(a = (b - a)/2 > 0\). Corresponding to Theorem 10, \(a \leq a + A < (a + b)/2\). For the power \(a_b\) of \(A\), \(a \leq a\) holds. Consequently, the assimilation of \(a\) from \(a\) \((8, \text{Theorem 13})\) is impossible because \(a < (a + b)/2\). We add \(b/2\) to both sides of \(a < c < b\) and we obtain \((a + b)/2 < b\). This gives \(b < c\) which contradicts \(b = c\).

(ii) If \(a - b \notin E\), then \((a, b, b, b) \in E_1\). Consequently, corresponding to Theorem 15, \(a, 2b \leq b < b\). If \(a - b \notin E\), then \((a, 2b, b, b, b) \in E_1\) and \(a < b\) implies \((a + b)/2 \leq b\). The theorem is proved.

Theorem 21. For every \(a, b \in A\), \(a < b\) there exists \(c \in A\) such that \(a < c < b\).

In other words, \(A\) is a dense set with respect to the order relation.

Proof.

(i) If \(a - b \notin E\), then, corresponding to the above theorem, \(a < c < b\) for \(c = (a + b)/2\);

(ii) Let \(a < b\) and \(a - b \notin E\) (i.e. \(a \leq b\)). Corresponding to theorem 5, \(v_a > v_b\) where \(v_a\) and \(v_b\) are the accuracies of \(a\) and \(b\) respectively. Let \(e\) be an arbitrary positive asymptotic number different from asymptotic zero, i.e. \(e \notin \mathbb{R}\), with power \(u_a = v_a\). Then \(e = a + e\) possesses the property (we are looking for) \(a < c < b\). Indeed, the positivity of \(e\) implies \(a < c\) and \(u_a = v_a\) implies \(a < c\), i.e. \(a < c\).

Besides, corresponding to Lemma 2, \(a < b\), which (because of \(v_a = v_b\)) leads to \(c < c\), i.e. \(e < b\). The proof is completed.

Remark. The condition \(e \notin \mathbb{R}\) was required in the above proof in order to comprise the cases \(a - b \notin E\) and \(v_a = v_b + 1\). If \(v_a > v_b + 1\), the choice \(e \notin \mathbb{R}\) is possible also.

The theorems exposed so far show that the properties of the order in \(A\) with respect to the algebraic operations in \(A\) are distinguished from the properties of any order ring or field (see, for instance, [1]). As we saw, however, the peculiarities of the order in \(A\) (in respect to \(A\)) refer to rather "narrow" subsets of AXAXA — the sets \(D, E_1\) and \(E_2\). This gives us a possibility to work in most cases with the inequalities in \(A\) by the usual (as in \(A\)) rules. In this respect the non-strict inequality "<" is more convenient because the peculiarities mentioned above refer only to the set \(D\).

We shall define some notions directly connected with order relation of the asymptotic numbers.

Definition 7 (Magnitude). By the term magnitude \(|a|\) of a given asymptotic number \(a\) we shall understand the extension of the function \(|x|\), \(x \in \mathbb{R}\) on \(A\) at the point \(x = a\) (8, Definition 8).
Remark. Corresponding to (8, Definition 8), in order to obtain \(|a|, a \in A\) we must form the set \(a^* = \{ |a| : a \in A \}\). The smallest asymptotic number with respect to the inclusion "\(\subset\)" which covers \(|a|^*\), i.e., \(a^* \subset |a|\), is the magnitude \(|a|\) of \(a\). We shall recall that we called a point "perfect" if \(a^* = a\) and "imperfect" if \(a^* \subset a\) (strict).

**Theorem 22.**

(i) \(a\) exists and \(|a| \in A\), for every choice of \(a \in A\). The numbers \(a^* \in \mathbb{C}_{\text{per}}\) are perfect and the numbers \(a \in \mathbb{C}_{\text{imper}}\) are imperfect points of the magnitude.

(ii) Moreover, \(|a| = \max (-a, a)\) holds for every \(a \in A\).

**Proof.**

(i) Let \(a \in A \setminus \mathbb{C}_{\text{per}}\) and let \(\mu\) and \(a_\mu\) be the power and the corresponding coefficient in the main part (8, (3)) of \(a\) (\(a^* \in \mathbb{C}_{\text{per}}\) implies \(\mu \in \mathbb{Z}\) and \(a_\mu \neq 0\)).

Every representative \(a \in a\) can be represented in the form \(\alpha(s) = a_\mu s^\mu + A(s)\), where \(\lim_{s \to 0} A(s) s^{\mu} = 0\). Hence, we draw the conclusion that the point \(s = 0\) is not a non-trivial adherent point of the set of all zeros of \(a\) (i.e., all points \(s\) at which \(a(s) = 0\)). Consequently, for every \(a \in a\) there exists \(E \in \mathbb{C}_{\text{per}}\) such that \(a(s)\) does not change its sign on \(E\). Besides, the sign of \(a(s)\) on \(E\) does not depend on the choice of \(a \in a\), namely, \(a(s) > 0\), \(s \in E\) if \(a_\mu > 0\) and \(a(s) < 0\), \(s \in E\) if \(a_\mu < 0\). That means \(a(s) \mid a(s)\) on \(E\) if \(a_\mu > 0\) and \(\mid a(s)\mid = -a(s)\) on \(E\) if \(a_\mu < 0\). The theorem is proved in the considered case;

(ii) Let \(a = 0^* \in \mathbb{C}_{\text{per}}\). We have (obviously) \(|0^*| \subset 0^*\) (strict), from which follow the existence of \(0^*\), as well as the imperfectness of \(a = 0^*\). Furthermore, we obtain \(|0^*| = 0^*\), i.e. \(0^* = \max (0^*, -0^*)\) holds too (because \(0^* > 0\) and \(-0^* < 0\)). The proof is completed.

**Theorem 23.** If \(a, b, c \in A\), then:

\[
\begin{align*}
|a| &\geq 0 \text{ and } |a| = 0 \text{ if and only if } a = 0, \\
|a - b| &\leq |a + b| \leq |a| + |b|, \\
|a \cdot b| &= |a| \cdot |b|, \\
|a / b| &= a / b, \quad b \neq 0.
\end{align*}
\]

The proof is analogous to that of Theorem 22 — we shall not give it.

**Remark.** Definition 7, Theorem 22, (i) and Theorem 23 are valid not only for real asymptotic numbers but also for the complex ones (8, sec. 2).

### 3. Order Relation in Some Subsets of \(A\)

In (8, Definition 10) we defined the sets of asymptotic numbers \(\mathbb{R}_0\), \(\mathbb{R}_\infty\) and \(\bar{\mathbb{R}}\) and we proved (8, Theorem 20) that \(\mathbb{R}_0\) and \(\mathbb{R}_\infty\) are isomorphic (with respect to the algebraic operations) to the field of real numbers \(\mathbb{R}\) and \(\bar{\mathbb{R}}\) maps homomorphically on \(\mathbb{R}\). The correspondences realizing the isomorphism and homomorphism mentioned above were given by:

\[
\begin{align*}
\mathbb{R}_0 \ni a = x + 0^r &\leftrightarrow x \in \mathbb{R}, \quad r = 0, \infty, \\
\mathbb{R}_\infty \ni a = x + 0^r &\leftrightarrow x \in \mathbb{R}, \quad r = 0, 1, \ldots, \infty
\end{align*}
\]

respectively, where the numbers, from \(\mathbb{R}_0\), \(\mathbb{R}_\infty\) and \(\bar{\mathbb{R}}\) are written in their normal additive form (8, Definition 13). Now we shall consider the properties of the sets \(\mathbb{R}_0\), \(\mathbb{R}_\infty\) and \(\bar{\mathbb{R}}\), as well as the correspondences (11) and (12) with respect to the order relation in \(A\) and \(\mathbb{R}\) respectively.

**Theorem 24.** \(\mathbb{R}_0\) and \(\mathbb{R}_\infty\) are order fields. The correspondence (11) preserves the order relation (strict and non-strict) in \(A\) and \(\mathbb{R}\) respectively, i.e.
8), in order to obtain \(|a|, a \in \mathbb{A}\) smallest asymptotic number with \(x^2\), i.e. \(a^2 \ll a\), is the magnitude point “perfect” if \(|a| = a\) and \(a \in \mathbb{A}\). The numbers \(a \notin \mathbb{A}\) are points of the magnitude:

\[
\text{If } a \in \mathbb{A}, \text{ and the corresponding coefllient } \mu \in \mathbb{Z} \text{ and } a_n = 0. \text{ Every form } a(s) = a_0 s^a + A(s), \text{ where } \text{that the point } s = 0 \text{ is not a } \alpha \text{ of } \alpha \text{ (i.e. all points } s \text{ at which exists } E \in \delta \text{ such that } (a(s) \text{ does not } s) \text{ on } E \text{ does not depend on the } 1 \text{ and } a(s) = 0, s \in E \text{ if } a_n = 0. \text{ That }
\]

\[
\text{if } a = 0, \text{ if only if } -0^r \text{ only if } a = 0,
\]

\[
\text{Proof. The first part of the theorem — that } \mathbb{R}^0 \text{ and } \mathbb{R}^\omega \text{ are order fields—}
\]

\[
\text{follows directly from Theorem 8. Further, let us take note that if } a, b \in \mathbb{R}, y = 0, \infty, \text{ then } a + b \text{ and } a - b \in \mathbb{A} \text{ are equivalent to each other. That proves the theorem.}
\]

\[
Theorem 25. \text{ The mapping (12) of } \mathbb{R} \text{ on } \mathbb{A} \text{ preserves the non-strict order, i.e. if } x, y \in \mathbb{A} \text{ are the images of } a, b \in \mathbb{R} \text{ by (12) respectively, then } a \leq b \text{ implies } x \leq y.
\]

\[
\text{The proof is analogous to that of Theorem 23. We notice that if } a, b \in \mathbb{R} \text{ and } a - b \in \mathbb{A}, \text{ then } a < b \text{ implies } x < y \text{ (but not } x < y), \text{ i.e. (12) does not preserve (in general) the strict inequality.}
\]

\[
\text{In (8, Theorem 23) we showed that the set } \mathbb{A}^\omega \text{ of all asymptotic numbers with infinite accuracies is a field which is isomorphic to the field of the formal power (asymptotic) series (with real coefficients in the case of real asymptotic numbers). With respect to the order, the following theorem holds:}
\]

\[
Theorem 26. \mathbb{A}^\omega \text{ is an order field. Furthermore, if } a \in \mathbb{A}^\omega, \text{ then } a = 0 \text{ if and only if } a_n = 0 \text{ respectively, where } \mu \text{ is the power and } a_n \text{ is the first coefficient of the main part of } a \text{ respectively.}
\]

\[
\text{Proof. The theorem is an immediate consequence of Theorem 8. In (8, Sec. 6) we introduced an additive (8, Definition 13):}
\]

\[
(a = q + 0^r, a \in \mathbb{A}, q \in \mathbb{A}^\omega, 0^r \in \mathbb{A})
\]

\[
\text{and a multiplicative (8, Definition 14):}
\]

\[
(a = r \cdot 1^s, a \in \mathbb{A} \setminus \mathbb{A}^\omega, r \in \mathbb{A}^\omega, 1^s \in \mathbb{A})
\]

\[
\text{form of } a. \text{ Further, we developed an algebraic technique (8, Sec. 6) with the symbols (11) and (12), which allowed us to express the algebraic operations and algebraic properties of } \mathbb{A}\text{ by the algebraic operations and algebraic properties of } \mathbb{A}^\omega, \mathbb{A}\text{ and } I. \text{ The sense of this approach lies in the fact that } \mathbb{A}^\omega \text{ is a field, (8, Theorem 23) and, as we know from Theorem 26, } \mathbb{A}^\omega \text{ is an order field, and } \mathbb{A} \text{ and } I \text{ have comparatively simple algebraic properties, including comparatively simple order (see Theorem 4, and Corollary of Theorem 5). The following theorems allow us to express the order in } \mathbb{A} \text{ by the order of } \mathbb{A}^\omega, \mathbb{A}\text{ and } I.
\]

\[
\text{Theorem 27. If } a = q + 0^r \text{ and } a' = q' + 0^r, \text{ then } a < a' \text{ if and only if:}
\]

\[
(i) q < q' \text{ in the case } a - a' \in \mathbb{A} \setminus \mathbb{A}^\omega;
\]

\[
(ii) 0^r < 0^s \text{ in the case } a - a' \in \mathbb{A}^\omega.
\]

\[
\text{Theorem 28. If } a = r \cdot 1^s \text{ and } a' = r' \cdot 1^t, \text{ where } a, a' \in \mathbb{A} \setminus \mathbb{A}^\omega, \text{ then } a < a' \text{ if and only if:}
\]

\[
(i) r < r' \text{ in the case } a / a' \in \mathbb{A} \setminus \mathbb{A}^\omega;
\]

\[
(ii) 1^t < 1^s \text{ in the case } a / a' \in \mathbb{A}^\omega.
\]

\[
\text{Theorem 29. Let } a = p + 0^r \text{ and } a = p \cdot 1^s \text{ be the normal additive (8, Definition 13) and multiplicative (8, Definition 14) forms of } a \text{ respectively. Then } a \text{ is negative, i.e. } a < 0, \text{ if and only if } p \text{ is negative, i.e. } p < 0.
\]

\[
\text{Theorems 27, 28, and 29 are an immediate periphrasis of material exhibited so far, therefore we do not rewrite their proofs.}
\]
4. Infinitesimals, Finite and Infinitely Large Asymptotic Numbers

Definition 8 (Infinitesimals, finite and infinitely large numbers):
(i) If $a \in A$, then $a$ will be called infinitely small or an infinitesimal if $a < x$ for all $x \in \mathbb{R}$, $x > 0$. The set of all infinitesimals will be denoted by $\Omega_0$.
(ii) The number $a \in A$ will be called finite if there exists $x \in \mathbb{R}$ such that $|a| < x$. The set of all finite numbers will be denoted by $\Omega$.
(iii) The asymptotic number $a$ will be called infinitely large if $|x| < a$ for all $x \in \mathbb{R}$. The set of all infinitely large numbers will be denoted by $\Omega_\infty$.

Remark 1. As we know from (8, Theorem 20), $\mathbb{R}$ is isomorphic to $\mathbb{R}^0$ and $\mathbb{R}^\infty$ (8, Definition 10), which, on their part, are subsets of $A$. So, as we write $x \in \mathbb{R}$, we have in mind either $x \in \mathbb{R}^0$ or $x \in \mathbb{R}^\infty$.

Remark 2. The asymptotic numbers: $a = s + 0^+$, $b = 2 + s + 0^+$ and $c = 1/s + 0^{-1}$ give us examples of an infinitesimal, finite and infinitely large number, respectively. In other words, $A$ possesses infinitesimals (different from zero), finite and infinitely large numbers. The latter could be formulated in the following way: $A$ is a non-archimedean order set. (A totally-ordered set $F$ which contains $N$ is called archimedean if for every $a \in F$ there exists $n \in N$ such that $|a| < n$, see, for instance, [6].) It is clear also that

$$\Omega_0 \subset \Omega, \Omega \cap \Omega_\infty = \emptyset, A = \Omega \cup \Omega_\infty.$$ 

Remark 3. The above definition makes sense also for the complex asymptotic numbers.

The following theorem establishes a connection between the notions just introduced and properties of the representatives of the asymptotic numbers.

Theorem 30.
(i) The asymptotic number $a$ is an infinitesimal, i.e. $a \in \Omega_0$, if and only if $\lim_{s \to 0} a(s) = 0$ for all $a \in A$.
(ii) The asymptotic number $a$ is finite, i.e. $a \in \Omega$, if and only if for every $a \in A$ there exists $x \in \mathbb{R}$ such that $|a| < x$. Besides, if $a \in \Omega$, then there exists $x \in \mathbb{R}$ such that $\lim_{s \to 0} a(s) = x$ for all $a \in A$.
(iii) The asymptotic number $a$ is infinitely large, i.e. $a \in \Omega_\infty$, if and only if there exists $a \in A$ such that $|a|$ is unbounded on every $E \in \mathbb{R}$. Moreover, if $a \in \Omega\infty$ and $a \notin \{0^{-n} : n \in N\}$, then $\lim_{s \to 0} a(s) = -\infty$ for all $a \in A$ in the case $a < 0$ and $\lim_{s \to 0} a(s) = \infty$ for all $a \in A$ in the case $a > 0$.

Proof. The theorem follows directly from (8, Theorem 24).

Theorem 32. The asymptotic zeros are either infinitesimals or infinitely large numbers. In particular, $0^+ \in \Omega_0$ for $\nu = 0, 1, \ldots, \infty$ and $0^\nu \in \Omega_\infty$, for $\nu = -1, -2, \ldots$. The asymptotic units $1^\nu, \lambda = 0, 1, \ldots, \infty$ are finite numbers and, more strictly, $I \subset \Omega \setminus \Omega_0 (I$ is the set of all asymptotic units).

Proof: Trivial.

Corollary:

$$\Omega_0 = \{ a \in A : |a| = 0^0 \},$$
$$\Omega = \{ a \in A : |a| < 0^{-1} \},$$
$$\Omega_\infty = \{ a \in A : |a| = 0^{-1} \}.$$
Large Asymptotic Numbers

Let us denote by \( \Omega \) the set of all infinitesimals (different from zero): for every \( \omega \in \Omega \), \( \omega \approx 0 \), \( \omega \approx 1 \), \( \omega \approx -1 \), \( \omega \approx 0^+ \), \( \omega \approx 0^- \), and \( \omega \approx 0^0 \).

**Theorem 33.** If \( a \in A \) and \( a \in \Omega \), then \( a \) is an infinitesimal or infinitely large if and only if \( a^{-1} \) is infinitely large or an infinitesimal, respectively, and \( a \in \Omega \setminus \Omega_0 \) if and only if \( a^{-1} \in \Omega \setminus \Omega_0 \).

**Proof.** The theorem follows directly from Definition 8.

**Theorem 34.**

(i) The set of finite numbers \( \Omega \) is closed with respect to the operations addition, subtraction and multiplication. With respect to the order relation, \( \Omega \) is an archimedean set. Similarly to \( \mathbb{R}^0 \), the real numbers, it is known as the asymptotic connection between the notions of the asymptotic numbers, i.e. \( a \in \Omega \), if and only if

\[ a \approx 0, \quad a \approx 1, \quad \text{or} \quad 0 \leq a < 1 \]

or \( a \approx 0 \), \( a \approx 1 \), \( a \approx -1 \), \( a \approx 0^+ \), \( a \approx 0^- \), and \( a \approx 0^0 \).

(ii) The relation \( \approx \) is an equivalence relation in the set of all finite numbers \( \Omega \);

(iii) If \( a, b, c, d \in \Omega \), then \( a \approx b \) and \( c \approx d \) implies \( a \pm c \approx b \pm d \) and \( a \cdot c \approx b \cdot d \).

(iv) The factor-set \( \Omega \) is isomorphic to the field of the complex numbers \( \mathbb{C} \) (or to the field of the complex numbers \( \mathbb{C} \) in the case of the complex asymptotic numbers).

**Proof.** Consequence of Theorem 30 (i).

**Remark.** As \( A \) (or \( \Omega \) too) is not a ring (6, Theorem 6) the properties of the ideals, we know from the theory of rings, do not hold (in general) in \( A \) (or in \( \Omega \)). In particular, if \( M \) is an ideal of \( A \) (or \( M \) is an ideal in \( \Omega \)), then the relation \( a \sim b \) if \( a-b \in M \) is not always an equivalence relation in \( A \) (in \( \Omega \)); for example, \( \mathbb{C} \) does not define such a relation (8, Theorem 18). Consequently, the assertion of the above theorem is not a standard corollary of Theorem 35. On the other hand, \( \Omega_0 \) is not a maximal ideal in \( \Omega \) (in spite of \( \Omega_0 \) being a field), e.g. \( \Omega_0 \cup (\Omega \cap A) \), \( k = 0, 1, \ldots, \infty \), are also ideals in \( \Omega \), where \( A \) were defined in (6, Definition 11).
Remark 2. Elements of \( \Omega/\Omega_0 \) (the equivalence classes) we shall denote by their images in \( \mathbb{R} \), i.e. as real numbers.

**Definition 10** (Real image \( r \)). The canonical homomorphism of \( \Omega \) onto \( \mathbb{R} \) (the homomorphism of \( \Omega \) onto \( \mathbb{R} \) with kernel \( \Omega_0 \)) will be denoted by \( r \). If \( a \) is a finite number, i.e. \( a \in \Omega \), then its image in \( \mathbb{R} \) will be denoted by \( r(a) \), i.e. for every \( a \in \Omega \). (In the case of complex asymptotic numbers \( r(a) \in \mathbb{C} \) for all finite complex asymptotic numbers \( a \).)

Corresponding to Theorem 30, if \( a \in \Omega \) and \( x \in \mathbb{R} \), then \( r(a) = x \) if and only if there exists a representative \( a = a' \) such that \( \lim_{s \to 0} a'(s) = x \). In fact, \( \lim_{s \to 0} a'(s) = x \) for all \( a \in a \).

**Lemma 4.**

(i) If \( a \in \Omega \), then \( r(a) = a_0 \), where \( a_0 \) is the corresponding (the first, i.e. for \( k = 0 \)) coefficient in the main part \((8, (3)) \) of \( a \);
(ii) \( r(a) = 0 \) for all \( a \in \Omega_0 \), i.e. for all infinitesimals.

**Proof:** Trivial.

**Theorem 37.** The homomorphism \( r \) possesses the following properties:

(i) If \( a, b \in \Omega \), then:
   (a) \( a \leq b \) implies \( r(a) \leq r(b) \);
   (b) \( a \leq b \) implies \( r(a) = r(b) \) if and only if \( a \sim b \);
   (c) \( r(a) = 0 \) if and only if \( a \sim a \).

**Proof.** We shall prove only the property (i); the remaining ones are proved analogously. Let \( a \leq b \), i.e. (by definition) there exist \( a \) and \( \beta \) such that \( \{ s : a(s) \leq \beta(s) \} \in \mathcal{E} \). The latter means \( \lim_{s \to 0} a(s) \leq \lim_{s \to 0} \beta(s) \), i.e. \( r(a) \leq r(b) \).

**Remark.** We proved that a unique real (or complex) number \( x = r(a) \) corresponds to every finite asymptotic numbers \( a \). On the other hand, corresponding to \((8, \text{Theorem 20}) \), \( \mathbb{R} \) is isomorphic to two subspaces \( \mathbb{R}^0 \) and \( \mathbb{R}^\infty \) of \( A \) \((8, \text{Definition 10}) \), i.e. \( \mathbb{R} \simeq \mathbb{R}^0 \) and \( \mathbb{R} \simeq \mathbb{R}^\infty \). Now, let us restrict ourselves to only one of these isomorphisms: either \( \mathbb{R} \simeq \mathbb{R}^0 \) or \( \mathbb{R} \simeq \mathbb{R}^\infty \). Then the following assertion holds: Every finite real asymptotic number \( a \) is infinitely close to a unique real number, namely, \( r(a) \), i.e. \( a \sim r(a) \) for all \( a \in \Omega \).

5. Interval Topology of \( A \)

**Definition 11** (Interval topology):

(i) A non-empty subset \( A \) of \( A \) will be called interval of \( A \) if \( a, b \in A, x \in A \) and \( a < x < b \) implies \( x \in A \);
(ii) The subset \( \sigma \) of \( A \) will be called an open set of \( A \) if for every \( x \in \sigma \) there exists an interval \( J \subseteq \sigma \) such that \( x \in J \). (The empty set \( \emptyset \) as well as \( A \) are open sets of \( A \));
(iii) The family of all open sets of \( A \) will be called the interval topology of \( A \);
(iv) By the term a neighbourhood of \( a \in A \) we shall understand every subset \( M \) of \( A \) for which there exists an open set \( \sigma \) of \( A \) so that \( a \in \sigma \subseteq M \);
(v) Let \( a \in A \) and \( a_n \in A, n \in N \). Then we shall say that the sequence \( \{a_n \in N \} \) is convergent (with respect to the interval topology) and \( a \) is its limit and we write this as \( \lim_{n \to \infty} a_n = a \) (in contrast to \(" \lim \" \), which will be preserved for the ordinary topology of \( \mathbb{R} \)) if for every neighbourhood \( M \) of \( a \) there exists \( n_0 \in N \) such that \( a_n \in M \) for all \( n \in N, n \geq n_0 \).
valence classes) we shall denote the nonical homomorphism of \( \Omega \) onto \( \Omega_n \) will be denoted by \( r \). If a in \( \mathfrak{a} \) will be denoted by \( r(a) \), i.e., complex asymptotic numbers \( r(a) \in C \). \( r(a) \) will be called real (or compact) numbers if \( \mathfrak{a} = \mathfrak{r} \), then \( r(a) = x \) if and only if \( x \rightarrow r(a) \). Let us restrict ourselves to (complex) number \( x = r(a) \) corresponding (the first, i.e., main) \( \mathfrak{a} \) simals.

The corresponding (the first, i.e., main) \( \mathfrak{a} \) simals.

asesses the following properties:

(i); the remaining ones are prov.

there exist \( a, b \in A \) with \( a \leq b \);

(ii) there exist \( a, b \in A \) such that \( (s) \leq \lim b(s) \), i.e. \( r(a) \leq r(b) \).

or complex number \( x = r(a) \) correspondence the first one of these two subspaces \( \mathfrak{a}^b \) and \( \mathfrak{a}^m \) of \( A \) if and only if \( \mathfrak{a} = \mathfrak{r} \) or \( \mathfrak{r} \simeq \mathfrak{a}^m \). Then the following number \( a \) is infinitely close to \( (a) = A \) for all \( a \in A \).

An interval of \( A \) if \( a, b \in A, x \in A \) set of \( A \) if for every \( x \in a \) there empty set \( \emptyset \) as well as \( A \) are called the interval topology of \( A \); we shall understand every subset of \( A \) so that \( a \in M \); say that the sequence \( \{a_n: n \in N\} \) (topology) and \( a \) is its limit and we which will be preserved for the neighborhood \( M \) of \( a \) there exists \( n_0 \in N \)

(vi) Let \( a_n \in A, n \in N \). The sequence \( \{a_n: n \in N\} \) will be called fundamental if for every \( \varepsilon > 0 \), \( a_n \in A_n \) is the set of all positive asymptotic numbers) there exists \( n_0 \in N \) such that \( a_n - a < \varepsilon \) for all \( p, q \in N \) such that \( p > n_0 \) and \( q > n_0 \) (see, for instance, [6]).

We ought to find a more convenient and simpler, if possible, base of neighbourhoods of every asymptotic number. In the case \( \lim_{n \to \infty} a_n = a \in A^m \) (Definition 11) the family of all intervals of type \( (a - \varepsilon, a + \varepsilon) = \{x \in A: x \in A \} \), where \( \varepsilon \in (0,1) \), turns out to be such a base. The following theorem is valid:

**Theorem 38** (The case \( \lim_{n \to \infty} a_n = A^m \)). Let \( a_n \in A, n \in N \) and \( A \subset \infty \). Then \( \lim_{n \to \infty} a_n = a \) if and only if for every \( \varepsilon \in A_+ \) there exists \( n_0 \in N \) such that \( |a_n - a| < \varepsilon \) for all \( n \geq n_0 \).

**Proof.** We must prove that the family of intervals \( (a - \varepsilon, a + \varepsilon), \varepsilon \in A_+ \) is a base of neighbourhoods of \( a \). Indeed, \( a', a'' \in A \), \( a \in A^m \) and \( a' \leq a < a'' \) implies \( a' = a - a' \in A \). Corresponding to Theorem 21, \( a < a - \varepsilon < a'' \) for \( \varepsilon = (a - a')/2 \) and \( \varepsilon'' = (a'' - a) \). Let \( \varepsilon = \min (\varepsilon', \varepsilon'') \). We obtain \( a' \leq a - \varepsilon < a < a'' \leq a + \varepsilon < a'' \). The theorem is proved.

The following theorem gives a test for convergence in the case \( \lim_{n \to \infty} a_n = A^m \).

**Theorem 39** (A test for convergence in the case \( \lim_{n \to \infty} a_n = A^m \)). Let \( a_n \in A, n \in N \) and \( A \in A^m \) be chosen arbitrarily and let

\[
d_n = \sum_{k=0}^{\infty} r_k \cdot s^k, \quad a = \sum_{n=0}^{\infty} a_n \cdot s^n
\]

be their normal additive forms (8, Definition 13), where \( a_n \) and \( r_n \) are the powers and accuracies of \( a_n \) and \( a_n \) and \( \infty \) are the power and accuracy of \( a \), respectively. Then \( \lim_{n \to \infty} a_n = a \) if and only if the following three conditions are valid:

(i) \( \lim_{n \to \infty} a_n = a \);

(ii) \( \lim_{n \to \infty} r_n = \infty \);

(iii) for every \( k \in \mathbb{Z}, k \geq \mu \) there exists \( n_0 \in N \) such that \( a_{kn} = a_k \) for all \( n \geq n_0 \).

**Remark.** As usual, \( \lim_{n \to \infty} a_n = a \) and \( \lim_{n \to \infty} r_n = r \) mean \( \mu \) and \( \nu \) are the unique adherent points of the sequences \( \{a_n: n \in N\} \) and \( \{r_n: n \in N\} \) respectively. In our case \( \mu_n \) and \( r_n \) \( n \in \mathbb{Z} \cup \{\infty\} \) and consequently, \( \mu \) and \( \nu \) or either adherent points of the above sequences, respectively (the sequences are trivial) or these sequences increase unboundedly.

**Proof.** Let us assume that (i), (ii) and (iii) hold and let \( \varepsilon \in A_+ \). If \( a_n \) are the powers of \( A = a_n - a \), \( n \in N \), then \( \lim_{n \to \infty} a_n = \infty \) and consequently, \( \mu \geq \mu_n \) holds for every sufficiently large \( n \), where \( \mu_n \) is the power of \( \varepsilon \). Hence, we obtain \( a_n - a < \varepsilon \) for all sufficiently large \( n \), i.e. \( \lim_{n \to \infty} a_n = a \). Let \( \lim_{n \to \infty} a_n = a \) and let us assume that \( \mu' \in Z \cup \{\infty\} \) is an adherent point of \( \{a_n: n \in N\} \) and \( \mu' = \mu \). If we choose \( \varepsilon \in A_+ \) such that \( \mu_n > \mu' \), then we obtain \( a_n - a > \varepsilon \) for infinitely many \( n \), which contradicts \( \lim_{n \to \infty} a_n = a \). If \( \varepsilon' \in Z \cup \{\infty\} \) is an adherent point of \( \{r_n: n \in N\} \) and \( \nu' + \nu \), then \( \lim_{n \to \infty} a_n = a \) we obtain a contradiction also. Let us assume that there exists \( k \in \mathbb{Z} \) such that \( k \geq \mu \) and \( a_n - a_k \) for infinitely many \( n \). The latter means \( a_n - a_k \leq k \) for infinitely many \( n \) and, consequently, if we choose \( \varepsilon \in A_+ \) such that \( \mu_n > k \), we obtain \( a_n - a > \varepsilon \) for infinitely many \( n \). The proof is completed.
**Corollary 1.** If \( \nu_n \in \mathbb{Z} \cup \{\infty\}, \nu \in \mathbb{N} \) and \( \lim \nu_n = \infty \), then

\[
\lim_{n \to \infty} \left( \sum_{k=\mu}^{n} \alpha_k s^k + O(n) \right) = \sum_{k=\mu}^{\infty} \alpha_k s^k
\]

for any \( \mu \in \mathbb{Z} \) and any \( \alpha_k \in \mathbb{R} \) (or \( \alpha_k \in \mathbb{C} \), \( k=\mu, \mu+1, \ldots \)).

**Corollary 2.** In the case “\( \nu_n = \infty \) for all \( n \)” (20) reduces to

\[
\lim_{n \to \infty} \left( \sum_{k=\mu}^{n} \alpha_k s^k \right) = \sum_{k=\mu}^{\infty} \alpha_k s^k
\]

for any \( \mu \in \mathbb{Z} \) and any \( \alpha_k \in \mathbb{R} \) (or \( \alpha_k \in \mathbb{C} \)).

**Corollary 3.**

\[
\lim_{n \to \infty} \alpha_n s^n = 0
\]

holds for arbitrarily chosen \( \alpha_n \in \mathbb{R} \) (or \( \alpha_n \in \mathbb{C} \), \( n \in \mathbb{N} \).

**Corollary 4.** \( \lim_{n \to \infty} \alpha_n = 0 \) holds. We notice that, corresponding to (8, Definition 12) and (8, Theorem 23), we treat every formal power series as a number of \( A^\infty \); as we did in (21) and (22), where the elements of the sequences on the lefthand sides of these formulas are numbers from \( A^\infty \).

Theorem 41. Let \( a_n \in A, n \in \mathbb{N} \). Then the sequence \( \{a_n : n \in \mathbb{N}\} \) is fundamental if and only if it is convergent and \( \lim_{n \to \infty} a_n \in A^\infty \).

**Proof.** The proof is quite analogous to that of Theorem 38. We shall not give it.

**Definition 12 (Series in \( A \).**

(i) Let \( \mu \in \mathbb{Z} \) and \( a_k \in \mathbb{A}, k=\mu, \mu+1, \ldots \) be an arbitrary asymptotic numbers. The sequence \( \{S_n : n=\mu, \mu+1, \ldots \} \), where

\[
S_n = \sum_{k=\mu}^{n} a_k
\]

will be called an infinite series, and for this sequence the notation

\[
\sum_{k=\mu}^{\infty} a_k
\]

will be used. The elements of the sequence \( S_n, n=\mu, \mu+1, \ldots \) are called the partial sums of the infinite series;

(ii) An infinite series will be called convergent if the sequence of its partial sums is convergent and the limit of this sequence will be called sum of the series. We shall not use different notations for convergent infinite series and their sum.

**Theorem 41.** For any \( \mu \in \mathbb{Z} \) and any \( a_k \in \mathbb{R} \) (or any \( a_k \in \mathbb{C} \), \( k=\mu, \mu+1, \ldots \), the series

\[
\sum_{k=\mu}^{\infty} a_k h^k
\]

is convergent on \( \Omega_0 \), i.e. for every infinitesimal \( h \), and (23) is the additive form (8, Definition 13) of its sum.

**Proof.** In fact, the above theorem is a trivial generalization of Corollary 2 of Theorem 38 (we could obtain the case described in this corollary by \( h=s \).
which is an infinitesimal). We should still notice that for infinitesimals \( h \) with finite accuracy \( v \) (i.e. for \( h \notin \Omega \setminus A^\infty \)) the sum (23) coincides with one of the partial sums of the series (23), namely, with \( S_v \).

Remark. (23) defines a mapping of \( \Omega \) into \( A \), i.e. (23) is an asymptotic function defined on \( \Omega \).

We considered the case \( \lim a_n \in \mathcal{A} \) only. The case \( \lim a_n \notin \mathcal{A} \) is more complicated and not too important for us. That is why we shall briefly expose the results in this case without giving the proofs.

Definition 13 (The set \( \mathcal{E} (a) \)). If \( a \in \mathcal{A} \), then we shall put

\[
\mathcal{E} (a) = \{ \varepsilon \in \mathcal{A}^+ \setminus \mathcal{O} : m \leq \varepsilon \} \cup \{ \varepsilon \in \mathcal{O} : \mu \leq \varepsilon \},
\]

where \( \mu \) is the accuracy of \( a \) and \( \mu_v \) is the power of \( \varepsilon \). (If \( v = \infty \), i.e. \( a \in \mathcal{A}^\infty \), then \( \mathcal{E} (a) = A^+ \)).

Theorem 42 (The case \( \lim a_n \notin \mathcal{A} \)). If \( a \notin \mathcal{A} \), then the accuracy \( v \) of the asymptotic number \( a \) is finite, i.e. \( v \in \mathbb{Z} \), then the family of intervals \( (a', a'+\varepsilon) \) obtained by all \( a' \in \mathcal{A} \) such that \( a' < a \) and \( a' - a \in \mathcal{O} \) (i.e. \( a' \subseteq a \)) and all \( \varepsilon \in \mathcal{E} (a) \) is a base (which is simple and convenient, in some ways) of the neighbourhoods of \( a \).

Theorem 43. \( A \) is not Dedekind complete. (An ordered set is called Dedekind complete if every non-empty subset, which is bounded above has a least upper bound.)

Proof. Indeed, the set \( A(\mu) \) of all asymptotic numbers with the same power \( \mu \) is bounded above, namely, every positive asymptotic number \( a \) with power \( \mu_a < \mu \) is an upper bound of \( A(\mu) \). But \( A(\mu) \) has not a least upper bound.

As we know (8, Theorem 20), the set of the real numbers \( \mathbb{R} \) is isomorphic to the sets \( \mathbb{R}^0 \) and \( \mathbb{R}^\infty \) (8, Definition 10). Consequently, the topology of \( A \) induces two topologies on \( \mathbb{R} : \mathbb{R}^0 \) — topology (by means of the isomorphism \( \mathbb{R}^0 \approx \mathbb{R} \)) and \( \mathbb{R}^\infty \) — topology (by means of the isomorphism \( \mathbb{R}^\infty \approx \mathbb{R} \)). The following theorems refer exactly to these two cases.

Theorem 45 (\( \mathbb{R}^0 \) — topology). Let \( a_n \in \mathbb{R}^0 \), \( n \in \mathbb{N} \), i.e. \( a_n = x_n + 0^0 \), where \( x_n \in \mathbb{R}, n \in \mathbb{N} \). Then the sequence \( \{ a_n : n \in \mathbb{N} \} \) is convergent (with respect to the topology of \( A \)) if and only if one of the following two conditions is valid:

(i) \( \{ x_n : n \in \mathbb{N} \} \) is convergent with respect to the topology of \( \mathbb{R} \) and \( x \leq x_n \) for all sufficiently large \( n \in \mathbb{N} \) where \( x = \lim x_n \).

(ii) \( \lim x_n = \infty \), i.e. \( \lim (x_n + 0^0) = \lim x_n + 0^0 \).

Theorem 46 (\( \mathbb{R}^\infty \) — topology). Let \( a_n \in \mathbb{R}^\infty \), \( n \in \mathbb{N} \), i.e. \( a_n = x_n + 0^0 \) where \( x_n \in \mathbb{R}, n \in \mathbb{N} \). Then the sequence \( \{ a_n : n \in \mathbb{N} \} \) is convergent (with respect to the topology of \( A \)) if and only if one of the following two conditions is valid:

(i) \( \{ x_n : n \in \mathbb{N} \} \) is trivial, i.e. there exists \( a \in \mathbb{R}^0 \) such that \( a_n = a \) for all sufficiently large \( n \in \mathbb{N} \).

(ii) \( \lim x_n = \infty \). In this case \( \lim a_n = 0^{-1} \).

Theorem 47. Let \( \{ a_n : n \in \mathbb{N} \} \) and \( \{ b_n : n \in \mathbb{N} \} \) be two convergent sequences of asymptotic numbers and let \( \lim a_n = a \) and \( \lim b_n = b \). Then the convergence of the sequences \( \{ a_n \pm b_n : n \in \mathbb{N} \}, \{ a_n \cdot b_n : n \in \mathbb{N} \} \) or \( \{ a_n / b_n : n \in \mathbb{N} \} \) implies \( \lim (a_n \pm b_n) = a \pm b, \lim (a_n \cdot b_n) = a \cdot b \) or \( \lim (a_n / b_n) = a / b \) respectively (in the case of division we ought to require still \( b_n \in \mathbb{R} \), \( n \in \mathbb{N} \)).
We shall stop investigating the properties of the set of asymptotic numbers \( A \). (We have done it perhaps in more details than was necessary for our purposes in physics.) Now, we are ready to define the asymptotic functions and, in particular, Dirac's delta-function, as well as its square, etc., in accordance with our promise made at the beginning of [8]. We shall put this off for a future paper.

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References

1. Bourbaki, N. Livre Il Algebra, Chap. VI, Groupes et corps ordonnes (Hermann 1965)