ALEXANDER'S SUBBASE LEMMA

S. SALBANY AND T. D. TODOROV

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This note provides a very simple proof of Alexander's subbase lemma, from the point of view of Nonstandard Analysis. There is no direct appeal to Zorn's lemma or equivalent principle (as in [1]). This set theoretic principle is, of course, embodied in the construction of the nonstandard extension *X.

Notation and terminology is that of A. Hurd and P. Loeb [2].

Lemma (Alexander). Let (X, T) be a topological space and $\mathcal{S}$ a subbase of closed sets. If every family of closed sets in $\mathcal{S}$ with the finite intersection property has nonempty intersection, then (X, T) is compact.

Proof. Recall that (X, T) is compact iff $\ast X = \bigcup_{x \in X} \mu(x)$ [2, Theorem (2.9), Chapter III] and that the monad of x is $\mu(x) = \bigcap\{ \ast G | x \in G, X - G \in \mathcal{S} \}$ [2, Proposition (1.4) of Chapter III]. Let $\alpha \in \ast X$. Consider $\mathcal{F} = \{ F | F \in \mathcal{S}, \alpha \in \ast F \}$. Then $\mathcal{F}$ has the finite intersection property and, by assumption, there is a point x such that $x \in \bigcap\{ F | F \in \mathcal{F} \}$. We show that $\alpha \in \mu(x)$: if $x \in G$ and $X - G \in \mathcal{S}$, then $\alpha \notin \ast (X - G)$, by our choice of x, hence $\alpha \in \ast G$, as required.

References


Department of Mathematics, University of Zimbabwe, Harare, Zimbabwe
International School for Advanced Studies (SISSA/ISAS), 34014 Trieste, Italy

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