Generalized Worpitzky Identities with Applications to Permutation Enumeration

DON RAWLINGS

The enumeration of permutations by inversions often leads to a $q$-analog of the usual generating function. In this paper, two generalizations of the Worpitzky identity for the Eulerian numbers are obtained and used to enumerate permutations by the descent number and the major index of their inverses. The resulting $(t, q)$-generating series do in fact generalize the $q$-series obtained when counting by inversions.

1. Introduction

The Eulerian numbers $A_{n,s}$ are defined by the identity

$$ r^n = \sum_{k=0}^{n} \binom{r-1-k+n}{n} A_{n,k} \tag{1.1} $$

due to Worpitzky. Combinatorially $A_{n,s}$ may be interpreted as follows. Let $\mathcal{S}_n$ denote the symmetric group on $\{1, 2, \ldots, n\}$. For $\sigma \in \mathcal{S}_n$ define the idescent number of $\sigma$, abbreviated by $\text{ides} \sigma$, to be the cardinality of the set $\{k : \sigma^{-1}(k) > \sigma^{-1}(k+1), 1 \leq k \leq n-1\}$. Then, it is well known that $A_{n,s}$ is the number of permutations $\sigma \in \mathcal{S}_n$ with $\text{ides} \sigma = s$.

One approach to proving (1.1) is to first interpret $r^n$ as the cardinality of the set $[r]^n$ consisting of all integer sequences $f = f(1)f(2) \cdots f(n)$ with $1 \leq f(k) \leq r$ for $1 \leq k \leq n$. A map $\Gamma$ is then defined (see Section 6) that associates with each $f \in [r]^n$ a pair $(g, \sigma)$ where $g$ is the non-decreasing rearrangement of $f$ and $\sigma \in \mathcal{S}_n$ tells how to reconstruct $f$ from $g$. The correspondence $\Gamma$ has been used several times, for instance, see [8, 13, 15]. The pairs $(g, \sigma)$ give rise to the right-hand side of (1.1): $\binom{r-1-k+n}{n}$ is the number of non-decreasing sequences $g \in [r]^n$ that may be paired with $\sigma \in \mathcal{S}_n$ having $\text{ides} \sigma = k$. The proof outlined here is essentially the one given by Foata and Schützenberger [13, p. 40].

The above proof may be generalized in a number of ways. First, $\Gamma$ carries much more information than was utilized. To be specific, the classic statistics des, maj, and inv defined in (3.1) are preserved by $\Gamma$, that is, if $\Gamma(f) = (g, \sigma)$ then $(\text{des} f, \text{maj} f, \text{inv} f) = (\text{des} \sigma, \text{mas} \sigma, \text{inv} \sigma)$.

Second, the method for counting the non-decreasing sequences $g$ that may be paired with $\sigma \in \mathcal{S}_n$ is generalized in Lemma 5.1. In addition to ides, another statistic known as the imaj (see (3.2)) is incorporated into the scheme. As will be seen, Lemma 5.1 may be thought of as a way of $q$-counting various sets of non-decreasing sequences.

Finally, it will be fruitful to restrict $\Gamma$ to various subsets of $[r]^n$. For instance, a sequence $f \in [r]^n$ is said to be up-down if

$$ f(1) \leq f(2) > f(3) \leq f(4) > \cdots. $$

It turns out that if $f$ is up-down and $\Gamma(f) = (g, \sigma)$, then $\sigma$ is also up-down.

The additional statistics (des, maj, inv), the $q$-counting lemma, and the restriction of $\Gamma$ combine to give the main result in Theorem 7.1. It is a generalization of (1.1) that relates the tri-variate distribution of (des, maj, inv) on subsets of sequences to the five-variate distribution of (des, ides, maj, imaj, inv) on subsets of permutations. Thus, in this context, a
distribution on sequences is equivalent to a \((t, q)\)-distribution on permutations. The two additional statistics in the permutation case both arise in connection with the inverse of the permutation.

Theorem 7.1 has a number of interesting applications. For instance, the generating function for \((\text{des}, \text{ides}, \text{maj}, \text{imaj})\) on \(S_n\), which was first given by Garsia and Gessel [15], may be readily obtained from the joint distribution of \((\text{des}, \text{maj})\) on sequences. The latter distribution is the solution to the classic Simon Newcomb problem.

In another application of Theorem 7.1, a \((t, q)\)-generating function for \((\text{ides}, \text{imaj})\) on up-down permutations is given (see Theorem 8.6). This generalizes a well known result of André [1]. He discovered that if \(B_n\) is the number of up-down permutations in \(S_n\), then

\[
\sum_{n=0}^{\infty} \frac{B_n u^n}{n!} = \sec u + \tan u. \tag{1.2}
\]

The two above results on permutations and others are presented in Section 8.

Now Stanley [23] and Gessel [16] have both developed systematic approaches for obtaining \(q\)-series that enumerate permutations by inversions. With the aid of a combinatorial correspondence due to Foata [11], it will be shown in Section 9 that the \((t, q)\)-series obtained when counting by \((\text{ides}, \text{imaj})\) do indeed generalize the \(q\)-series obtained when enumerating by inversions.

2. Terminology and Notation

Let \(\mathbb{N} = \{0, 1, 2, \ldots \}\). A finite sequence \(f = (f(k))_{1 \leq k \leq n}\) of \(\mathbb{N}\) will be written as a word \(f = f(1)f(2) \cdots f(n)\) where \(f(1), f(2), \ldots, f(n)\) are the \(n\) letters and \(n\) is defined to be the length of \(f\), denoted by \(\lambda f\). Let \(\mathbb{N}^*\) be the set of finite words with letters in \(\mathbb{N}\). The down set of \(f \in \mathbb{N}^*\) is defined to be

\[
\text{down } f = \{k : f(k) > f(k + 1), 1 \leq k \leq \lambda f - 1\}.
\]

Certain expressions involving a sequence \((j_1, j_2, \ldots, j_r)\) of \(\mathbb{N}\) will occur with such frequency that it will be convenient to abbreviate them.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Abbreviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>((j_1, j_2, \ldots, j_r))</td>
<td>(f(r))</td>
</tr>
<tr>
<td>(j_1 + j_2 + \cdots + j_r = n)</td>
<td>(f(r) = n)</td>
</tr>
<tr>
<td>(\sum_{k=1}^{r} (r-k)j_k)</td>
<td>(r \cdot j(r))</td>
</tr>
</tbody>
</table>

For each \(n \in \mathbb{N}\) let \([n] = \{1, 2, \ldots, n\}\) if \(n\) is positive and \([0] = \emptyset\). Also, set

(a) \([r]^n = \{f \in \mathbb{N}^* : \lambda f = n, f(k) \in [r] \text{ for } 1 \leq k \leq n\}\),
(b) \(ND_n(r) = \{f \in [r]^n : f(1) \leq f(2) \leq \cdots \leq f(n)\}\),
(c) \(\mathcal{R}(f(r) = n) = \{f \in \mathbb{N}^* : \lambda f = n, f \text{ has exactly } j_k \text{ letters equal to } k \text{ for } 1 \leq k \leq r\}\),
(d) \(S_n = \{\sigma : \sigma \text{ is a permutation of } [n]\}\).

Words of \(ND_n(r)\) are said to be non-decreasing. Observe that \(\mathcal{R}(f(r) = n)\) is a subset of \([r]^n\) and, in the case \(j_k = 1\) for \(1 \leq k \leq r\), \(S_n = \mathcal{R}(f(r) = n)\).
Furthermore, for $d$ a positive integer let $\mathcal{U}[r]_d^+$ (respectively $\mathcal{D}[r]_d^+$) be the subset of $[r]^n$ consisting of sequences $f$ satisfying
\[ \text{down } f = \{ ds: 1 \leq ds \leq n - 1 \} \tag{2.3} \]
(respectively down $f = [n-1]-\{ ds: 1 \leq ds \leq n - 1 \}$). The subsets $\mathcal{U}_d(f(r) = n)$, $\mathcal{D}_d(f(r) = n)$, and $\mathcal{D}_d^+$ of $\mathcal{P}(f(r) = n)$ and $\mathcal{S}_n$ are similarly defined. In the case $d = 2$, $\mathcal{U}_n^+$ and $\mathcal{D}_n^+$ are respectively the up-down and the down-up permutations in $\mathcal{S}_n$.

Finally, the cardinality of a finite set $A$ is denoted by $|A|$. The $\chi$ function is defined by setting $\chi$ (statement) equal to 1 if the statement is true and 0 otherwise.

3. The Five Statistics

For a sequence $f \in \mathbb{N}^n$ let
\[ \text{(a)} \quad \text{des } f = |\text{down } f|, \]
\[ \text{(b)} \quad \text{maj } f = \sum_{k \in \text{down } f} k, \]
\[ \text{(c)} \quad \text{inv } f = \sum_{l=2}^{n} \sum_{k<l} \chi(f(k) > f(l)). \]

Parts (a), (b) and (c) respectively define what are commonly referred to as the descent number, the major or greater index, and the inversion number of $f$.

For $\sigma \in \mathcal{S}_n$ there are two more statistics to be defined. Let $\sigma^{-1}$ denote the inverse of $\sigma$ and set
\[ \text{(a)} \quad \text{idown } \sigma = \text{down } \sigma^{-1}, \]
\[ \text{(b)} \quad \text{ides } \sigma = \text{des } \sigma^{-1}, \]
\[ \text{(c)} \quad \text{Imaj } \sigma = \text{maj } \sigma^{-1}. \tag{3.2} \]

4. Preliminary Identities

The following $q$-identities will be needed. The $q$-analog of $n \in \mathbb{N}$ is defined to be the polynomial $(n)_q = 1 + q + q^2 + \cdots + q^{n-1}$ and $(0)_q = 0$. The $q$-factorial of $n$ is given by $(n)_q! = (1)_q(2)_q \cdots (n)_q$ where $(0)_q! = 1$. Finally, the $q$-multinomial coefficient is defined to be
\[ \binom{n}{f(r)}_q = \frac{(n)_q!}{(j_1)_q!(j_2)_q! \cdots (j_r)_q!} \]
where $f(r) = n$.

For convenience, set
\[ (t; q)_{n+1} = (1-t)(1-tq) \cdots (1-tq^n) \]
and recall that
\[ \text{(a)} \quad (t; q)_{n+1} = \sum_{k \geq 0} (-1)^k \binom{n+1}{k} q^k t^k, \]
\[ \text{(b)} \quad \frac{1}{(t; q)_{n+1}} = \sum_{k \geq 0} \binom{n+k}{k} t^k. \tag{4.1} \]

The identities in (4.1) may be found in Andrews [2, p. 36].

*Richard Askey has recently informed me that (4.1(a)) dates back to a book of Rothe published in 1811.
The two \( q \)-analog for the exponential function presented in

\[
\begin{align*}
(\text{a}) & \quad e_q(u) = \sum_{n \geq 0} \frac{u^n}{(n)_q!}, \\
(\text{b}) & \quad E_q(u) = \sum_{n \geq 0} q^{(q)} \frac{u^n}{(n)_q!},
\end{align*}
\]  

will also turn up in the study.

In addition to the above basic \( q \)-identities, a number of generating functions on sequences will be used in deriving results concerning permutations. They are stated

\[
\begin{align*}
(\text{a}) & \quad \sum_{f \in \mathcal{R}(f(r) = n)} q^{\text{maj}} = \sum_{f \in \mathcal{R}(f(r) = n)} q^{\text{inv}} = \binom{n}{j(r)}_q, \\
(\text{b}) & \quad \sum_{f \in \mathcal{R}(f(r) = n)} t^{\text{des}} q^{\text{maj}} = (t; q)_{n+1} \sum_{s \geq 0} t^s \prod_{k=1}^{r} \left[ \left( s - i k \right) \right]_q, \\
(\text{c}) & \quad \sum_{n \geq 0} \sum_{|\mathcal{R}_2(f(r) = n)|} |X^{(r)} = \frac{2 \Pi(X, r) \Pi(-X, r) - i[\Pi(X, r) - \Pi(-X, r)]}{\Pi(X, r) + \Pi(-X, r)}}, \\
(\text{d}) & \quad \sum_{n \geq 0} \sum_{|\mathcal{R}_2(f(r) = n)|} |X^{(r)} = \frac{2 - i[\Pi(X, r) - \Pi(-X, r)]}{\Pi(X, r) + \Pi(-X, r)}}, \\
(\text{e}) & \quad \sum_{n \geq 0} \sum_{j(r) = d(n)} |X^{(r)} = d \left[ \prod_{m=0}^{d-1} \prod_{k=1}^{r} (1 - w^{2m+1} x_k)^{-1} \right],
\end{align*}
\]

where \( i = \sqrt{-1}, \ w = e^{i \pi/d}, \ X^{(r)} = \prod_{k=1}^{r} x_k^h, \) \( \Pi(\pm X, r) = \prod_{k=1}^{r} (1 \pm i x_k). \)

Parts (a) and (b) of (4.3) are due to MacMahon [20]. Part (a) points out that the statistics \( \text{maj} \) and \( \text{inv} \) are identically distributed on \( R(f(r) = n) \) and (b) gives a \( q \)-analog of the solution to the classic Simon Newcomb problem. For the case \( q = 1 \) see [10, 2, p. 62].

Identities (c) and (d) are due to Carlitz [4, 5]. From (c) and (d) it is seen that enumeration of up-down sequences leads to a slightly different generating function than does enumeration of down-up sequences.

Part (e) may be found in Gessel [16, p. 51] or in Stanley [23]. Note that in the case \( d = 2 \), (e) does reduce to the even part of (c).

5. The \( q \)-Counting Lemma

The key counting tool is now presented. Let \( f \in \mathbb{N}^* \) with \( \lambda f = n \geq 1 \) and define

\[ \mathcal{H}(r, f) = \{ h \in \mathcal{N}D_n(r) : \text{if } k \in \text{dom } f \text{ then } h(k) < h(k + 1) \}. \]  

Observe that \( \mathcal{H}(r, f) \) depends only on the down set of \( f \) and that if \( h \in \mathcal{H}(r, f) \) then

\[ (h(1), f(1))(h(2), f(2)) \cdots (h(n), f(n)) \]

is in lexicographic order.

For counting purposes, with \( h \in \mathcal{H}(r, f) \) associate the word

\[ h^0 = 0^{m_0} 1 \ 0^{m_1} 1 \cdots 1 \ 0^{m_n} \]

where

\[
\begin{align*}
(\text{a}) & \quad m_0 = h(1) - 1, \\
(\text{b}) & \quad m_k = h(k + 1) - h(k) - \chi(k \in \text{down } f) \text{ for } 1 \leq k \leq n - 1, \\
(\text{c}) & \quad m_n = r - h(n).
\end{align*}
\]
Then \( \sum_{k=0}^{n} m_k = r - 1 - \text{des } f \) and \( \text{inv } h^0 = \sum_{k=1}^{n} km_k \). Actually, the map \( h \to h^0 \) is a bijection of \( \mathcal{H}(r, f) \) onto the set of sequences

\[
\mathcal{S}(r, f) = \left\{ 0^{m_0} 10^{m_1} \cdots 10^{m_n} : m_k \in \mathbb{N}, \sum_{k=0}^{n} m_k = r - 1 - \text{des } f \right\}.
\]

Certainly, \( h^0 \in \mathcal{S}(r, f) \) uniquely determines the corresponding \( h \in \mathcal{H}(r, f) \) by (a), (b) and (c) of (5.2). The map \( h \to h^0 \) and (a) of (4.3) imply

\[
\sum_{h \in \mathcal{H}(r, f)} q^{\text{inv } h^0} = \sum_{h^0 \in \mathcal{S}(r, f)} q^{\text{inv } h^0} = \binom{r - 1 - \text{des } f + n}{n}. \tag{5.3}
\]

Information concerning the major index of \( f \) may be obtained from (5.2) as follows:

\[
\text{inv } h^0 = \sum_{k=1}^{n} km_k = n(r - h(n)) + \sum_{k=1}^{n-1} k[h(k+1) - h(k) - \chi(k \in \text{down } f)]
=
= nr - \text{maj } f - \sum_{k=1}^{n} h(k). \tag{5.4}
\]

For later use this identity is reformulated. As \( h \) is non-decreasing, \( h \) may be written in the form \( 1^{j_1} 2^{j_2} \cdots r^{j_r} \) where \( j(r) = n \). Then (5.4) becomes

\[
\text{maj } f + \text{inv } h^0 = (j_1 + j_2 + \cdots + j_r)r - \sum_{k=1}^{r} k j_k
=
= \sum_{k=1}^{r} (r - k) j_k = r \cdot j(r). \tag{5.5}
\]

The last equality in (5.5) comes from definition (2.1). Identities (5.3) and (5.5) are recorded in

**Lemma 5.1.** If \( f \in \mathbb{N}^* \) with \( \lambda f \geq 1 \), then

(a)

\[
\sum_{h \in \mathcal{H}(r, f)} q^{\text{inv } h^0} = \binom{r - 1 - \text{des } f + \lambda f}{\lambda f}.
\]

Moreover,

(b)

If \( h = 1^{j_1} 2^{j_2} \cdots r^{j_r} \in \mathcal{H}(r, f) \) then \( \text{maj } f + \text{inv } h^0 = r \cdot j(r) \).

Part (a) of Lemma 5.1 may be thought of as a way of "\( q \)-counting" various classes of non-decreasing words in \([r]^n\) where \( n = \lambda f \). In particular, if \( \text{des } f = 0 \) and \( q = 1 \), then \( \mathcal{H}(r, f) = \text{ND}_n(r) \) and (a) reduces to \( |\text{ND}_n(r)| = \binom{r - 1 + n}{n} \).

6. **The Correspondence \( \Gamma \)**

The map \( \Gamma \) is easily understood when described in terms of readings. Let \( f \in [r]^n \) and \([n]\) be our set of labels. Using the labels in order, on the \( k \)th reading from left to right place a label above each \( k \) in \( f \). Continue until all labels are used. Then \( \Gamma(f) \) is defined to be the pair \((g, \sigma)\) where \( g \) is the non-decreasing rearrangement of \( f \) and \( \sigma \) is the word that appears
above \( f \). Note that \( \sigma \in \mathcal{S}_n \). For example, if \( f = 5 \ 1 \ 1 \ 3 \ 5 \ 4 \ 1 \ 4 \in [5]^8 \) then
\[
\begin{align*}
\text{1st reading,} &\quad f = 5 \ 1 \ 1 \ 3 \ 5 \ 4 \ 1 \ 4 \\
\text{2nd reading} &\quad f = 5 \ 1 \ 1 \ 3 \ 5 \ 4 \ 1 \ 4 \\
\text{3rd reading,} &\quad f = 5 \ 1 \ 1 \ 3 \ 5 \ 4 \ 1 \ 4 \\
\text{4th and 5th readings,} &\quad f = 5 \ 1 \ 1 \ 3 \ 5 \ 4 \ 1 \ 4 
\end{align*}
\]
and \( \Gamma(f) = (g, \sigma) \) where
\[
g = 1 \ 1 \ 1 \ 3 \ 4 \ 4 \ 5 \ 5 \in ND_{8}(5)
\]
and
\[
\sigma = 7 \ 1 \ 2 \ 4 \ 8 \ 5 \ 3 \ 6 \in \mathcal{S}_{8}.
\]
Note that the label placed above \( f(m) \) is \( \sigma(m) \) for \( 1 \leq m \leq n \). The important observation to make about \( \Gamma \) is the following proposition.

**PROPOSITION 6.1.** If \( \Gamma(f) = (g, \sigma) \) and \( k < l \), then \( f(k) > f(l) \) if and only if \( \sigma(k) > \sigma(l) \).

The proof of Proposition 6.1 is not difficult and is omitted.

**COROLLARY 6.2.** If \( \Gamma(f) = (g, \sigma) \) then
\[
(a) \quad \text{down } f = \text{down } \sigma,
\]
\[
(b) \quad \text{for } 2 \leq l \leq n, \sum_{k<l} \chi(f(k) > f(l)) = \sum_{k<l} \chi(\sigma(k) > \sigma(l)),
\]
\[
(c) \quad f\sigma^{-1} = f(\sigma^{-1}(1))f(\sigma^{-1}(2)) \cdots f(\sigma^{-1}(n)) \in \mathcal{H}(r, \sigma^{-1}).
\]

**PROOF.** Parts (a) and (b) follow immediately from Proposition 6.1 and the definition of the down set. By (5.1), part (c) is equivalent to
\[
(a) \quad f(\sigma^{-1}(k)) \leq f(\sigma^{-1}(k + 1)) \text{ for } 1 \leq k \leq n - 1,
\]
\[
(b) \quad \text{if } k \in \text{idown } \sigma \text{ then } f(\sigma^{-1}(k)) < f(\sigma^{-1}(k + 1)).
\]

First, (a) of (6.2) is clear by the labeling: the label \( \sigma(\sigma^{-1}(k + 1)) = k + 1 \) placed above \( f(\sigma^{-1}(k + 1)) \) is greater than the label \( \sigma(\sigma^{-1}(k)) = k \) placed above \( f(\sigma^{-1}(k)) \). As labels are used in order, this implies \( f(\sigma^{-1}(k + 1)) > f(\sigma^{-1}(k)) \). Finally, (6.2(b)) follows from Proposition 6.1: \( \sigma^{-1}(k + 1) < \sigma^{-1}(k) \) and \( \sigma(\sigma^{-1}(k + 1)) > \sigma(\sigma^{-1}(k)) \) imply \( f(\sigma^{-1}(k + 1)) > f(\sigma^{-1}(k)) \).

There are a number of consequences to be noted. First, it follows from Corollary 6.2 that if \( \Gamma(f) = (g, \sigma) \) then
\[
(a) \quad \text{des } f = \text{des } \sigma,
\]
\[
(b) \quad \text{maj } f = \text{maj } \sigma,
\]
\[
(c) \quad \text{inv } f = \text{inv } \sigma,
\]
\[
(d) \quad g = f\sigma^{-1} \in \mathcal{H}(r, \sigma^{-1}).
\]
Corollary 6.2(b) shows that $f$ uniquely determines $\sigma$. This fact together with Corollary 6.2(a) and (6.3(d)) imply the following.

**Corollary 6.3.** The map $\Gamma$ is a bijection from $\{f \in [r]^n : \text{down } f = D\}$ to $\{(g, \sigma) : \sigma \in \mathcal{F}_n, \text{down } \sigma = D, g \in \mathcal{K}(r, \sigma^{-1})\}$ where $D$ is any subset of $[n-1]$.

### 7. The Worpitzky Identities

In the identities that follow there will be three different $q$-multinomial coefficients appearing. For convenience, let

\[
\binom{n}{j(r)}_{q_1} = \binom{n}{j(r)}, \quad \binom{n}{j(r)}_{q_2} = \binom{n}{j(r)}', \quad \binom{n}{j(r)}_p = \binom{n}{j(r)}.
\]

Suppose that $\mathcal{G}(j(r) = n)$ is a subset of $\mathcal{R}(j(r) = n)$ and that the restriction of $\Gamma$ to $\bigcup_{j(r) = n} \mathcal{G}(j(r) = n)$ is a bijection onto $\{(g, \sigma) : \sigma \in \mathcal{D}_n, g \in \mathcal{K}(r, \sigma^{-1})\}$ where $\mathcal{D}_n$ is some subset of $\mathcal{S}_n$. Then we have the following theorem.

**Theorem 7.1.** The distribution of $(\text{des}, \text{ides}, \text{maj}, \text{imaj}, \text{inv})$ on $\mathcal{D}_n$ is related to the distribution of $(\text{des}, \text{maj}, \text{inv})$ on $\bigcup_{j(r) = n} \mathcal{G}(j(r) = n)$ by

\[
\begin{align*}
\sum_{\sigma \in \mathcal{D}_n} & \sum_{g \in \mathcal{K}(r, \sigma^{-1})} \binom{n}{j(r)}_{q_1} \binom{n}{j(r)}_{q_2} \binom{n}{j(r)}_p \prod_{i \in \{0, 1\}} \binom{n}{j(r)}_{2i} \prod_{i \in \{0, 1\}} \binom{n}{j(r)}_{2i+1} \\
& = \sum_{g \in \mathcal{K}(r, \sigma^{-1})} \binom{n}{j(r)}_{q_1} \binom{n}{j(r)}_{q_2} \binom{n}{j(r)}_p \sum_{\sigma \in \mathcal{D}_n} \prod_{i \in \{0, 1\}} \binom{n}{j(r)}_{2i} \binom{n}{j(r)}_{2i+1}.
\end{align*}
\]

**Proof.** Lemma 5.1(b) and (6.3) together imply that the right-hand side of (a) is equal to

\[
\sum_{\sigma \in \mathcal{D}_n} \prod_{g \in \mathcal{K}(r, \sigma^{-1})} \binom{n}{j(r)}_{2i} \binom{n}{j(r)}_{2i+1}.
\]

Applying (a) of lemma 5.1 then establishes Theorem 7.1(a). To prove (b), first observe that if $F(f) = (g, \sigma)$, then (5.1) and Corollary 6.2(a) imply that $\mathcal{H}(s, f) = \mathcal{H}(s, \sigma)$. This fact, Lemma 5.1 and (6.3) combine to yield (b).

In the case $\mathcal{G}(j(r) = n) = \mathcal{R}(j(r) = n)$ and $t_1 = q_1 = q_2 = p = 1$, identity (a) of Theorem 7.1 reduces immediately to the Worpitzky identity (1.1). Part (b) may be thought of as a bi-variate Worpitzky identity. When $\mathcal{G}(j(r) = n) = \mathcal{R}(j(r) = n)$ and $q_1 = q_2 = p = 1$, then (b) reduces to

\[
\begin{align*}
\sum_{k=0}^n & \sum_{l=0}^n \binom{n}{j(r) - k + n} \binom{n}{j(r) - l + n} \binom{n}{j(r)_{2i}} \prod_{i \in \{0, 1\}} \binom{n}{j(r)_{2i+1}} \\
& = \sum_{k=0}^n \sum_{l=0}^n \binom{n}{j(r) - k + n} \binom{n}{j(r) - l + n} \binom{n}{j(r)_{2i}} \prod_{i \in \{0, 1\}} \binom{n}{j(r)_{2i+1}}.
\end{align*}
\]

where $A_{n,l,k} = |\{\sigma \in \mathcal{S}_n : \text{des } \sigma = n, \text{ides } \sigma = l, \text{maj } \sigma = k\}|$. Identity (7.3) may be found in Carlitz, Roselle and Scoville [8].
As a final remark here, it is conceivable that there are other statistics besides des, maj and inv that are preserved by \( \Gamma \). Of course, any such statistic may be easily added into Theorem 7.1.

8. PERMUTATION ENUMERATION

Let

\[ \mathcal{P}_{n,k} = \{ \sigma \in \mathcal{S}_n : \text{ides } \sigma = k \}. \]  

(8.1)

The subsets \( \mathcal{U} \mathcal{P}_{n,k} \) and \( \mathcal{D} \mathcal{P}_{n,k} \) of \( \mathcal{U} \mathcal{P}_n \) and \( \mathcal{D} \mathcal{P}_n \) are similarly defined. The basic polynomials involved in the results of this section are

(a) \[ A_{n,k}(t_1, q_1, q_2, p) = \sum_{\sigma \in \mathcal{P}_{n,k}} t_1^{\text{des } \sigma} q_1^{\text{maj } \sigma} q_2^{\text{imaj } \sigma} p^{\text{inv } \sigma}, \]

(b) \[ A_n(t_1, t_2, q_1, q_2, p) = \sum_{k \geq 0} A_{n,k}(t_1, q_1, q_2, p)t_2^k, \]

(c) \[ B_{n,k}^d(q_2, p) = \sum_{\sigma \in \mathcal{P}_{n,k}} q_2^{\text{maj } \sigma} p^{\text{inv } \sigma}, \]

(d) \[ B_n^d(t_2, q_2, p) = \sum_{k \geq 0} B_{n,k}(q_2, p)t_2^k, \]

(e) \[ C_n^d(t_2, q_2, p) = \sum_{\sigma \in \mathcal{P}_{n,k}} t_2^{\text{ides } \sigma} q_2^{\text{maj } \sigma} p^{\text{inv } \sigma}. \]

It will be convenient to use the definitions of (8.2) with a certain flexibility. For instance, if \( t_2 = p = 1 \) we will omit \( t_2 \) and \( p \) from the expression as in

\[ A_n(t_1, 1, q_1, q_2, 1) = A_n(t_1, q_1, q_2). \]

In other words, the variables \( t_1, t_2, q_1, q_2 \) and \( p \) are respectively associated with the statistics des, ides, maj, imaj and inv. Also, note that the index \( k \) is associated with ides.

Combination of (4.3) with Theorem 7.1 will yield several generating functions for permutations. The first one presented is due to Gessel [16, p. 99].

**Theorem 8.1.** The tri-variate distribution of \( (\text{ides}, \text{maj}, \text{inv}) \) on \( \mathcal{S}_n \) is given by

\[ \sum_{n \geq 0} A_n(t_2, q_2, p)u^n = \sum_{r \geq 0} t_2^r \prod_{k=0}^{r} e_r(q_2^ku). \]

**Proof.** Theorem 7.1(a) with \( \mathcal{G}(j(r) = n) = \mathcal{R}(j(r) = n) \) and \( t_1 = q_1 = 1 \) combined with Corollary 6.3 and (4.3(a)) yield

\[ \sum_{k=0}^{n-1} \binom{n-1-k+n}{n} A_{n,k}(q_2, p) = \sum_{j(r) = n} q_2^{j(r)} \binom{n}{j(r)}. \]  

(8.3)

Then, using (4.1(b)), it is not difficult to verify that (8.3) is equivalent to Theorem 8.1.

**Theorem 8.2.** The distributions of \( (\text{ides}, \text{maj}, \text{inv}) \) and \( (\text{ides}, \text{maj}, \text{maj}) \) are identical on \( \mathcal{S}_n \), that is,

\[ \sum_{\sigma \in \mathcal{P}_n} t_2^{\text{ides } \sigma} q_2^{\text{maj } \sigma} p^{\text{inv } \sigma} = \sum_{\sigma \in \mathcal{P}_n} t_2^{\text{ides } \sigma} q_2^{\text{maj } \sigma} p^{\text{maj } \sigma}. \]

**Proof.** This follows immediately from (4.3(a)) combined with Theorem 7.1(a).

The symmetry described in Theorem 8.2 and the following one mentioned in Theorem 8.3 were first observed in [12] and [14] respectively. They will be further discussed in Section 9.
THEOREM 8.3. The distribution of \((\text{des, ides, maj, imaj})\) on \(S_n\) satisfies the symmetry relation

\[ A_n(t_1, t_2, q_1, q_2) = A_n(t_1 q_1^{-1}, t_2, q_1^{-1}, q_2). \]

**Proof.** By direct calculation, it follows from (4.3(b)) that

\[ \sum_{f \in \mathcal{R}(j(r)-n)} t_{f}^{\text{des}} q^{\text{maj}} = \sum_{f \in \mathcal{R}(j(r)-n)} (t q^n)^{\text{des}} q^{-\text{maj}}. \] 

Then Theorem 8.3 is a consequence of (8.4) and Theorem 7.1(a) with \(p = 1\).

Of course, the calculation involved in proving (8.4) is a bit lengthy. In the next section, (8.4) will be proven using a combinatorial correspondence due to Foata and Schützenberger [14].

**Theorem 8.4.** The four-variate distribution of \((\text{des, ides, maj, imaj})\) on \(S_n\) is given by

\[ \sum_{n \geq 0} A_n(t_1, t_2, q_1, q_2) u^n = \sum_{s \geq 0} \sum_{r \geq 0} t_1^s t_2^r \prod_{l=0}^{s} \prod_{k=0}^{r} (1 - u q_1^l q_2^k)^{-1}. \]

**Proof.** Theorem 7.1(a) with \(p = 1\) and \(\mathcal{K}(j(r)=n) = \mathcal{K}(j(r)=n)\) together with (4.3(b)) imply

\[ \sum_{k=0}^{r-1} \left[ \frac{r-1-k+n}{n} \right] A_{n,k}(t_1, q_1, q_2) = (t_1; q_1)_{n+1} \sum_{l(r)=n} q_2^{l(r)} \sum_{s \geq 0} \sum_{k=1}^{r} \left[ \frac{s+k}{f_k} \right]. \]

Then, using the basic \(q\)-identities given in (4.1), it may be verified that (8.5) is equivalent to Theorem 8.4.

Theorem 8.4 was obtained by Garsia and Gessel [15] using the so-called bipartite partitions of Gordon [18]. Multiplying both sides of the generating function in Theorem 8.4 by \((1-t_1)(1-t_2)\) and the setting \(t_1 = t_2 = 1\) yields the following corollary.

**Corollary 8.5.** The generating function for \((\text{maj, imaj})\) on \(S_n\) is

\[ \sum_{n \geq 0} (1-q_1)(1-q_1^2) \cdots (1-q_1^n) = \prod_{l=0}^{\infty} \prod_{k=0}^{\infty} (1 - u q_1^l q_2^k)^{-1}. \]

This result may be found in [9, 22] or implicitly in [18].

Now consider the restriction of \(\Gamma\) to \(\mathcal{U}[r]_n^d\) or \(\mathcal{D}[r]_n^d\). By Corollary 6.3, \(\Gamma\) is a bijection onto \((\{g, \sigma\}: \sigma \in \mathcal{S}_n, g \in \mathcal{K}(r, \sigma^{-1})\) or \((\{g, \sigma\}: \sigma \in \mathcal{C}_n, g \in \mathcal{K}(r, \sigma^{-1})\) respectively. For convenience, let

\[ \Pi(u, r) = \prod_{k=0}^{r} (1 + i u q^k). \]

(4.3(c)–(e)) may then be used to prove the following theorem.

**Theorem 8.6.** The generating functions for \((\text{ides, imaj})\) on \(\mathcal{U}^2_n, \mathcal{D}^2_n, \text{ and } \mathcal{U}^d_n\) are

\[ a) \sum_{n \geq 0} B^{2}_{n}(t, q) u^n = \sum_{r \geq 0} t^r \left[ \frac{2 |\Pi(u, r)|^{2} |\Pi(-u, r)}{2 |\Pi(u, r)|^{2} + |\Pi(-u, r)|} \right], \]

\[ b) \sum_{n \geq 0} C^{2}_{n}(t, q) u^n = \sum_{r \geq 0} t^r \left[ \frac{2 |\Pi(u, r)|^{2} |\Pi(-u, r)}{2 |\Pi(u, r)|^{2} + |\Pi(-u, r)|} \right], \]

\[ c) \sum_{n \geq 0} B^{d}_{n}(t, q) u^n = d \sum_{r \geq 0} t^r \left[ \frac{\sum_{m=0}^{r-1} \prod_{k=0}^{r} (1 - w^{2m+1} u q^k)^{-1}}{\sum_{k=0}^{r} \prod_{k=0}^{r} (1 - w^{2m+1} u q^k)^{-1}} \right]. \]
**Proof of (a).** From Theorem 7.1 with \( \mathcal{G}(j(r) = n) = \mathcal{U}_2(j(r) = n) \), \( t_1 = q_1 = p > 1 \) and \( q_2 = q \) it follows that
\[
B_{n,k}^2(q) = \sum_{i(r) = n} q^{r(i)} \mathcal{U}_2(j(r) = n).
\]
(8.7)

By equating coefficients of \( t' \), identity (a) of Theorem 8.6 is seen to be equivalent to
\[
\sum_{n \geq 0} u^n \sum_{k=0}^r \binom{r-k+n}{n} B_{n,k}^2(q) = \frac{2\Pi(u,r)\Pi(-u,r) - i[\Pi(u,r) - \Pi(-u,r)]}{\Pi(u,r) + \Pi(-u,r)}.
\]
(8.8)

Replacing \( r \) by \( r-1 \) in (8.8) and then using (4.3(c)) with \( uq^{r-1} = x_k \) to expand the right-hand side of (8.8) out leads to
\[
\sum_{n \geq 0} u^n \sum_{k=0}^r \binom{r-k+n}{n} B_{n,k}^2(q) = \sum_{n \geq 0} \sum_{i(r) = n} \mathcal{U}_2(j(r) = n) \prod_{k=1}^r (uq^{k-1})^k
\]
\[
= \sum_{n \geq 0} u^n \sum_{i(r) = n} q^{r(i)} \mathcal{U}_2(j(r) = n).
\]
(8.9)

Comparison of (8.7) and (8.9) yields (a) of the theorem. Proofs of (b) and (c) are similar.

Of course, this theorem does not exhaust all possibilities. From Corollary 6.3 it is clear that any generating function for sequences with a prescribed down set may be used to obtain a \((t, q)\)-generating function for permutations with the same prescribed down set. Other such generating functions for sequences may be found in [16, p. 52] or [23].

**Corollary 8.7.** The distribution of \( \text{imaj} \) on \( \mathcal{U}^2_n, \mathcal{F}_n^2, \) and \( \mathcal{U}^d_\text{ad} \) gives rise to
\[
(a) \quad \sum_{n \geq 0} B_n^2(q)u^n = \frac{2E_u(iu)E_u(-iu) - i[E_u(iu) - E_u(-iu)]}{E_u(iu) + E_u(-iu)}
\]
\[
= 2 - \frac{i[E_u(iu) - E_u(-iu)]}{E_u(iu) + E_u(-iu)}
\]
\[
(b) \quad \sum_{n \geq 0} C_n^2(q)u^n = \frac{2 - i[E_u(iu) - E_u(-iu)]}{E_u(iu) + E_u(-iu)}
\]
\[
(c) \quad \sum_{n \geq 0} B_{dn}^d(q)u^n = \frac{2 - i[E_u(iu) - E_u(-iu)]}{E_u(iu) + E_u(-iu)}
\]
\[
= 2 - \frac{i[E_u(iu) - E_u(-iu)]}{E_u(iu) + E_u(-iu)}
\]
\[
= \sum_{m=0}^{d-1} e_u(w^{2m+1}u^{d-1})^{-1}.
\]

**Proof of (a).** Multiplying both sides of Theorem 8.6(a) by \((1-t)\) and then setting \( t = 1 \) yields
\[
\sum_{n \geq 0} \frac{B_n^2(q)u^n}{(1-q)(1-q^2) \cdots (1-q^n)} = \frac{2\Pi(u, \infty)\Pi(-u, \infty) - i[\Pi(u, \infty) - \Pi(-u, \infty)]}{\Pi(u, \infty) + \Pi(-u, \infty)}.
\]
(8.10)

By [2, Identity (2.2.6), p. 19] we have
\[
\prod_{n \geq 0} (1 + uq^n) = 1 + \sum_{n \geq 1} q^{\binom{n}{p}} \frac{u^n}{(1-q)(1-q^2) \cdots (1-q^n)}.
\]
(8.11)

With (8.11) in mind, the transformation \( u \to (1-q)u \) applied to (8.10) yields Corollary 8.7(a). The second equality in part (a) follows from the fact that \( E_u(u)e_u(-u) = 1 \). Proofs of (b) and (c) are similar.
In the case $q = 1$ both (a) and (b) of Corollary 8.7 reduce to André's [1] result recorded in (1.2). Furthermore, (a)-(c) agree with the $q$-series obtained when counting by inversions (see [16, 23]).

9. Symmetries

A bijection $\Omega$ of $\mathcal{S}_n$ to itself is said to preserve the idown if $\text{idown}(\sigma) = \text{idown}(\sigma)$ for all $\sigma \in \mathcal{S}_n$. Such a map naturally induces a bijection $\Gamma^{-1}\Omega \Gamma: \mathcal{R}(j(r) = n) \rightarrow \mathcal{R}(j(r) = n)$ defined by

$$f \rightarrow (g, \sigma) \rightarrow (g, \Omega(\sigma)) \rightarrow \Gamma^{-1}\Omega \Gamma(f). \quad (9.1)$$

Since $\Gamma$ is a bijection of $[n]^\ast$ onto the set of pairs $\{(g, \sigma): \sigma \in \mathcal{S}_n, g \in \mathcal{R}(\mathcal{R}(\sigma^-1))\}$, the fact that $\Omega$ preserves the idown is essential. Two idown preserving bijections are now discussed.

In [11] Foata constructed a bijection $\Phi: \mathbb{N}^\ast \rightarrow \mathbb{N}^\ast$ with the properties

(a) $\Phi(f)$ has the same letters as $f$

(b) $\text{maj} f = \text{inv} \Phi(f)$

for all $f \in \mathbb{N}^\ast$. When restricted to $\mathcal{S}_n$ it turns out that

$$\text{idown} \sigma = \text{idown} \Phi(\sigma) \quad (9.3)$$

for all $\sigma \in \mathcal{S}_n$. The map $\Phi$ was used in [12] to observe the symmetry proven by calculation in Theorem 8.2 of this paper. The map $\Phi$ may also be used to show that $q$-counting permutations by inversions leads to the same $q$-series as $q$-counting by the major index of the inverse. More precisely stated, we have the following theorem.

**Theorem 9.1.** The statistics $\text{imaj}$ and $\text{inv}$ are identically distributed on $\{\sigma \in \mathcal{S}_n: \text{down } \sigma = \mathcal{B}\}$ where $\mathcal{B}$ is any subset of $[n - 1]$.

**Proof.** Let $I: \mathcal{S}_n \rightarrow \mathcal{S}_n$ be defined by $I(\sigma) = \sigma^{-1}$. Then the map $I\Phi I: \mathcal{S}_n \rightarrow \mathcal{S}_n$ satisfies the conditions

(a) $\text{imaj} \sigma = \text{inv} I\Phi I(\sigma)$

(b) $\text{down} \sigma = \text{down} I\Phi I(\sigma)$

for all $\sigma \in \mathcal{S}_n$. This follows from (9.2), (9.3) and the fact that $\text{inv} \sigma = \text{inv} I(\sigma)$. By (9.4(b)), the restriction of $I\Phi I$ to $\{\sigma \in \mathcal{S}_n: \text{down } \sigma = \mathcal{B}\}$ is a bijection onto $\{\sigma \in \mathcal{S}_n: \text{down } \sigma = \mathcal{B}\}$. The assertion now follows from (9.4(a)).

Another idown preserving bijection $J: \mathcal{S}_n \rightarrow \mathcal{S}_n$ was constructed in [14]. The map $J$ has the properties

(a) $\text{down} J(\sigma) = \{n - l: l \in \text{down } \sigma\}$

(b) $\text{idown} J(\sigma) = \text{idown } \sigma$

for all $\sigma \in \mathcal{S}_n$. Roughly speaking, (9.5) says that the down set is "reversed" by $J$ while the idown set is unaltered. By (9.1) and (9.5), $\Gamma^{-1}J\Gamma: \mathcal{R}(j(r) = n) \rightarrow \mathcal{R}(j(r) = n)$ is a well defined bijection and

$$\text{down } \Gamma^{-1}J\Gamma(f) = \{n - l: l \in \text{down } f\} \quad (9.6)$$

for all $f \in \mathcal{R}(j(r) = n)$. Thus, $\Gamma^{-1}J\Gamma$ provides a combinatorial proof of Identity (8.4). Foata and Schützenberger [14] used $J$ to prove the result listed in Theorem 8.3. The map $J$ may also be used to observe analogous symmetries on the polynomials $B_n^q(t, q)$ and $C_n^q(t, q)$. 
**Theorem 9.2.** The polynomials $B_n^d(t, q)$ and $C_n^d(t, q)$ satisfy

(a) $B_n^d(t, q) = B_n^d(tq^n, q^{-1})$

(b) $C_n^d(t, q) = C_n^d(tq^n, q^{-1})$.

**Proof.** The bijection $III: \mathcal{S}_n \rightarrow \mathcal{S}_n$ has the properties

(a) $\text{down } III(\sigma) = \text{down } \sigma$

(b) $\text{idown } III(\sigma) = \{n - k : k \in \text{idown } \sigma\}$. \hfill (9.7)

The assertion now follows from (9.7).

**References**

17. I. Gessel, Exponential generating functions (mod $p$) and their $q$-analogs, in preparation.