Necessary and Sufficient Condition that the Limit of Stieltjes Transforms is a Stieltjes Transform

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Abstract

The pointwise limit S of a sequence of Stieltjes transforms (S_n) of real Borel probability measures (P_n) is itself the Stieltjes transform of a Borel p.m. P if and only if $iy\,S(iy) \to -1$ as $y \to \infty$, in which case P_n converges to P in distribution. Applications are given to several problems in mathematical physics.

Key words and phrases: real Borel probability measure, convergence in distribution, Stieltjes transform, Lévy continuity theorem, Akhiezer-Krein theorem, weak convergence of probability measures.

Lévy's classical continuity theorem says that if the pointwise limit of the characteristic functions of a sequence of real Borel probability measures (P_n) exists, then the limit function φ is itself the characteristic function for a probability measure P if and only if φ is continuous at zero, in which case $P_n \to P$ in distribution. The purpose of this note is to prove a direct analog of Lévy's theorem for Stieltjes transforms, complementing those for other representing functions in [HS] and [HK], and to give several examples of applications.

Throughout this note, $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, respectively; p.m. and s.p.m. denote Borel probability measures, and sub-probability (mass ≤ 1) measures, respectively, on $\mathbb{R}$; and s.p.m.'s (μ_n) *converge vaguely* to a s.p.m. μ [C, p. 80], if there exists a dense subset D of $\mathbb{R}$ such that for all $a, b \in D$, $a < b$, $\mu_n((a,b]) \to \mu((a,b])$. (Thus if (μ_n), μ are p.m.'s, vague convergence is equivalent to convergence in distribution.)

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[1] Research partially supported by NSF Grant DMS-9970613.
[2] Research partially supported by NSF Grant DMS-9971146 and Göttingen Academy of Sciences Gauss Professorship (Fall 2000).

Definition 1. The *Stieltjes transform* S_P of a p.m. P is the function $S_P : \{\mathrm{Im}(z) > 0\} \to \mathbb{C}$ given by

$$S_P(z) = \int_{-\infty}^{\infty} \frac{1}{w - z}\, dP(w).$$

A basic property of Stieltjes transforms, which has important applications in the theory of moments (cf. [A], [S1], [ST]) and in mathematical physics ( ), is that they are a representing class for finite measures.

Lemma 1. *For s.p.m.'s P and Q, $P = Q$ iff $S_P = S_Q$.*

Proof. Follows immediately from the Stieltjes transform inversion formula [A, p. 125]. □

Just as limits of characteristic functions of p.m.'s are in general not characteristic functions, and limits of Hardy-Littlewood functions or expected-extrema functions are not in general Hardy-Littlewood or expected-extrema functions [HK], limits of Stieltjes transforms are not always Stieltjes transforms, as the next easy example shows.

Example 1. For $n = 1, 2, \ldots$, let $P_n = \delta_{(n)}$, the Dirac point mass at n. Then $S_{P_n}(z) = (n - z)^{-1}$ for all n and all z with $\mathrm{Im}(z) > 0$, so $\lim_{n\to\infty} S_{P_n}(z) \equiv 0$, which is clearly not the Stieltjes transform for any p.m. P (see Lemma 2 below).

On the other hand, just as with Lévy's theorem, the limit of Stieltjes transforms is itself a Stieltjes transform if and only if it satisfies one single universal limit condition. The next theorem is the main result of this note.

Theorem 1. *Suppose that (P_n) are real Borel probability measures with Stieltjes transforms (S_n), respectively. If $\lim_{n\to\infty} S_n(z) = S(z)$ for all z with $\mathrm{Im}(z) > 0$, then there exists a Borel probability measure P with Stieltjes transform $S_P = S$ if and only if*

$$\lim_{y\to\infty} iyS(iy) = -1, \tag{1}$$

in which case $P_n \to P$ in distribution.

Corollary 1. *If P, (P_n) are real Borel p.m.'s with Stieltjes transforms S, (S_n), respectively, then $P_n \to P$ in distribution if and only if $S_n \to S$ pointwise.*

Proof of Corollary. If $S_n \to S$, then $P_n \to P$ in distribution by Theorem 1. Conversely, suppose that $P_n \to P$ in distribution. Since $f_z(w) := (w-z)^{-1}$ is continuous and bounded in w for fixed z in $\{\text{Im}(z) > 0\}$, then $\text{Im}(f_z)$ and $\text{Re}(f_z)$ are also continuous and bounded, so by the basic equivalence of convergence in distribution of p.m.'s and convergence of integrals of bounded continuous functions [C, Theorem 4.4.2], $\int \text{Im}(f_z)dP_n \to \int \text{Im}(f_z)dP$ and $\int \text{Re}(f_z)dP_n \to \int \text{Re}(f_z)dP$, so $S_n(z) \to S(z)$. $\square$

To facilitate the proof of Theorem 1, two additional lemmas are useful, which are stated here for ease of reference.

Lemma 2. *Let $S : \{\text{Im}(z) > 0\} \to \mathbb{C}$ be analytic. Then there exists a p.m. P with $S_P(z) = S(z)$ for all z with $\text{Im}(z) > 0$ if and only if (1) holds and*

$$\text{Im}(S(z)) > 0 \qquad \text{for all } z \text{ with } \text{Im}(z) > 0. \tag{2}$$

Proof. By the classical Akhiezer-Krein theorem [A, p. 93], $S = S_P$ for some finite Borel measure P if and only if: S is analytic in $\{\text{Im}(z) > 0\}$; S satisfies (2); and

$$\sup_{y \geq 1} |yS(iy)| < \infty. \tag{3}$$

Suppose P is a p.m. with $S = S_P$. The Akhiezer-Krein theorem implies that (2) holds, and (1) follows immediately from the definition of S_P. Conversely, suppose that S is analytic and satisfies (1) and (2). Since $yS(iy)$ is continuous in y, (1) easily implies (3), so by the Akhiezer-Krein theorem again, there is a finite Borel measure P with $S_P = S$. Since clearly $\lim_{y \to \infty} [-iyS_P(iy)] = \text{mass}(P)$, (1) implies that P is a p.m. $\square$

Lemma 3. *Let $\mathcal{F}$ be a family of functions analytic in a connected open domain D. If for each compact $K \subset D$ there exists a constant $M(K) < \infty$ such that*

$$(4) \qquad |f(z)| \leq M(K) \qquad \text{for all } z \in K \text{ and all } f \in \mathcal{F},$$

then all pointwise limits of functions in $\mathcal{F}$ are also analytic in D.

Proof. ([H, Theorem 15.2.3]). □

Proof of Theorem 1. If $\lim S_n = S = S_P$ for some p.m. P, then (1) follows by Lemma 2.

Conversely, suppose that $S = \lim_n S_n$ satisfies (1). Let $\mathcal{F} = \{\bigcup S_n\}$, and for $K \subset D := \{\mathrm{Im}(z) > 0\}$, let $d(K) = \inf\{\|y - z\| : y \in \mathbb{R}, z \in K\}$, the smallest distance from K to the real line. Clearly $0 < d(K) < \infty$ for all compact $K \subset D$, and $M(K) = 1/d(K)$ satisfies (4), so Lemma 3 implies that $S = \lim S_n$ is analytic in D. By Lemma 2, $\mathrm{Im}(S_n(z)) > 0$ for all $z \in D$, so $\mathrm{Im}(S(z)) \geq 0$ for all $z \in D$. Suppose, by way of contradiction to (2), that $\mathrm{Im}(S(z_0)) = 0$ for some $z_0 \in D$. Since S is analytic, $\mathrm{Im}(S)$ and $\mathrm{Re}(S)$ are harmonic on D [K, p. 590]. By the maximum principle [K, p. 760], a non-constant function which is harmonic in a simply connected bounded open set G has neither a maximum nor a minimum in G, so since $\mathrm{Im}(S(z)) \geq 0$ on G for every simply connected open bounded set G with $z_0 \in G \subset D$, it follows (taking $G_t = \{z \in D : \|z\| < t\}$, and letting $t \to \infty$) that $\mathrm{Im}(S(z)) \equiv 0$ for all $z \in D$, which contradicts (1). Thus (2) holds, and since S is analytic and (1) holds by assumption, Lemma 2 implies there exists a real Borel p.m. P with $S_P = S$.

For the convergence in distribution conclusion, suppose that $S_n = S_{P_n} \to S_P$ pointwise in D for p.m.'s (P_n), P. By the Helly selection theorem [C, Theorem 4.3.3], there exists a s.p.m. Q and a subsequence (P_{n_k}) of (P_n) such that $P_{n_k} \to Q$ vaguely. Fix z in D, and let $f_z : \mathbb{R} \to \mathbb{C}$ be given by $f_z(w) = (w - z)^{-1}$. Since f_z is continuous in w and vanishes at infinity, $\mathrm{Re}(f_z)$ and $\mathrm{Im}(f_z)$ are continuous and vanish at infinity, so it follows by the equivalence of vague convergence of s.p.m.'s and convergence of integrals of continuous functions which vanish at infinity [C, Theorem 4.4.1] that $S_{P_{n_k}}(z) \to S_Q(z)$ as $k \to \infty$ for all $z \in D$. By hypothesis, $S_{P_n} \to S_P$, so $S_P = S_Q$, which by Lemma 1 implies that $P = Q$. Since every vaguely convergent subsequence of (P_n) thus converges to P, this implies [C, Theorem 4.3.4] that P_n converges vaguely to P, that is, since (P_n) and P are p.m.'s, P_n converges to P in distribution. □

Sketch of Alternative Proof. (B. Simon [S2]). The functions $\{S_{P_n}\}$ are Herglotz functions, so the limit S is Herglotz, and pointwise convergence $S_n \to S$ implies weak convergence for the measures $(1+x^2)^{-1}dP_n$ to a measure P on $[-\infty, \infty]$, where $(1+x^2)^{-1}dP$ is finite. Given that $S(iy) \to 0$ as $y \to \infty$, $P(\{-\infty, \infty\}) = 0$, so it follows using the fact that $S_{P_n} \to S_P$, the Herglotz representation theorem, and the monotone convergence theorem, that P is a p.m. Then weak convergence of P_n to P can be shown given the weak convergence of the measures when multiplied by $(1+x^2)^{-1}$. (For related ideas, see pp. 129–130 in [S1]). $\square$

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