Adjacencies in Words

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Based on two inversion formulas for enumerating words in the free monoid by adjacencies, we present a new approach to a class of permutation problems having Eulerian-type generating functions. We also show that a specialization of one of the inversion formulas gives Diekert’s lifting to the free monoid of an inversion theorem due to Cartier and Foata.

1. INTRODUCTION

There are a number of powerful theories of inversion [9, 10, 13, 16] for dealing with combinatorial objects having generating functions of Eulerian-type

\[
\frac{1}{1 + \sum_{n \geq 1} (-1)^n (1 - t)^{n-1} c_n z^n}.
\]

Using two such inversion formulas, we present new derivations of Stanley’s [13] generating functions for generalized $q$-Eulerian and $q$-Euler polynomials on $r$-tuples of permutations. We further indicate how one of the inversion formulas gives Diekert’s [5] lifting to the free monoid of an
inversion theorem of Cartier and Foata [4]. The inversion theorems we use enumerate words in the free monoid by adjacencies.

An alphabet \( X \) is a non-empty set whose elements are referred to as letters. A finite sequence (possibly empty) \( w = x_1x_2 \cdots x_n \) of \( n \) letters is said to be a word of length \( n \). The empty word will be denoted \( 1 \). The set of all words formed with letters in \( X \) along with the concatenation product is known as the free monoid generated by \( X \) and is denoted by \( X^* \). We let \( X^+ \) be the set of words having positive length.

From \( X \), we construct the adjacency alphabet \( A = \{a_{xy} : (x, y) \in X \times X\} \). The adjacency monomial and the sieve polynomial for \( w = x_1x_2 \cdots x_n \in X^* \) of length \( n \geq 2 \) are defined respectively as \( a(w) = a_{x_1x_2}a_{x_2x_3} \cdots a_{x_{n-1}x_n} \) and \( \bar{a}(w) = (a_{x_1x_2} - 1)(a_{x_2x_3} - 1) \cdots (a_{x_{n-1}x_n} - 1) \). For \( 0 \leq n \leq 1 \), we set \( a(w) = \bar{a}(w) = 1 \). In \( \mathbb{Z}[A] \ll X \gg \), the algebra of formal series of words in \( X^* \) with coefficients from the commutative ring of polynomials in \( A \) having integer coefficients, the following inversion formulas hold:

**Theorem 1.** According to adjacencies, the words in \( X^* \) are generated by

\[
\sum_{w \in X^*} a(w)w = \left(1 - \sum_{w \in X^+} \bar{a}(w)w\right)^{-1}.
\]  

**Theorem 2.** For non-empty subsets \( U, V \subseteq X \), the words according to adjacencies in \( U^*V = \{uw : u \in U^*, v \in V\} \) are generated by

\[
\sum_{w \in U^*V} a(w)w = \left(1 - \sum_{w \in U^+} \bar{a}(w)w\right)^{-1}\left(\sum_{w \in U^*V} \bar{a}(w)w\right).
\]  

Theorem 1 may be deduced from Stanley's [14, p. 266] synthesis of an inversion formula on clusters due to Goulden and Jackson [10, p. 131] with a related result of Zeilberger's [16] that enumerates words by mistakes. Theorem 2 bears comparison to (but is not equivalent to either) Viennot's [15] formula that counts heaps of pieces with restricted maximal elements and with a theorem of Goulden and Jackson [10, p. 238] for strings with distinguished final string. Proofs of Theorems 1 and 2 are deferred to Section 6. In passing, we mention that Hutchinson and Wilf [11] have given a closed formula for counting words by adjacencies.

The applications we give rely on the fact that setting \( a_{xy} = 1 \) eliminates all words containing \( xy \) as a factor from the right-hand sides of (1) and (2).
For instance, suppose that $X = \{x, y, z\}$. Set $a_{xx} = a$, $a_{xy} = b$, and the remaining $a_{ij} = 1$. Theorem 1 yields

$$\sum_{w \in \{x, y, z\}^*} a(w)w$$

$$= \frac{1}{1 - y - z - \sum_{n \geq 1} (a - 1)^{n-1} x^n - \sum_{n \geq 1} (a - 1)^{n-1} (b - 1) x^n y}$$

$$= (1 + x - ax)(1 - ax - y - z + (a - b)xy + (a - 1)xz)^{-1}.$$  

2. A KEY BIJECTION

In applying Theorems 1 and 2 to the enumeration of permutations, we make repeated use of a bijection that associates a pair $(\sigma, \lambda)$, where $\sigma$ is a permutation and $\lambda$ is a partition, to a finite sequence $w$ of non-negative integers. Let $N = \{0, 1, 2, \ldots \}$ and $N^n$ be the set of words of length $n$ in $N^*$. The rise set, rise number, inversion number, and norm of $w = i_1 i_2 \cdots i_n \in N^n$ are respectively defined to be

$$\text{Ris } w = \{ k : 1 \leq k < n, i_k \leq i_{k+1} \}, \quad \text{ris } w = |\text{Ris } w|,$$

$$\text{inv } w = |\{(k, m) : 1 \leq k < m \leq n, i_k > i_m \}|, \quad ||w|| = i_1 + \cdots + i_n.$$  

The set of non-decreasing words in $N^n$ (i.e., partitions with at most $n$ parts) will be denoted by $P_n$. A permutation $\sigma$ in the symmetric group $S_n$ on $\{1, 2, \ldots, n\}$ will be viewed as the word $\sigma(1)\sigma(2)\cdots\sigma(n)$. The key bijection used in Sections 3 and 4 may be described as follows.

LEMMA 1. For $n \geq 1$, there exists a bijection $f_n : S_n \times P_n \rightarrow N^n$ such that $\text{Ris } \sigma = \text{Ris } w$ and $\text{inv } \sigma + ||\lambda|| = ||w||$ whenever $f_n(\sigma, \lambda) = w$.

Proof. First, for $\sigma \in S_n$ and $1 \leq k \leq n$, let $c_k$ be the cardinality of the set $\{ j : k + 1 \leq j \leq n, \sigma(j) > \sigma(k) \}$. The number $c_k$ counts the inversions in $\sigma$ due to $\sigma(k)$. The word $c = c_1 c_2 \cdots c_n$ is known as the Lehmer code [12] of $\sigma$. Note that $\text{inv } \sigma = c_1 + \cdots + c_n = ||c||$ and that $\text{Ris } \sigma = \text{Ris } c$. As an illustration, the Lehmer code of $\sigma = 51342 \in S_5$ is $c = 40110$. Also, $\text{inv } \sigma = 6 = ||c||$ and $\text{Ris } \sigma = [2, 3] = \text{Ris } c$.

Next, for $(\sigma, \lambda) = (\sigma(1)\sigma(2)\cdots\sigma(n), \lambda_1 \lambda_2 \cdots \lambda_n) \in S_n \times P_n$, define $f_n(\sigma, \lambda)$ to be the word $w = i_1 i_2 \cdots i_n \in N^n$, where $i_k = c_k + \lambda_{\sigma(k)}$ for $1 \leq k \leq n$. When $f_n(\sigma, \lambda) = w$, we clearly have the properties

$$k \in \text{Ris } \sigma \text{ iff } c_k + \lambda_{\sigma(k)} \leq c_{k+1} + \lambda_{\sigma(k+1)} \text{ iff } k \in \text{Ris } w,$$

$$\text{inv } \sigma + ||\lambda|| = c_1 + \cdots + c_n + \lambda_1 + \cdots + \lambda_n = ||w||.$$
For example, the map $f_5$ sends the pair $(\sigma, \lambda) = (5 \ 1 \ 3 \ 4 \ 2, 1 \ 1 \ 1 \ 1 \ 2) \in S_5 \times P_5$ to the word $w = 6 \ 1 \ 2 \ 2 \ 1 \in N^5$. Note that $\text{Ris} \ \sigma = \{2, 3\} = \text{Ris} \ \lambda$ and that $\text{inv} \ \sigma + \|\lambda\| = 6 + 6 = \|w\|$. The inverse of $f_n$ may be realized by applying the insertion-shift bijection presented in [6] to the word $w$ to obtain $(\sigma^{-1}, \lambda)$. The description of $f_n$ given above was suggested by Foata (personal communication).

3. $q$-EULERIAN POLYNOMIALS

As the first application of Theorem 1, we derive a generating function for the sequence

$$A_n(t, q) = \sum_{\sigma \in S_n} t^{\text{Ris} \ \sigma} q^{\text{inv} \ \sigma}.$$ 

The polynomial $A_n(t, 1)$ is the $n$th Eulerian polynomial. We further obtain the generating function for Stanley’s [13] generalized $q$-Eulerian polynomials on $r$-tuples of permutations.

The first step in obtaining a generating function for the distribution of $(\text{Ris}, \text{inv})$ on $S_n$ is to appropriately define the adjacency monomial and sieve polynomial for the alphabet $N$. Toward this end, we set $a_{i,j} = t$ if $i \leq j$ and $a_{i,j} = 1$ otherwise. For $w = i_1 i_2 \cdots i_n$, note that $a(w) = t^{\text{Ris} \ w}$ and that

$$\bar{a}(w) = \begin{cases} (t-1)^{n-1} & \text{if } i_1 \leq i_2 \leq \cdots \leq i_n \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1 reduces to

$$\sum_{w \in N^*} t^{\text{Ris} \ w} W(w) = \frac{1}{1 - \sum_{n \geq 1} (t-1)^n \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_n} i_1 i_2 \cdots i_n}. \quad (3)$$

Next, we assign the weight $W(i) = z q^i$ to each $i \in N$ and extend $W$ to a multiplicative homomorphism on $N^*$. Let $(q; q)_0 = 1$ and, for $n \geq 1$, set $(q; q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n)$. Then, Lemma 1 and (3) justify the calculation

$$\sum_{n \geq 0} \frac{A_n(t, q) z^n}{(q; q)_n} = \sum_{n \geq 0} z^n \sum_{(\sigma, \lambda) \in S_n \times P_n} t^{\text{Ris} \ \sigma} q^{\text{inv} \ \sigma + \|\lambda\|} = \sum_{w \in N^*} t^{\text{Ris} \ w} W(w) = \frac{1}{1 - \sum_{n \geq 1} (t-1)^n \sum_{0 \leq i_1 \leq \cdots \leq i_n} q^{i_1 + \cdots + i_n} \sum_{0 \leq i_1 \leq \cdots \leq i_n} i_1 i_2 \cdots i_n}.$$
The common rise number of an $r$-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_r)$ of permutations in $S^n = S_n \times \cdots \times S_n$ is defined to be $\text{cris}(\sigma_1, \sigma_2, \ldots, \sigma_r) = |\cap_{j=1}^r \text{Ris} \sigma_j|$. The argument in (4) is readily adapted to deriving Stanley's [13] generating function for the polynomials

$$A_{n,r}(t, q_1, q_2, \ldots, q_r) = \sum_{(\sigma_1, \sigma_2, \ldots, \sigma_r) \in S^n} t^{\text{cris}(\sigma_1, \sigma_2, \ldots, \sigma_r)} q_1^{\text{inv} \sigma_1} q_2^{\text{inv} \sigma_2} \cdots q_r^{\text{inv} \sigma_r}.$$

(5)

We sketch the details for $r = 2$ and then state the general result.

For letters $i = (i_1, i_2)$ and $j = (j_1, j_2)$ in the alphabet $N \times N$, we define

$$a_{ij} = \begin{cases} t & \text{if } i_1 \leq i_2 \text{ and } j_1 \leq j_2 \\ 1 & \text{otherwise.} \end{cases}$$

For $(v, w) = (i_1 i_2 \cdots i_n, j_1 j_2 \cdots j_n) \in (N \times N)^n$, we have $a(v, w) = t^{\text{cris}(v, w)}$, where $\text{cris}(v, w) = |\text{Ris} v \cap \text{Ris} w|$. Also,

$$\bar{a}(v, w) = \begin{cases} (t-1)^{n-1} & \text{if } i_1 \leq i_2 \leq \cdots \leq i_n \text{ and } j_1 \leq j_2 \leq \cdots \leq j_n \\ 0 & \text{otherwise.} \end{cases}$$

The map of Lemma 1 applied component-wise to $(S_n \times P_n) \times (S_n \times P_n)$,

$$f_n \times f_n(\sigma_1, \lambda; \sigma_2, \mu) = (f_n(\sigma_1, \lambda), f_n(\sigma_2, \mu)) = (v, w),$$

is a bijection to $N^n \times N^n$ with $\text{cris}(\sigma_1, \sigma_2) = \text{cris}(v, w)$, $\text{inv} \sigma_1 + \|\lambda\| = \|v\|$, and $\text{inv} \sigma_2 + \|\mu\| = \|w\|$. Repeating (4) with appropriate modifications gives

$$\sum_{n \geq 0} A_{n,2}(t, q_1, q_2) z^n = \frac{1-t}{J(z(1-t), q_1, q_2) - t},$$

where $J(z, q_1, q_2) = \sum_{n \geq 0} (-1)^n z^n/(q_1; q_1)_n(q_2; q_2)_n$ is a bibasic Bessel function. We note that replacing $z$ by $z(1-q_1)(1-q_2)$ and letting $q_1, q_2 \to 1^-$ give the original result of Carlitz, Scoville, and Vaughan [3] that initiated the study of statistics on $r$-tuples of permutations.
If we let $q = (q_1, q_2, \ldots, q_r)$ and $(q; q)_{n, r} = (q_1; q_1)(q_2; q_2) \cdots (q_r; q_r)_n$, it follows in general that

**THEOREM 3 (Stanley).** For $r \geq 1$, the sequence $\{A_{n, r}(t, q)\}_{n \geq 0}$ is generated by

$$\sum_{n \geq 0} A_{n, r}(t, q) z^n = \frac{1 - t}{F_r(z(1 - t), q) - t},$$

where $F_r(z, q) = \sum_{n \geq 0} (-1)^n z^n / (q; q)_{n, r}$.

Further consideration of statistics on $r$-tuples of permutations is given in [7, 8]. In [7], we extend the technique of Carlitz et al. [3] and present recurrence relationships that refine Theorem 3. We also discuss several related distributions. In [8], we obtain a stronger version of Theorem 3 by using Theorems 1 and 2 in combination with a map that carries more information than does the bijection of Lemma 1.

4. **q-EULER POLYNOMIALS**

André [1] shows that if $E_n$ is the number of up-down alternating permutations in $S_n$ (that is, $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \cdots$), then

$$\sum_{n \geq 0} E_n z^n = \frac{1 + \sin z}{\cos z}. \quad (6)$$

The number $E_n$ is known as the $n$th Euler number.

We now apply Theorems 1 and 2 to the more general problem of counting the set of odd-up permutations

$$\mathcal{O}_n = \{ \sigma \in S_n : \sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \ldots \}$$

by inversion number and by the number of even indexed rises

$$\text{ris}_2 \sigma = |\{ k \in \text{Ris} \sigma : k \text{ is even} \}|.$$

Toward this end, let

$$E_n(t, q) = \sum_{\sigma \in \mathcal{O}_n} t^{\text{ris}_2 \sigma} q^{\text{inv} \sigma}.$$

Note that $E_n(0, 1) = E_n$. The analysis is split into two cases: $n$ odd and $n$ even. We only present the odd case, which requires use of Theorem 2.
Let $U = \{i = i_1i_2 : i_1, i_2 \in \mathbb{N} \text{ with } i_1 \leq i_2\}$, $V = \mathbb{N}$, and $X$ be the union of $U$ and $V$. For $i = i_1i_2$, $j = j_1j_2 \in U$, and $k \in V$, we set

$$a_{ij} = \begin{cases} t & \text{if } i_2 \leq j_1 \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad a_{ik} = \begin{cases} t & \text{if } i_2 \leq k \\ 1 & \text{otherwise} \end{cases}$$

Viewing a word $w \in U^*V$ as being in $N^*$, let $\mathrm{ris}_2 w$ denote the number of rises in $w$ having even index. Theorem 2 implies that

$$\sum_{w \in U^*V} t^{\mathrm{ris}_2 w} w = \frac{\sum_{m \geq 0} (t - 1)^m \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_{2m + 1}} i_1i_2 \cdots i_{2m + 1}}{1 - \sum_{m \geq 1} (t - 1)^m \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_{2m}} i_1i_2 \cdots i_{2m}}. \quad (7)$$

Again set $W(i) = zq^i$ for $i \in \mathbb{N}$ and multiplicatively extend $W$ to $N^*$. Let $U^+_mV = \{uv : u \in U^* \text{ is of length } m, v \in V\}$. From Lemma 1, the bijection $f_{2m+1} : \Theta_{2m+1} \times P_{2m+1} \to U^+_mV$ satisfies the properties $\mathrm{ris}_2 \sigma = \mathrm{ris}_2 w$ and $\mathrm{inv} \sigma + ||\lambda|| = ||w||$ whenever $f_{2m+1}((\sigma, \lambda)) = w$. It then follows from (7) that

$$\sum_{m \geq 0} E_{2m+1}(t, q) \frac{z^{2m+1}}{q^{2m+1}} = \sum_{w \in U^*_mV} t^{\mathrm{ris}_2 w} W(w)$$

$$= \sum_{m \geq 0} \sum_{(\sigma, \lambda) \in \Theta_{2m+1} \times P_{2m+1}} t^{\mathrm{ris}_2 \sigma} \frac{\mathrm{inv} \sigma + ||\lambda||}{q^{2m+1}}$$

$$= \sum_{w \in U^*V} t^{\mathrm{ris}_2 w} W(w)$$

$$= \frac{\sum_{m \geq 0} (t - 1)^m z^{2m+1} \sum_{0 \leq i_1 \leq \cdots \leq i_{2m+1}} q^{i_1 + \cdots + i_{2m+1}}}{1 - \sum_{m \geq 1} (t - 1)^m \frac{z^{2m} \sum_{0 \leq i_1 \leq \cdots \leq i_{2m}} q^{i_1 + \cdots + i_{2m}}}{1 - \sum_{m \geq 1} (t - 1)^m \frac{z^{2m}}{q} \frac{1}{q^{2m+1}}}}$$

$$= \frac{(1 - t)^{1/2} \sin_q(z\sqrt{1-t})}{\cos_q(z\sqrt{1-t})} - t, \quad (8)$$

where $\cos_q z = \sum_{n \geq 0} (-1)^n z^{2n} / (q; q)_{2n}$ and $\sin_q z = \sum_{n \geq 0} (-1)^n z^{2n+1} / (q; q)_{2n+1}$. As the even case is essentially contained in the analysis above, we have

$$\sum_{n \geq 0} \frac{E_n(t, q) z^n}{(q; q)_n} = \frac{(1 - t)(1 + (1 - t)^{-1/2} \sin_q(z\sqrt{1-t}))}{\cos_q(z\sqrt{1-t})} - t.$$
Setting \( t = 0 \), replacing \( z \) by \( z(1 - q) \), and letting \( q \to 1^- \) give (6).

Generalization to \( r \)-tuples of \( m \)-permutations is relatively straightforward. Let \( S_{n,m} \) denote the set of \( \sigma \in S_n \) satisfying the property that \( \sigma(k) > \sigma(k + 1) \) implies \( k \) is a multiple of \( m \). Note that \( S_{n,2} = \emptyset_n \). For \( (\sigma_1, \sigma_2, \ldots, \sigma_r) \in S_{n,m} \), define \( \text{cris}_m(\sigma_1, \sigma_2, \ldots, \sigma_r) \) to be the number of \( k \in \bigcap_{j=1}^r \text{Ris} \sigma_j \) such that \( k \) is a multiple of \( m \). Combining the ideas behind Theorem 3 and (8) gives

**Theorem 4.** For \( m, r \geq 1 \), the sequence of polynomials

\[
E_{n,m,r}(t, q) = \sum_{(\sigma_1, \sigma_2, \ldots, \sigma_r) \in S_{n,m}'} t^{\text{cris}_m(\sigma_1, \sigma_2, \ldots, \sigma_r)} q_1^{\text{inv} \sigma_1} q_2^{\text{inv} \sigma_2} \cdots q_r^{\text{inv} \sigma_r}
\]

is generated by

\[
\sum_{n \geq 0} \frac{E_{n,m,r}(t, q) z^n}{(q; q)_n} = \frac{(1 - t) \left( 1 + \sum_{\rho=1}^{m-1} (1 - t)^{-\rho/m} \Phi_{m, \rho, r} \left( z \sqrt{1 - t}, q \right) \right)}{\Phi_{m, 0, r} \left( z \sqrt{1 - t}, q \right) - t},
\]

where \( \Phi_{m, \rho, r}(z, q) = \sum_{\nu \geq 0} (-1)^\nu z^\nu \Phi_{\nu + \rho}(q; q)_\nu \).

Theorem 4 is essentially due to Stanley [13]. Note that \( E_{n,1,r}(t, q) \) is equal to the generalized \( q \)-Eulerian polynomial defined in (5). Thus, taking \( m = 1 \) in Theorem 4 gives Theorem 3 as a corollary. We further remark that \( \Phi_{m, \rho, 1}(z, q) \) is a \( q \)-Olivier function. When \( r = 1 \) and \( t = s = 0 \), replacing \( z \) by \( z(1 - q) \) and letting \( q \to 1^- \) give the initial result of Carlitz [2] on \( m \)-permutations.

5. FROM THE TRACE TO THE FREE MONOID

As the final application, we use Theorem 1 to obtain Diekert's [5, pp. 96–99] lifting to the free monoid of an inversion formula due to Cartier and Foata [4] from a partially commutative monoid (or trace monoid) in which the defining binary relation admits a transitive orientation.

Let \( \theta \) be an irreflexive symmetric binary relation on \( X \). Define \( \equiv_{\theta} \) to be the binary relation (induced by \( \theta \)) on \( X^* \) consisting of the set of pairs \((w, v)\) of words such that there is a sequence \( w = w_0, w_1, \ldots, w_m = v \), where each \( w_i \) is obtained by transposing a pair of letters in \( w_{i-1} \) that are consecutive and contained in \( \theta \). For instance, if \( X = \{x, y, z\} \) and \( \theta = \{(x, y), (y, x)\} \), then the sequence \( zyx, zyx, zyy \) implies that \( zyx \equiv_{\theta} zyy \).
Clearly, $\equiv_\theta$ is an equivalence relation on $X^*$. The quotient of $X^*$ by $\equiv_\theta$ gives the partially commutative monoid induced by $\theta$ and is denoted by $M(X, \theta)$. The equivalence class $\hat{w}$ of $w \in X^*$ is referred to as the trace of $w$.

A word $w = x_1x_2 \cdots x_n \in X^*$ is said to be a basic monomial if $x_i \theta x_j$ for all $i \neq j$. A trace $\hat{w}$ is said to be $\theta$-trivial if any one of its representatives is a basic monomial. If one lets $\mathcal{T}^+(X, \theta)$ be the set of $\theta$-trivial traces, the inversion formula of Cartier and Foata reads as follows.

**Theorem 5** (Cartier and Foata). For $\theta$ an irreflexive symmetric binary relation on $X$, the traces in $M(X, \theta)$ are generated by

$$
\sum_{\hat{w} \in M(X, \theta)} \hat{w} = \frac{1}{1 + \sum_{\mathbf{i} \in \mathcal{T}^+(X, \theta)} (-1)^{l(\hat{i})} \hat{i}},
$$

where $l(\hat{i})$ denotes the length of any representative of $\hat{i}$.

A natural question to ask is whether $\hat{w}$ and $\hat{i}$ can be replaced by some canonical representatives so that Theorem 5 remains true as a formula in the free monoid $X^*$. As resolved by Diekert [5], such canonical representatives exist if and only if $\theta$ admits a transitive orientation.

To be precise, a subset $\tilde{\theta}$ of $\theta$ is said to be an orientation of $\theta$ if $\tilde{\theta}$ is a disjoint union of $\theta$ and $\{(x, y) : (y, x) \in \theta\}$. The set of $t = t_1t_2 \cdots t_n \in X^*$ satisfying $t_1 \tilde{\theta} t_2 \tilde{\theta} \cdots \tilde{\theta} t_n$ is denoted by $T^+(X, \tilde{\theta})$. Note that $T^+(X, \tilde{\theta})$ is a set of representatives for the $\theta$-trivial traces $\mathcal{T}^+(X, \theta)$ whenever $\tilde{\theta}$ is transitive. A word $w = x_1x_2 \cdots x_n \in X^*$ is said to have a $\tilde{\theta}$-adjacency in position $k$ if $x_k \tilde{\theta} x_{k+1}$. We denote the number of $\theta$-adjacencies of $w$ by $\tilde{\theta}\text{adj} w$. Although Diekert did not explicitly introduce the notion of a $\tilde{\theta}$-adjacency, his lifting theorem may be paraphrased as follows.

**Theorem 6** (Diekert). Let $\theta$ be an irreflexive symmetric binary relation on $X$ and let $\tilde{\theta}$ be an orientation of $\theta$. Then, $\tilde{\theta}$ is transitive if and only if there exists a complete set $W$ of representatives for the traces of $M(X, \theta)$ such that

$$
\sum_{w \in W} w = \frac{1}{1 + \sum_{t \in T^+(X, \tilde{\theta})} (-1)^{l(t)} t},
$$

Moreover, $W = \{w \in X^* : \tilde{\theta}\text{adj} w = 0\}$.

To see how Theorem 1 intervenes in the matter, suppose that $\tilde{\theta}$ is an orientation of $\theta$ (not necessarily transitive for now). If for $x, y \in X$ we set $a_{xy} = a$ when $x \theta y$ and $a_{xy} = 1$ otherwise, then Theorem 1 reduces to

$$
\sum_{w \in X^*} a^{\tilde{\theta}\text{adj} w} x_1x_2 \cdots x_n = \frac{1}{1 + \sum_{t \in T^+(X, \tilde{\theta})} (-1)^{l(t)} (1 - a)^{l(t)} - 1 t}.
$$

(9)
When $\theta$ is transitive, setting $a = 0$ in (9) gives the lifting of Theorem 5 to the free monoid as stated in Diekert's theorem. We close this section with two examples.

**Transitive Example.** Let $X = \{x, y, z\}$ with $\theta = \{(x, y), (y, x), (x, z), (z, x)\}$. Among other possibilities, $\theta = \{(y, x), (z, x)\}$ is a transitive orientation of $\theta$. The $\theta$-adjacencies of a word correspond to factors $yx$ and $zx$. Note that $T^+(X, \theta) = \{x, y, z, yx, zx\}$ is a complete set of representatives for the $\theta$-trivial traces $\mathcal{T}^+(X, \theta)$. Also, the only word in

$$\overrightarrow{xyzxy} = \{xzxy, xzyxy, zzyyx, xzyyx, xzyyx, zyxxy, zyyxx\}$$

having no $\theta$-adjacencies is $xzyyx$. From (9), we have

$$\sum_{w \in \{x, y, z\}^*} a_{\theta \text{adj} w} = \frac{1}{1 - (x + y + z) + (1 - a)(yx + zx)}.$$

Setting $a = 0$ gives an identity that can be viewed as having been lifted from the trace monoid as in Theorem 6.

**Non-transitive Example.** Let $X$ and $\theta$ be as in the previous example. The orientation $\theta = \{(y, x), (x, z)\}$ is not transitive. Observe that the word $yxz$ in $T^+(X, \theta) = \{x, y, z, yx, zx, yxz\}$ is not a $\theta$-trivial trace. Also, $\overleftarrow{yxz} = \{yxz, xzy, yzx\}$ contains two words having no $\theta$-adjacencies. Nevertheless, (9) implies

$$\sum_{w \in \{x, y, z\}^*} a_{\theta \text{adj} w} = \frac{1}{1 - (x + y + z) + (1 - a)(yx + zx) - (1 - a)^2 yxz}.$$

6. PROOFS FOR THEOREMS 1 AND 2

To establish Theorem 1, we begin by noting that (1) is equivalent to

$$\sum_{w \in X^*} a(w)w - \sum_{w \in X^*} \left( \sum_{w = uv, \nu \neq 1} a(u)\overline{a}(\nu) \right)w = 1.$$

Thus, by equating coefficients, it suffices to show that

$$a(w) = \sum_{w = uv, \nu \neq 1} a(u)\overline{a}(\nu)$$

(10)
for all $w \in X^*$. We proceed by induction on the length $l(w)$ of $w$. For $l(w) = 1$, (10) is trivially true. Suppose $l(w) \geq 2$. Then $w$ factorizes as $w = w_1xy$, where $w_1 \in X^*$ and $x, y \in X$. Assuming (10) holds for words of length smaller than $w$, it follows that

$$a(w) = a_{xy}a(w_1x) = a(w_1x) + (a_{xy} - 1)a(w_1x)$$

$$= a(w_1x)\tilde{a}(y) + \tilde{a}(xy) \sum_{w, x = w_1x} a(u)\tilde{a}(v_1x)$$

$$= a(w_1x)\tilde{a}(y) + \sum_{w_1, xy = w_1xy} a(u)\tilde{a}(v_1xy)$$

$$= \sum_{w = uv, v \neq 1} a(u)\tilde{a}(v),$$

and the proof is complete.

We use an alternate approach to prove Theorem 2. Let $W$ denote the left-hand side of (2) and define

$$W_{n+1} = \sum_w a(w)w,$$

where the sum is over words $w = x_1 x_2 \cdots x_{n+1} \in U^*V$ of length $(n + 1)$. Note that $W = \sum_{n \geq 0} W_{n+1}$. Since $\tilde{a}(x_1) = 1$ and $\tilde{a}(x_1 x_2) = a_{x_1 x_2} - 1$, it is a triviality that

$$W_{n+1} = \sum_w \tilde{a}(x_1)a(x_2, \ldots, x_{n+1})w + \sum_w \tilde{a}(x_1 x_2)a(x_2, \ldots, x_{n+1})w$$

for $n \geq 1$. Similarly, the second sum on the above right may be split as

$$\sum_w \tilde{a}(x_1 x_2)a(x_3, \ldots, x_{n+1})w + \sum_w \tilde{a}(x_1 x_2 x_3)a(x_3, \ldots, x_{n+1})w$$

so that

$$W_{n+1} = \sum_{k=1}^2 \sum_w \tilde{a}(x_1 \cdots x_k)a(x_{k+1} \cdots x_{n+1})w$$

$$+ \sum_w \tilde{a}(x_1 x_2 x_3)a(x_3, \ldots, x_{n+1})w.$$
ADJACENCIES IN WORDS

Iterating the above argument and then factoring give

\[ W_{n+1} = \sum_{k=1}^{n} \sum_{w} \bar{a}(x_1, \ldots, x_k) a(x_{k+1} \cdots x_{n+1}) w + \sum_{w} \bar{a}(w) w \]

\[ = \sum_{k=1}^{n} \left( \sum_{u} \bar{a}(u) u \right) W_{n+1-k} + \sum_{w} \bar{a}(w) w, \]

where the sum to the immediate left of \( W_{n+1-k} \) is over words \( u = x_1 \cdots x_k \in U^+ \) of length \( k \). As the above recurrence relationship for \( W_{n+1} \) is valid for \( n \geq 0 \), it follows that

\[ W = \left( \sum_{w \in U^+} \bar{a}(w) w \right) W + \sum_{w \in U^* \cup \{v\}} \bar{a}(w) w, \]

which implies Theorem 2.

Either of the preceding arguments may be easily modified to give an inversion formula for words in \( X^* \) that end in a fixed word \( v \). Without giving the details, we have

\textbf{THEOREM 7.} According to adjacencies, words ending in a word \( v = b_1 b_2 \cdots b_m \in X^* \) of length \( m \) are generated by

\[ \sum_{w \in X^*} a(wv) wv = \left( 1 - \sum_{w \in X^*} \bar{a}(w) w \right)^{-1} \left( a(v) \sum_{w \in X^*} \bar{a}(wb_1) wv \right). \]

\textbf{REFERENCES}