Lifted $p$-Adic Homology with Compact Supports of the Weierstrass Family and Its Zeta Endomorphism

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The relations among the generators for the lifted $p$-adic homology with compact supports of the various subfamilies of the Weierstrass family in characteristic $p > 0$ ($p \neq 2, 3$) are explicitly given in Section 2. Then, the universal coefficient spectral sequence and the zeta endomorphism in Section 3 enable one to compute explicitly the lifted $p$-adic homology with compact supports of all fibres, including all the elliptic curves and all their singular degenerations in the family.
$W_{Z/p}$ which is isomorphic to $\text{Spec} \left( \mathbb{Z}/p\mathbb{Z} \right) [g_2, g_3]$. Therefore, each fibre of $W_{Z/p}$ has exactly one point at $\infty$, which is a rational point in the fibre.

Let $A = \mathbb{Z}/p\mathbb{Z} [g_2, g_3]$ and let $A = (\mathbb{Z}/p\mathbb{Z})[g_2, g_3]$. Then, from the long exact sequence corresponding to the triple $(\text{points at } \infty, W_{Z/p}, U)$, $\cdots \to H^i_\text{c}(W_{Z/p}, d^1 \otimes \mathbb{Z}/p\mathbb{Z}) \to H^i_\text{c}(U, d^1 \otimes \mathbb{Z}/p\mathbb{Z}) \to \cdots$ and the first group being zero for $i \neq 2$, we have $H^i_\text{c}(W_{Z/p}, d^1 \otimes \mathbb{Z}/p\mathbb{Z}) \cong H^i_\text{c}(U, d^1 \otimes \mathbb{Z}/p\mathbb{Z})$ for $i = 1$. By the definition in [5], we have

\[ H^i_\text{c}(U, d^1 \otimes \mathbb{Z}/p\mathbb{Z}) = H^2_\text{c}(\mathbb{Z}/p\mathbb{Z} [g_2, g_3]), \quad H^2_\text{c}(\mathbb{Z}/p\mathbb{Z} [g_2, g_3]) \times [g_2, g_3]) - U, \quad \Gamma^*_2(\mathbb{A}^2(\text{Spec}(\mathbb{Z}/p\mathbb{Z} [g_2, g_3]))) \otimes \mathbb{Z}/p\mathbb{Z}). \] (1)

Note also that a unique singular point of each fibre over $p$ on the closed subscheme $\mathcal{A} = (g_2^3 - 27g_3^2 = 0)$ lies in the affine open $U$.

If one knows

(i) the lifted $p$-adic homology with compact supports of $U$ and

(ii) the zeta endomorphism of the homology group,

then one can determine the lifted $p$-adic homology with compact supports of all the fibres in the family. This is because the zeta function of a fibre is given by

\[ Z_p(T) = \frac{\prod_{p | q - \text{odd}} P_{p,q}(T)}{\prod_{p | q - \text{even}} P_{p,q}(T)}. \]

where $P_{p,q}$ is the reverse characteristic polynomial of the endomorphism of the $E_{p,q}$-term of the universal coefficient spectral sequence $\text{Tor}^d_\mathbb{Z}(\mathbb{Z}/p\mathbb{Z} [g_2, g_3], W(k(\mu)) \otimes \mathbb{Z}/p\mathbb{Z})$, where $W(k(\mu))$ is the complete discrete valuation ring, e.g., for a perfect field $k(\mu)$. Furthermore, the above universal coefficient spectral sequence abuts upon the finitely generated lifted $p$-adic homology with compact supports of the fibre, which gives the zeta function of that fibre (see [5, 6] Chaps. 5 and 6). See [4] also.

The topics of this paper are (i) and (ii) above for the Weierstrass family. The preimage of $\text{Spec}((\mathbb{Z}/p\mathbb{Z} [g_2, g_3, A^{-1}])$ of $U$ is the open subfamily considered in [1].

2. Module Structure

Let $A = (\mathbb{Z}/p\mathbb{Z} [g_2, g_3]) = \mathbb{A}^2(\text{Spec}(A))$. One can use the covering \{$A^2(\text{Spec}(A)), A^2(\text{Spec}(A)) - (Y^2 - 4X^3 + g_2X + g_3 = 0)$\} to compute the cohomology group (1) in the Introduction. Then the long sequence $\cdots \to H^0(A^2(\text{Spec}(A)), A^2(\text{Spec}(A)) - U, \quad \Gamma^*_2(A^2(\text{Spec}(A))) \otimes \mathbb{Z}/p\mathbb{Z})$
\[ \rightarrow H^n(A^{2}(\text{Spec}(A))), \Gamma_3^n(A^{2}(\text{Spec}(A)))^\dagger \otimes \mathbb{Z} \mathbb{Q} \rightarrow H^n(A^{2}(\text{Spec}(A))) \rightarrow U, \]
\[
\Gamma_3^n(A^{2}(\text{Spec}(A)))^\dagger \otimes \mathbb{Z} \mathbb{Q} \rightarrow 0 \quad \text{is induced. The second and third groups are cohomologies of the global sections.}
\]

**Theorem 2.1.** The \( A^\dagger \otimes \mathbb{Z} \mathbb{Q} \)-module \( H_1^2(U, A^\dagger \otimes \mathbb{Z} \mathbb{Q}) \) has the recursive cohomologous relations among the generators

\[
2(i-1) d C^{-i} dX \wedge dY \sim (6i-13) 6g_2 X C^{-i-1} dX \wedge dY
\]
\[
-(6i-11) 9g_3 C^{-i-1} dX \wedge dY
\]
\[
4(i-1) \Delta XC^{-i} dX \wedge dY \sim (6i-11) g_3^3 C^{-i-1} dX \wedge dY
\]
\[
-(6i-13) 18g_3 X C^{-i-1} dX \wedge dY,
\]

\( i \geq 2 \), where \( C = Y^2 - 4X^3 + g_2 X + g_3 \) and \( d \) is the discriminant \( g_3^2 - 27g_3^3 \). In particular, \( H_1^2(U, A^\dagger \otimes \mathbb{Z} \mathbb{Q}) \) is generated by \{\( C^{-i} dX \wedge dY, X C^{-i} dX \wedge dY \} \text{ for } i \geq 1 \text{ over } A^\dagger \otimes \mathbb{Z} \mathbb{Q}.

**Proof.** The cohomology group (1) in the Introduction is the abutment of the spectral sequence \( H^q(A^2(A), A^2(A)-U, \Gamma_3^n(A^2(d)))^\dagger \otimes \mathbb{Z} \mathbb{Q} \). Then we have the isomorphisms by Lemma 1 in [2]:

\[
H_1^2(U, A^\dagger \otimes \mathbb{Z} \mathbb{Q}) \cong H^2(A^2(A)-U, \Gamma_3^n(A^2(d)))^\dagger \otimes \mathbb{Z} \mathbb{Q})
\]
\[
\cong \text{Coker}(\Gamma_3^2(A[X, Y, C^{-1}])) \quad (2)
\]

The cohomologous relations, induced by the map \( d_{1,0}^1 \) in (2), among the elements of \( \Gamma_3^2(A[X, Y, C^{-1}]) \) are given by

\[
2iX^k Y^{i+1} C^{-i-1} dX \wedge dY \sim jX^k Y^{i-1} C^{-i} dX \wedge dY
\]
\[
12iX^k Y^{i} C^{-i-1} dX \wedge dY + kX^{k-1} Y^{i} C^{-i} dX \wedge dY
\]
\[
\sim g_2 iX^k Y^{i} C^{-i-1} dX \wedge dY.
\]

Then we have

\[
\frac{(6i-11)}{6(i-1)} C^{-i-1} dX \wedge dY \sim \frac{2g_2}{3} X C^{-i} dX \wedge dY + g_3 C^{-i} dX \wedge dY
\]
\[
\frac{6i-13}{6(i-1)} X C^{-i-1} dX \wedge dY \sim \frac{g_2}{18} C^{-i} dX \wedge dY + g_3 X C^{-i} dX \wedge dY.
\]

The equations (4) plainly imply Eq. (CR) in Theorem 2.1. The generation of \( H_1^2(U, A^\dagger \otimes \mathbb{Z} \mathbb{Q}) \) by the elements \{\( C^{-i} dX \wedge dY, X C^{-i} dX \wedge dY \} \text{ for } i \geq 1 \text{ over } A^\dagger \otimes \mathbb{Z} \mathbb{Q}.

can be shown in the same way as in the case of characteristic zero (see [2]). The universal coefficient spectral sequence implies, if $U_d$ is the open subfamily of non-singular fibres,

$$H^*_*(U, A^+ \otimes \mathbb{Q}) \otimes \mathbb{Z}_p \cong H^*_*(U_d, (A^{-1}A)^+ \otimes \mathbb{Q}), \quad (5)$$

where $A^{-1}A$ denotes the localization of $A$ at the discriminant $A$. (The latter was computed in [1] to be free of rank two over $(A^{-1}A)^+ \otimes \mathbb{Q}$.) Applying the long exact sequence for $k = 2, 1, 0$ in the following Note 1, we have the exact sequence

$$0 \to A^+ \otimes \mathbb{Q} \to A^+ \otimes \mathbb{Q} \to A^+ \otimes \mathbb{Q}/A \cdot A^+ \otimes \mathbb{Q} \to 0.$$

From this, we extract the short exact sequence

$$0 \to H^*_*(U, A^+ \otimes \mathbb{Q}) \to H^*_*(U, A^+ \otimes \mathbb{Q}) \to H^*_*(U, A^+ \otimes \mathbb{Q}) \to 0.$$

That is, $H^*_*(U, A^+ \otimes \mathbb{Q})$ has no non-zero $A$-torsion; i.e., $H^*_*(U, A^+ \otimes \mathbb{Q})$ is torsion free. Therefore the equations (CR) tell us that there is no cohomologous relation among the set of generators in spite of the "$t$" completion of the base ring $A$; hence, the inclusion (6) in Section 3 follows. Otherwise, the homology groups on the left in (5) would become free of rank one over $(A^{-1}A)^+ \otimes \mathbb{Q}$.

Remark 1. The isomorphism in (5) can be given by $C^{-1} dX \wedge dY \to Y dX$ and $X C^{-1} dX \wedge dY$ as $(A^{-1}A)^+ \otimes \mathbb{Q}$-modules (see [1]).

Corollary 2.2. Let $U'$ be the closed Weierstrass subfamily over $A/(A \cdot A)$, where $A = (\mathbb{Z}/p \mathbb{Z})[g_2, g_3]$ and $\Delta = g_2^3 - 27g_3^2$, i.e., $U'$ is the closed subscheme over $A/\Delta \cdot A$ consisting of all the singular fibres of $U$. Then, the lifted $p$-adic homology with compact supports of this Weierstrass subscheme is generated by $\{C^{-1} dX \wedge dY, X C^{-1} dX \wedge dY\}_{i \geq 1}$ over $A^+ \otimes \mathbb{Q}/A \cdot A^+ \otimes \mathbb{Q}$.
Note 1. Since generally we have, for a non-zero-divisor \( \Delta \in \mathcal{A} \),
\[
\text{Tor}_i^\Delta(H^i_\Delta(U, \mathcal{A} \otimes \mathbb{Z} \mathbb{Q})), \mathcal{A} / \Delta \mathcal{A})
\]
\[
= \begin{cases} 
H^i_\Delta(U, \mathcal{A} \otimes \mathbb{Z} \mathbb{Q}) / \Delta \cdot H^i_\Delta(U, \mathcal{A} \otimes \mathbb{Z} \mathbb{Q}), & i = 0, \\
\ker(\text{mult. by } \Delta), & i = 1, \\
0, & i \geq 2,
\end{cases}
\]
respectively, we have the long exact sequence
\[
\begin{array}{cccc}
\text{---} & \to & H^i_\Delta(U, \mathcal{A} \otimes \mathbb{Z} \mathbb{Q}) & \to H^i_\Delta(U, \mathcal{A} \otimes \mathbb{Z} \mathbb{Q}) \\
& & & \to H^i_\Delta(U', \mathcal{A} \otimes \mathbb{Z} \mathbb{Q}) / \Delta \cdot \mathcal{A} \otimes \mathbb{Z} \mathbb{Q} \to \text{---}
\end{array}
\]
from the corresponding universal coefficient spectral sequence (see [6, Chap. 5]).

Proof. Since \( H^i_\Delta(U', \mathcal{A} \otimes \mathbb{Z} \mathbb{Q}) / \Delta \cdot \mathcal{A} \otimes \mathbb{Z} \mathbb{Q} \) is obtained by taking the cohomology of the cochain complex, which is obtained by tensoring the cochain complex
\[
C^*(\mathcal{A}^2(A), \mathcal{A}^2(A) - U, \mathcal{A}^* \otimes \mathbb{Z} \mathbb{Q})
\]
with \( \mathcal{A} \otimes \mathbb{Z} \mathbb{Q} / \Delta \cdot \mathcal{A} \otimes \mathbb{Z} \mathbb{Q} \) over \( \mathcal{A} \otimes \mathbb{Z} \mathbb{Q} \) (see [5]), the assertion of Corollary 2.2 is obtained from (CR) in Theorem 2.1 by substituting \( \Delta = 0 \); i.e.,
\[
(6i - 13) 2g_2XC^{-(i-1)} dX \wedge dY \sim (6i - 11) 3g_3XC^{-(i-1)} dX \wedge dY
\]
\[
(6i - 11) g_2^3XC^{-(i-1)} dX \wedge dY \sim (6i - 13) 18g_3XC^{-(i-1)} dX \wedge dY.
\]

Remark 2. Let \( \tilde{\mathcal{A}} = \mathcal{A} \otimes \mathbb{Z} \mathbb{Q} / \Delta \cdot \mathcal{A} \otimes \mathbb{Z} \mathbb{Q} \). One can observe that
\[
H^i(U, \tilde{\mathcal{A}}) \otimes \mathcal{A} \tilde{\mathcal{A}}_g \quad \text{and} \quad H^i(U, \tilde{\mathcal{A}}) \otimes \mathcal{A} \tilde{\mathcal{A}}_g,
\]
where \( \tilde{\mathcal{A}}_g \) and \( \tilde{\mathcal{A}}_g \) are localizations at \( g_2 \) and \( g_3 \), respectively, are generated by \( \{ C^{-i} dX \wedge dY \}_{i \geq 1} \) or, since \( g_2 \neq 0 \) implies \( g_3 \neq 0 \), \( \{ XC^{-i} dX \wedge dY \}_{i \geq 1} \) over \( \tilde{\mathcal{A}}_g \), \( i = 2, 3 \). Note also that if \( g_2 = 0 \) (then \( g_3 = 0 \)), (4) computes the homology of the singular fibre over \( \mu = (g_2 = g_3 = 0) \); i.e., the corresponding homology group is trivial.

Note 2. We have the short exact sequence
\[
0 \to H^i_\Delta(U, \mathcal{A} \otimes \mathbb{Z} \mathbb{Q}) \to H^i_\Delta(U, \mathcal{A} \otimes \mathbb{Z} \mathbb{Q}) \to H^i_\Delta(U', \tilde{\mathcal{A}}) \to 0
\]
from the universal coefficient spectral sequence (see the proof of Theorem 2.1 and [6, Chap. 3]) induced by the short exact sequence

$$0 \to \mathbb{A}^1 \otimes \mathbb{Q} \xrightarrow{\text{mult. by } d} \mathbb{A}^1 \otimes \mathbb{Z} \otimes \mathbb{Q} \to \mathbb{Q} \to 0;$$

see Note 1.

**Corollary 2.3.** Let $U^0$ and $U^3$ be the corresponding closed subfamilies of the Weierstrass obtained by pulling back the Weierstrass family to the closed subsets $(g_2 = 0)$ and $(g_2 = 3)$ of the base scheme, respectively. Then, the module structures of $H_c^i(U^0, \mathbb{Z}_p[g_3] \otimes \mathbb{Q})$ and $H_c^i(U^3, \mathbb{Z}_p[g_3] \otimes \mathbb{Q})$ over $\mathbb{Z}_p[g_3] \otimes \mathbb{Z} \otimes \mathbb{Q}$ are given from (CR) or equations (4) by substituting $g_1 = 0$ and $g_1 = 3$, respectively. That is, the equations (4) become (4$^3$) and (4$^0$) for $g_2 = 3$ and $g_2 = 0$, respectively,

$$\frac{6i - 11}{6(i - 1)} \, C^{-i} dX \wedge dY - 2XC^{-i} dX \wedge dY + g_3 C^{-i} dX \wedge dY - 2(6i - 13) \, XC^{-i} dX \wedge dY - 2g_3 XC^{-i} dX \wedge dY + C^{-i} dX \wedge dY$$

$$(4^3)$$

$$\frac{6i - 11}{6(i - 1)} \, C^{-i} dX \wedge dY - g_3 C^{-i} dX \wedge dY$$

$$(4^0)$$

If $g_3 = +1$, i.e., the singular fibre over $\mu = (g_2 = 3, g_3 = 1)$, then the corresponding homology of this projective line with an ordinary double point is free of rank one; one can choose, for example, $C^{-1} dX \wedge dY$ as a basis element. For $g_3 = -1$ use (CR) to have the corresponding statement. If $g_3 \neq 0$ in (4$^0$), then the open subfamily of $U^0$ defined by "$g_3 \neq 0"$ has the homology generated by two elements $C^{-1} dX \wedge dY$ and $XC^{-1} dX \wedge dY$

**Proof.** The above statements can be observed plainly from (CR) and (4).

### 3. Zeta Endomorphism

Define a ring endomorphism $F^*: \mathbb{Z}_p[g_2, g_3]^+ \to \mathbb{Z}_p[g_2, g_3]^+$ over $\mathbb{Z}_p$ as $F(g_2) = g_2^p$ and $F(g_3) = g_3^p$ and let $f$ be the $p$th power map of the Weierstrass scheme in characteristic $p$. Then the first zeta endomorphism $H_c^*(F, f)$ is induced on the lifted $p$-adic homology with compact supports of the Weierstrass family $M = H_c^*(U, \mathbb{Z}_p[g_2, g_3]^+ \otimes \mathbb{Z} \otimes \mathbb{Q})$. The homology
group $H^j(U, \mathbb{Z}_p[\mathbb{Z}_2, \mathbb{Z}_3]) \otimes \mathbb{Q}$ becomes a vector space of dimension two after being tensored with the quotient field of the ring $\mathbb{Z}_p[\mathbb{Z}_2, \mathbb{Z}_3]$. Since $M$ is torsion free (see, the proof of Theorem 2.1, Notes 1 and 2 in Section 2) we have the inclusion

$$M \hookrightarrow M \otimes \mathbb{Q}_p(\mathbb{Z}_2, \mathbb{Z}_3)$$  \hspace{1cm} (6)

Let $\mathcal{K}^+$ be the quotient field of $\mathbb{Z}_p[\mathbb{Z}_2, \mathbb{Z}_3]^\dagger$. Then $M \otimes \mathbb{Q}_p(\mathbb{Z}_2, \mathbb{Z}_3) \otimes \mathcal{O}_{\mathcal{M}(\mathcal{K}, \mathcal{K})}, \mathcal{K}^+$ is $H^j(U_\mathcal{K}, \mathcal{K}^+)$, where $U_\mathcal{K}$ is the generic fibre of $U$, i.e., the $p$-adic homology with compact supports of an elliptic curve. Therefore, the zeta matrix, like the one computed in [1], induces a semi-linear endomorphism of the free module

$$M \otimes (A^{-1}\mathcal{O}_{\mathcal{M}(\mathbb{Z}_2, \mathbb{Z}_3)})^\dagger$$  \hspace{1cm} (7)

of rank two over $(A^{-1}\mathcal{O}_{\mathcal{M}(\mathbb{Z}_2, \mathbb{Z}_3)})^\dagger$. The zeta endomorphism of $M$ is obtained by restricting the zeta matrix of (7) on $M$ by the inclusion (6). We now compute it explicitly as follows:

$$H^j(F, f)(C^{-1} dX \land dY),$$  \hspace{1cm} (B_1)

$$H^j(F, f)(XC^{-1} dX \land dY).$$  \hspace{1cm} (B_2)

(B_1) equals

$$\frac{1}{Y^{2p} - 4X^{3p} + g_2^pX^p + g_3^p} \frac{dX^p \land dY^p}{C^p - pT} dX \land dY,$$

where $C = Y^2 - 4X^3 + g_2X + g_3$ and $T$ is a polynomial in $g_2, g_3, X$, and $Y$. Similarly (B_2) equals

$$\frac{p^3XC^{2p-1}Y^{p-1}}{C^p - pT} dX \land dY.$$

Rewrite $C^p - pT$ as $C^p(1 - pT/C^p)$. Then (B_1) and (B_2) become

$$\sum_{i \geq 0} p^{i+2}T^iX^{p-1}Y^{p-1}C^{-p(i+1)} dX \land dY$$ \hspace{1cm} (B'_1)

$$\sum_{i \geq 0} p^{i+2}T^iX^{2p-1}Y^{p-1}C^{-p(i+1)} dX \land dY.$$ \hspace{1cm} (B'_2)

Let $2j = p' - 1$, where $p'$ is uniquely determined by $p$ and the even power
of $Y$ in $T$. Then the first equation of (3) implies that the terms $X'Y^2/C^{-p(l+1)}dX \wedge dY$ in $(B_1')$ and $(B_2')$ can be written as

$$
\frac{(2j-1) \cdots (2j-2\beta+1) \cdots 3 \cdot 1}{k(k-1) \cdots (k-j+1)} X'C^{-p(l+1)+\gamma}dX \wedge dY.
$$

Now the second equation in (3) implies that the term $X'C^{-p(l+1)+\gamma}dX \wedge dY$ can be written as a linear combination of $\{XC^{-k}dX \wedge dY, C^{-k}dX \wedge dY\}_{k \geq 1}$ by Theorem 2.1 over $\mathbb{Z}_p[g_2, g_3]^1 \otimes \mathbb{Q}$ (see [2]). Consequently, over $d^{-1}\mathbb{Z}_p[g_2, g_3]^1 \otimes \mathbb{Q}$, $(B_1)$ and $(B_2)$ can be written as linear combinations of $C^{-1}dX \wedge dY$ and $XC^{-1}dX \wedge dY$ by (CR) in Theorem 2.1 (see also Remark 1). Therefore, the zeta matrix of the $(d^{-1}\mathbb{Z}_p[g_2, g_3]^1)^*$-module (7) is obtained. As a consequence, we have the first zeta endomorphism of $M = H^*(U, \mathcal{O} \otimes \mathbb{Q})$.

Remark 3. The zeta matrix via bounded Witt cohomology for the Weierstrass open subfamily, i.e., over $(\mathbb{Z}/p\mathbb{Z})[g_2, g_3, d^{-1}]$, will be published in [3] when it has been made elegant enough. See [7] also.

REFERENCES

7. S. Lubkin, Generalization of $p$-adic cohomology; Bounded Witt vectors. A canonical lifting of a variety in characteristic $p \neq 0$ back to characteristic zero, Compositio Math. 34 (1977), 225–277.