Application of an Itô-Based Approximation Method to Random Vibration of a Pinching Hysteretic System

MOHAMMAD N. NOORI  
Department of Mechanical Engineering, Worcester Polytechnic Institute, Worcester, MA 01609, U.S.A.;

MARK D. PADULA  
GTE Government Systems Corporation, Needham, MA 02194, U.S.A.;

and

HAMID DAVOODI  
Department of Mechanical Engineering, University of Puerto Rico, Mayaguez, PR 00708, Puerto Rico.

Abstract. In this paper, an extension of the Cumulant-Neglect closure scheme is utilized for the random vibration analysis of a single degree of freedom system with a general pinching hysteresis restoring force. The hysteresis element used in the system model can simulate commonly observed forms of stiffness, strength and pinching degradations. The second order statistics of the system response to a stationary Gaussian white noise input are derived using an Itô-based approximation technique. The validity of these response statistics are then verified by comparing them to Monte Carlo simulation results. The numerical studies performed for different combinations of degradation parameters and excitation levels show that the response estimates obtained by this solution method are in good agreement with Monte Carlo simulation. These studies also indicate the applicability of this technique for response analysis of complicated forms of non-linearities.

Key words: Pinching, hysteresis, Itô, random vibrations.

1. Introduction

This paper is part of an overall study to develop a practical and efficient means for determining the response of highly non-linear degrading hysteretic systems to random vibrations. The solution method presented herein is based on an extension of the cumulant-neglect method developed independently by Ibrahim [1], and Wu and Lin [2], which was recently proposed by Noori and Davoodi [3, 4] for response analysis of non-linear hysteretic systems.

Random vibration of hysteretic systems, particularly degrading hysteretic systems, has been the subject of much investigation in recent years. This interest is due to the significance of random vibration and hysteresis behavior in the evaluation and design of structures subjected to events such as earthquakes, floods, hurricanes an atmospheric turbulence. The response of a structure to these severe loading conditions frequently involves numerous excursions into the inelastic range. This results in deterioration of the system and significant energy dissipation due to hysteresis.

Predicting the response of non-linear hysteretic systems is difficult and time consuming, and is frequently avoided in practice. The mathematical models for hysteretic behavior are generally more complex than those for elastic behavior due to the inclusion of the energy dissipation and the various forms of system degradations which usually accompany hysteretic behavior. Elastic models
such as hardening and softening springs are often used when the energy dissipation due to the deformation of the material is small and can be neglected or approximated by a linear viscous damping element. These types of models have been applied extensively in the area of dynamic package cushioning [5, 6].

Since most structural damage is associated with plastic deformations, it is often necessary to use hysteretic models to evaluate structural safety and reliability. In the past three decades, there has been much research directed towards the development and improvement of both the mathematical models used to represent observed forms of hysteretic behavior, and the solution techniques used for random vibration response analysis. Recent reviews for much of this work are presented in [7–16].

Recently, an approximation method for Gaussian and non-Gaussian response analysis of hysteretic systems subjected to Gaussian white noise excitations was proposed by Noori et al. [3, 4]. In this approach, a set of differential equations for the expected values, or moments, of the system response variables are derived from an Itô stochastic differential formulation of the equation of motion. Because the equation of motion is non-linear, the set of differential equations is not closed in the sense that the equation derived for the $n$th moment contains moments of a higher order than $n$. In particular, when the non-linearity is hysteretic the resulting moment equations contain expected value functions defined implicitly in terms of higher order moments. An infinite hierarchy of equations is created and closure technique must be used to reduce the equations to a finite set from which the response moments can be calculated.

The closure technique developed in this Itô-based solution method is an extension of the scheme proposed independently by Ibrahim [1, 17], and Wu and Lin [2]. In this approach, the set of moment equations is closed by selecting a general multi-dimensional probability density function for the unknown density function of the response variables, and expressing the higher order expected values in terms of the lower order moments using the definition of a moment. The selected density function can be Gaussian, if only the Gaussian response statistics are desired, or non-Gaussian, if non-Gaussian responses are to be evaluated.

Noori et al. [4] demonstrated this solution method for the Gaussian response analysis of a single degree of freedom (SDOF) hysteretic system. The authors also compared these results to results obtained from the Equivalent Linearization solution method [6]. This comparison showed a very good agreement between the two solution methods, as was expected and discussed by Wu and Lin [2], Orabi and Ahmadi [18], and Noori and Davoodi [19].

The non-Gaussian response analysis of the same SDOF system was also demonstrated by Noori and Davoodi [3]. In this analysis a multi-dimensional Edgeworth expansion was selected as the density function for the response variables. Response moments up to and including the fourth order were obtained for low and moderate excitation levels. These results also compared well with MCS.

In the aforementioned studies, the non-linear hysteretic behavior was modeled using the hysteretic restoring force model proposed by Wen and Baber [7, 20] based on previous work by Bouc [21], referred to as the Bouc–Baber–Wen (BBW) hysteresis model. The BBW model can simulate both hardening and softening systems with a wide variety of cyclic energy dissipation characteristics and stiffness and strength degradations.

In this paper, the Itô-based solution method is demonstrated for the Gaussian response analysis, referred to as the Itô–Gaussian method, of a SDOF pinching hysteresis model proposed by Baber and Noori [8, 22]. The development of this versatile model, referred to as the Single Element Pinching (SEP) hysteresis model, was based on a general modeling technique proposed
by the above authors [11]. This model is capable of simulating commonly observed forms of pinching degradations in addition to the behavior simulated by the BBW model.

The term “pinching” is used to describe a type of physical degradation commonly encountered during the inelastic response of a structural system. Pinching behavior most commonly occurs in reinforced concrete structures during load reversals and has been observed in numerous experimental tests of various structural systems. This behavior is best illustrated by an example given by Sozen [23] for reinforced concrete beams.

The procedure for applying the solution method to this system is similar to that used in the previous studies by Noori et al. [3, 4]. However, unlike the previous studies, in this work the integral expressions for the unknown expected values are solved in a closed form using classical integration procedures. The derived expressions are incorporated directly into the differential equations for the response moments and the moments are then calculated using an iterative numerical technique. Response estimates obtained for high, moderate and low excitation levels are compared to MCS.

Although a main advantage of using the Itô calculus approach is when parametric excitation is present, with the exception of Monte Carlo simulation, this approach seems to be the first and only analysis technique utilized for non-Gaussian analysis of hysteretic systems [3]. A future extension of this study will be the application of the Itô-based solution method to the non-Gaussian analysis of the SEP hysteresis model.

2. Application of the Itô–Gaussian Solution Method

The differential equation of motion for a single degree of freedom non-linear system may be given in general form by

$$M \ddot{U} + Q(U, \dot{U}, t) = F(t),$$  \hspace{1cm} (1)

where $M$ is the mass of the system; $Q$ is the non-linear restoring force; $F(t)$ is the forcing function; and the system displacement, velocity and acceleration are represented by $U$, $\dot{U}$ and $\ddot{U}$ respectively. By excluding the non-linear velocity dependent forces, equation (1) may be simplified and written as

$$M \ddot{U} + C \dot{U} + q(U, t) = F(t),$$  \hspace{1cm} (2)

where $C$ is the linear viscous damping coefficient. The simplified non-linear restoring force is represented by the function $q$. For a general hysteretic system $q$ is given by

$$q(U, t) = \alpha K U + (1 - \alpha) K z,$$  \hspace{1cm} (3)

where $K$ is the linear pre-yield stiffness, $\alpha$ is the ratio of post-yield to pre-yield stiffness and $z$ is the hysteretic component. Equation (3) is comprised of two displacement dependent forces. A non-hysteretic force $\alpha K U$ that is a function of the instantaneous displacement, and a hysteretic force $(1 - \alpha) K z$ that is a function of the time history of the displacement. The hysteretic force models the non-linear deterioration characteristics typical of some structural elements, particularly reinforced concrete elements, under severe dynamic loads. An extensive study of this model and the physical meaning of various parameters can be found in reference [11]. The total restoring force for the system can be schematically represented by a linear spring element $\alpha K U$, a linear
viscous damping element and a hysteretic element \((1 - \alpha)Kz\) in parallel, as shown in Figure 1. The non-linear equation of motion for the system is

\[ M\ddot{U} + C\dot{U} + \alpha KU + (1 - \alpha)Kz = F(t) \]  

and can be written as

\[ \ddot{U} + 2\xi\omega_0\dot{U} + \omega_0^2U + (1 - \alpha)\omega_0^2z = f(t), \]  

where \(\xi\) is the viscous damping factor, \(\xi = C/[2\omega_0 M]\); \(\omega_0\) is the natural frequency in the linear elastic region, \(\omega_0 = \sqrt{K/M}\); and \(f(t)\) is the force per unit mass. In the SEP hysteresis model, the hysteretic component \(z\) is given by a non-linear differential equation that has the form

\[ z = \left[1 - \xi \exp\left(\frac{-z^2}{\xi \omega_0^2}\right)\right] \left[A\dot{U} - n(\beta|\dot{U}|z|^{n-1}z + \gamma \dot{U}|z|^n)\right] \left(\frac{1}{\eta}\right), \]

where the first part of the expression is the pinching inducing function and the second part is the BBW hysteresis model [8, 22]. The general shape of the hysteresis loop is determined by the selection of parameters \(\beta, \gamma, A\) and \(n\). The parameter \(\beta\) controls the total energy dissipation for a cycle (area inside the loop). The parameter \(A\) controls the initial slope of the force-displacement curve, and \(n\) controls the sharpness of the transition from the elastic to plastic regions. A large number of softening and hardening models, and narrow and wide band models can be obtained by varying these parameters. This model can be used for damage prediction of a variety of reinforced concrete structures under earthquake loads. The specific types of hysteretic models obtained for different values of the shape parameters is presented elsewhere [24]. This hysteresis model can be utilized for various engineering applications where hysteretic damping along with sudden deterio-

\[ \text{Pinching Inducing} \quad \text{BBW Model} \]

\[ \text{Function} \]

\[ \text{Fig. 1. A SDOF system with SEP hysteresis restoring force.} \]
ation of stiffness is observed in the dynamic response of a non-linear system. Applications can vary from modeling pinching type deterioration of reinforced concrete elements to modeling several base isolation mechanisms for primary structures or secondary systems.

Stiffness and strength degradation are introduced into equation (6) by the parameters \( \nu, \eta \) and \( A \). These parameters were chosen by Baber and Wen to vary as linear functions of the total energy dissipated by hysteresis, \( \varepsilon(t) \), and are given by

\[
A = A_0 - \delta_A \varepsilon(t) ; \quad \eta = \eta_0 + \delta_\eta \varepsilon(t) ; \quad \nu = \nu_0 + \delta_\nu \varepsilon(t),
\]  

(7)

where \( A_0, \eta_0 \) and \( \nu_0 \) are the initial values and \( \delta_A, \delta_\eta \) and \( \delta_\nu \) determine the degradation rates. Different combinations of strength and stiffness degradations may be obtained by varying these parameters. The energy dissipation \( \varepsilon(t) \) of the hysteresis loop is

\[
d\varepsilon(t) = (1 - \alpha_0) \omega_0^2 z dU \quad \text{or} \quad \dot{\varepsilon}(t) = \frac{d\varepsilon(t)}{dt} = (1 - \alpha_0) \omega_0^2 z \dot{U}.
\]

(8)

In the pinching inducing function the parameters \( \xi_1 \) and \( \xi_2 \) control the severity and the sharpness of the pinching respectively. Since pinching behavior is a function of the response severity and duration, similar to strength and stiffness degradation, \( \varepsilon(t) \) is used as an index for varying \( \xi_1 \) and \( \xi_2 \). In order to simulate commonly observed forms of pinching behavior \( \xi_1 \) and \( \xi_2 \) must be non-linear functions of \( \varepsilon(t) \) and are given by

\[
\xi_1 = \xi_{10}[1 - \exp(-p\varepsilon(t))] \quad \text{and} \quad \xi_2 = [\xi_0 + \delta_\xi \varepsilon(t)](\lambda + \xi_1),
\]

(9)

where \( \xi_{10} \) is the maximum value of \( \xi_1 \); \( p \) controls the development rate of \( \xi_1 \); and \( \lambda, \xi_0 \) and \( \delta_\xi \) control the evolution of \( \xi_2 \) [8]. A typical SEP force-displacement curve is shown in Figure 2. In depth studies of this model are given by Baber and Noori [8, 9].

![Fig. 2. A typical SEP smooth hysteresis force-displacement curve with combined strength, stiffness and pinching degradations.](image)
As mentioned previously, the Itô-based solution method is based on a stochastic differential formulation of the equation of motion. By using a coordinate transformation where \( Y_1 = U, Y_2 = \tilde{U} \) and \( Y_3 = z \), equation (5) can be written as a set of first order differential equations:

\[
\begin{align*}
\dot{Y}_1 &= Y_2; \\
\dot{Y}_2 &= \tilde{U} = \frac{d(t)}{2\omega_0}Y_2 - \alpha\omega_0^2 Y_1 - (1 - \alpha)\omega_0^2 Y_3; \\
\dot{Y}_3 &= \dot{z} = \left[ 1 - \xi_1 \exp\left(\frac{-Y_3^2}{2\xi_2^2}\right) \right] (AY_2 - \nu[|Y_2|^a]Y_3 - |Y_3|^a - \gamma Y_2 Y_3). 
\end{align*}
\]  

(10)

The equations of motion for a system subjected to a random excitation may then be written as a set of stochastic differential equations of the form

\[
\frac{dY}{dt} = F(Y, t) + G(Y, t)\xi(t). 
\]  

(11)

In this equation \( Y = [Y_1(t), Y_2(t), \ldots, Y_n(t)]^T \) is an \( n \)-dimensional state vector for the system response variables \( Y_n(t) \); \( F(Y, t) \) is an \( n \)-dimensional vector that represents the linear or non-linear system behavior due to deterministic forces; \( \xi(t) \) is an \( m \)-dimensional vector that represents a random force whose influence on the system depends on the state of the system, which is governed by \( G(Y, t) \); \( G(Y, t) \) is an \( n \times m \) linear or non-linear matrix whose elements are functions of the system variables.

When the random force \( \xi(t) \) is approximated as a Gaussian white noise process, \( W(t) \), the state vector \( Y \) constitutes a Markov vector and the rules of Itô stochastic calculus can be applied. This approximation provides meaningful results when the physical excitation spectrum being approximated varies slowly in the vicinity of the system's natural frequency \([24, 25]\).

The Gaussian white noise \( W(t) \) can be defined by the formal derivative of the Brownian motion process \( B(t), W(t) = dB(t)/dt \), and equation (11) can be written in the form

\[
\frac{dY}{dt} = F(Y, t)dt + G(Y, t)dB(t). 
\]  

(12)

This equation is known as the Itô stochastic differential equation. The differential equations for the response moments can be generated by using the differential rule of Itô calculus. Consider a scalar valued real function of the Markov state vector \( Y \) and time \( t, \psi(Y, t) \), that is continuously differentiable in time and twice continuously differentiable with respect to \( Y \). Using the Itô differential rule \([1, 26, 27]\) the differential of \( \psi(Y, t) \) can be written as

\[
\frac{d\psi(Y, t)}{dt} = \left\{ \frac{\partial \psi}{\partial t} + \frac{1}{2} \text{Trace} \left[ GQG^T \psi_{YY} \right] \right\} dt + \psi_Y^T dY, 
\]  

(13)

where ‘Trace’ is the matrix operation for the summation of the diagonal elements, \( G \) is \( G(Y, t) \) as given in equation (12), \( \psi_{YY} \) represents the Jacobian matrix for \( \psi(Y, t) \), and \( \psi_Y^T \) and \( Q \) are given by

\[
\psi_Y^T = \left\{ \frac{\partial \psi}{\partial Y_1}, \frac{\partial \psi}{\partial Y_2}, \ldots, \frac{\partial \psi}{\partial Y_n} \right\},
\]  

(14)

\[
Q dt = E[\{dB(t)\} dB(t)]^T.
\]  

(15)
The function \( \psi(Y, t) \) can be replaced by \( \Phi(Y) = \Phi_{Y}(Y) \) where \( \Phi(Y) \) is an arbitrary scalar function of the response coordinates \( Y_1, Y_2, \) and \( Y_3 \). The choice of \( \Phi(Y) \) depends on the type of statistical function to be evaluated. This function is used if the joint moments of the response are desired [1, 3, 4]. Equation (13) can be rewritten as

\[
\frac{d\Phi(Y)}{dt} = \left( \frac{\partial \Phi(Y)}{\partial Y_n} \right)^T dY + \frac{1}{2} \text{Trace}\{QG}_n(Y)\} dt .
\]

By substituting equation (12) for \( dY \), equation (16) can be written as

\[
\frac{d\Phi(Y)}{dt} = \left( \frac{\partial \Phi(Y)}{\partial Y_n} \right)^T \{F(Y, t) dt + G(Y, t) dB(t)\} + \frac{1}{2} \text{Trace}\{QG}_n(Y)\} dt .
\]

Taking the expected value of both sides yields

\[
E[\frac{d\Phi(Y)}{dt}] = E\left[ \left( \frac{\partial \Phi(Y)}{\partial Y_n} \right)^T \{F(Y, t) dt + G(Y, t) dB(t)\} + \frac{1}{2} \text{Trace}\{QG}_n(Y)\} dt \right] .
\]

In this study the system excitation is approximated by a zero mean white noise process, \( E[B(t)] = E[dB(t)] = 0 \). After dividing by \( dt \), equation (18) reduces to the general form of the differential equation for the response moments;

\[
\dot{M}_k = \frac{d}{dt} E[\Phi(Y)] = E\left[ \left( \frac{\partial \Phi(Y)}{\partial Y_n} \right)^T F(Y, t) \right] + \frac{1}{2} \text{Trace}\{QG}_n(Y)\} dt ,
\]

where \( \dot{M}_k \) is the time derivative of the \( k \)th order response moment \( M_k \) and \( K = i + j + k \).

For the SEP hysteretic system shown in Figure 1, the parameters in equation (19) are given by

\[
F(Y, t) = \begin{bmatrix}
-\alpha \omega_0^2 Y_1 - 2 \xi \omega_0 Y_2 - (1 - \alpha) \omega_0^2 Y_3 \\
1 - \xi \exp\left(-\frac{Y_2^2}{2 \xi^2}\right) \{AY_2 - \nu [\beta |Y_2|^n |Y_3|^{n-1} Y_3 + \gamma Y_2 |Y_3|^n] / \eta \}
\end{bmatrix}
\]

\[
G(Y, t) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\left\{ \frac{\partial \Phi(Y)}{\partial Y_n} \right\}^T = \begin{bmatrix}
iY_1^{i-1} Y_2^i Y_3, jY_1^{i-1} Y_2^i Y_3, kY_1^{i-1} Y_2^i Y_3^{k-1} \\
kY_1^{i-1} Y_2^i Y_3, jY_1^{i-1} Y_2^i Y_3, iY_1^{i-1} Y_2^i Y_3^{k-1}
\end{bmatrix}
\]

\[
\Phi_{Y}(Y) = \frac{\partial^2 \Phi}{\partial Y_i \partial Y_j \partial Y_k} \text{ (Jacobian Matrix)}
\]

and \( Q dt = E[\{dB(t)\}^T dB(t)] = 2 \pi D \), where \( D \) is the power spectral density (PSD) of the Gaussian white noise input.

By performing the matrix operations specified in equation (19), the differential moment
equation can be written as

\[
M_{i,j,k} = \frac{1}{2} j(j-1)v^2 M_{i-2,j} + iM_{i-1,j+1,k} - j\alpha \omega_0^2 M_{i+1,j-1,k} - j2\xi \omega_0 M_{i,j,k}
\]

\[
+ \frac{k}{\eta} AM_{i,j+1,k-1} - \frac{k}{\eta} \nu\beta E[|Y_2||Y_3|^{n-1}Y_1^iY_2^jY_3^k]
\]

\[
- \frac{k}{\eta} \nu\gamma E[|Y_3|^nY_1^iY_2^{j+1}Y_3^{k-1}] - \frac{k}{\eta} A\xi E[\exp\left(\frac{-Y_3^2}{2\xi_1^2}\right)Y_1^iY_2^jY_3^{k-1}]
\]

\[
+ \frac{k}{\eta} \nu\beta \xi E\left[\exp\left(\frac{-Y_2^2}{2\xi_2^2}\right)|Y_2||Y_3|^{n-1}Y_1^iY_2^jY_3^k\right]
\]

\[
+ \frac{k}{\eta} \nu\gamma \xi E\left[\exp\left(\frac{-Y_3^2}{2\xi_2^2}\right)|Y_3|^nY_1^iY_2^{j+1}Y_3^{k-1}\right] \tag{24}
\]

In this equation, \(M_{i,j,k} = E[Y_1^iY_2^jY_3^k]\) and represents the \(K\)th order moment for \(K = i + j + k\); and \(v^2 = 2\pi D\).

The degradation parameters \(A, \nu, \eta, \zeta_1, \zeta_2\) which are functions of the response history, are not included in the ensemble averages. Closed form incorporation of these parameters is difficult, and it has been found that a reasonable approximation is to update the parameter at each time step using their mean values \([7, 8, 9, 10, 20, 24]\). This is done by replacing \(\varepsilon(t)\) by its expected value \(\mu_\varepsilon\), which is given by

\[
\mu_\varepsilon = (1 - \alpha)\omega_0^2E[\dot{z}\dot{U}] \quad \text{or} \quad \mu_\varepsilon = (1 - \alpha)\omega_0^2M_{011} \tag{25}
\]

This equation is integrated in parallel with the differential moment equations to allow updating the degradation parameters and completing the evaluation of the moments at each time step. This treatment of the degradation parameters is based on the assumption that the degradation takes place slowly.

When the probability distribution of the response variables is approximated by a Gaussian distribution the system response is defined by the first and second order moments. Differential equations for the first order moments are obtained from equation (24) by setting \(i + j + k = 1\). In this analysis the hysteresis loop shaping parameter \(n\) is set equal to one, which gives a very smooth transition from the elastic to plastic regions of the force-displacement curve. This results in the following set of first order equations;

\[
\dot{M}_{001} = \frac{1}{\eta} \left\{ AM_{010} - \nu\beta E[|Y_2||Y_3|] - \nu\gamma E[|Y_3|Y_2] \right. \\
- A\xi E\left[\exp\left(\frac{-Y_3^2}{2\xi_1^2}\right)Y_2\right] + \nu\beta \xi E\left[\exp\left(\frac{-Y_2^2}{2\xi_2^2}\right)|Y_2||Y_3|\right] \\
- \nu\xi E\left[\exp\left(\frac{-Y_3^2}{2\xi_2^2}\right)|Y_3||Y_2|\right] \right\}, \tag{26}
\]

\[
\dot{M}_{010} = -2\xi \omega_0 M_{010} - \alpha \omega_0^2 M_{100} - (1 - \alpha)\omega_0^2 M_{001},
\]

\[
\dot{M}_{100} = M_{010}.
\]
The following set of second order differential moment equations is obtained by setting $i + j + k = 2$ and $n = 1$;

$$M_{200} = 2M_{110},$$

$$M_{020} = \nu^2 - 4\xi\omega_0M_{020} - 2\alpha\omega_0^2M_{110} - 2(1 - \alpha)\omega_0^2M_{001},$$

$$M_{002} = \frac{2}{\eta} \left\{ A\omega_{011} - \beta\nu E[|Y_2|Y_3] - \gamma\nu E[|Y_3|Y_2Y_3] - A\xi E\left[\exp\left(-\frac{Y_3^2}{2\xi^2}\right)|Y_2Y_3\right] + \nu\beta\xi E\left[\exp\left(-\frac{Y_3^2}{2\xi^2}\right)|Y_3|Y_2Y_3\right] 
+ \nu\gamma\xi E\left[\exp\left(-\frac{Y_3^2}{2\xi^2}\right)|Y_3|Y_2Y_3\right]\right\},$$

$$M_{110} = M_{020} - \alpha\omega_2^2M_{200} - 2\xi\omega_0M_{110} - (1 - \alpha)\omega_0^2M_{001},$$

$$M_{101} = M_{011} + \frac{1}{\eta} \left\{ A\omega_{011} - \beta\nu E[|Y_2|Y_1Y_3] - \gamma\nu E[|Y_3|Y_1Y_2] - A\xi E\left[\exp\left(-\frac{Y_3^2}{2\xi^2}\right)|Y_2Y_3\right] + \nu\beta\xi E\left[\exp\left(-\frac{Y_3^2}{2\xi^2}\right)|Y_3|Y_2Y_3\right] 
+ \nu\gamma\xi E\left[\exp\left(-\frac{Y_3^2}{2\xi^2}\right)|Y_3|Y_1Y_2\right]\right\},$$

$$M_{011} = -\alpha\omega_0^2M_{101} - 2\xi\omega_0M_{001} - (1 - \alpha)\omega_0^2M_{002}$$

$$\quad + \frac{1}{\eta} \left\{ A\omega_{020} - \beta\nu E[|Y_3|Y_2Y_3] - \gamma\nu E[|Y_3|Y_2^2] - A\xi E\left[\exp\left(-\frac{Y_3^2}{2\xi^2}\right)|Y_2^2\right] 
+ \nu\beta\xi E\left[\exp\left(-\frac{Y_3^2}{2\xi^2}\right)|Y_2Y_3Y_2\right] + \nu\gamma\xi E\left[\exp\left(-\frac{Y_3^2}{2\xi^2}\right)|Y_3Y_2^2\right]\right\}.$$ 

As can be seen from equations (26) and (27), the differential moment equations contain expected values that cannot be directly written in terms of $M_{i,j,k}$, but are defined implicitly in terms of higher order moments. The equations are closed by assuming a multi-dimensional Gaussian probability density function for the joint probability density of the response variables in each unknown expected value. The unknown expected values are then solved for in terms of the first and second order moments by using the definition of a moment. For example;

$$E[|Y_3|Y_1Y_2] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Y_3|Y_1Y_2P(Y_1, Y_2, Y_3) dY_1 dY_2 dY_3, \quad (28)$$

where $P(Y_1, Y_2, Y_3)$ is a three-dimensional Gaussian density function. In this study $P(\cdot)$ is a zero mean Gaussian density function, which in three-dimensional form is given by

$$P(Y_1, Y_2, Y_3) = \frac{(2\pi)^{-3/2}}{\sqrt{\Delta}} \exp\left\{ \frac{-1}{2\Delta} \sum_{i=1}^{3} \sum_{j=1}^{3} \text{cof}(\Delta)_{ij}Y_i Y_j \right\}, \quad (29)$$
where $|\Delta|$ is the determinant of the covariance matrix; and $\text{cof}(\Delta)_{ij}$ is the cofactor of the covariance matrix element $C_{ij}$.

Since the probability distribution of the response variables is approximated by a zero mean Gaussian distribution, the first order moments given by equations (26) are equal to zero and only the fifteen unknown expected values in equations (27) must be evaluated. In this study the integral expressions for these expected values, which are typified by equation (28), are solved in an exact form using classical integration procedures. The solutions are given in the Appendix. The second order response moments are then obtained from equations (27) by applying an iterative numerical integration procedure that was developed for this purpose.

### 3. Numerical Studies

The objectives of the following numerical studies are: (1) to investigate the validity of the presented solution approach for a hysteretic system that has a typical form of pinching behavior; (2) to explore any limitations in the approach and, if possible, identify the source of these limitations. To accomplish these objectives, response estimates are predicted for the system shown in Figure 1 at various excitation levels. These response estimates are then compared to similar statistics obtained using a 300 sample Monte Carlo Simulation.

For the input excitation, four different levels of the power spectral density are considered in the numerical studies; PSD = 0.02, 0.1, 0.5 and 1.0. Two levels of linear damping are also investigated; $\zeta = 0.0$ and 0.2. The other two parameters in equation (5) are assigned to be $\omega_0 = 1.0$ and $\alpha = 1/21$. The hysteresis shape parameters are assigned to be $\beta = \gamma = 0.5$, $n = 1$ and $A_0 = \eta_0 = \nu_0 = 1.0$. Since the main interest of the studies is the pinching behavior, for convenience the stiffness and strength degradation rates are set equal to zero (i.e., $\delta_A = \delta_\eta = \delta_\nu = 0.0$). The use of nonzero degradation rates does not cause any difficulties in the computations and was investigated by Noori et al. [4].

Two levels of pinching severity are investigated. A moderate pinching behavior is obtained by setting $\zeta_{10} = 0.8$ and $\lambda = 0.05$. A more severe pinching behavior is obtained by setting $\zeta_{10} = 0.9$ and $\lambda = 0.15$. The other parameters in equations (9) are assigned to be $\zeta_0 = 0.2$, $\delta_\zeta = 0.01$ and $p = 1.0$.

Figures 3 through 6 show the rms displacement, rms velocity, rms of the restoring force and the mean dissipated hysteretic energy for the case of moderate pinching ($\zeta_{10} = 0.8$, $\lambda = 0.5$) and zero linear damping ($\zeta = 0$). It can be seen from Figure 3 that the rms displacement predicted by the Itô–Gaussian method compares well with Monte Carlo simulation for PSD = 0.02 and 0.1. For PSD = 0.5 and 1.0 the deviations between the two methods are more evident. However, these deviations are expected at the higher excitation levels. At low excitation levels, the system behaves more linearly and the response behavior can be assumed to be Gaussian. At high levels of excitation, the system is highly non-linear and the response behavior can no longer be assumed Gaussian. Thus the Itô–Gaussian method will not lead to accurate predictions of response statistics, and a non-Gaussian approach would be more suitable in this case.

Similar observations can also be made for the rms velocity (Figure 4) and the rms of the restoring force (Figure 5). However, as can be seen from Figure 5 the Itô–Gaussian method significantly underestimates the restoring force for PSD = 0.5 and PSD = 1.0. As a result, the mean value for the dissipated hysteretic energy (Figure 6) predicted by the Itô–Gaussian method is also underestimated.

Figures 7 through 10 show the response statistics for the cases of moderate pinching with
Fig. 3. RMS displacement response of the SEP system for a zero mean white noise input; $\xi = 0.0$, $\xi_{10} = 0.8$, $\lambda = 0.05$.

Fig. 4. RMS velocity response of the SEP system for a zero mean white noise input; $\xi = 0.0$, $\xi_{10} = 0.8$, $\lambda = 0.05$. 
Fig. 5. RMS prediction of restoring force response for a zero mean white noise input; $\zeta = 0.0, \zeta_{10} = 0.8, \lambda = 0.05$

Fig. 6. Mean of the dissipated hysteretic energy for a zero mean white noise input; $\zeta = 0.0, \zeta_{10} = 0.8, \lambda = 0.05$. 
Fig. 7. RMS displacement response of the SEP system for a zero mean white noise input; $\zeta = 0.2$, $\xi_{10} = 0.8$, $\lambda = 0.05$.

Fig. 8. RMS velocity response of the SEP system for a zero mean white noise input; $\zeta = 0.2$, $\xi_{10} = 0.8$, $\lambda = 0.05$. 
Fig. 9. RMS prediction of restoring force response for a zero mean white noise input; $\zeta = 0.2$, $\zeta_0 = 0.8$, $\lambda = 0.05$.

Fig. 10. Mean of the dissipated hysteretic energy for a zero mean white noise input; $\zeta = 0.2$, $\zeta_0 = 0.8$, $\lambda = 0.05$. 
nonzero linear damping ($\zeta = 0.2$). In general, as the damping increases there is better agreement between the Itô–Gaussian and Monte Carlo response predictions. This is expected since an increase in damping results in a decrease in the system response levels.

Figures 11 through 18 show the response statistics for the cases of increased pinching ($\xi_{10} = 0.9, \lambda = 0.15$) with zero and nonzero linear damping ($\zeta = 0.0$ and 0.2). It can be seen from these figures that the comparison between the response statistics predicted by the Itô–Gaussian method and Monte Carlo simulation is similar to the comparison made for the cases of moderate pinching. The results from the two solution methods compare well for PSD = 0.02 and 0.1, with more deviation occurring when PSD = 0.5 and 1.0.

The standard deviations of the stationary displacement and velocity under stationary white noise are shown in Figures 19 and 20 respectively, as predicted by the Itô–Gaussian approach. These figures represent the response for a wide range of input PSD levels. These studies have been performed for a system with no viscous damping ($\zeta = 0.0$) and with high pinching characteristics ($\xi_{10} = 0.9, \lambda = 0.15$). As Figure 19 indicates, at low levels of excitation (PSD < 0.002) the response increases very slowly as the excitation increases. This is due to the fact that at low levels of excitation, the hysteretic restoring force is effectively present. The resulting response is wider band and energy is dissipated through hysteretic action, resisting the growth of the response. As the excitation level increases further, the response increases more rapidly due to the presence of significant system deterioration. The resulting response is more narrow band with less energy dissipation through hysteresis action. At the higher levels of excitation (PSD > 0.2), the hysteretic restoring force is much less effective and therefore, the rate of increase in the response is even greater. As can be seen, the response predicted by this Itô–Gaussian approach is not in
Fig. 12. RMS velocity response of the SEP system for a zero mean white noise input; \( \zeta = 0.0, \zeta_{10} = 0.9, \lambda = 0.15 \)

Fig. 13. RMS prediction of restoring force response for a zero mean white noise input; \( \zeta = 0.0, \zeta_{10} = 0.9, \lambda = 0.15 \)
Fig. 14. Mean of the dissipated hysteretic energy for a zero mean white noise input; $\zeta = 0.0$, $\xi_{\infty} = 0.9$, $\lambda = 0.15$.

Fig. 15. RMS displacement response of the SEP system for a zero mean white noise input; $\zeta = 0.2$, $\xi_{\infty} = 0.9$, $\lambda = 0.15$. 
Fig. 16. RMS velocity response of the SEP system for a zero mean white noise input; $\zeta = 0.2$, $\zeta_{10} = 0.9$, $\lambda = 0.15$

Fig. 17. RMS prediction of restoring force response for a zero mean white noise input; $\zeta = 0.2$, $\zeta_{10} = 0.9$, $\lambda = 0.15$
Fig. 18. Mean of the dissipated hysteretic energy for a zero mean white noise input; $\zeta = 0.2$, $\zeta_0 = 0.9$, $\lambda = 0.15$.

Fig. 19. RMS of the stationary displacement response for zero mean white noise inputs; $\zeta = 0.0$, $\zeta_0 = 0.9$, $\lambda = 0.15$. 
Fig. 20. RMS of the stationary velocity response for zero mean white noise inputs; $\zeta = 0.0$, $\zeta_0 = 0.9$, $\lambda = 0.15$.

good agreement with simulation results at very high PSD levels when the system acts highly non-linear. Similar behavior is observed in Figure 20.

In order to study the effect of pinching parameters on the RMS response, several studies were performed using different pinching parameters. The trend of the response was very similar to those of Figures 19 and 20. Also, the effect of changing pinching parameters was not significant on the magnitude of the RMS responses. Since there was no significant effect on the magnitude or trend of the response for the different pinching parameters, these results are not included in Figures 19 and 20.

4. Summary and Conclusions

The main objective of this study was to extend the Itô–Gaussian solution method to a general smooth hysteresis that incorporates pinching behavior. Based on the numerical studies presented, it can be concluded that the extension of the Itô–Gaussian solution method presented herein is capable of accurately predicting the response statistics of the SEP single degree of freedom system. The response estimates predicted for this system compare well with Monte Carlo simulation at low and moderate response levels. At high response levels, larger deviations occur between the two methods. These deviations are expected since the response estimates from the Itô–Gaussian method were obtained by assuming a Gaussian distribution for the response variables, and the actual response is strongly non-Gaussian at the high response levels. However, the response estimates for the restoring force and dissipated energy are significantly underestimated at the higher levels.
It should be noted, that in general the Itô–Gaussian response estimates are nonconservative compared to Monte Carlo simulation. This should be taken into consideration when applying this solution method to problems where the expected response levels are high.

The main advantage of the Itô–Gaussian solution method over Monte Carlo simulation is that the required computer processing effort, and resulting expenses, are less. This difference in efficiency was very noticeable in the numerical studies performed on the single degree of freedom system in this study. These numerical studies were performed on an Encore Multimax 520 computer. The elapsed run time for the Itô–Gaussian computer program was about 17 cpu seconds, and for the 300 sample Monte Carlo simulation it was 936 cpu seconds. This difference in computation time is expected to be even greater for multi-degree of freedom systems. Also, if the number of samples used for the Monte Carlo simulation are increased, the computation time will increase accordingly.

Appendix

The following is a summary of the solutions for the higher order moments in equations (27). These solutions were obtained by approximating the probability density function for the response variables \( Y_1 \), \( Y_2 \) and \( Y_3 \) with a zero mean Gaussian function.

**VARIABLE DEFINITION:**

- \( \sigma_2 \) – standard deviation of \( Y_2 \)
- \( \sigma_3 \) – standard deviation of \( Y_3 \)
- \( \rho \) – correlation coefficient for \( Y_2 \) and \( Y_3 \)
- \( |\Delta|_{23} \) – determinant of the zero mean covariance matrix for \( Y_2 \) and \( Y_3 \) where: \( |\Delta|_{23} = \sigma_2^2 \sigma_3^2 (1 - \rho^2) \)
- \( |\Delta| \) – determinant of the zero mean covariance matrix \( C \) for \( Y_1 \), \( Y_2 \) and \( Y_3 \), where:

\[
C = \begin{bmatrix}
E[Y_1^2] & E[Y_1Y_2] & E[Y_1Y_3] \\
E[Y_1Y_2] & E[Y_2^2] & E[Y_2Y_3] \\
E[Y_1Y_3] & E[Y_2Y_3] & E[Y_3^2]
\end{bmatrix}
\]

\[
|\Delta| = C_{11} E[Y_1^2] + C_{12} E[Y_1 Y_2] + C_{13} E[Y_1 Y_3]
\]

- \( C_{ij} \) – cofactor of the covariance matrix element \( c_{ij} \)

**Solutions:**

1. \( E[|Y_2| Y_3] = \frac{2\sigma_2 \sigma_3^2}{\sqrt{2\pi}} (1 + \rho^2) \)

2. \( E[|Y_3| Y_2 Y_3] = \frac{4\rho \sigma_3 \sigma_3^2}{\sqrt{2\pi}} \)

3. \( E[|Y_2| Y_2 Y_3] = \frac{4\rho \sigma_2 \sigma_3^2}{\sqrt{2\pi}} \)

4. \( E[|Y_3| Y_2^2] = \frac{2\sigma_3 \sigma_3^2}{\sqrt{2\pi}} (1 + \rho^2) \)

5. \( E \left[ \exp \left( \frac{-Y_3^2}{2\xi_2^2} \right) |Y_2| Y_3 \right] = \frac{\rho \sigma_3 \xi_3 \xi_2^{3/2}}{\left( \sigma_3^2 + \xi_2^2 \right)^{3/2}} \)

6. \( E \left[ \exp \left( \frac{-Y_3^2}{2\xi_2^2} \right) |Y_2| Y_3 \right] = \frac{1}{4 P_4 X_5 \sqrt{|\Delta|_{23} \pi P}} \left( 1 + \frac{\rho \sigma_3^2 \xi_2^3}{2|\Delta|_{23} P_4 X_5} \right) \)
7. \[ E \left[ \exp \left( \frac{-Y_3^2}{2\xi_2^2} \right) \right] = \exp \left( \frac{4\rho_2\sigma_3\xi_2}{\sqrt{2\pi} (\sigma_3^2 + \xi_2^2)} \right) \]

8. \[ E[Y_2|Y_1, Y_3] = \frac{1}{2C_{11}P_1X_1\sqrt{C_{11}P_1}} \left( \frac{-C_{12}R}{X_1} - \frac{C_{13}R}{2P_1X_1} \right) \]

9. \[ E[Y_3|Y_1, Y_2] = \frac{1}{2C_{11}P_2X_2\sqrt{C_{11}P_2}} \left( \frac{-C_{12}R}{X_2} - \frac{C_{13}R}{2P_2X_2} \right) \]

10. \[ E \left[ \exp \left( \frac{-Y_3^2}{2\xi_2^2} \right) \right] = \frac{1}{4C_{11}P_3X_3\sqrt{C_{11}P_3}} \left( \frac{-C_{12}R}{X_3} - \frac{C_{13}R}{2P_3X_3} \right) \]

11. \[ E \left[ \exp \left( \frac{-Y_2^2}{2\xi_2^2} \right) \right] = \frac{1}{2C_{11}P_2X_2\sqrt{C_{11}P_2}} \left( \frac{-C_{13}R}{X_2} - \frac{C_{12}R}{2P_2X_2} \right) \]

12. \[ E \left[ \exp \left( \frac{-Y_2^2}{2\xi_2^2} \right) \right] = \frac{1}{2C_{11}P_3X_3\sqrt{C_{11}P_3}} \left( \frac{-C_{13}R}{X_3} - \frac{C_{12}R}{2P_3X_3} \right) \]

13. \[ E \left[ \exp \left( \frac{-Y_2^2}{2\xi_2^2} \right) \right] = \frac{\xi_2\sigma_2}{\sqrt{\sigma_3^2 + \xi_2^2}} \left[ 1 - \rho^2 + \frac{\rho^2\xi_2^2}{(\sigma_3^2 + \xi_2^2)} \right] \]

14. \[ E \left[ \exp \left( \frac{-Y_3^2}{2\xi_2^2} \right) \right] = \frac{\rho_2\sigma_3\xi_2}{4|\Delta|\sigma_3^2\xi_2} \left[ 1 - \rho^2 + \frac{2\rho_2\xi_2^2}{(\sigma_3^2 + \xi_2^2)} \right] \]

The dummy variables used in the above solutions for the expected values are defined as follows:

\[ R = \frac{-C_{23}}{2|\Delta|} - \frac{C_{12}C_{13}}{2|\Delta|C_{11}} ; \quad X_5 = \frac{\sigma_3^2}{2|\Delta|} - \frac{\rho^2\sigma_2^2\sigma_3^2}{4|\Delta|^2P_4} \]

\[ P_1 = \frac{C_{33}}{2|\Delta|} - \frac{C_{13}^2}{2|\Delta|C_{12}} ; \quad X_1 = \frac{C_{22}}{2|\Delta|} - \frac{C_{12}^2}{2|\Delta|C_{11}} - \frac{R^2}{P_1} \]

\[ P_2 = \frac{C_{22}}{2|\Delta|} - \frac{C_{12}^2}{2|\Delta|C_{11}} ; \quad X_2 = \frac{C_{33}}{2|\Delta|} - \frac{C_{13}^2}{2|\Delta|C_{11}} - \frac{R^2}{P_2} \]

\[ P_3 = \frac{C_{33}}{2|\Delta|} - \frac{C_{13}^2}{2|\Delta|C_{11}} + \frac{1}{2\xi_2^2} ; \quad X_3 = \frac{C_{22}}{2|\Delta|} - \frac{C_{12}^2}{2|\Delta|C_{11}} - \frac{R^2}{P_3} \]

\[ P_4 = \frac{1}{2\xi_2^2} + \frac{\sigma_2^2}{2|\Delta|} ; \quad X_4 = \frac{C_{33}}{2|\Delta|} - \frac{C_{13}^2}{2|\Delta|C_{11}} - \frac{R^2}{P_4} + \frac{1}{2\xi_2^2} \]

References


