

## PARTITIONING GENERAL PROBABILITY MEASURES<sup>1</sup>

BY THEODORE P. HILL

*Georgia Institute of Technology*

Suppose  $\mu_1, \dots, \mu_n$  are probability measures on the same measurable space  $(\Omega, \mathcal{F})$ . Then if all atoms of each  $\mu_i$  have mass  $\alpha$  or less, there is a measurable partition  $A_1, \dots, A_n$  of  $\Omega$  so that  $\mu_i(A_i) \geq V_n(\alpha)$  for all  $i = 1, \dots, n$ , where  $V_n(\cdot)$  is an explicitly given piecewise linear nonincreasing continuous function on  $[0, 1]$ . Moreover, the bound  $V_n(\alpha)$  is attained for all  $n$  and all  $\alpha$ . Applications are given to  $L_1$  spaces, to statistical decision theory, and to the classical nonatomic case.

**1. Introduction.** The underlying space of any nonatomic probability measure may always be partitioned into  $n$  measurable subsets each having measure exactly  $1/n$ . More generally, if there are  $k$  nonatomic probability measures on the same space, Neyman [6] showed there is a measurable partition of the space into  $n$  subsets so that *each* probability assigns measure exactly  $1/n$  to each subset, thereby solving Fisher's "Problem of the Nile" [4]. In the case of  $n$  continuous probability measures, Steinhaus, Banach and Knaster [7] gave a practical method for determining a partition into  $n$  sets with the property that the  $i$ th measure of the  $i$ th subset is *at least*  $1/n$ . Extensions of these results, many using Lyapounov's convexity theorem [5] ("the range of every nonatomic finite-dimensional, vector valued (finite) measure is convex (and compact)") and generalizations were obtained by Dvoretzky, Wald and Wolfowitz [2] and Dubins and Spanier [1].

In general, all of the above-mentioned results fail if the measures have atoms, and it is the purpose of this paper to determine some best possible partitioning bounds as a function of the maximum size of the atoms.

Throughout this paper  $(\Omega, \mathcal{F}) = (\mathbb{R}, \text{Borels})$ , but any measurable space admitting nonatomic probability measures will do; this particular choice is mainly for notational convenience since a measure  $\mu$  on  $(\mathbb{R}, \text{Borels})$  is nonatomic if and only if  $\mu(\{x\}) = 0$  for all  $x \in \mathbb{R}$ .

**DEFINITION.** For each  $\alpha \in [0, 1]$ ,

$$\mathcal{P}(\alpha) = \{ \mu : \mu \text{ is a probability measure on } (\Omega, \mathcal{F}) \\ \text{with } \mu(\{x\}) \leq \alpha \text{ for all } x \in \Omega \}.$$

**DEFINITION.**  $V_n: [0, 1] \rightarrow [0, n^{-1}]$  is the unique nonincreasing function (see Figure 1) satisfying

$$(1) \quad V_n(\alpha) = 1 - k(n-1)\alpha$$

---

Received April 1985.

<sup>1</sup>Research partially supported by NSF Grant DMS-84-01604.

AMS 1980 subject classifications. Primary 60A10, 28A99; secondary 60E15, 62C20.

Key words and phrases. Optimal-partitioning inequalities, atomic probability measures, cake-cutting, fair division problems, minimax decision rules.

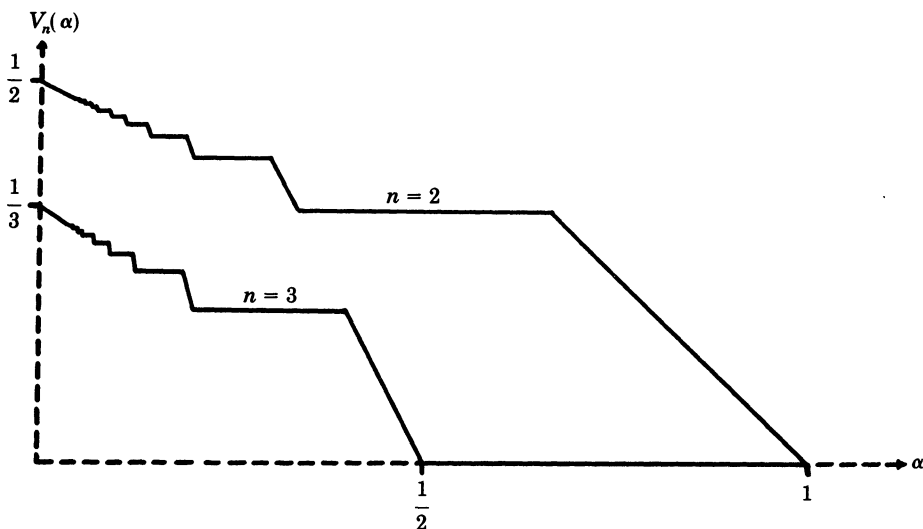


FIG. 1. Graphs of  $V_2$  and  $V_3$ .

for

$$\alpha \in \left[ (k + 1)k^{-1}((k + 1)n - 1)^{-1}, (kn - 1)^{-1} \right],$$

for all  $k \geq 1$ .

The main results of this paper are the following two closely related theorems.

**THEOREM 1.1.** *Let  $\mu \in \mathcal{P}(\alpha)$ . Then for each  $n > 1$  there exists a measurable partition  $\{A_i\}_{i=1}^n$  of  $\Omega$  satisfying*

$$(2) \quad \mu(A_i) \geq V_n(\alpha), \quad \text{for all } i = 1, \dots, n;$$

moreover,  $V_n$  is the best possible bound in (2), and is attained for all  $\alpha$ .

**THEOREM 1.2.** *Let  $\mu_1, \dots, \mu_n \in \mathcal{P}(\alpha)$ . Then there exists a measurable partition  $\{A_i\}_{i=1}^n$  of  $\Omega$  satisfying*

$$(3) \quad \mu_i(A_i) \geq V_n(\alpha), \quad \text{for all } i = 1, \dots, n;$$

again,  $V_n$  is the best possible bound in (3), and is attained for all  $\alpha$ .

**REMARK.** Theorems 1.1 and 1.2 are “dual” in the following sense: the bound (2) in Theorem 1.1 follows from (3) of Theorem 1.2 by taking  $\mu_1 = \dots = \mu_n$ , whereas the sharpness of the bound (3) in Theorem 1.2 follows similarly from the sharpness of (2) in Theorem 1.1.

A “cake-cutting” interpretation of Theorem 1.2 based on a description by Dubins and Spanier [1] is this. Suppose a cake  $\Omega$  is to be divided among  $n$

people whose values  $\{\mu_i\}_{i=1}^n$  of different portions of the cake may differ [here  $\mu_i(A)$  represents the value of piece  $A$  to person  $i$ ]. Then if no one values any *crumb* (indivisible portion of the cake) more than  $\alpha$ , the cake may be divided so that each person receives a piece he himself values at least  $V_n(\alpha)$ , and in general it is not possible to do better.

**EXAMPLE 1.3.** Suppose three people must divide a cake, and each agrees that no crumb is worth more than  $10^{-3}$  the value of the whole cake. Then there is a way of cutting the cake into three pieces, and giving each person a piece, in such a way that each person values his own piece at least  $V_n(\alpha) = V_3(10^{-3}) = 83/250$  and in general it is not possible to do better.

(A similar interpretation of Theorem 1.1 is also possible. Suppose a cake of total volume (or weight) one is to be cut into  $n$  pieces so that the smallest piece has as large a volume as possible. If each atom (or molecule, or crumb, or other indivisible piece) has volume  $\alpha$  or less, then in an optimal partitioning the smallest piece has volume at least  $V_n(\alpha)$ , and in general this is the best possible bound.)

Intuitively, it is clear that the nonatomic case is the limit of the general case as the maximum atom size approaches zero.

**COROLLARY 1.4** ([1], [2], [7]). *Suppose  $\mu_1, \dots, \mu_n$  are nonatomic measures on  $(\Omega, \mathcal{F})$ . Then there exists a measurable partition  $\{A_i\}_{i=1}^n$  of  $\Omega$  so that*

$$\mu_i(A_i) \geq n^{-1}, \quad \text{for all } i = 1, \dots, n.$$

This paper is organized as follows: Section 2 contains the proof of Theorem 1.1; Section 3 the proof of Theorem 1.2; Section 4 further observations about the upper bound function  $V_n(\alpha)$ ; and Section 5 contains several applications to  $L_1$  function spaces and statistical decision theory.

**2. Partitioning a single probability measure.** The main objective of this section is to prove Theorem 1.1. Throughout this paper,  $\Pi_{\mathcal{G}}$  will denote the collection of  $\mathcal{G}$ -measurable partitions of  $\Omega$ , where  $\mathcal{G}$  is a sub- $\sigma$ -algebra of  $\mathcal{F}$ , and  $\sigma(\mathcal{G})$  will denote the  $\sigma$ -algebra generated by  $\mathcal{G}$ .

**DEFINITION 2.1.** Suppose  $\mu$  is a probability measure on  $(\Omega, \mathcal{F})$ . Then

$$U_n(\mu) = \sup \left\{ \min_{1 \leq i \leq n} \{\mu(A_i)\} : \{A_i\}_{i=1}^n \in \Pi_{\mathcal{F}} \right\}$$

and

$$U_n(\alpha) = \inf \{ U_n(\mu) : \mu \in \mathcal{P}(\alpha) \}.$$

**LEMMA 2.2.** *Fix  $\alpha \in (0, 1]$ . For each  $\mu \in \mathcal{P}(\alpha)$  there exists a purely atomic  $\hat{\mu} \in \mathcal{P}(\alpha)$  having at most  $2\alpha^{-1}$  atoms, and satisfying*

$$(4) \quad U_n(\hat{\mu}) \leq U_n(\mu), \quad \text{for all } n \geq 1.$$

**PROOF.** The idea of the proof is simply that collapsing mass to atoms reduces the partitioning options available, and thus reduces  $U_n$ ; for completeness the first step will be given in some detail. Let  $A = \{x_1, x_2, \dots\} \subset \Omega$  denote the atoms of  $\mu$  and  $A^c = \Omega \setminus A$ . If  $\mu(A^c) > 0$ , let  $A_1, A_2, \dots$  be a measurable partition of  $A^c$  satisfying  $0 < \mu(A_i) \leq \alpha$  for all  $i$ , which is possible since  $\mu$  is nonatomic on  $A^c$ . For each  $i$ , fix  $y_i \in A_i$ , and let  $\mu_1 \in \mathcal{P}(\alpha)$  be the purely atomic probability measure defined by  $\mu_1(\{x_i\}) = \mu(\{x_i\})$  and  $\mu_1(\{y_i\}) = \mu(A_i)$ . Since  $\mu$  restricted to  $\sigma(\{x_1\}, A_1, \{x_2\}, A_2, \dots)$  is isomorphic to  $\hat{\mu}$  restricted to  $\sigma(\{x_1\}, \{y_1\}, \{x_2\}, \{y_2\}, \dots)$ , and since (recall  $\{x_1\}, A_1, \{x_2\}, A_2, \dots$  are disjoint)  $\sigma(\{x_1\}, A_1, \{x_2\}, A_2, \dots) \subset \mathcal{F}$ , it follows that  $U_n(\mu_1) \leq U_n(\mu)$  for all  $n \geq 1$ .

The next step is to replace  $\mu_1$  by a purely atomic measure with each atom having mass at least  $\alpha 2^{-1}$  (and hence having at most  $2\alpha^{-1}$  atoms). This is done by first combining the tail  $\{x_N\}, \{y_N\}, \{x_{N+1}\}, \{y_{N+1}\}, \dots$  into one atom (where  $\sum_{i=N}^{\infty} [\mu_1(\{x_i\}) + \mu_1(\{y_i\})] \leq \alpha$ ) to reduce to a finite number of atoms, and then by repeatedly combining any two atoms with mass  $\leq \alpha/2$ .  $\square$

**LEMMA 2.3.** *For each  $\alpha \in [0, 1]$  and  $n \geq 1$ , there exists a  $\mu \in \mathcal{P}(\alpha)$  and a partition  $\{A_i\}_{i=1}^n \in \Pi_{\mathcal{F}}$  satisfying*

$$(5) \quad U_n(\alpha) = \mu(A_1) \leq \mu(A_2) \leq \dots \leq \mu(A_n).$$

**PROOF.** For  $\alpha = 0$  (which will not be needed in this paper) the result is an easy consequence (taking  $\mu_1 = \dots = \mu_n$ ) of Lyapounov's convexity theorem [5].

Fix  $\alpha \in (0, 1]$  and  $k > \max\{n, 2\alpha^{-1}\}$ , and choose  $k$  distinct points  $x_1, \dots, x_k$  in  $\Omega$ . By the definition of  $U_n(\alpha)$  and Lemma 2.2,  $U_n(\alpha) = \inf\{U_n(\mu) : \mu \in \mathcal{P}(\alpha, k)\}$ , where  $\mathcal{P}(\alpha, k) = \{\mu \in \mathcal{P}(\alpha) : \sum_{i=1}^k \mu(\{x_i\}) = 1\}$ . Since  $\mathcal{P}(\alpha, k)$  is compact, and since  $U_n$  is a continuous function of  $\mu \in \mathcal{P}(\alpha, k)$ ,  $\inf\{U_n(\mu) : \mu \in \mathcal{P}(\alpha, k)\}$  is attained by some  $\hat{\mu} \in \mathcal{P}(\alpha, k)$ . Since the support of  $\hat{\mu}$  is a finite set (subset of  $\{x_1, \dots, x_k\}$ ), it is clear that there is a partition  $\{A_i\}_{i=1}^n \in \Pi_{\mathcal{F}}$  satisfying (5) with  $\hat{\mu}$  in place of  $\mu$ .  $\square$

**PROOF OF THEOREM 1.1.** Fix  $n > 1$  and  $k \geq 1$  and let  $\alpha \in I(n, k)$ , where  $I(n, k) = [(k + 1)k^{-1}((k + 1)n - 1)^{-1}, (kn - 1)^{-1}] \subset (0, 1)$ .

It first will be shown that on  $I(n, k)$ ,  $V_n = U_n$ . By Lemmas 2.3 and 2.2 there exists a purely atomic measure  $\mu \in \mathcal{P}(\alpha)$  with at most  $2\alpha^{-1}$  atoms, and a partition  $\{A_i\}_{i=1}^n \in \Pi_{\mathcal{F}}$  satisfying (5).

Suppose, by way of contradiction, that  $\mu(A_1) < 1 - k(n - 1)\alpha$ . Since  $\mu$  is a probability measure,  $\mu(\cup_{i=2}^n A_i) > k(n - 1)\alpha$ , and since the  $\{A_i\}$  are disjoint, this implies that for some  $j \in \{2, 3, \dots, n\}$ ,  $\mu(A_j) > k\alpha$ . Since  $\mu$  is purely atomic and in  $\mathcal{P}(\alpha)$ ,  $A_j$  must contain at least  $k + 1$   $\mu$ -atoms. Let  $\{x_j\} \in A_j$  be the smallest atom in  $A_j$  (which exists since  $\mu$  has only a finite number of atoms) and observe that

$$(6) \quad \mu(A_1 \cup \{x_j\}) > \mu(A_1).$$

Since  $\{x_j\}$  is the smallest atom in  $A_j$ , and there are at least  $k + 1$  atoms in  $A_j$ , this implies

$$(7) \quad \mu(A_j \setminus \{x_j\}) \geq k(k + 1)^{-1}\mu(A_j) > k^2(k + 1)^{-1}\alpha \geq 1 - k(n - 1)\alpha,$$

where the last inequality in (7) follows since  $\alpha \geq (k + 1)k^{-1}[(k + 1)n - 1]^{-1}$ .

If  $\mu(A_2) > \mu(A_1)$ , then together (6) and (7) contradict the assumed optimality (5) of  $\mu$  and the partition  $\{A_i\}_{i=1}^n$ ; otherwise [i.e., if  $\mu(A_2) = \mu(A_1)$ ], repeat the procedure with  $A_2$ , etc. Since there are only a finite number of sets in the partition, eventually such a contradiction is reached. This implies that  $U_n(\mu) \geq V_n(\alpha)$ , and hence that  $U_n \geq V_n$  on  $I(n, k)$ .

To show  $U_n(\alpha) \leq V_n(\alpha)$ , let  $\hat{\mu} \in \mathcal{P}(\alpha)$  be a purely atomic measure with  $kn - 1$  atoms of mass  $\alpha$ , and one atom of mass  $1 - \alpha(kn - 1)$ . [Since  $\alpha \in I(n, k)$ , it follows that  $0 \leq 1 - \alpha(kn - 1) \leq \alpha$ .] Clearly an optimal partition for  $\hat{\mu}$  has

$$\begin{aligned} \hat{\mu}(A_1) &= (k - 1)\alpha + 1 - \alpha(kn - 1) \\ &= 1 - k(n - 1)\alpha \leq k\alpha \\ &= \hat{\mu}(A_2) = \dots = \hat{\mu}(A_n), \end{aligned}$$

which shows that  $U_n = V_n$  on  $I(n, k)$ , and in fact that  $V_n(\alpha)$  is attained (by  $\hat{\mu}$ ).

To complete the proof, observe that the value of  $V_n$  at the left endpoint of  $I(n, k)$  is the same as the value of  $V_n$  at the right endpoint of  $I(n, k + 1)$ , that is,  $1 - k(n - 1)x = 1 - (k + 1)(n - 1)y$  for  $x = (k + 1)k^{-1}[(k + 1)n - 1]^{-1}$  and  $y = ((k + 1)n - 1)^{-1}$ . Then since  $V_n$  was defined to be nonincreasing, it must be constant on  $[0, 1] \setminus \cup_{k=1}^\infty I(n, k)$ .

That  $V_n(0) = n^{-1}$  and  $V_n(1) = 0$  are also attained is easy.  $\square$

**3. Partitioning several probability measures.** The main objective of this section is to prove Theorem 1.2; the first two results (Lemma 3.2 and Proposition 3.3) concern stochastic matrices and are purely combinatorial in nature.

Throughout this section, the following notation is used:

- $\mathcal{S}_{n, k}$  is the set of  $n \times k$  stochastic matrices;
- $\Pi_k$  is the collection of partitions of the set  $\{1, 2, \dots, k\}$ ; and
- $P_k$  is the set of permutations of  $\{1, 2, \dots, k\}$ .

**DEFINITION 3.1.** Suppose  $A = (a_{i, j}) \in \mathcal{S}_{n, k}$ . Then

$$W_n(A) = \max \left\{ \min_{1 \leq i \leq n} \left\{ \sum_{j \in J_i} a_{i, j} \right\} : \{J_i\}_{i=1}^n \in \Pi_k \right\}.$$

**LEMMA 3.2.** For each  $A = (a_{i, j}) \in \mathcal{S}_{n, n}$  there exist  $\pi \in P_n$  and  $j \in \{1, \dots, n\}$  satisfying both

$$(8) \quad W_n(A) = \min_{1 \leq i \leq n} \{a_{i, \pi(i)}\}$$

and

$$(9) \quad a_{j, \pi(j)} = \max_{1 \leq k \leq n} \{a_{k, \pi(j)}\}.$$

PROOF. Since  $A \in \mathcal{S}_{n,n}$ , it is easy to see that

$$W_n(A) = \max \left\{ \min_{1 \leq i \leq n} \{a_{i, \pi(i)}\} : \pi \in P_n \right\}.$$

Let  $\pi^* \in P_n$  satisfy (10) and (11),

$$(10) \quad W_n(A) = \min_{1 \leq i \leq n} \{a_{i, \pi^*(i)}\},$$

$$(11) \quad \sum_{i=1}^n a_{i, \pi^*(i)} = \max \left\{ \sum_{i=1}^n a_{i, \pi(i)} : \pi \in P_n, \min_{1 \leq i \leq n} \{a_{i, \pi(i)}\} = W_n(A) \right\}.$$

Renumbering if necessary, assume  $\pi^*(i) = i$  for all  $i = 1, \dots, n$ , and  $W_n(A) = a_{1,1} \leq a_{2,2} \leq \dots \leq a_{n,n}$ . It will now be shown that

$$(12) \quad a_{jj} = \max_{1 \leq k \leq n} a_{k,j}, \text{ for some } j \in \{1, \dots, n\},$$

which, with (10), will complete the proof.

To establish (12), suppose by way of contradiction that for each  $j \in \{1, \dots, n\}$ ,  $a_{jj} < \max_{1 \leq k \leq n} \{a_{k,j}\}$ . Then there exist  $i_1, i_2, \dots, i_n \in \{1, \dots, n\}$  satisfying (with  $i_0 := 1$ )

$$(13) \quad \begin{aligned} W_n(A) &= a_{i_0, i_0} < a_{i_1, i_0}, \\ & a_{i_1, i_1} < a_{i_2, i_1}, \\ & \vdots \\ & a_{i_{n-1}, i_{n-1}} < a_{i_n, i_{n-1}}. \end{aligned}$$

Since  $i_0, \dots, i_n \in \{1, \dots, n\}$ , the ordered  $(n + 1)$ -tuple  $(i_0, i_1, \dots, i_n)$  contains a primitive cycle, that is, there exists  $j \in \{0, \dots, n - 1\}$  and  $k \in \{0, \dots, n - j\}$  such that  $i_j, i_{j+1}, \dots, i_{j+k}$  are distinct and  $i_j = i_{j+k+1}$ .

Next consider the permutation  $\tilde{\pi} \in P_n$  defined by  $\tilde{\pi}(i_{j+m+1}) = i_{j+m}$  for  $m = 0, 1, \dots, k$ , and  $\tilde{\pi} = \pi^*$  otherwise. By (13),

$$a_{i_{j+m+1}, \tilde{\pi}(i_{j+m+1})} = a_{i_{j+m+1}, i_{j+m}} > a_{i_{j+m}, i_{j+m}} \geq a_{1,1} = W_n(A),$$

for  $m = 0, 1, \dots, k$ , so the definition of  $\tilde{\pi}$  implies that  $W_n(A) \leq \min_{1 \leq i \leq n} a_{i, \tilde{\pi}(i)}$ , and hence by the definition of  $W_n(A)$  that

$$(14) \quad W_n(A) = \min_{1 \leq i \leq n} a_{i, \tilde{\pi}(i)}.$$

But (13) and the definition of  $\tilde{\pi}$  also imply that  $\sum_{i=1}^n a_{i, \tilde{\pi}(i)} > \sum_{i=1}^n a_{i, \pi^*(i)}$  which, with (14), contradicts (11). This completes the proof of (12), and the lemma.  $\square$

The next proposition states that there is always an optimal partitioning of a stochastic matrix in which the “cooperative value,” that is, the sum of the partition-assignment values, is at least one.

PROPOSITION 3.3. For each  $A = (a_{i,j}) \in S_{n,m}$  there is a partition  $\{J_i\}_{i=1}^n \in \Pi_m$  satisfying both

$$(15) \quad W_n(A) = \min_{1 \leq i \leq n} \left\{ \sum_{j \in J_i} a_{i,j} \right\}$$

and

$$(16) \quad \sum_{i=1}^n \sum_{j \in J_i} a_{i,j} \geq 1.$$

**PROOF.** Fix  $A = (a_{i,j}) \in S_{n,m}$ . By the definition of  $W_n$ , there exists a partition  $\{J_i\}_{i=1}^n$  of  $\Pi_m$  satisfying (15). If  $m > n$ , let  $\hat{A} = (\hat{a}_{i,j}) \in S_{n,n}$  be the matrix defined by  $\hat{a}_{i,j} = \sum_{k \in J_j} a_{i,k}$ , and observe that both

$$(17) \quad W_n(A) = W_n(\hat{A}) = \min_{1 \leq i \leq n} \hat{a}_{i,i}$$

and

$$(18) \quad \sum_{i=1}^n \hat{a}_{i,i} = \sum_{i=1}^n \sum_{j \in J_i} a_{i,j}.$$

By (17) and (18), it is enough to establish the proposition for  $n \times n$  stochastic matrices  $A$  (if  $m < n$ , simply add  $n - m$  columns of zeros to  $A$ ). The proof will proceed by induction on  $n$ ; for  $n = 1$  the conclusion is trivial, so assume it holds for  $1, 2, \dots, n - 1$  and let  $A = (a_{i,j}) \in S_{n,n}$ .

By Lemma 3.2 there exists  $\pi \in P_n$  and  $j \in \{1, \dots, n\}$  satisfying (8) and (9). Reordering if necessary, assume  $j = n = \pi(n)$ , and observe that by (9) the  $(n - 1) \times (n - 1)$  matrix  $\tilde{A}$  obtained from  $A$  by deleting the  $n$ th row and column is substochastic with row sums  $\sum_{j=1}^{n-1} a_{i,j} \geq 1 - a_{n,n}$  for all  $i = 1, \dots, n - 1$ . It follows easily from the induction hypothesis that there exists  $\tilde{\pi} \in P_{n-1}$  satisfying both

$$(19) \quad W_{n-1}(\tilde{A}) = \min_{1 \leq i \leq n-1} \{a_{i, \tilde{\pi}(i)}\}$$

and

$$(20) \quad \sum_{i=1}^{n-1} a_{i, \tilde{\pi}(i)} \geq 1 - a_{n,n}.$$

Defining  $\hat{\pi} \in P_n$  by  $\hat{\pi}(i) = \tilde{\pi}(i)$  for  $i < n$  and  $\hat{\pi}(n) = \pi(n) = n$ , (8) and (19) together imply that

$$(21) \quad W_n(A) = \min\{W_{n-1}(\tilde{A}), a_{n,n}\} = \min_{1 \leq i \leq n} \{a_{i, \hat{\pi}(i)}\},$$

and (20) and the definition of  $\hat{\pi}$  imply that

$$(22) \quad \sum_{i=1}^n a_{i, \hat{\pi}(i)} \geq 1.$$

The induction conclusion then follows from (21) and (22) by taking  $J_i = \{\hat{\pi}(i)\}$  for  $i = 1, \dots, n$ .  $\square$

Not all optimal partitions [partitions achieving  $W_n(A)$ ] satisfy (16).

**EXAMPLE 3.4.** Let

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}.$$

The partition  $\{J_i\} = \{i\}$ ,  $i = 1, 2, 3$ , satisfies

$$W_3(A) = 0.3 = \min_{1 \leq i \leq 3} \left\{ \sum_{j \in J_i} a_{i,j} \right\} \\ = \min\{\alpha_{1,1}, \alpha_{2,2}, \alpha_{3,3}\},$$

but  $\sum_{i=1}^n \sum_{j \in J_i} a_{i,j} = 0.9 < 1$ .

**PROOF OF THEOREM 1.2.** That  $V_n(\alpha)$  is attained for all  $\alpha$  follows from Theorem 1.1 by taking  $\mu_1 = \mu_2 = \dots = \mu_n$ .

Fix  $\alpha \in (0, 1]$  and  $\mu_1, \dots, \mu_n \in \mathcal{P}(\alpha)$ . By an argument directly analogous to that in the proof of Lemma 2.2, it may be assumed without loss of generality that  $\{\mu_i\}_{i=1}^n$  are purely atomic each with at most  $m \leq 2n\alpha^{-1}$  atoms. In other words, it suffices to show that if

$$(23) \quad A = (a_{i,j}) \in S_{n,m} \quad \text{and} \quad a_{i,j} \leq \alpha \\ \text{for all } i = 1, \dots, n \quad \text{and} \quad j = 1, \dots, m,$$

then

$$(24) \quad W_n(A) \geq V_n(\alpha).$$

Fix  $A$  satisfying (23). By Proposition 3.3 there exists a partition  $\{J_i\}_{i=1}^n \in \Pi_m$  satisfying (15) and (16). To prove (24), fix  $n > 1$ ,  $k \geq 1$ , and

$$\alpha \in I(n, k) = \left[ (k+1)k^{-1}((k+1)n-1)^{-1}, (kn-1)^{-1} \right].$$

Suppose, by way of contradiction, that  $\sum_{j \in J_i} a_{i,j} < 1 - k(n-1)\alpha$ . By (16),  $\sum_{i=2}^n \sum_{j \in J_i} a_{i,j} > k(n-1)\alpha$ , so for some  $i \in \{2, \dots, n\}$ ,  $\sum_{j \in J_i} a_{i,j} > k\alpha$ . The argument now proceeds as in the proof of Theorem 1.1, the key difference having been the use of Proposition 3.3 (which is trivial for the  $\mu_1 = \mu_2 = \dots = \mu_n$  context of Theorem 1.1).  $\square$

**4. Several remarks concerning  $V_n(\alpha)$ .** The following proposition is an easy consequence of the definition of  $V_n(\alpha)$ .

**PROPOSITION 4.1.** *For each  $n \geq 1$ ,  $V_n(\cdot)$  is continuous and nonincreasing on  $[0, 1]$ , piecewise linear on  $(0, 1]$ , and satisfies*

- (i)  $V_n(0) = n^{-1}, \quad V_n(1) = 0;$
- (ii)  $V_{n+1}(\alpha) < V_n(\alpha), \quad \text{if } V_n(\alpha) > 0, \quad \text{and}$   
 $V_{n+1}(\alpha) = V_n(\alpha), \quad \text{if } V_n(\alpha) = 0;$

and

- (iii)  $V_n(\alpha) \geq n^{-1} - (n-1)n^{-1}\alpha.$

The critical points at the left-hand endpoints of the intervals where  $V_n$  is constant are local minima. For example,  $V_2$  has local minima at  $1/3, 1/5, 1/7, \dots$ ; and for the first of these, one interpretation is that in the case of



bisection ( $n = 2$ ), atoms of mass exactly  $1/3$  are locally the worst—in general atoms slightly less than or slightly greater than  $1/3$  allow better partitions.

**5. Applications to  $L_1$  spaces and statistical decision theory** It is easy to translate the settings of Theorems 1.1 and 1.2 to the theory of  $L_1$  spaces; the next theorem is the analog of Theorem 1.2.

**THEOREM 5.1.** *Suppose  $\lambda$  is a Borel measure on  $\mathbb{R}$ . If  $f_1, f_2, \dots, f_n \in L_1(\lambda)$  satisfy*

- (i)  $f_i \geq 0, i = 1, \dots, n$ ;
- (ii)  $\int f_i d\lambda = 1, i = 1, \dots, n$ ; and
- (iii)  $\lambda(\{x\})f_i(x) \leq \alpha$ , for all  $x \in \mathbb{R}$ ,

*then there exists a measurable partition  $\{A_i\}_{i=1}^n$  of  $\mathbb{R}$  satisfying*

$$\int_{A_i} f_i d\lambda \geq V_n(\alpha), \quad \text{for all } i = 1, \dots, n.$$

*Moreover, this bound is best possible, and is attained for all  $\alpha$  and  $n$ .*

The final theorem is an application of Theorem 1.2 to statistical decision theory which is related to similar applications of partitioning inequalities in [2] and [3].

Suppose there is an  $\Omega$ -valued random variable  $X$  which has one of the known distributions  $\mu_1, \dots, \mu_n$  (but it is not known which one). A single observation  $X(\omega)$  of  $X$  is made, and then it is to be guessed from which of the distributions  $\mu_1, \dots, \mu_n$  the observation came. A *decision rule* is simply a (measurable) partition  $\{A_i\}_{i=1}^n$  of  $\Omega$  ("if  $X(\omega) \in A_i$ , then guess distribution  $\mu_i$ "). A *minimax* decision rule is a partition which attains the "minimax risk"  $R$  given by

$$(25) \quad R(\mu_1, \dots, \mu_n) = \inf \left\{ \max_{1 \leq i \leq n} P(X \notin A_i | \text{dist}(X) = \mu_i) : \{A_i\}_{i=1}^n \in \Pi_{\mathcal{F}} \right\}.$$

Since

$$\begin{aligned} R(\mu_1, \dots, \mu_n) &= \inf \left\{ \max_{1 \leq i \leq n} \{1 - \mu_i(A_i)\} : \{A_i\}_{i=1}^n \in \Pi_{\mathcal{F}} \right\} \\ &= 1 - \sup \left\{ \min_{1 \leq i \leq n} \{\mu_i(A_i)\} : \{A_i\}_{i=1}^n \in \Pi_{\mathcal{F}} \right\}, \end{aligned}$$

Theorem 1.2 has the following immediate consequence.

**THEOREM 5.2.** *Let  $\mu_1, \dots, \mu_n \in \mathcal{P}(\alpha)$ . Then*

$$R(\mu_1, \dots, \mu_n) \leq 1 - V_n(\alpha),$$

*and this bound is attained for all  $\alpha$  and all  $n$ .*

A similar application (see [2]) can also be made to the theory of zero-sum two-person games.

## REFERENCES

- [1] DUBINS, L. and SPANIER, E. (1961). How to cut a cake fairly. *Amer. Math. Monthly* **68** 1–17.
- [2] DVORETZKY, A., WALD, A. and WOLFOWITZ, J. (1951). Relations among certain ranges of vector measures. *Pacific J. Math.* **1** 59–74.
- [3] ELTON, J., HILL, T. and KERTZ, R. (1986). Optimal-partitioning inequalities for non-atomic probability measures. *Trans. Amer. Math. Soc.* **296** 703–725.
- [4] FISHER, R. (1936). Uncertain inference. *Proc. Amer. Acad. Arts Sci.* **71** 245–257.
- [5] LYAPOUNOV, A. (1940). Sur les fonctions-vecteurs complètement additives. *Bull. Acad. Sci. URSS* **4** 465–478.
- [6] NEYMAN, J. (1946). Un théorème d'existence. *C.R. Acad. Sci. Paris Sér. A–B* **222** 843–845.
- [7] STEINHAUS, H. (1949). Sur la division pragmatique. *Econometrica* (supplement) **17** 315–319.

SCHOOL OF MATHEMATICS  
GEORGIA INSTITUTE OF TECHNOLOGY  
ATLANTA, GEORGIA 30332