LARGE SETS OF ZERO ANALYTIC CAPACITY

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ABSTRACT. We prove that certain Cantor sets with non-sigma-finite one-dimensional Hausdorff measure have zero analytic capacity.

1. INTRODUCTION

In this paper we consider a Cantor set $K$ similar to the $\frac{1}{4}$-Cantor set of [G70] and [I84]. Fix $p > 2$ and for $n > 0$ define

$$\sigma_n = 4^{-n}\alpha_n = 4^{-n}[\log(n + 1)]^{1/p}.$$ 

Set $K_0 = [0, 1] \times [0, 1]$ and $K_1 = \bigcup_{j=1}^{4} K_{1,j}$, where $K_{1,j} \subset K_0$ is a square of sidelength $\sigma_1$ having sides parallel to the axis and containing one of the four corners of $K_0$. Next take $4^2$ squares $K_{2,j}$ of sidelength $\sigma_2$, one in each corner of each square $K_{1,j}$, and define $K_2 = \bigcup_{j=1}^{4^2} K_{2,j}$. Continuing we obtain $K_n = \bigcup_{j=1}^{4^n} K_{n,j}$, and the Cantor set we study is

$$K = K(p) = \bigcap_{n=1}^{\infty} K_n.$$ 

If $E$ is a compact plane set define

$$A(E, 1) = \{ f : f \text{ analytic on } E^c, \quad f(\infty) = 0, \quad \| f \|_{L^\infty(E^c)} \leq 1 \}$$

and define the analytic capacity of $E$ by

$$\gamma(E) = \sup\{ |f'|(\infty) : f \in A(E, 1) \},$$

where

$$f'(\infty) = \lim_{z \to \infty} zf(z).$$

If $\gamma(E) = 0$, then the only function in $A(E, 1)$ is the constant $f \equiv 0$ and in this case $E$ is removable for bounded analytic functions. For more details about analytic capacity see [G72].

**Theorem 1.** Let $p > 2$, and let $K$ be the four-corner Cantor set $K(p)$. Then $\gamma(K) = 0$ but $K$ does not have $\sigma$-finite one-dimensional measure.
The proof of Theorem 1 depends on a lemma of Jones [J89] used for a proof different from [G70] that the $\frac{1}{4}$-Cantor set has zero analytic capacity.

Let $h(t)$ be an increasing continuous function on $t \geq 0$ with $h(0) = 0$, and write $\Lambda_{h(t)}(E)$ for the Hausdorff $h$-measure of $E$. Now define an increasing function $h(t)$ so that $h(0) = 0$ and $h(\sigma_n) = 4^{-n}$ for all $n$. We say $h(t)$ is a measure function corresponding to the Cantor set $K$. For every $n$ define a measure $\mu_n$ on $K_n$ by $\mu_n(K_{n,j}) = 4^{-n}$ for all $j$. Then $\{\mu_n\}$ converges weak-star to a measure $\mu$ supported on $K$ and satisfying $\mu(K_{n,j}) = 4^{-n}$. Suppose $\frac{\sqrt{2}}{2}\sigma_n < r < \frac{\sqrt{2}}{4}\sigma_{n-1}$ and let $D(z, r)$ be a disk of radius $r$ and center $z \in K$. Then $D(z, r)$ can meet at most 4 squares of sidelength $\sigma_n$. Hence

$$\mu(D(z, r)) \leq 4\mu(K_{n,j}) = 4 \cdot 4^{-n} = 4h(\sigma_n) \leq 4h(r),$$

so that $\mu(D(z, r)) \leq 16h(r)$ for any disk $D(z, r)$. Therefore $\Lambda_{h}(K) > 0$ by Frostman’s Theorem [G72]. Since

$$\lim_{t \to 0} \frac{h(t)}{t} = 0,$$

if follows that $K$ has non-$\sigma$-finite 1-dimensional measure.

If $h(t)$ is a measure function corresponding to $K$, then

$$\int_0^1 \frac{h(t)^2}{t^3} dt \sim \sum_{n=1}^{\infty} \frac{1}{(a_n)^2} = \sum_{n=1}^{\infty} \frac{1}{(\log n)^2} = \infty.$$

On the other hand, Mattila [M96] proved that $\gamma(K) > 0$ if $K$ is a Cantor set built with squares of side $\sigma_n$ and if

$$\int_0^1 \frac{h(t)^2}{t^3} dt < \infty,$$

where $h$ is any measure function for corresponding to $K$. Mattila’s proof used Menger curvature (see [Me95] and [MMV96]). However, if the Cantor set $K$ has corresponding measure function $h$ satisfying

$$\int_0^1 \frac{h(t)^2}{t^3} dt = \infty,$$

then Eiderman [E98] proved that $\gamma^{+}(K) = 0$, where

$$\gamma^{+}(E) = \sup \left\{ \int_E d\mu : \left| \int_E \frac{d\mu(\zeta)}{\zeta - z} \right| < 1, \forall z \in \mathbb{C}\backslash E, \mu > 0, \text{spt}(\mu) \subset E \right\}.$$

Since $\gamma^{+}(E) \leq \gamma(E)$, our result is a partial improvement of Eiderman’s theorem. Mattila [M96] has conjectured that for Cantor sets of this type $\gamma(K) = 0$ if and only if

$$\int_0^1 \frac{h(t)^2}{t^3} dt = \infty,$$

when $h$ corresponds to $K$. This latter condition holds if and only if

$$\sum_{n=1}^{\infty} \frac{1}{(a_n)^2} = \infty.$$

If Mattila’s conjecture is true, then together with Eiderman’s theorem it gives Cantor set evidence supporting the more ambitious conjecture that $\gamma(E) > 0$ implies $\gamma^{+}(E) > 0$. 

In [G72] it was incorrectly claimed that $\gamma(K) > 0$ if and only if
\[ \int_0^1 \frac{h(t)}{t^2} dt < \infty. \]
Eiderman, however, found a mistake in the proof. In fact the result in [M96] shows that the claim is false. See L.D. Ivanov [I84] for the first example of a Cantor set of non-$\sigma$-finite linear measure and zero analytic capacity.

2. TWO LEMMAS OF PETER JONES

We need the following two lemmas from [J89]. The proofs we give are small variations on [J89] and [C90].

Define $\gamma_j^n = \partial cK_{n,j}$, where $cK_{n,j}$ is the square concentric to $K_{n,j}$ with sidelength $c\sigma_n$ and where $c > 1$ is chosen so that the $\gamma_j^n$ do not overlap. We refer to $\gamma_k^m$ as a square, although it is only the boundary of a square. Notice that
\[ A_1(\gamma_k^m) = cA_1(\partial K_{m,k}) \]
for the same constant $c$. We associate to each $\gamma_k^m$ a “square annulus"
\[ A_k^n = \{ w : \text{dist}(w, \gamma_k^m) \leq \varepsilon_0 \sigma_m \} \]
and we choose $\varepsilon_0 > 0$ so small that the annuli $A_k^n$ are pairwise disjoint.

Define $\Omega = \mathbb{C} \setminus K$. Since $K$ has positive logarithmic capacity, Green’s function $G(z, \zeta)$ exists for $\zeta, z \notin K$, and harmonic measure $\omega(\zeta, E)$ exists for $\zeta \notin K$ and $E \subset K$. We write $\omega(\zeta, K_{m,k})$ for $\omega(\zeta, K_{m,k} \cap K)$.

We also define the slightly larger “squares”
\[ S_{m,k} = \{ w : \text{dist}(w, K_{m,k}) \leq \varepsilon_1 \sigma_m \} \]
and set
\[ S_m = \bigcup_{k=1}^{4^m} S_{m,k}, \]
where $\varepsilon_1 > 0$ is so small that $S_{m,k} \cap A_k^n = \emptyset$. Then $K = \bigcap_{m=1}^{\infty} S_m$. Green’s function and harmonic measure also exist for the domain $\Omega_m = \mathbb{C} \setminus S_m$. Denote these by $G_m(z, \zeta)$ and $\omega_m(\zeta, E)$ respectively.

Lemma 2. Let $z \in A_k^n$.
(a) There are constants $0 < c_1 < c_2 < 1$, independent of $k$ and $m$, such that
\[ c_1 \leq \omega_m(z, \partial S_{m,k}) \leq c_2. \]
(b) If $\zeta \in \Omega$ and $1 \geq \text{dist}(\zeta, K) \geq 2 \text{dist}(z, K)$, then
\[ G_m(z, \zeta) \sim \omega_m(\zeta, \partial S_{m,k}). \]

Proof. For (a) note that there is $c' > 0$ such that there exists a second square annulus $B_k^n$ so that $A_k^n \subset B_k^n \subset \Omega_m$ and $\text{dist}(z, \partial B_k^n) \geq c' \sigma_m$. The lower bound then follows by a comparison with $B_k^n$. There is $S_{m,j}$ with $j \neq k$ such that $\text{dist}(S_{m,j}, S_{m,k}) \leq c_4 \sigma_m$ and the upper bound follows by a comparison with $\mathbb{C} \setminus (S_{m,k} \cup S_{m,j})$, using symmetry and Harnack’s inequality.

To prove (b) note first that as in the proof of (a) there are constants $C_1$ and $C_2$ such that by Harnack’s inequality and a comparison
\[ C_1 \leq G_m(z, w) \leq C_2 \]
for \( w \in \partial B^m_k \). Then using the symmetry of Green’s function and (a) for a larger square we obtain
\[
C_1 \omega_m(\zeta, \partial S_{m,k}) \leq G_m(\zeta, z) \leq C_2 \omega_m(\zeta, \partial S_{m,k}).
\]

We write \( \gamma^m_k \prec \gamma^n_j \) and say \( \gamma^m_k \) is subordinated to \( \gamma^n_j \) if \( \gamma^n_j \) has winding number one about \( \gamma^m_k \). If the winding number is zero, we write \( \gamma^m_k \not\prec \gamma^n_j \). For any \( f \in A(K, 1) \) and \( \gamma^m_k \) define
\[
D(\gamma^m_k) = \sup_{w \in \gamma^m_k} |f'(w)| \sigma_n.
\]
We say a square \( \gamma^m_k \) has condition \textbf{J} if
\[
D(\gamma^m_k) \leq \delta
\]
for some previously defined \( f \) and \( \delta > 0 \).

**Lemma 3.** Let \( f \in A(K, 1) \). For every \( \delta > 0 \) there exists a \( C_0 > 0 \) such that for every \( \gamma^n_j \) there exists \( \gamma^m_k \prec \gamma^n_j \) such that \( m \leq n + C_0 \delta^{-2} \) and such that \( \gamma^m_k \) has condition \textbf{J}.

**Proof.** Observe that by Harnack’s inequality
\[
\sup_{\gamma^m_{n,j}} |f'(z)|^2 \sim \int \int_{A^m_k} |f'|^2 \frac{dxdy}{\sigma_n^2}.
\]
Suppose the lemma is false. Choose \( \zeta \) with \( \text{dist}(\zeta, K) = 1 \). Then by Green’s theorem and the above observation
\[
4 \geq \int_{\partial \Omega_n} |f(z) - f(\zeta)|^2 d\omega_n(\zeta, z)
\]
\[
= \int_{\Omega_n} |f'(z)|^2 G_n(z, \zeta) dxdy
\]
\[
\geq \sum_{t=m+1}^n \sum_j \int_{A^t_j} |f'(z)|^2 G_n(z, \zeta) dxdy
\]
\[
\geq C_0 \delta^2 \sum_{t=m+1}^n \sum_j \omega(\zeta, S_{t,j} \cap K)
\]
\[
\geq C'(n - m) \delta^2
\]
and we have a contradiction.

3. A STOPPING-TIME ARGUMENT

We choose \( n_\delta = 4^M q \) where \( q > 1 \) and \( M = \left[ 1 + \frac{C_0}{\delta^2} \right] \). Then because \( p > 2 \) in the definition of \( a_n = (\log(n + 1))^{1/p} \) we have
\[
\lim_{\delta \to 0^+} \delta \cdot a_{n\delta} M = 0
\]
and
\[
\lim_{\delta \to 0^+} \left( 1 - 4^{-M} \right)^{n_\delta} a_{n\delta} M = 0.
\]

By construction, either \( \gamma^m_k \prec \gamma^n_j \), \( \gamma^n_j \prec \gamma^m_k \), or neither is subordinate to the other. We also write \( \gamma^m_k \not\prec F \) if \( \gamma^m_k \not\prec \gamma^n_j \) for all \( \gamma^n_j \in F \) where \( F \) is some family of \( \gamma^n_j \).
Lemma 4. For every $\varepsilon > 0$, there exists $\delta > 0$, integer $m$ and two families of sets $F_1$ and $F_2$, such that for some constant $c$:

(a) $F_1 = \{\gamma^n_j : \gamma^n_j \text{ has condition } J\}$,
(b) $\delta \Lambda_1(\bigcup F_1 \gamma^n_j) < c\varepsilon$,
(c) $F_2 = \{\gamma^m_k : \gamma^m_k \not\in F_1\}$,
(d) $\Lambda_1(\bigcup F_2 \gamma^m_k) < c\varepsilon$,
(e) $F_1 \cup F_2$ has winding number 1 about $K$.

Proof. Given $\varepsilon > 0$, choose $\delta > 0$ so that $\delta a_{n_3 M} < \varepsilon$ and $(1 - 4^{-M})n_4 a_{n_4 M} < \varepsilon$. Fix $m = n_5 M$.

Now define $F_1$ to be the set of $\gamma^n_n$ such that $n \leq m$, $\gamma^n_n$ has condition $J$, and $\gamma^n_n$ is maximal, i.e., if $K_{n,k} \subset K_{t,j}$ with $t < n$, then $\gamma^n_j$ does not have condition $J$. Then (a), (c) and (e) hold for $F_1$ and $F_2$.

To prove (b) consider $\gamma^n_j \in F_1$. Since $0 \leq n \leq m$ we may replace $\gamma^n_j$ by $4^{m-n}$ squares of the form $\gamma^m_k$. Consequently,

$$\Lambda_1(\gamma^n_j) \leq 4^{m-n} \sigma_m = 4^{-n}a_m.$$ 

Since the $\gamma^n_j \in F_1$ have pairwise disjoint $K_{n,j}$, $\bigcup F_1 \gamma^n_j$ has smaller $\Lambda_1$ measure than $\bigcup_{k=1}^m \gamma^m_k$ and therefore

$$\delta \Lambda_1(\bigcup_{F_1} \gamma^n_j) \leq \delta \Lambda_1(\bigcup_{k=1}^m \gamma^m_k) \leq c\delta \cdot 4^m \cdot 4^{-m}a_m = c\delta a_{n_3 M} \leq c\varepsilon,$$

where $c$ is a universal constant.

To prove (d) we use Lemma 3 to obtain

$$\Lambda_1(\bigcup_{F_2} \gamma^m_k) \leq c(4^M - 1)^m 4^{-m}a_m \leq c(1 - 4^{-M})n_4 a_{n_4 M} \leq c\varepsilon.$$

4. PROOF OF THEOREM 1

Suppose $f \in A(K,1)$ and $\varepsilon > 0$ are arbitrary. Let $F_1$ and $F_2$ be the two families provided by Lemma 4. Let $z^m_k$ be an arbitrary point in $\gamma^m_k$. Then

$$2\pi |f'(\infty)| = \left| \sum_{\gamma^n_j \in F_1} \int_{\gamma^n_j} f(z)dz + \sum_{\gamma^m_k \in F_2} \int_{\gamma^m_k} f(z)dz \right| \leq \left| \sum_{\gamma^n_j \in F_1} \int_{\gamma^n_j} f(z)dz \right| + \left| \sum_{\gamma^m_k \in F_2} \int_{\gamma^m_k} f(z)dz \right| \leq \sum_{\gamma^n_j \in F_1} \int_{\gamma^n_j} |f(z) - f(z^m_k)|dz + \Lambda_1(\bigcup F_2 \gamma^m_k)$$
\[ \leq c \sum_{\gamma_m^k \in F_1} \sup_{w \in \gamma_m^k} |f'(w)| 4^{-m} a_n \, dz + \varepsilon \]
\[ = c \sum_{\gamma_m^k \in F_1} D(\gamma_m^k) A_1(\gamma_m^k) + \varepsilon \]
\[ \leq c \delta \sum_{\gamma_m^k \in F_1} A_1(\gamma_m^k) + \varepsilon \]
\[ \leq c \delta a_{\alpha_5} M + \varepsilon \]
\[ \leq c \varepsilon. \]

Since \( \varepsilon \) was chosen arbitrarily and \( c \) is a universal constant, \( f'(\infty) = 0 \). Therefore, \( \gamma(K) = 0 \).

5. REMARK

We could obtain a better result if we could improve the estimate in Jones’ lemma (Lemma 3). For example, if we could only replace \( M = \frac{C_2}{\delta^2} \) by \( \frac{C_2}{\delta} \) for \( q < 2 \), then in the theorem \( a_n \) could grow like \((\log n)^{\frac{1}{2}}\). As noted above, Mattila [M96] conjectured that \( \gamma(K) = 0 \) if the Cantor set \( K \) has \( \sum \frac{1}{(\alpha_n)^q} = +\infty \). Mattilla’s conjecture would follow from the method here if the Jones’ lemma could be established with \( M = c \log(\frac{1}{\delta}) \) with \( c \) constant.

REFERENCES


