LARGE SETS OF ZERO ANALYTIC CAPACITY

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ABSTRACT. We prove that certain Cantor sets with non-sigma-finite one-dimensional Hausdorff measure have zero analytic capacity.

1. INTRODUCTION

In this paper we consider a Cantor set \( K \) similar to the \( \frac{1}{4} \)-Cantor set of [G70] and [I84]. Fix \( p > 2 \) and for \( n > 0 \) define

\[
\sigma_n = 4^{-n}a_n = 4^{-n}[\log(n + 1)]^{1/p}.
\]

Set \( K_0 = [0, 1] \times [0, 1] \) and \( K_1 = \bigcup_{j=1}^4 K_{1,j} \), where \( K_{1,j} \subset K_0 \) is a square of sidelength \( \sigma_1 \) having sides parallel to the axis and containing one of the four corners of \( K_0 \). Next take \( 4^2 \) squares \( K_{2,j} \) of sidelength \( \sigma_2 \), one in each corner of each square \( K_{1,j} \), and define \( K_2 = \bigcup_{j=1}^{4^2} K_{2,j} \). Continuing we obtain \( K_n = \bigcup_{j=1}^{4^n} K_{n,j} \), where \( K_{n,j} \) is a square of sidelength \( \sigma_n \). The Cantor set we study is

\[
K = K(p) = \bigcap_{n=1}^\infty K_n.
\]

If \( E \) is a compact plane set define

\[
A(E, 1) = \{f : f \text{ analytic on } E^c, \ f(\infty) = 0, \ |f|_{L^\infty(E^c)} \leq 1\}
\]

and define the analytic capacity of \( E \) by

\[
\gamma(E) = \sup\{|f'(\infty)| : f \in A(E, 1)\},
\]

where

\[
f'(\infty) = \lim_{z \to \infty} zf(z).
\]

If \( \gamma(E) = 0 \), then the only function in \( A(E, 1) \) is the constant \( f \equiv 0 \) and in this case \( E \) is removable for bounded analytic functions. For more details about analytic capacity see [G72].

Theorem 1. Let \( p > 2 \), and let \( K \) be the four-corner Cantor set \( K(p) \). Then \( \gamma(K) = 0 \) but \( K \) does not have \( \sigma \)-finite one-dimensional measure.
The proof of Theorem 1 depends on a lemma of Jones [J89] used for a proof different from [G70] that the $\frac{1}{4}$-Cantor set has zero analytic capacity.

Let $h(t)$ be an increasing continuous function on $t \geq 0$ with $h(0) = 0$, and write $\Lambda_h(t)(E)$ for the Hausdorff $h$-measure of $E$. Now define an increasing function $h(t)$ so that $h(0) = 0$ and $h(\sigma_n) = 4^{-n}$ for all $n$. We say $h(t)$ is a measure function corresponding to the Cantor set $K$. For every $n$ define a measure $\mu_n$ on $K_n$ by $\mu_n(K_{n,j}) = 4^{-n}$ for all $j$. Then $\{\mu_n\}$ converges weak-star to a measure $\mu$ supported on $K$ and satisfying $\mu(K_{n,j}) = 4^{-n}$. Suppose $\sqrt[3]{2}\sigma_n \leq r < \sqrt[3]{2}\sigma_{n-1}$ and let $D(z,r)$ be a disk of radius $r$ and center $z \in K$. Then $D(z, r)$ can meet at most 4 squares of sidelength $\sigma_n$. Hence

$$\mu(D(z, r)) \leq 4\mu(K_{n,j}) = 4 \cdot 4^{-n} = 4h(\sigma_n) \leq 4h(r),$$

so that $\mu(D(z, r)) \leq 16h(r)$ for any disk $D(z, r)$. Therefore $\Lambda_h(K) > 0$ by Frostman’s Theorem [G72]. Since

$$\lim_{t \to 0} \frac{h(t)}{t} = 0,$$

if follows that $K$ has non-$\sigma$-finite 1-dimensional measure.

If $h(t)$ is a measure function corresponding to $K$, then

$$\int_0^1 \frac{h(t)^2}{t^3} \, dt \sim \sum_{n=1}^{\infty} \frac{1}{(a_n)^2} = \sum_{n=1}^{\infty} \frac{1}{(\log n)^2} = \infty.$$  

On the other hand, Mattila [M96] proved that $\gamma(K) > 0$ if $K$ is a Cantor set built with squares of side $\sigma_n$ and if

$$\int_0^1 \frac{h(t)^2}{t^3} \, dt < \infty,$$

where $h$ is any measure function for corresponding to $K$. Mattila’s proof used Menger curvature (see [Me95] and [MMV96]). However, if the Cantor set $K$ has corresponding measure function $h$ satisfying

$$\int_0^1 \frac{h(t)^2}{t^3} \, dt = \infty,$$

then Eiderman [E98] proved that $\gamma^+(K) = 0$, where

$$\gamma^+(E) = \sup \left\{ \int_E d\mu : \int_E d\mu(\zeta) < 1, \forall z \in \mathbb{C}\backslash E, \mu > 0, \text{spt}(\mu) \subset E \right\}. $$

Since $\gamma^+(E) \leq \gamma(E)$, our result is a partial improvement of Eiderman’s theorem. Mattila [M96] has conjectured that for Cantor sets of this type $\gamma(K) = 0$ if and only if

$$\int_0^1 \frac{h(t)^2}{t^3} \, dt = \infty,$$

when $h$ corresponds to $K$. This latter condition holds if and only if

$$\sum_{n=1}^{\infty} \frac{1}{(a_n)^2} = \infty.$$

If Matilla’s conjecture is true, then together with Eiderman’s theorem it gives Cantor set evidence supporting the more ambitious conjecture that $\gamma(E) > 0$ implies $\gamma^+(E) > 0$. 

In [G72] it was incorrectly claimed that $\gamma(K) > 0$ if and only if
\[
\int_0^1 \frac{h(t)}{t^2} dt < \infty.
\]
Eiderman, however, found a mistake in the proof. In fact the result in [M96] shows that the claim is false. See L.D. Ivanov [I84] for the first example of a Cantor set of non-$\sigma$-finite linear measure and zero analytic capacity.

2. TWO LEMMAS OF PETER JONES

We need the following two lemmas from [J89]. The proofs we give are small variations on [J89] and [C90].

Define $\gamma_j^n = \partial cK_{n,j}$, where $cK_{n,j}$ is the square concentric to $K_{n,j}$ with sidelength $c\sigma_n$ and where $c > 1$ is chosen so that the $\gamma_j^n$ do not overlap. We refer to $\gamma_k^m$ as a square, although it is only the boundary of a square. Notice that
\[
\Lambda_1(\gamma_k^m) = c\Lambda_1(\partial K_{m,k})
\]
for the same constant $c$. We associate to each $\gamma_k^m$ a “square annulus”
\[
A_k^m = \{ w : \text{dist}(w, \gamma_k^m) \leq \varepsilon_0 \sigma_m \}
\]
and we choose $\varepsilon_0 > 0$ so small that the annuli $A_k^m$ are pairwise disjoint.

Define $\Omega = \overline{\mathbb{C}} \setminus K$. Since $K$ has positive logarithmic capacity, Green’s function $G(z, \zeta)$ exists for $\zeta, z \notin K$, and harmonic measure $\omega(\zeta, E)$ exists for $\zeta \notin K$ and $E \subseteq K$. We write $\omega(\zeta, K_{m,k})$ for $\omega(\zeta, K_{m,k} \cap K)$.

We also define the slightly larger “squares”
\[
S_{m,k} = \{ w : \text{dist}(w, K_{m,k}) \leq \varepsilon_1 \sigma_m \}
\]
and set
\[
S_m = \bigcup_{k=1}^{4^m} S_{m,k},
\]
where $\varepsilon_1 > 0$ is so small that $S_{m,k} \cap A_k^m = \emptyset$. Then $K = \bigcap_{m=1}^{\infty} S_m$. Green’s function and harmonic measure also exist for the domain $\Omega_m = \overline{\mathbb{C}} \setminus S_m$. Denote these by $G_m(z, \zeta)$ and $\omega_m(\zeta, E)$ respectively.

**Lemma 2.** Let $z \in A_k^m$.

(a) There are constants $0 < c_1 < c_2 < 1$, independent of $k$ and $m$, such that
\[
c_1 \leq \omega_m(z, \partial S_{m,k}) \leq c_2.
\]

(b) If $\zeta \in \Omega$ and $1 \geq \text{dist}(\zeta, K) \geq 2 \text{dist}(z, K)$, then
\[
G_m(z, \zeta) \sim \omega_m(\zeta, \partial S_{m,k}).
\]

**Proof.** For (a) note that there is $c' > 0$ such that there exists a second square annulus $B_k^m$ so that $A_k^m \subseteq B_k^m \subseteq \Omega_m$ and $\text{dist}(z, \partial B_k^m) \geq c' \sigma_m$. The lower bound then follows by a comparison with $B_k^m$. There is $S_{m,j}$ with $j \neq k$ such that $\text{dist}(S_{m,j}, S_{m,k}) \leq c_4 \sigma_m$ and the upper bound follows by a comparison with $\overline{\mathbb{C}} \setminus (S_{m,k} \cup S_{m,j})$, using symmetry and Harnack’s inequality.

To prove (b) note first that as in the proof of (a) there are constants $C_1$ and $C_2$ such that by Harnack’s inequality and a comparison
\[
C_1 \leq G_m(z, w) \leq C_2
\]
for \( w \in \partial B_k^m \). Then using the symmetry of Green’s function and (a) for a larger square we obtain

\[
C_1 \omega_m(\zeta, \partial S_{m,k}) \leq G_m(\zeta, z) \leq C_2 \omega_m(\zeta, \partial S_{m,k}).
\]

We write \( \gamma_k^m < \gamma_j^n \) and say \( \gamma_k^m \) is \textbf{subordinate} to \( \gamma_j^n \) if \( \gamma_j^n \) has winding number one about \( \gamma_k^m \). If the winding number is zero, we write \( \gamma_k^m \not< \gamma_j^n \). For any \( f \in A(K, 1) \) and \( \gamma_k^m \) define

\[
D(\gamma_k^m) = \sup_{w \in \gamma_k^m} |f'(w)| \sigma_n.
\]

We say a square \( \gamma_k^m \) has condition \textbf{J} if

\[
D(\gamma_k^m) \leq \delta
\]

for some previously defined \( f \) and \( \delta > 0 \).

\begin{lemma}
Let \( f \in A(K, 1) \). For every \( \delta > 0 \) there exists a \( C_0 > 0 \) such that for every \( \gamma_j^n \) there exists \( \gamma_k^m < \gamma_j^n \) such that \( m \leq n + C_0 \delta^{-2} \) and such that \( \gamma_k^m \) has condition \textbf{J}.
\end{lemma}

\textbf{Proof.} Observe that by Harnack’s inequality

\[
\sup_{\gamma_{n,j}} |f'(z)|^2 \sim \int \int_{A_k^n} |f'|^2 \frac{dz \, dy}{\sigma_n^2}.
\]

Suppose the lemma is false. Choose \( \zeta \) with \( \text{dist}(\zeta, K) = 1 \). Then by Green’s theorem and the above observation

\[
4 \geq \int_{\partial \Omega_n} |f(z) - f(\zeta)|^2 d\omega_n(\zeta, z)
\]

\[
= \int_{\Omega_n} |f'(z)|^2 G_n(z, \zeta) \, dz \, dy
\]

\[
\geq \sum_{t=m+1}^{n} \sum_{j} \int_{A_j^n} |f'(z)|^2 G_n(z, \zeta) \, dz \, dy
\]

\[
\geq C \delta^2 \sum_{t=m+1}^{n} \sum_{j} \omega(\zeta, S_{t,j} \cap K)
\]

\[
\geq C'(n-m) \delta^2
\]

and we have a contradiction.

3. A \textbf{STOPPING-TIME ARGUMENT}

We choose \( n_\delta = 4^{Mq} \) where \( q > 1 \) and \( M = \left[ 1 + \frac{C_0}{\delta^2} \right] \). Then because \( \nu > 2 \) in the definition of \( a_n = (\log(n+1))^{\frac{1}{\nu}} \) we have

\[
\lim_{\delta \to 0^+} \delta \cdot a_{n_\delta M} = 0
\]

and

\[
\lim_{\delta \to 0^+} \left( 1 - 4^{-M} \right)^{n_\delta} a_{n_\delta M} = 0.
\]

By construction, either \( \gamma_k^m < \gamma_j^n \), \( \gamma_j^n < \gamma_k^m \), or neither is subordinate to the other. We also write \( \gamma_k^m \not< F \) if \( \gamma_k^m \not< \gamma_j^n \) for all \( \gamma_j^n \in F \) where \( F \) is some family of \( \gamma_j^n \).
Lemma 4. For every $\varepsilon > 0$, there exists $\delta > 0$, integer $m$ and two families of sets $F_1$ and $F_2$, such that for some constant $c$:

(a) $F_1 = \{ \gamma^n_j : \gamma^n_j \text{ has condition } J \}$,
(b) $\delta \Lambda_1(\bigcup_{F_1} \gamma^n_j) < c\varepsilon$,
(c) $F_2 = \{ \gamma^n_k : \gamma^n_k \notin F_1 \}$,
(d) $\Lambda_1(\bigcup_{F_2} \gamma^n_k) < c\varepsilon$,
(e) $F_1 \cup F_2$ has winding number 1 about $K$.

Proof. Given $\varepsilon > 0$, choose $\delta > 0$ so that $\delta a_{n\delta M} < \varepsilon$ and $(1 - 4^{-M}) a_{n\delta M} < \varepsilon$. Fix $m = n\delta M$.

Now define $F_1$ to be the set of $\gamma^n_k$ such that $n \leq m$, $\gamma^n_k$ has condition $J$, and $\gamma^n_k$ is maximal, i.e. if $K_{n,k} \subset K_{t,j}$ with $t < n$, then $\gamma^n_j$ does not have condition $J$. Then (a), (c) and (e) hold for $F_1$ and $F_2$.

To prove (b) consider $\gamma^n_j \in F_1$. Since $0 \leq n \leq m$ we may replace $\gamma^n_j$ by $4^{m-n}$ squares of the form $\gamma^n_k$. Consequently,

$$\Lambda_1(\gamma^n_j) \leq 4^{m-n} \cdot \sigma_m = 4^{-n} a_m.$$

Since the $\gamma^n_j \in F_1$ have pairwise disjoint $K_{n,j}$, $\bigcup_{F_1} \gamma^n_j$ has smaller $\Lambda_1$ measure than $\bigcup_{k=1}^k \gamma_k^m$ and therefore

$$\delta \Lambda_1(\bigcup_{F_1} \gamma^n_j) \leq \delta \Lambda_1(\bigcup_{k=1}^m \gamma_k^m) \leq c\delta \cdot 4^m \cdot 4^{-m} a_m = c\delta a_{n\delta M} \leq c\varepsilon,$$

where $c$ is a universal constant.

To prove (d) we use Lemma 3 to obtain

$$\Lambda_1(\bigcup_{F_2} \gamma_k^m) \leq c(4^M - 1)^m 4^{-m} a_m \leq c(1 - 4^{-M}) a_{n\delta M} \leq c\varepsilon.$$

4. Proof of Theorem 1

Suppose $f \in A(K,1)$ and $\varepsilon > 0$ are arbitrary. Let $F_1$ and $F_2$ be the two families provided by Lemma 4. Let $z_k^m$ be an arbitrary point in $\gamma_k^m$. Then

$$2\pi |f'(\infty)| = \left| \sum_{\gamma_k^m \in F_1} \int_{\gamma_k^m} f(z)dz + \sum_{\gamma_k^m \in F_2} \int_{\gamma_k^m} f(z)dz \right| \leq \left| \sum_{\gamma_k^m \in F_1} \int_{\gamma_k^m} f(z)dz \right| + \left| \sum_{\gamma_k^m \in F_2} \int_{\gamma_k^m} f(z)dz \right| \leq \sum_{\gamma_k^m \in F_1} \int_{\gamma_k^m} |f(z) - f(z_k^m)|dz + \Lambda_1(\bigcup_{F_2} \gamma_k^m)$$
\[ \leq c \sum_{\gamma_k^m \in F_1} \int_{\gamma_k^m} \sup_{w \in \gamma_k^m} |f'(w)| \cdot 4^{-m} a_n \, dz + \varepsilon \]

\[ = c \sum_{\gamma_k^m \in F_1} D(\gamma_k^m) \Lambda_1(\gamma_k^m) + \varepsilon \]

\[ \leq c \delta \sum_{\gamma_k^m \in F_1} \Lambda_1(\gamma_k^m) + \varepsilon \]

\[ \leq c \delta \delta n^s M + \varepsilon \]

\[ \leq c \varepsilon. \]

Since \( \varepsilon \) was chosen arbitrarily and \( c \) is a universal constant, \( f'(\infty) = 0 \). Therefore, \( \gamma(K) = 0. \)

\[ \square \]

5. Remark

We could obtain a better result if we could improve the estimate in Jones’ lemma (Lemma 3). For example, if we could only replace \( M = \frac{C_\delta}{\delta^2} \) by \( \frac{C_\delta}{\delta^2} \) for \( q < 2 \), then in the theorem \( a_n \) could grow like \( (\log n)^{1/2} \). As noted above, Mattila [M96] conjectured that \( \gamma(K) = 0 \) if the Cantor set \( K \) has \( \sum (a_n)^{1/2} = +\infty \). Mattilla’s conjecture would follow from the method here if the Jones’ lemma could be established with \( M = c \log(\frac{1}{\delta}) \) with \( c \) constant.

References


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