
Garland Durham\textsuperscript{a,b,*}, John Geweke\textsuperscript{c,d,e}, Pulak Ghosh\textsuperscript{f}

\textsuperscript{a} Quantos Analytics, LLC, USA  
\textsuperscript{b} California Polytechnic State University, USA  
\textsuperscript{c} University of Technology Sydney, Australia  
\textsuperscript{d} Erasmus University Rotterdam, Netherlands  
\textsuperscript{e} Colorado State University, USA  
\textsuperscript{f} Indian Institute of Management, Bangalore, India

Abstract

Christoffersen, Jacobs, and Ornthanalai (2012) (CJO) propose an interesting and useful class of generalized autoregressive conditional heteroskedasticity (GARCH)-like models with dynamic jump intensity, and find evidence that the models not only fit returns data better than some commonly used benchmarks but also provide substantial improvements in option pricing performance. While such models pose difficulties for estimation and analysis, CJO propose an innovative approach to filtering intended to addresses them. However, some statistical issues arise that their approach leaves unresolved, with implications for the option pricing results. This note proposes a solution based on using the filter and estimator proposed by CJO but interpreted in the context of an alternative model. With respect to this model, the estimator is consistent, and likelihood-based model comparisons and hypothesis tests are valid.

1. Introduction

Much interest exists in models for asset returns that include dynamic jump intensity, going back to seminal work by Chan and Maheu (2002) and Maheu and McCurdy (2004). In more recent work, Rangel (2011) examines the effects of news events on jump intensity, and Christoffersen, Jacobs, and Ornthanalai (2012) (CJO) and Santa-Clara and Yan (2010) find that dynamic jump intensity plays an important role in option pricing. In a somewhat different vein, Wright and Zhou (2009) find that evidence extracted from high-frequency stock index returns supports the premise of time variation in jump mean, variance, and intensity and that jump variance (but not intensity) has strong predictibility for excess bond returns. Aït-Sahalia, Cacho-Diaz and Laeven (2013) suggest an innovative modeling framework with origins in epidemiology to explain the presence of time-varying jump intensity based on mutually exciting jump processes (Hawkes processes).

CJO propose an interesting and useful class of generalized autoregressive conditional heteroskedasticity (GARCH)-like models with dynamic jump intensity. They find evidence that the models not only fit returns data better than some commonly used benchmarks but also provide substantial improvements in option pricing performance. In the model of primary interest in that paper, the returns process is driven
by two dynamic state variables: one is closely related to standard GARCH volatility factors, and the other generates time-varying jump intensities. We refer to this as the GARCH-DJI model.

While such models pose difficulties for estimation and analysis, CJO propose an innovative approach to filtering that addresses these issues. However, the filter they propose lacks a key property. That is, the filtered states are not equal to their expected values conditional on the relevant information set. Furthermore, while CJO refer to the estimator they propose as a maximum likelihood estimator (MLE), it is not the MLE for the GARCH-DJI model they study.

A model, however, can be constructed that represents a well-defined data generating process based on the CJO filtering algorithm. We refer to this as the FILTER-DJI model. For this model, the states are trivially identified. And it is this model for which the estimator proposed by CJO is in fact the MLE.

In this note, we investigate some characteristics of GARCH-DJI and FILTER-DJI models. We find that for a given parameter vector the two classes of models represent similar data generating processes, but some clear differences exist. When applied to simulated data generated from a GARCH-DJI model, we show that the filter proposed by CJO is biased. The estimator they propose is also biased, and hypothesis tests have incorrect size. In a Monte Carlo study, we find that the coverage ratio of a nominal 95% confidence region for the model parameter vector is only 39%. Furthermore, the log likelihood values reported by CJO and used for model comparisons are not valid.

With respect to the FILTER-DJI model, in contrast, the estimator maintains the usual attractive properties of maximum likelihood estimation. In particular, when interpreted in the context of this model (instead of the GARCH-DJI model proposed by CJO), parameter estimates (including those reported by CJO) are consistent, hypothesis tests are correctly sized, and model comparison results are valid.

There are implications for option pricing. The approach proposed by CJO involves simulating option prices conditional on an estimated parameter vector and filtered states. While the estimator is consistent for the parameters of the FILTER-DJI model and the filter is unbiased (conditional on a parameter vector), neither of these properties holds for GARCH-DJI, the model under which option prices are simulated. Option prices implied by the two models are close at short times to maturity, but they diverge as time to maturity increases.

A more fundamental issue underlying the one involved here is a subtle but important one for economists working with state space models. Even if a filter that provides unbiased estimates for the states is available (a condition not satisfied by the GARCH-DJI model), treating the extracted states as known and using them for maximum likelihood estimation will not in general yield a consistent estimator or correct values for the log likelihood. For valid statistical analysis, either the state uncertainty must be integrated out or the model reformulated in such a way as to eliminate it. (In special cases, such as the linear Gaussian state space model, the integral may be computable analytically.) This point has been noted in a related setting by Fleming and Kirby (2003).

Because the class of models proposed by CJO, and others like it, have properties that are of considerable interest for applied work, it is important that subsequent work have a solid theoretical foundation to build upon. The FILTER-DJI model proposed here provides one possible workaround for the estimation issues left unresolved by CJO. While the estimation results and model comparisons reported by CJO are not valid in the context of the GARCH-DJI model, they are valid if interpreted in the context of the FILTER-DJI model. This note thus provides a constructive solution that reaffirms the usefulness of CJO’s empirical findings and helps open the way for further research building on their work.

### 2. Two dynamic jump intensity models

The GARCH-DJI model proposed by CJO is given by

\[ R_t = \mu_t + z_t + y_t \]

and

\[ \mu_t = \alpha_t + \left( \lambda_z - \frac{1}{2} \right) h_{zt} + (\lambda_y - \xi) y_t \]

where \( R_t \) is a log return, \( \lambda_z \) and \( \lambda_y \) are risk premia, \( r_t \) is the risk-free rate, \( z_t \sim N(0, h_{zt}) \), and \( y_t \) is a Poisson jump process with intensity \( h_{yt} \), mean \( \mu_y \), and variance \( \sigma_y^2 \). The dynamics of variance (\( h_{zt} \)) and jump intensity (\( h_{yt} \)) are given by

\[ h_{zt,t+1} = w_z + b_z h_{zt} + \frac{a_z}{h_{zt}} (z_t - c_z h_{zt})^2 + d_z (y_t - e_z)^2 \]

and

\[ h_{yt,t+1} = w_y + b_y h_{yt} + \frac{a_y}{h_{yt}} (z_t - c_y h_{zt})^2 + d_y (y_t - e_y)^2 \]

with initial conditions \( h_{zt,0} \) and \( h_{yt,0} \). The terms \( h_{zt,2} \) and \( \xi \) in \( h_{zt} \) in Eq. (2) are convexity adjustments.

The full model has parameter vector \( \theta = (\lambda_z, \lambda_y, \mu_y, \sigma_y, a_z, b_z, c_z, d_z, e_z, w_z, a_y, b_y, c_y, d_y, e_y, w_y) \).

Let \( n_t \) denote the number of jumps at time \( t \). Then, \( (R_t | R_{t-1}, h_{zt}, h_{yt}, n_t) \sim N(\mu_t + n_t \mu_y, h_{zt} + n_t \sigma_y^2) \) and \( (n_t | h_{yt}) \sim \text{Poisson} (h_{yt}) \). Integrating across \( n_t \), \( R_t | R_{t-1}, h_{zt}, h_{yt} \) is a mixture of normals with density

\[ p(R_t | R_{t-1}, h_{zt}, h_{yt}) = \sum_{j=0}^{\infty} p(j | h_{yt}) \phi(R_t | \mu_t + j \mu_y, h_{zt} + j \sigma_y^2) \]

where \( p(j | h_{yt}) \) is the Poisson(h_{yt}) density and \( \phi \) is the Gaussian density. For future reference, note that

\[ p(n_t | R_t, R_{t-1}, h_{zt}, h_{yt}) \propto p(R_t | R_{t-1}, h_{zt}, h_{yt}, n_t) p(n_t | h_{yt}) \]

by Bayes’ rule.

If \( n_t, z_t, y_t, h_{zt} \) and \( h_{yt} \) are not observable. To estimate the model, CJO propose the filter:

\[ \tilde{\mu}_t = \alpha_t + \left( \lambda_z - \frac{1}{2} \right) \tilde{h}_{zt} + (\lambda_y - \xi) \tilde{h}_{yt} \]

\[ \tilde{z}_t = \sum_{j=0}^{\infty} \frac{\tilde{h}_{zt}}{h_{zt} + j \sigma_y^2} (R_t - \tilde{\mu}_t - j \mu_y) p(n_t = j | R_t, R_{t-1}, \tilde{h}_{zt}, \tilde{h}_{yt}) \]

\[ \tilde{y}_t = R_t - \tilde{\mu}_t - \tilde{z}_t \]
\[ h_{zt+1} = w_2 + b_y h_{yt} + \frac{\varphi}{h_{zt}} (\tilde{z}_t - c_y h_{zt})^2 + d_y (\tilde{y}_t - e_y)^2 \]  
(10)

and

\[ h_{yt+1} = w_y + b_y h_{yt} + \frac{\varphi}{h_{zt}} (\tilde{z}_t - c_y h_{zt})^2 + d_y (\tilde{y}_t - e_y)^2, \]  
(11)

with initial conditions \( h_{zt0} \) and \( h_{yt0} \). CJO show that Eqs. (1)–(2) with \( h_{zt} = \hat{h}_{zt} \) and \( h_{yt} = \hat{h}_{yt} \) imply that \( \tilde{z}_t = E(z_t | r_t, R_t, \hat{h}_{zt}, \hat{h}_{yt}) \) and \( \tilde{y}_t = E(y_t | r_t, r_{zt-1}, \hat{h}_{zt}, \hat{h}_{yt}) \).

Using this filter, it is straightforward to back out implied values of \( \hat{h}_{zt} \) and \( \hat{h}_{yt} \) conditional on data \( (R_t, r_{zt-1})\), parameter vector \( \theta \), and initial conditions \( \hat{h}_{zt0} \) and \( \hat{h}_{yt0} \). The parameter vector is then estimated by optimization,

\[ \hat{\theta} = \text{argmax}_{\theta \in \Theta} \sum_{t=1}^{T} p(R_t | \hat{h}_{zt}, \hat{h}_{yt}, \theta), \]  
(12)

where the summands are given by Eq. (5) but with \( \hat{h}_{zt} \) and \( \hat{h}_{yt} \) in place of \( h_{zt} \) and \( h_{yt} \). CJO refer to this as a maximum likelihood estimator Eq. (12) with respect to the parameters of the GARCH-DJI model (1)–(4) but instead for the model defined by (7)–(11) in conjunction with

\[ R_t = \mu_t + \epsilon_t, \]  
(13)

where \( \epsilon_t \) is a mixture of normals with density \( \sum_{j=1}^{\infty} p(j; \hat{h}_{yt}, \hat{h}_{zt}, -j^2) \) and \( p(j; \hat{h}_{yt}) \) is the Poisson \( (\hat{h}_{yt}) \) density. This model, which we refer to as FILTER-DJI, is motivated by GARCH-DJI but not equivalent to it.

While CJO show that \( \tilde{z}_t = E(z_t | r_t, R_t, \hat{h}_{zt}, \hat{h}_{yt}) \) and \( \tilde{y}_t = E(y_t | R_t, r_{zt-1}, \hat{h}_{zt}, \hat{h}_{yt}) \), it does not follow that \( \hat{h}_{zt} = E(h_{zt} | F_{t-1}) \) or that \( \hat{h}_{yt} = E(h_{yt} | F_{t-1}) \), where \( F_t \) is the \( \sigma \)-algebra generated by \( (R_t, r_{zt-1}) \). So Eqs. (7)–(10) does not possess a key feature typically desired of a filter. And while Eq. (12) is the MLE for FILTER-DJI, it is not the MLE for GARCH-DJI. Furthermore, even if the state \( \hat{h}_{zt} \) were unbiased, (12) would not be the MLE for GARCH-DJI. Computing the likelihood for this model requires integrating across state uncertainty. A plug-in estimate of the states, even an unbiased one, is not sufficient. The properties of this estimator with respect to the parameters of the GARCH-DJI model are unknown.

3. Findings

We performed a number of experiments to assess the extent of bias in state and parameter estimates associated with using the CJO filter. Some of the results are reported here.

3.1. Comparison of true and filtered states

This subsection investigates the extent to which the filtered states \( \hat{h}_{zt} \) and \( \hat{h}_{yt} \) are informative about the true states \( h_{zt} \) and \( h_{yt} \). The experiment performed here uses simulated data generated using the GARCH-DJI model with the parameter vector reported by CJO in Table 1 of that paper. We simulate one million observations with a burn-in period of one thousand observations to minimize the effects of initial conditions. The filter (7)–(11) is then applied to the simulated returns to extract filtered states \( \hat{h}_{zt} \) and \( \hat{h}_{yt} \) and innovations \( \tilde{z}_t \) and \( \tilde{y}_t \). Because the data are simulated, the true states \( h_{zt} \) and \( h_{yt} \) and innovations \( z_t \) and \( y_t \), which are latent in empirical applications, can be observed. Thus, comparison of the true and filtered states and innovations is possible.

Table 1 reports summary statistics for the true and filtered states and innovations. Relative to the true states, the filtered states \( \hat{h}_{zt} \) and \( \hat{h}_{yt} \) are systematically biased downward, have smaller standard deviation, and are less skewed and less leptokurtic.

3.2. Comparison of true and estimated parameters

This subsection reports the results of a Monte Carlo study investigating the issue of potential bias in the estimator Eq. (12) with respect to the parameters of the GARCH-DJI model. Each replication in the study involves generating \( N = 11,979 \) observations of simulated data (equal to the sample size used in the application provided by CJO) using the GARCH-DJI model with the parameters from Table 1 of CJO (as described above) and then estimating the model using Eq. (12) (as proposed by CJO). We perform one thousand replications and report the bias and root mean square error of the resulting parameter estimates. For comparison, we then repeat this procedure using data generated from the FILTER-DJI model with the same parameter vector. In this case, Eq. (12) represents the true MLE.

In the full model, some of the parameters are difficult to pin down accurately, with correlations implied by

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Summary statistics.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>True</td>
</tr>
<tr>
<td>GARCH state ((h_j))</td>
<td>Mean ((1,000))</td>
</tr>
<tr>
<td></td>
<td>SD ((1,000))</td>
</tr>
<tr>
<td></td>
<td>Skewness</td>
</tr>
<tr>
<td></td>
<td>Kurtosis</td>
</tr>
<tr>
<td>Jump intensity ((h_j))</td>
<td>Mean ((1,000))</td>
</tr>
<tr>
<td></td>
<td>SD ((1,000))</td>
</tr>
<tr>
<td></td>
<td>Skewness</td>
</tr>
<tr>
<td></td>
<td>Kurtosis</td>
</tr>
<tr>
<td>GARCH innovation ((z))</td>
<td>Mean ((1,000))</td>
</tr>
<tr>
<td></td>
<td>SD ((1,000))</td>
</tr>
<tr>
<td></td>
<td>Skewness</td>
</tr>
<tr>
<td></td>
<td>Kurtosis</td>
</tr>
<tr>
<td>Jump innovation ((y))</td>
<td>Mean ((1,000))</td>
</tr>
<tr>
<td></td>
<td>SD ((1,000))</td>
</tr>
<tr>
<td></td>
<td>Skewness</td>
</tr>
<tr>
<td></td>
<td>Kurtosis</td>
</tr>
</tbody>
</table>
estimates of the asymptotic covariance matrix as high as 0.995. This is analogous to the situation of near-multicollinearity in linear regression and results in very large standard errors. In addition to being very flat in some dimensions, the likelihood surface has multiple local maxima, making global optimization problematic. For the parameters of the FILTER-DJI model, the estimator shows evidence of bias with a magnitude equal to about one standard error for several of the parameters, notably $\alpha_2$, $\sigma_1$, $\mu_1$, and $\gamma$. No evidence of bias is apparent when the estimator is applied to data generated from the FILTER-DJI model.

The columns of the table labeled “CR” report the coverage rate of the nominal 95% confidence interval for each parameter. Confidence intervals for all but the first four parameters are very narrow and are contained in the interval [0,1]. For the first four parameters, the nominal 95% confidence interval for each parameter contains the true value about 95% of the time. The columns labeled “Bias” report the coverage rate for the confidence region constructed by the estimator. The columns labeled “SD” report the standard deviation for each parameter. The columns labeled “RMSE” report the root mean square error of the estimator. The columns labeled “CR” report the coverage rate of the nominal 95% confidence interval for each parameter. SD = standard deviation. For GARCH-DJI, coverage ratio for 95% confidence region centered at $\theta_0$: 94.9%. For FILTER-DJI, coverage ratio for 95% confidence region centered at $\theta_0$: 94.9%.

4. Conclusions

CJO provide evidence that including dynamic jump intensity provides a significant improvement in fitting stock returns relative to some models without that feature. They also provide evidence that the models they propose perform better than some alternatives in explaining option prices. Although the GARCH-DJI model is the primary object of interest in that paper, the estimator that CJO propose is the MLE for a different model, which we refer to as FILTER-DJI. The estimator is not consistent with respect to the parameters of the GARCH-DJI model, the associated filter is biased, and implied log likelihood values (and corresponding model comparisons) are invalid. These issues are resolved by the FILTER-DJI model proposed here.

References