THE CONSTRUCTION OF KHOVANOV HOMOLOGY

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Knot theory is a rich topic in topology that studies the how circles can be embedded in Euclidean 3-space. One of the main questions in knot theory is how to distinguish between different types of knots efficiently. One way to approach this problem is to study knot invariants, which are properties of knots that do not change under a standard set of deformations. We give a brief overview of basic knot theory, and examine a specific knot invariant known as Khovanov homology. Khovanov homology is a homological invariant that refines the Jones polynomial, another knot invariant that assigns a Laurent polynomial to a knot. Dror Bar-Natan wrote a paper in 2002 that explains the construction of Khovanov homology and proves that it is an invariant. We follow his lead and attempt to clarify and explain his formulation in more precise detail.
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Chapter 1

INTRODUCTION

1.1 Knots

To define a knot, we first define an embedding.

**Definition 1.1.1.** Let $X, Y$ be topological spaces. An embedd*ing* of $X$ into $Y$ is a continuous function $f: X \to Y$ such that $f$ yields a homeomorphism from $X$ onto $f(X)$.

Now suppose that $X, Y$ are smooth manifolds. A smooth embedding is an embedding which is infinitely differentiable.

Note that in particular, an embedding $f$ is injective. An embedding is a way to view a given topological space $X$ as a subspace of an ambient space $Y$. Depending on the specific embedding, $X$ may take on various properties in relation to $Y$ that can be more complex than just taking $X$ as an object.

**Definition 1.1.2.** Fix a parameterization $\gamma$ of $S^1 \subseteq \mathbb{C}$ by

$$
\gamma: [0, 1] \to S^1 \\
x \mapsto e^{2\pi ix}
$$

An (oriented) knot is a smooth embedding $k: S^1 \to \mathbb{R}^3$. The orientation of the knot is the direction induced by the parameterization of $S^1$ [4].
Put in plain terms, a knot is a circle $S^1$, viewed as living in Euclidean 3-space. On its own, $S^1$ has a very simple topology. However, when we embed it into $\mathbb{R}^3$, we may introduce any number of twists and turns that cause the complement of the knot to have complex topological properties, and that is what makes knots interesting. It is standard to abuse terminology and conflate a knot (a type of map) with its image (a set in $\mathbb{R}^3$). Figure 1.1 provides a few examples of knots.

The orientation of a knot can be thought of as a choice of direction of travel along the knot, and is often denoted by an arrow indicating direction of travel. This choice is defined for us by the way the underlying space $S^1$ was oriented. If $k$ is a knot, then we may define a knot $k'$ with opposite orientation, which has $\bar{\gamma}(x) = e^{-2\pi i x}$ as the parameterization (see figure 1.2). That is, $k = k' \circ \bar{\gamma}$.

The reason we specify that knots must be smooth embeddings is to avoid cases such as in figure 1.3.

Knots with these non-smooth behavior at one or more points are known as wild knots, and the non-smooth behavior at these points causes many theorems to be inapplicable. Smoothness of the embedding prevents this from occurring; we may think of smoothness as being able to “thicken” the knot into a smoothly embedded
solid torus. With this condition, we can happily talk about knots, confident that they behave as we might expect a physical string would.

We may also talk about links.

**Definition 1.1.3.** A link is a smooth embedding $L$ of a disjoint union of copies of $S^1$ into $\mathbb{R}^3$. That is, $L: \bigvee_{i}^{n} S^1_{i} \to \mathbb{R}^3$ for some $n$. We call the restriction of $L$ to a single $S^1$ a link component.

In other words, a link is multiple knots considered together. All of the ideas we have introduced so far apply to links as well, and we will note where definitions and theorems differ for links. Note that any knot is also a link, with $n = 1$. For this reason we will often switch between link and knot interchangeably.
The first question one naturally asks is, “When are two knots the same?” For example, introducing a single twist to an unknot to make it into a figure eight does not meaningfully change the properties of the knot. The ability to deform a knot into another is a sensible way to think about equivalent knots. To formalize this deformation, we introduce the notion of an isotopy

Definition 1.1.4. Let \( f, g : X \to Y \) be smooth embeddings.

1. \( f \) and \( g \) are smoothly isotopic if there is a smooth map

\[
H : X \times [0, 1] \to Y
\]

such that for all \( t \in [0, 1] \), \( H(x, t) \) is a smooth embedding, and for all \( x \in X \), \( H(x, 0) = f(x) \), and \( H(x, 1) = g(x) \). We call \( H \) a smooth isotopy.

2. \( f \) and \( g \) are smoothly ambiently isotopic if there is a smooth isotopy

\[
H : Y \times [0, 1] \to Y, \quad H(y, t) = h_t(y),
\]

with \( h_0 = id_Y \) and \( h_1 \circ f = g \). \( H \) is called a smooth ambient isotopy.

Definition 1.1.5. Two knots are equivalent if they are smoothly ambiently isotopic.
An isotopy $H$ is a continuous transformation from an embedding $f$ to another embedding $g$, through a family of embeddings. That is, across the time interval $[0, 1]$, $H$ transforms from $f$ into $g$ while remaining an embedding at any given time. An ambient isotopy distorts one submanifold of a space to another; in our case, this works on knots. Note that the orientation of a knot is preserved by an isotopy, so the orientation of a knot matters for equivalence. Some knots are not smoothly ambiently isotopic to their opposite orientation. Equipped with this notion of equivalence between knots, we naturally form equivalence classes. Common knot equivalence classes are given names; so for example, in figure 1.4 we have the unknot, the (left-handed) trefoil, and the figure 8 knot. At this point we commit another standard abuse of terminology and use the term “knot” to refer to an equivalence class of knots.

1.1.1 Knot Diagrams

While we define knots as living in 3D space, most often we work on the page or blackboard, meaning we must project the knot into a 2D plane. We have already done this a few times. Formally, we define this projection as follows.

**Definition 1.1.6.** Let $V \subseteq \mathbb{R}^3$ be a 2-dimensional vector subspace, and $\pi_V: \mathbb{R}^3 \rightarrow V$ be the projection to $V$. Given a knot $k: S^1 \rightarrow \mathbb{R}^3$, let $p = \pi_V \circ k$. We call $s \in p(S^1)$ a double point if $|p^{-1}(s)| = 2$, and denote the set of all double points of a projection $W$. Then $p$ is a regular projection if:

1. $\forall s \in p(S^1), |p^{-1}(s)| \leq 2$, and

2. At every double point $s$, the tangents of the curves forming the neighborhood of $s$ in $p(S^1)$ span $\mathbb{R}^2$. 


Figure 1.5: Non-regular projections

A knot diagram $D$ of a knot $K$ is a regular projection of $K$, along with a map $C: W \to \{-1, 1\}$. For $s \in W$, call the pair $(s, C(s))$ a crossing of $D$. Additionally, we say $(s, C(s))$ is a right-handed (resp. left-handed) crossing if $C(s) = 1$ (resp. $-1$). The orientation of $K$ may be denoted on the diagram with an arrow pointing in the direction of travel.

Figure 1.5 shows what is NOT allowed by regular projections. The left image has 3 curves meeting at a point. The middle image has two curves tangent to each other, and the right image has two curves overlap on an interval.

In layman’s terms, a regular projection is a projection onto a plane such that the only overlaps look like X’s. A knot diagram is a regular projection, together with information about which strand is “in front” of another at each crossing point. This information is usually shown by deleting a short segment from the strand that is “behind” on the page. The orientation of the knot also induces an orientation on each crossing, as in figure 1.6

**Theorem 1.1.7.** Every knot admits a regular projection. That is, given any knot $K$ there exists a knot diagram $D$.

It is not immediately obvious that a knot diagram exists for every possible knot. The fact that this is possible comes as a consequence of the Whitney Immersion Theorem. For a proof, see [5, Prop. 70.19].

6
Knot diagrams are the main way we represent a knot on the page. There are clearly many different diagrams we could use for a single knot, depending on the plane to which we project and the representative of the knot equivalence class we choose. We want different diagrams of the same knot equivalence class to be equivalent. This is no problem; we just need to define an equivalence.

**Definition 1.1.8 (Reidemeister Moves).** Below are Reidemeister moves Ω₁, Ω₂, and Ω₃. Two knot diagrams are equivalent if there is a finite sequence of Reidemeister moves that transform one into the other.

![Reidemeister Moves](image)

The Reidemeister moves are 2D representations of the ways we can deform a knot in space. In fact, knot equivalence and knot diagram equivalence agree, as the following theorem states.

**Proposition 1.1.9 (See e.g. [4, 1.14]).** Two knots are equivalent iff any diagram of one can be transformed into a diagram of the other by a sequence of Reidemeister moves.
With all these definitions in place, note that we often abuse terminology and use the word “knot” to mean a map from $S^1$ to $\mathbb{R}^3$, the image of that map, a diagram, or most often an equivalence class of knots. Also, unless otherwise specified, the word “knot” will mean “knot or link”, as many operations we perform on knots will cause one to become another.

1.2 Knot Properties and Invariants

A large area of focus in knot theory is to catalog every knot (up to knot equivalence). It is difficult to look at any two given knots or diagrams and come up with a way to transform one into the other (or to prove that no such transformation exists). Instead, to study knots efficiently, we study knot invariants - properties of knots that do not change when a knot is transformed. In this section, we will give an overview of some basic knot invariants.

1.2.1 Crossing Number

**Definition 1.2.1.** The crossing number of a knot is the minimum number of crossings among all diagrams of that knot.

The trefoil has a crossing number of 3, as can be proven by examining the different ways to connect two crossings.

Typically knots are cataloged by their crossing number in a standardized fashion; so for example, the trefoil is also referred to as “3_1”. Since crossing number is defined as a minimum over all knot diagrams for a knot, it is clearly an invariant; deforming the knot does not change which diagrams are associated with it. However, given an arbitrary knot diagram, in order to know the crossing number, one must already know
the “simplest diagram” of the given knot. Since there are an infinite number of ways the Reidemeister moves can be used to find equivalent diagrams, it can be difficult to find the crossing number (and therefore narrow down which knot equivalence class the diagram represents) using the definition alone.

**Definition 1.2.2.** The unknotting number of a knot is the minimum, across all diagrams of the knot, number of crossing changes required to unknot it.

Recall that each crossing is assigned a right- or left-handedness. A crossing change is changing a right-handed crossing to a left-handed one, or vice versa. This changes which strand is “in front” and changes which knot is represented by the diagram.

### 1.3 Knot crossings

Analyzing specific areas of a knot diagram is an important way to analyze knots. We can focus our attention to a local part of the knot and ask how we can modify it without changing the rest of the knot. We have already seen one example of this - the Reidemeister moves are local transformations of the knot.

**Definition 1.3.1.** Let $D$ be a knot diagram. A tangle (diagram) is an embedded disc $B \subset \mathbb{R}^2$ such that $\partial B \cap \text{im} \, D$ contains no double points.

![Figure 1.8: A tangle diagram](image)
Figure 1.8 shows an example of a tangle. Sometimes we omit showing the boundary of the disk, as we have done already in figures 1.4 and 1.6. These diagrams are useful for zooming in on a specific part of interest in a knot.
Chapter 2

THE JONES POLYNOMIAL

One historically important knot invariant is the Jones polynomial. The Jones polynomial is a polynomial invariant related to the braid group of a knot discovered by Vaughan Jones during his study of operator algebras.[8] However, these topics are far out of the scope of this thesis. Instead, we will first define a polynomial on a given knot diagram (which is not quite an invariant) called the Kauffman bracket, and use it to define the Jones polynomial.

Definition 2.0.1. Suppose we have three knot diagrams $D_+, D_0,$ and $D_-$ that are identical except at a tangle $T$. A diagram that displays how they differ is called a skein diagram. If $p(D)$ is a polynomial associated with diagram $D$, a skein relation for $p$ is an equation $F(D_+, D_0, D_-) = 0$ that describes how $p(D_+)$, $p(D_0)$, and $p(D_-)$ are related [8].

We use a skein relation to define the Kauffman bracket.

Definition 2.0.2. Let $L$ be an oriented link and $D$ an unoriented diagram of $L$. The Kauffman bracket of $D$ is a finite Laurent polynomial $\langle D \rangle \in \mathbb{Z}[q, q^{-1}]$ that satisfies the following conditions:

1. If $D$ is the empty diagram $\emptyset$ then

\[ \langle \emptyset \rangle = 1. \]
2. If $D$ consists of the disjoint union of the trivial diagram $\bigcirc$ and a diagram $D_0$ of a link $L_0$, then

$$\langle D \rangle = (q + q^{-1}) \langle D_0 \rangle.$$

3. Suppose $\hat{D}$, $\bar{D}$ are diagrams that differ from $D$ by the skein diagram given in Figure 2.1.

![Figure 2.1: Jones skein relation](image)

Then the following skein relation holds:

$$\langle D \rangle = \langle \hat{D} \rangle - q \langle \bar{D} \rangle.$$

We call $\hat{D}$ the 0-smoothing and $\bar{D}$ the 1-smoothing of $D$.

**Theorem 2.0.3.** Let $D$ be a diagram for a link $L$. Then $\langle D \rangle$ exists and is unique.

**Proof.** Let $n$ be the number of crossings of $D$, and label each crossing from 1 to $n$. Let $\hat{D}_1$ and $\bar{D}_1$ be the diagrams that fulfill the skein diagram at crossing 1, so that $\langle D \rangle = \langle \hat{D}_1 \rangle - q \langle \bar{D}_1 \rangle$. Notice that $\hat{D}_1$ and $\bar{D}_1$ have $n - 1$ crossings; We may repeat this process on each diagram, untangling each crossing in one of two ways. We will end up with $2^n$ diagrams $D_\alpha$ with 0 crossings each, called smoothings, where $\alpha \in \{0, 1\}^n$ is the string of 0- or 1-smoothings used in order of the crossings. Since each diagram
has 0 crossings, they are each \( k_\alpha \) disjoint copies of the trivial diagram \( \circ \), and so contribute \((q + q^{-1})^{k_\alpha}\) to \( \langle D \rangle \). Each \( D_\alpha \) has a “height”, \( r := |\alpha| \), which is the number of 1-smoothings used. Then

\[
\langle D \rangle = \sum_\alpha (q + q^{-1})^{k_\alpha} (-q)^r.
\]

This sum is independent of the choice of labelling, since the labelling does not affect \(|\alpha|\) and every 0-crossing diagram will be represented by a single term.

The Kauffman bracket itself is not a knot invariant; it is not invariant under Reidemeister move 1.

To see this, compare \( \langle \circ \rangle \) with \( \langle \bigcirc \bigcirc \rangle \):

\[
\langle \circ \rangle = q + q^{-1} \\
\langle \bigcirc \bigcirc \rangle = \langle \circ \rangle - q\langle \bigcirc \bigcirc \rangle \\
= q + q^{-1} - q(q + q^{-1})^2 \\
= -q^3 - q - q^{-1}
\]

In order to extract a knot invariant from the Kauffman bracket, we reintroduce the orientation of the knot to the polynomial, modifying it accordingly.

**Definition 2.0.4.** The (normalized) Jones polynomial of a link \( L \) is

\[
J(L) := \frac{(-1)^n q^{n+2n} - \langle D \rangle}{q + q^{-1}}
\]
where $D$ is a knot diagram of $L$, $n_+$ is the number of right-handed crossings in $D$ and $n_-$ is the number of left-handed crossings in $D$.

We have been using diagrams of knots inside the bracket notation to represent the Kauffman bracket of the indicated knot. We may also sometimes use the diagram of a tangle. When $T$ is a specific tangle in $D$, we write $\langle T \rangle$ as an abbreviation of $\langle D \rangle$. This allows us to also compare diagrams $D$ and $D'$ that differ in one tangle: we may refer to $\langle T \rangle$ and $\langle T' \rangle$, where $T$ and $T'$ are the specified tangle and the diagrams are identical otherwise. A similar notation is used for the Jones polynomial as well, so we may write $J(T)$. This is particularly useful when we want to compare the invariants for the Reidemeister moves, as we only have a single tangle of information to work with.

**Theorem 2.0.5.** The Jones polynomial is a knot invariant.

*Proof.* We show $J(L)$ is invariant under Reidemeister moves and thus is a knot invariant. To do so, we examine how the Kauffman brackets on each side of a Reidemeister move relate to each other, and show that after normalizing, they produce the same polynomial.

$\Omega_1$:

The Kauffman bracket of the loop evaluates as follows:

$$
\langle \bigcirc \rangle = \langle \bigcirc \rangle - q\langle \bigcirc \rangle \\
= (q + q^{-1})\langle \{ \} \rangle - q\langle \{ \} \rangle \\
= q^{-1}\langle \{ \} \rangle
$$
Now if $L$ has $n_+$ right-handed crossings and $n_-$ left-handed crossings, then $\Omega_1(L)$ has $n_+ + 1$ right-handed crossings and $n_-$ left-handed crossings. So

$$J(\langle \varnothing \rangle) = \frac{(-1)^{n_+ - q^{n_+ + 1}} - 2n_- q^{-1} \langle \varnothing \rangle}{q + q^{-1}} = \frac{(-1)^{n_+ - q^{n_+ + 1}} - 1 \langle \varnothing \rangle}{q + q^{-1}} = \frac{(-1)^{n_+ - q^{n_+ + 1}} - 1 \langle \varnothing \rangle}{q + q^{-1}} = J(\langle \varnothing \rangle).$$

$\Omega_2$:

$$\langle \varnothing \rangle = \langle \varnothing \rangle - q\langle \varnothing \rangle$$

$$= \langle \varnothing \rangle - q\langle \varnothing \rangle - q\left(\langle \varnothing \rangle - q\langle \varnothing \rangle\right)$$

$$= \langle \varnothing \rangle - q(q + q^{-1})\langle \varnothing \rangle - q\langle \varnothing \rangle + q^2\langle \varnothing \rangle$$

$$= -q\langle \varnothing \rangle$$

No matter the orientation of the link $L$, $\Omega_2(L)$ has exactly one more right-handed and left-handed crossing than $L$. So we get

$$J(\langle \varnothing \rangle) = \frac{(-1)^{n_+ + 1} - 2(n_- + 1)(-q\langle \varnothing \rangle)}{q + q^{-1}} = \frac{(-1)^{n_+ + 2} q^{n_+ - 2n_-} - 1 \langle \varnothing \rangle}{q + q^{-1}} = \frac{(-1)^{n_+ - q^{n_+ - 2n_-}} \langle \varnothing \rangle}{q + q^{-1}} = J(\langle \varnothing \rangle).$$

$\Omega_3$:
\[ \langle \gamma \rangle = \langle \gamma \rangle - q \langle \gamma \rangle \]
\[ = \langle \gamma \rangle - q \langle \gamma \rangle - q \langle \gamma \rangle + q^2 \langle \gamma \rangle \]
\[ = \langle \gamma \rangle - q \langle \gamma \rangle - q \langle \gamma \rangle + q^2 \langle \gamma \rangle - q \langle \gamma \rangle + q^2 \langle \gamma \rangle - q^3 \langle \gamma \rangle \]
\[ = \langle \gamma \rangle - q \langle \gamma \rangle - q \langle \gamma \rangle + q^2 \langle \gamma \rangle - q \langle \gamma \rangle + q^2 \langle \gamma \rangle - q^3 \langle \gamma \rangle \]
\[ = \langle \gamma \rangle - q \langle \gamma \rangle - q \langle \gamma \rangle + q^2 \langle \gamma \rangle - q \langle \gamma \rangle + q^2 \langle \gamma \rangle - q^3 \langle \gamma \rangle \]
\[ = \langle \gamma \rangle - q \langle \gamma \rangle - q \langle \gamma \rangle + q^2 \langle \gamma \rangle + q^2 \langle \gamma \rangle - q^3 \langle \gamma \rangle \]

Either side of \( \Omega_3 \) evaluates to the same bracket as shown above, so even without normalizing, the Kauffman bracket is invariant under \( \Omega_3 \). Either side of \( \Omega_3 \) will have the same crossing types no matter orientation, and so the Jones polynomial is invariant as well.

\[ \langle \gamma \rangle = \langle \gamma \rangle - q \langle \gamma \rangle \]
\[ = \langle \gamma \rangle - q \langle \gamma \rangle - q \langle \gamma \rangle + q^2 \langle \gamma \rangle \]
\[ = \langle \gamma \rangle - q \langle \gamma \rangle - q \langle \gamma \rangle + q^2 \langle \gamma \rangle - q \langle \gamma \rangle + q^2 \langle \gamma \rangle - q^3 \langle \gamma \rangle \]
\[ = \langle \gamma \rangle - q \langle \gamma \rangle - q \langle \gamma \rangle + q^2 \langle \gamma \rangle - q \langle \gamma \rangle + q^2 \langle \gamma \rangle - q^3 \langle \gamma \rangle \]
\[ = \langle \gamma \rangle - q \langle \gamma \rangle - q \langle \gamma \rangle + q^2 \langle \gamma \rangle + q^2 \langle \gamma \rangle - q^3 \langle \gamma \rangle \]

\[ \text{Either side of } \Omega_3 \text{ evaluates to the same bracket as shown above, so even without normalizing, the Kauffman bracket is invariant under } \Omega_3. \text{ Either side of } \Omega_3 \text{ will have the same crossing types no matter orientation, and so the Jones polynomial is invariant as well.} \]

### 2.1 Calculating the Jones Polynomial

**Definition 2.1.1.** The cube of resolutions for a knot diagram \( D \) which has \( n \) crossings is an \( n \)-dimensional cube with the following:
1. Each of the $2^n$ vertices is labelled with a smoothing $D_\alpha$, $\alpha \in \{0,1\}^n$.

2. If $\alpha$ and $\beta$ differ at a single position, an edge connects the vertices labelled $D_\alpha$ and $D_\beta$ and is labelled $e_\gamma$, where $\gamma = \alpha$ but with a $\ast$ at the position where $\alpha$ and $\beta$ differ.

Figure 2.2 shows the cube for the Hopf link, as an example.

![Cube of resolutions for the Hopf link](image)

Figure 2.2: Cube of resolutions for the Hopf link

The cube of resolutions is a useful way to visualize the smoothings to aid in the calculation of the Jones polynomial. Notice that we have arranged the smoothings in columns from left to right according to how many 1-smoothings have been used, referred to as the height of a smoothing. Edges connect each pair of smoothings that differ at exactly one crossing. Each edge in the cube can be labelled $e_{\alpha_1\ldots\alpha_{i-1}\ast\alpha_{i+1}\ldots\alpha_n}$, where $i$ is the crossing at which the two vertices connected by the edge differ. To realize the Kauffman bracket from such a cube, each smoothing $D_\alpha$ contributes a term
of the form \((-q)^r(q + q^{-1})^k\), where \(r\) is the height of a smoothing and \(k\) is the number of copies of \(S^1\) in the smoothing. Summing everything together and expanding the resulting polynomial gives us the Kauffman bracket, after which the orientation of the knot is reintroduced to form the Jones polynomial by adjusting for the number of positive and negative crossings.

**Example 1.** We calculate the Jones polynomial for the Hopf link. The Kauffman bracket is

\[
\langle \bigcirc \bigcirc \rangle = (q + q^{-1})^2 - 2q(q + q^{-1}) + q^2(q + q^{-1})^2 = q^{-2} + 1 + q^2 + q^4.
\]

Using the orientation shown in figure 2.2, we have two right-handed crossings. This gives us

\[
J(\bigcirc \bigcirc) = \frac{q^2 \langle \bigcirc \bigcirc \bigcirc \rangle}{q + q^{-1}} = \frac{1 + q^2 + q^4 + q^6}{q + q^{-1}} = q^5 + q.
\]

**Example 2.** Figure 2.3 shows the cube of resolutions for a figure eight knot, and the corresponding Jones polynomial is calculated below.

\[
\langle \bigcirc \bigcirc \bigcirc \rangle =
(q + q^{-1})^3 - 4q(q + q^{-1})^2 + (5q^2(q + q^{-1}) + q^2(q + q^{-1})^3) - 4q^3(q + q^{-1})^2 + q^4(q + q^{-1})^3
= q^3 + q^7
\]

The figure 8 knot has 2 left-handed crossings and 2 right-handed crossings, so

\[
J(\bigcirc \bigcirc \bigcirc) = \frac{(-1)^2 q^2 - 4q + q^7}{q + q^{-1}} = \frac{q^5 + q^{-5}}{q + q^{-1}} = q^4 - q^2 + 1 - q^{-2} + q^{-4}.
\]
Figure 2.3: Cube of resolutions for the figure 8 knot
Chapter 3

KHOVANOV HOMOLOGY

Where we have gone from a cube of diagrams to a polynomial, Khovanov’s breakthrough was in recognizing that such a cube existed [6]. Informally, the Jones Polynomial is a sort of “projection” of the cube, and as with most projections, some of the structure of the original is lost. The relationship between the Jones polynomial and Khovanov homology is similar to that of the Betti number and simplicial homology for topological spaces. In order to retain more of this structure, we will introduce some more concepts and use them to create a knot invariant that keeps more of the cube intact. This formulation of Khovanov homology comes from Bar-Natan’s paper [3], and was the main topic of our research.

3.1 Background material

Definition 3.1.1. Let $M, N$ be $n$-dimensional manifolds. A cobordism $W: M \to N$ from $M$ to $N$ is a manifold with boundary, whose boundary is $M \sqcup N$.

We specify that $W$ is $M$ to $N$ and not between $M$ and $N$ because we would like to form a correspondence between cobordisms (on smoothings) and linear maps (between vector spaces).

Figure 3.1 shows the three cobordisms that will occur between smoothings. The first and second cobordisms are strictly speaking identical. However, because we associate them to maps, we consider cobordisms as morphisms from $M$ to $N$. The cobordisms are referred to as “merge, “split, “identity” because of how they affect $M$. 

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Definition 3.1.2. Let $V$, $W$ be vector spaces over a field $\mathbb{F}$. The tensor product of $V$ and $W$ is the vector space $V \otimes W := \{v \otimes w \mid v \in V, w \in W\}$ that has a bilinear map $V \times W \to V \otimes W$ defined by $(v, w) \mapsto v \otimes w$ with the following properties:

i. For all $\alpha \in \mathbb{F}$, $(\alpha v) \otimes w = \alpha (v \otimes w) = v \otimes (\alpha w),$

ii. $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w,$

iii. $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2.$
If \( e_1, e_2, \ldots, e_m \) is a basis for \( V \) and \( f_1, f_2, \ldots, f_n \) is a basis for \( W \), then \( \{ e_i \otimes f_j \mid i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \} \) is a basis for \( V \otimes W \). So \( V \otimes W \) has dimension \( mn \).

**Definition 3.1.3.** A graded vector space is a vector space \( V \) viewed as a direct sum \( V = \bigoplus_{m \in \mathbb{Z}} V_m \). The graded dimension of \( V \) is the Laurent series \( \dim_q V := \sum_m q^m \dim V_m \).

A graded vector space is a direct sum of vector spaces, where each component space has been assigned a certain power of \( q \). The graded dimension of a finite dimensional vector space, then, is a Laurent polynomial in \( q \) where the coefficient of \( q^m \) is the dimension of the corresponding space. In particular, \( \dim_q V = \dim V \) when \( q = 1 \).

We define the degree shift \( \cdot \{ l \} \) such that \( V \{ l \}_m := V_{m-l} \) (i.e. add \( l \) to the degrees of each component space). In this case, \( \dim_q V \{ l \} = q^l \dim_q V \).

**Definition 3.1.4.** A chain complex \( C \) is a sequence \( \ldots, C_0, C_1, C_2, \ldots \) of vector spaces together with linear maps \( d_i : C_i \to C_{i+1} \) called differentials such that \( d_{i+1} \circ d_i = 0 \) \([10]\).

We may occasionally omit the indices of the differential and write \( d \), if the specific maps are clear from context. Chain complexes are often represented by diagrams similar to the following.

\[
\cdots \rightarrow C_{i-1} \xrightarrow{d_{i-1}} C_i \xrightarrow{d_i} C_{i+1} \rightarrow \cdots
\]

**Remark.** Note that we follow Khovanov’s notation, in which the differentials of the complex increase the index. This non-standard notation is so that the chain complex we will produce matches the conventions we have used to construct the Jones polynomial.
Definition 3.1.5. A chain subcomplex \( S \) of a chain complex \( C \) (whose differential is \( d \)) is a sequences of vector subspaces \( S_i \subseteq C_i \) such that \( d_i(S_i) \subseteq S_{i+1} \).

Since \( d^2 = 0 \) on \( C \), it is also 0 on \( S \) and so \( S \) is also a chain complex where the differential is \( d|_S \), the restriction of \( d \) to \( S \). We almost always use the term “chain subcomplex” to mean this chain complex as well.

Definition 3.1.6. The \( i \)-th homology group of a chain complex \( C \) is \( H_i(C) = \ker d_i / \im d_{i-1} \).

The homology of a chain complex is \( H_*(C) := \bigoplus_{i \in \mathbb{Z}} H_i(C) \). If \( H_i(C) = 0 \) for all \( i \), then we say \( C \) is acyclic.

The definition of \( H_*(C) \) implies that the homology groups form a graded vector space. Indeed, any chain complex is a graded vector space as well, when viewed as being \( \bigoplus_{i \in \mathbb{Z}} C_i \). Khovanov homology will be a bi-graded vector space, with the homology groups being graded twice. One of these gradings comes from the chain complex structure, and we refer to it as the homological grading. The other is known as the quantum grading, and will be explained in the next section.

Definition 3.1.7. Let \( C, \bar{C} \) be chain complexes. A chain map \( f : C \to \bar{C} \) is a sequence of homomorphisms \( \ldots, f_0, f_1, f_2, \ldots \) where \( f_i : C_i \to \bar{C}_i \) commutes with the differential. In other words, the following diagram commutes.

\[
\begin{array}{ccccccccc}
\ldots & \longrightarrow & C_{i-1} & \xrightarrow{d_{i-1}} & C_i & \xrightarrow{d_i} & C_{i+1} & \longrightarrow & \ldots \\
\downarrow{f_{i-1}} & & \downarrow{f_i} & & \downarrow{f_{i+1}} & & & & \\
\ldots & \longrightarrow & \bar{C}_{i-1} & \xrightarrow{\bar{d}_{i-1}} & \bar{C}_i & \xrightarrow{\bar{d}_i} & \bar{C}_{i+1} & \longrightarrow & \ldots
\end{array}
\]

We define the height shift \( [s] \) such that if \( C \) is a chain complex \( \ldots \to C_i \to C_{i+1} \to \ldots \), then \( C[s] \) is the chain complex where \( C_{i-s} = C[s]_i \) (i.e. all the chain groups are shifted up by \( s \)).
The Jones polynomial can be computed from a certain chain complex (whose homology will also be a knot invariant). The construction of this chain complex is a similar process to the way we constructed the Jones polynomial in chapter 2.

Armed with the definitions above, we can turn a cube of resolutions for a knot diagram to a chain complex. By carefully defining tensor spaces and differential maps according to the cube, the chain complex will have a homology that is a knot invariant.

Given a link diagram $D$ with $n$ crossings, the construction of this chain complex proceeds as follows.

1. We begin with the cube of resolutions for $D$ (See 2.1). Recall that each smoothing is labelled by $\alpha \in \{0, 1\}^n$ and each edge is labelled $e_\gamma$, where $\gamma \in \{0, 1, \ast\}^n$. We now also label each strand in $D$ in an arbitrary manner. Each circle in a smoothing is then labelled by the smallest strand $j$ it contains.

2. For $i \in \mathbb{Z}$, let $V_i = \mathbb{F}\langle v_i^+, v_i^- \rangle$ be a graded vector space with two basis elements over a field $\mathbb{F}$. The graded degrees of these $v_i^\pm$ are defined to be $\pm 1$ respectively, so that $\dim_q V_i = q + q^{-1}$. Each vertex $\alpha$ of the cube is then assigned the tensor product $V_\alpha := (V_{i_1} \otimes \ldots \otimes V_{i_k})\{r\}$ where $k$ is the number of circles in the corresponding smoothing and $r$ is the height of the smoothing. Specifically, $V_i \subseteq V_\alpha$ is the vector space corresponding to the circle labelled $i$. $V_\alpha$ may sometimes be denoted as $V^\otimes k\{r\}$, when we want to emphasize the height and dimension of the tensor space (Recall $\dim V^\otimes k = 2^k$). Note also that the degree of an element in one of these spaces is the sum of the degrees of each component, plus the height shift. For example, $v_+ \otimes v_+$ is a degree 2 element. If the height shift is $\{2\}$, this becomes a degree 4 element.
3. Each edge $e_\gamma = e_{\gamma_1 \cdots \gamma_{i-1} \ast \gamma_{i+1} \cdots \gamma_\ell}$ corresponds to a cobordism between smoothings that is either a merge or a split on the circle(s) that touched the $i$-th crossing, and the identity on every other circle. We can associate these cobordisms to linear maps between vector spaces as follows, where $i < j$:

(a) Define the **merge map** $m: V_i \otimes V_j \to V_i$ by
\[
\begin{align*}
m(v_i^+ \otimes v_j^+) &= v_i^+ \\
m(v_i^+ \otimes v_j^-) &= v_i^-\\
m(v_i^- \otimes v_j^+) &= v_i^-\\
m(v_i^- \otimes v_j^-) &= 0
\end{align*}
\]
(b) Define the **split map** $\Delta: V_i \to V_i \otimes V_j$ by
\[
\begin{align*}
\Delta(v_i^+) &= v_i^+ \otimes v_j^- + v_i^- \otimes v_j^+ \\
\Delta(v_i^-) &= v_i^- \otimes v_j^-
\end{align*}
\]

If the cobordism is a merge between circles $i$ and $j$ (in $\alpha$) to circle $i$ (in $\beta$), then define $d_\gamma: V_\alpha \to V_\beta$ to be $m: V_i \otimes V_j \to V_i$, extended by the identity to the remaining tensor factors. Similarly, if the cobordism is a split from circle $i$ to circles $i$ and $j$, then define $d_\gamma: V_\alpha \to V_\beta$ to be $\Delta: V_i \to V_i \otimes V_j$, again extended by the identity to the remaining tensor factors. Note that the definition of the maps above each send an element in $V_\alpha$ of degree $t$ to an element of degree $t - 1$, but the height of $V_\beta$ is one more than the height of $V_\alpha$. Recall that we defined the $V_\alpha$ with a height shift, which cancels the step down in degree from the maps. Thus, these maps respect the grading we have put on our vector spaces; elements of a certain degree get mapped to elements of the same degree.

4. The definition of the chain complex $[D]$ is as follows. Let the $i$-th chain space be $[D]^i := \bigoplus V_\alpha$ i.e. the direct sum of all vector spaces $V_\alpha$ of height $i$. The differential $d_i: [D]^i \to [D]^{i+1}$ is defined to be $\sum_{|\gamma| = i} (-1)^{|\gamma|} d_\gamma$, where $|\gamma| :=$ the
number of 1's in $\gamma$ to the left of the $\star$. This is the sum of all the edge maps that begin at column $i$, with a sign term to force $d_i \circ d_{i+1} = 0$, as we will prove.

Finally, just like with the Jones polynomial, we apply a normalizing shift to the chain complex and define the **Khovanov complex**

$$CKh(D) = [D][-n_-]{n_+ - 2n_-}$$

where $n_+$ ($n_-$) is the number of right-handed (left-handed) crossings in the knot diagram $D$. 

Figure 3.2: Khovanov complex construction for a trefoil number of 1's in $\gamma$ to the left of the $\star$. This is the sum of all the edge maps that begin at column $i$, with a sign term to force $d_i \circ d_{i+1} = 0$, as we will prove. Finally, just like with the Jones polynomial, we apply a normalizing shift to the chain complex and define the **Khovanov complex**

$$CKh(D) = [D][-n_-]{n_+ - 2n_-}$$

where $n_+$ ($n_-$) is the number of right-handed (left-handed) crossings in the knot diagram $D$. 

Figure 3.2: Khovanov complex construction for a trefoil
Example 3. Figure 3.2 shows an example of this process on the right-handed trefoil knot.

It is not immediately clear that this produces a chain complex. In particular, we need to show that each face of the cube of resolutions commutes, and so that the differentials are well-defined and cancel once the signs are introduced.

**Theorem 3.2.1.** The construction described above is a chain complex. That is, the maps $m$ and $\Delta$ commute on each square of the diagram, and $d_i \circ d_{i+1} = 0$.

**Proof.** The edge maps are either a merge or split on two tensor factors, and the identity on the rest. We show that each of the combinations of two maps commute on the appropriate factor, and it should be obvious that any factor that is unchanged by at least one of the two commutes. In the following computations, we assume $i < j < k$. Note, however, that the indexing of the involved vector spaces does not affect whether the maps commute. Since the manner in which we assigned indices to the strands of the diagram was arbitrary, we may omit the indices of the vector spaces for the purposes of this proof. We will go through each of the possible combinations of elements for each of the combinations of maps. These come from the possible ways the crossings can resolve to produce circles.

\[
(m, m)
\]

\[
V_i \otimes V_j \otimes V_k \xrightarrow{m} V_i \otimes V_j \xrightarrow{m} V_i \otimes V_k \xrightarrow{m} V_i
\]
\[ m(m(v_i^+ \otimes v_j^+ \otimes v_k^+) = m(v_i^+ \otimes m(v_j^+ \otimes v_k^+)) = v_i^+ \]
\[ m(m(v_i^+ \otimes v_j^- \otimes v_k^+) = m(v_i^+ \otimes m(v_j^- \otimes v_k^+)) = v_i^- \]
\[ m(m(v_i^- \otimes v_j^+ \otimes v_k^+) = m(v_i^- \otimes m(v_j^+ \otimes v_k^+)) = 0 \]
\[ m(m(v_i^- \otimes v_j^- \otimes v_k^+) = m(v_i^- \otimes m(v_j^- \otimes v_k^-)) = 0 \]

\[ (\Delta, \Delta) \]

\[ \Delta(\Delta(v_i^+)) = \Delta(v_i^+ \otimes v_j^+ + v_i^- \otimes v_k^+) \]
\[ = v_i^+ \otimes v_j^+ \otimes v_k^- + v_i^- \otimes v_j^- \otimes v_k^+ + v_i^+ \otimes v_j^- \otimes v_k^+ + v_i^- \otimes v_j^+ \otimes v_k^- \]
\[ = \Delta(v_i^+ \otimes v_i^+ + v_i^- \otimes v_i^-) = \Delta(\Delta(v_i^+)) \]

\[ \Delta(\Delta(v_i^-)) = \Delta(v_i^- \otimes v_j^+) \]
\[ = v_i^- \otimes v_i^- \otimes v_k^- \]
\[ = \Delta(v_i^- \otimes v_i^-) = \Delta(\Delta(v_i^-)) \]

\[ (m, \Delta) \]

\[ m(m(v_i \otimes v_j \otimes v_k) = m(m(v_i \otimes v_j \otimes v_k | m(v_i \otimes v_j \otimes v_k) = v_i \]

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\[ \Delta(m(v^i_+ \otimes v^j_+)) = \Delta(v^i_+) \]
\[ = v^i_+ \otimes v^k_- + v^i_- \otimes v^k_+ \]
\[ = m((v^i_+ \otimes v^k_- + v^i_- \otimes v^k_+) \otimes v^i_+) = m(\Delta(v^i_+) \otimes v^i_+) \]

\[ \Delta(m(v^i_+ \otimes v^j_-)) = \Delta(v^i_-) \]
\[ = v^i_- \otimes v^k_- \]
\[ = m((v^i_+ \otimes v^k_- + v^i_- \otimes v^k_+) \otimes v^i_-) = m(\Delta(v^i_+) \otimes v^i_-) \]

\[ \Delta(m(v^i_- \otimes v^j_+)) = \Delta(v^i_-) \]
\[ = v^i_- \otimes v^k_- \]
\[ = m((v^i_- \otimes v^k_- \otimes v^j_+) = m(\Delta(v^i_-) \otimes v^j_+) \]

\[ \Delta(m(v^i_- \otimes v^j_-)) = \Delta(0) \]
\[ = 0 \]
\[ = m((v^i_- \otimes v^k_- \otimes v^j_-) = m(\Delta(v^i_-) \otimes v^j_-) \]
Note that there are two more possibilities for the square, as in the following diagrams:

\[
\begin{array}{cccc}
V_i \otimes V_j & \xrightarrow{\Delta} & V_i \otimes V_j \otimes V_k & \xrightarrow{m} & V_i \otimes V_j \\
V_i \otimes V_j \otimes V_{k'} & \xrightarrow{m} & V_i \otimes V_j \\
V_i \otimes V_j & \xrightarrow{m} & V_i \otimes V_j \\
V_i \otimes V_j & \xrightarrow{m} & V_i \otimes V_j \\
\end{array}
\]

But both of these clearly commute, again because the indexing is arbitrary.

In all the above cases, the maps commute and so we only need show that the sign of \((-1)^{|e|}\) ensures \(d_i \circ d_{i+1} = 0\). To see this, note that in any square of the cube, we need exactly one of the 4 maps to have a different sign than the rest; so that the combined sign along the top edges of the square is opposite that of the bottom edges. Recall that \(|e|\) is the number of 1’s in the edge label positioned before the \(\star\). Since each square only has 2 crossings involved, the edge maps differ in exactly 2 places, where they are one of 0\(\star\), 1\(\star\), \(\star\)0, or \(\star\)1. We can see that only 1\(\star\) adds an extra \(-1\) to the product, and so we have cancellation. Any of the letters between the differing places do not matter since they will be considered exactly once along either path. \(\square\)

**Definition 3.2.2.** The Khovanov homology of a link \(\mathcal{H}_*(L)\) is the homology of the chain complex \(\mathcal{C}K\mathcal{H}(L)\). The \(r\)th homology groups of this chain complex are denoted \(\mathcal{H}_r(L)\).

We will show Khovanov homology is a link invariant in the following section, justifying the terminology as being the homology of a link \(L\) rather than of a diagram \(D\).
Remark. Note that the grading on each chain group $CKh_i(D)$ allows us to view each one as a direct sum $CKh_i(L) = \bigoplus_j C_{i,j}$, where each $C_{i,j}$ is the subspace of $C_i$ with graded dimension $j$. Because the differentials are grading preserving, our chain complex splits along the $q$ grading into chain subcomplexes

$$\ldots \to C_{i-1,j} \xrightarrow{d_{i-1,j}} C_{i,j} \xrightarrow{d_{i,j}} C_{i+1,j} \to \ldots$$

for each $j \in \mathbb{Z}$. The homology $\mathcal{H}_r(L)$ respects this structure (again, since the differentials are degree preserving) and can also be viewed as a graded vector space, whose grading is induced from the grading of the chain groups. This allows us to construct the graded polynomial

$$p(L) := \sum_r t^r \dim_q \mathcal{H}_r(L).$$

One may verify that $p(L)$ evaluates to the unnormalized Jones polynomial when $t = -1$, showing that Khovanov homology indeed refines the Jones polynomial.

### 3.3 Invariance of Khovanov Homology

**Theorem 3.3.1** (Khovanov [6]). The graded dimensions of the homology groups $\mathcal{H}_r(L)$ are link invariants, and hence $p(L)$, a polynomial in the variables $t$ and $q$, is a link invariant that specializes to the unnormalized Jones polynomial at $t = -1$.

We prove this theorem similarly to the way we proved the Jones polynomial was a link invariant: examine both sides of the Reidemeister moves and show they have the same homology. There is a “cancellation principle” for chain complexes that makes this process easier:

**Lemma 3.3.2.** Let $\mathcal{C}$ be a chain complex and let $\mathcal{C}' \subset \mathcal{C}$ be a chain subcomplex.
1. If $C'$ is acyclic (has no homology), then it can be “cancelled”. That is, in this case the homology $H_r(C)$ of $C$ is isomorphic to the homology $H_r(C/C')$ of $C/C'$ for all $r \in \mathbb{Z}$.

2. Likewise, if $C/C'$ is acyclic, then $H_*(C) \cong H_*(C')$.

Proof. The short exact sequence

$$0 \rightarrow C' \xrightarrow{\text{inclusion}} C \xrightarrow{\text{quotient}} C/C' \rightarrow 0$$

gives us the associated diagram of homologies

$$\begin{array}{ccc}
0 & \rightarrow & H_r(C') \\
\downarrow d_{r-1} & & \downarrow d_{r-1} \\
H_r(C) & \rightarrow & H_r(C/C') \\
\downarrow d_r & & \downarrow d_r \\
H_{r+1}(C') & \rightarrow & H_{r+1}(C/C') \\
\downarrow d_{r+1} & & \downarrow d_{r+1} \\
\vdots & \vdots & \vdots \\
\end{array}$$

which, from the Snake Lemma, produces the long exact sequence

$$\ldots \rightarrow H^r(C') \rightarrow H^r(C) \rightarrow H^r(C/C') \rightarrow H^{r+1}(C') \rightarrow \ldots.$$ 

If $H(C') = 0$, then this becomes

$$\ldots \rightarrow 0 \rightarrow H^r(C) \rightarrow H^r(C/C') \rightarrow 0 \rightarrow \ldots,$$

in which case $H^r(C) \cong H^r(C/C')$ for all $r$, since the sequence is exact. Similarly, when $H(C/C') = 0$, we get that $H^r(C) \cong H^r(C')$ for all $r$. \qed
Just as we did for the Kauffman bracket, we will also use tangles in the Khovanov notation. When $T$ is a specific tangle in $D$, we write $[T]$ as an abbreviation of $[D]$. This allows us to also compare diagrams $D$ and $D'$ that differ in one tangle: we may refer to $[T]$ and $[T']$, where $T$ and $T'$ are the specified tangle and the diagrams are identical otherwise. One can verify that when we restrict the cube of an entire link to one with the crossing resolved in this manner, we still get an appropriate chain complex. To denote chain subcomplexes, we may also add a subscript to the notation, such as $[\{\bigcirc\}]_{v_+}$. To explain which subcomplex we mean, recall that the tensor factors in each chain group correspond to a circle in the smoothing. The notation of $[\{\bigcirc\}]$ marks a particular circle (the one in the tangle representation). So $[\{\bigcirc\}]_{v_+}$ means “the subspace of $[\{\bigcirc\}]$ in which the tensor factor corresponding to the marked circle is always $v_+$”. We will be careful to use this notation only when such a specified circle is in the tangle representation. Similarly, $[\{\bigcirc\}]_{v_+}=0$ means to quotient the complex by that subspace, “modding out by $v_+ = 0$”.

### 3.3.1 Invariance under the first Reidemeister move

To find $H(\{\bigcirc\})$, we produce the complex

$$C = [\{\bigcirc\}] = \left( [\{\bigcirc\}] \xrightarrow{m} [\{\bigcirc\}\{1\}] \right).$$

This has the subcomplex

$$C' = \left( [\{\bigcirc\}]_{v_+} \xrightarrow{m} [\{\bigcirc\}\{1\}] \right).$$

Every basis element of this subcomplex is of the form $v_+ \otimes w$, and recall that $m(v_+ \otimes w) = w$ (see the definition of $m$), so $m$ is an isomorphism when restricted

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to the subcomplex. This then means that \( \ker d = \im d \), so \( \mathcal{C}' \) is acyclic. By the first cancellation principle, \( H_*(\mathcal{C}) \cong H_*(\mathcal{C}/\mathcal{C}') \), and we can study the quotient complex

\[
\mathcal{C}/\mathcal{C}' = \left( \left[ \{ \} \right]_{/v_+=0} \rightarrow 0 \right).
\]

Now \( V/(v_+ = 0) = \mathbb{F}\langle v_+, v_- \rangle/(v_+ = 0) \cong \mathbb{F}\langle v_- \rangle \), which is one-dimensional. When considered as part of the tensor product, we note that \( \mathbb{F}\langle v_- \rangle \otimes U \) is isomorphic to \( U \) via the map that sends \( v_- \otimes u \mapsto u \). Thus, \( \left[ \{ \} \right]_{/v_+=0} \) is isomorphic to \( \left[ \{ \} \right] \). So their homologies are the same, except that the extra tensor factor in \( \left[ \{ \} \right] \) causes a degree shift. This is accounted for when we construct \( CKh(D) = [D][-n_-]\{n_+ - 2n_-\} \): \( \left[ \{ \} \right] \) has one more right-handed crossing than \( \left[ \{ \} \right] \), meaning \( CKh(\{ \} \) = \( CKh(\{ \} \)\{1\}. This degree shift up by 1 exactly cancels out the \(-1\) degree from the \( v_- \) factor we get in each element of \( \left[ \{ \} \right]_{/v_+=0} \), ensuring that the homologies agree on their \( q \) dimensions.

### 3.3.2 Invariance under the second Reidemeister move

![Figure 3.3: Incomplete cube of resolutions for the second Reidemeister move](image)
We begin with the crossed side of $\Omega_2$ and look to simplify it. The cube of resolutions (as far as we can smooth it) gives us the corresponding diagram $\mathcal{C}$:

$$
\begin{array}{ccc}
\mathcal{C} : \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\end{array}
$$

This has the subcomplex $\mathcal{C}'$:

$$
\begin{array}{ccc}
\mathcal{C}' : \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\end{array}
$$

For the same reasons as in Section 3.3.1, $[\mathcal{P}]_{v+k} \{1\}$ is isomorphic to $[\mathcal{Q}] \{2\}$ with a degree shift, and so $\mathcal{C}'$ is acyclic. So by the first cancellation principle, we reduce to the complex $\mathcal{C}/\mathcal{C}'$, which is

$$
\begin{array}{ccc}
\mathcal{C}/\mathcal{C}' : \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\end{array}
$$

Continuing to search for simplification, we produce the chain subcomplex $\mathcal{C}''$
which contains the chain complex of the other side of $\Omega_2$, albeit with some extra baggage: the shifts in degree of both $\{1\}$ and $[1]$. Now we would like to use the second cancellation principle, and so we look at $(C/C')/C'':$

\[
(C/C')/C'' : \quad \begin{array}{c}
\{ v \}\text{ } \overset{\Delta}{\longrightarrow} \quad 0 \\
\oplus \quad \quad \quad \oplus \\
0 \quad \longrightarrow \quad 0
\end{array}
\]

Examining the map induced by $\Delta$ onto the quotient space, $\Delta(v_+) = 0 \otimes v_- + v_- \otimes v_+ = v_- \otimes v_+$ and $\Delta(v_-) = v_- \otimes v_+$. So $\Delta$ is an isomorphism between these spaces, via the map that sends $u$ to $v_- \otimes u$. Thus, $(C/C')/C''$ is acyclic and we use both parts of the cancellation principle to state definitively that $H_*(C) \cong H_*(C'')$.

The extra null spaces in $C''$ do not affect the homology, and so the homologies of either side of $\Omega_2$ agree, but $\{ v \}$ has a height shift of 1 (from the shift in location of the chain) and a degree shift of 1 (since $C''$ has an extra $v_+$ term in each element). Once again, the normalization term $[-n_-] \{ n_+ - 2n_- \}$ solves this issue. We can see that $\{ v \}$ has one extra crossing of each handedness than $\{ \| \}$, so $CKh(\{ v \}) = CKh(\{ \| \})[-1] \{ -1 \}$. 

$\blacksquare$
3.3.3 Invariance under the third Reidemeister move

The resolution of each side of the third Reidemeister move is shown in Figure 3.4. This time, because both sides of $\Omega_3$ have crossings, we must work with two different chain
complexes $\mathcal{C}$ and $\mathcal{D}$. We will again search for subcomplexes and quotient complexes of these that simplify the cubes. If we can find an isomorphism between these simpler complexes, then they will also have the same Khovanov homology. This will then in turn, be passed onto the original complexes via our cancellation principle. We present the chain complexes $\mathcal{C}$ and $\mathcal{D}$, which have been rearranged to emphasize the cubical structure:

$\mathcal{C}$:

```
\[
\begin{array}{c}
\left[\gamma\gamma\right]\{1\} \quad \left[\gamma\gamma\right]\{2\} \quad \left[\gamma\gamma\right]\{3\} \\
\left[\gamma\gamma\right]\{1\} \quad \left[\gamma\gamma\right]\{2\} \quad \left[\gamma\gamma\right]\{3\}
\end{array}
\]
```

We can simplify the icons by a small isotopy to make it clearer where these two cubes differ.
Now we can see that the bottom squares of the cubes are exactly the same, and that the top squares are similar to the complex $\mathcal{C}$ we found in Section 3.3.2, but with an extra strand in each smoothing. However, trying to use the same chain subcomplex we used for $\Omega_2$ to quotient the top poses a problem, because the top squares are mirror images of one another. This mirroring causes the differential from the bottom face to the top face to be different in each cube, because the vector spaces they link together are different: for example, compare the $d_\ast 10$ in each diagram and note the destination space have different structures. Therefore, in order to find an isomorphism between these cubes, we must find a different chain subcomplex. We do so by introducing a symmetrizing map between vector spaces in the top layer.
We will describe the map as it works on $C$, and once defined it should be obvious how to form the same map on $D$. Also, because the bottom layers of the cube are already isomorphic, we use the trivial subspace $0$ for the bottom layer of each subcomplex, and omit that part of the diagram.

We start similarly to $\Omega_2$, and produce the subcomplex $C'$ and quotient complex $C/C'$.

\[
C' : \begin{array}{c}
\begin{tikzpicture}[baseline=0]
  \node (a) {$\left[\sum \right]_{v=1} \{2\}$};
  \node (b) [right of=a] {$\left[\sum \right] \{3\}$};
  \node (c) [below of=a] {$0$};
  \draw[->] (a) -- (b);
  \draw[->] (a) -- (c);
  \draw[->] (b) -- (c);
  \node at (a) {$\oplus$};
\end{tikzpicture}
\end{array}
\]

\[
C/C' : \begin{array}{c}
\begin{tikzpicture}[baseline=0]
  \node (a) {$\left[\sum \right]_{v=0=0} \{2\}$};
  \node (b) [right of=a] {$0$};
  \node (c) [below of=a] {$\left[\sum \right] \{1\}$};
  \node (d) [right of=c] {$\left[\sum \right] \{2\}$};
  \draw[->] (a) -- (b);
  \draw[->] (a) -- (c);
  \draw[->] (c) -- (d);
  \node at (a) {$\Delta$};
  \node at (b) {$\Delta$};
\end{tikzpicture}
\end{array}
\]

Note that in $C/C'$, the map $\Delta$ is a bijection, as we showed in Section 3.3.2. Thus we may define a new map $\tau : \left[\sum \right]_{v=0=0} \{2\} \to \left[\sum \right] \{2\}$ where $\tau = d_{1*0} \circ \Delta^{-1}$. Let $T = \{ (\beta, \tau \beta) \mid \beta \in \left[\sum \right]_{v=0=0} \{2\} \}$ and note that $T \cong \left[\sum \right]_{v=0=0} \{2\}$. Since $T$ is a subspace of $\left[\sum \right]_{v=0=0} \{2\} \oplus \left[\sum \right] \{2\}$, we may define $C'''$ to be the subcomplex of $C/C'$ as follows:

\[
\left[\sum \right] \{1\} \xrightarrow{(\Delta, d_{1*0})} T \xrightarrow{} 0.
\]

$C'''$ is acyclic because $\Delta$ is a bijection, so we may use the cancellation principle to study $(C/C')/C'''$.

\[
(C/C')/C''' : \begin{array}{c}
\begin{tikzpicture}[baseline=0]
  \node (a) {$\left[\sum \right]_{v=0=0} \{2\}$};
  \node (b) [right of=a] {$\left[\sum \right] \{2\}$};
  \node (c) [below of=a] {$0$};
  \node (d) [right of=c] {$0$};
  \draw[->] (a) -- (b);
  \draw[->] (a) -- (c);
  \draw[->] (b) -- (d);
  \draw[->] (c) -- (d);
\end{tikzpicture}
\end{array}
\]

40
We record an observation about representatives of elements in this quotient space. Any element of $[\gamma]\{2\} \oplus \bigoplus \begin{array}{c} \gamma' \\ \gamma \end{array}$ is a pair $(\beta, \gamma)$ with $\beta \in [\gamma]\{2\} \oplus \bigoplus \begin{array}{c} \gamma' \\ \gamma \end{array}$, $\gamma \in \bigoplus \begin{array}{c} \gamma' \\ \gamma \end{array}$. In the quotient space, there is an equivalence $(\beta, \gamma') \sim (0, \gamma - \tau \beta)$.

Claim. The map $f: [\gamma]\{2\} \rightarrow \bigoplus \begin{array}{c} \gamma' \\ \gamma \end{array} \oplus \begin{array}{c} \gamma' \\ \gamma \end{array} / T$ given by $\gamma \mapsto (0, \gamma) + T$ is an isomorphism.

Proof. From the above paragraph, we know that any $(\beta, \gamma) + T = (0, \gamma - \tau \beta) + T$. Then $\gamma - \tau \beta \mapsto (\beta, \gamma) + T$, so $f$ is surjective. Suppose $(\beta, \gamma) + T = T$. Then $(\beta, \gamma) \in T \implies \gamma = \tau \beta$ and $(\beta, \tau \beta) \sim (0, \tau \beta - \tau \beta) = (0, 0) = f(0)$, so $f$ is injective.

Now we return to the full cubes to see how the new quotient complexes $(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$ and $(\mathcal{D}/\mathcal{D}')/\mathcal{D}'''$ are isomorphic.

$(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$:
Here, in both cubes the top-right chain space is isomorphic (as a vector space) to $\left[ \gamma \right] \{2\}$ by the claim we just proved. In particular, in $(C/C')/C''$ this space looks like all equivalence classes $(\beta_1, \gamma_1) \sim (0, \gamma_1 - \tau_1 \beta_1)$, and we let $\nu_C$ be the isomorphism to $\left[ \gamma \right] \{2\}$ given by $[\beta_1, \gamma_1] \mapsto \gamma_1$. In $(D/D')/D''$ we instead have the relation $(\gamma_2, \beta_2) \sim (\gamma_2 - \tau_2 \beta_2, 0)$, and we let $\nu_D$ be the isomorphism to $\left[ \gamma \right] \{2\}$ given by $[\beta_2, \gamma_2] \mapsto \gamma_2$. Using these maps, we define an isomorphism

$$\Upsilon: \left( \left[ \gamma \right] \{2\} \oplus \left[ \frac{\gamma}{\nu+0} \right] \{2\} \right) / T_C \to \left( \left[ \frac{\gamma}{\nu+0} \right] \{2\} \oplus \left[ \frac{\gamma}{\nu+0} \right] \{2\} \right) / T_D$$

where $\Upsilon := -\nu_D^{-1} \circ \nu_C$, so that $\Upsilon([\beta_1, \gamma_1]) = [-\gamma_2, -\beta_2]$ whenever $\gamma_1 = -\gamma_2$. We extend $\Upsilon$ to the rest of the chain complex by the identity map, and continue to call this new map $\Upsilon$. We claim $\Upsilon$ is a chain isomorphism between $(C/C')/C''$ and $(D/D')/D''$.

Claim. The following diagram is commutative.

$$(C/C')/C'' \xrightarrow{\Upsilon} (D/D')/D'':$$

$$\begin{array}{c}
\left[ \frac{\gamma}{\nu+0} \right] \{2\} \oplus \left[ \frac{\gamma}{\nu+0} \right] \{2\} \oplus 0 \\
\downarrow \Upsilon=\text{id} \quad \downarrow \Upsilon=\text{id} \quad \downarrow \Upsilon=\text{id} \\
\left[ \frac{\gamma}{\nu+0} \right] \{2\} \oplus \left[ \frac{\gamma}{\nu+0} \right] \{2\} \oplus 0 \\
\end{array}$$

$$\begin{array}{c}
d_1 \\
\downarrow \Upsilon=\text{id} \\
d_2 \\
\end{array}$$

$$\begin{array}{c}
\left[ \frac{\gamma}{\nu+0} \right] \{2\} \oplus \left[ \frac{\gamma}{\nu+0} \right] \{2\} \oplus 0 \\
\downarrow \Upsilon=\text{id} \\
\left[ \frac{\gamma}{\nu+0} \right] \{2\} \oplus \left[ \frac{\gamma}{\nu+0} \right] \{2\} \oplus 0 \\
\end{array}$$

$$\begin{array}{c}
d_1 \\
\downarrow \Upsilon=\text{id} \\
d_2 \\
\end{array}$$

$$\begin{array}{c}
\left[ \frac{\gamma}{\nu+0} \right] \{2\} \oplus \left[ \frac{\gamma}{\nu+0} \right] \{2\} \oplus 0 \\
\downarrow \Upsilon=\text{id} \\
\left[ \frac{\gamma}{\nu+0} \right] \{2\} \oplus \left[ \frac{\gamma}{\nu+0} \right] \{2\} \oplus 0 \\
\end{array}$$

$$\begin{array}{c}
d_1 \\
\downarrow \Upsilon=\text{id} \\
d_2 \\
\end{array}$$
Note that we have suppressed the height shifts in each chain space for brevity. The right square commutes because the destination is trivial. In the left square, $d_1 = d_2$ since the smoothings involved are identical, and so the square commutes. The middle square needs a more thorough examination. $\Upsilon$ is the identity on the final $\left\lceil \frac{\gamma}{\gamma} \right\rceil$ component of the 3rd column as well. So for the same reason as the left square, and because the maps interact component-wise, we only need to show that $\Upsilon \circ d_1(x, y) = d_2(x, y)$ for $(x, y) \in \left\lceil \frac{\gamma}{\gamma} \right\rceil \oplus \left\lceil \frac{\lambda}{\lambda} \right\rceil$.

Evaluating the maps, we get that $\Upsilon \circ d_1(x, y) = \Upsilon(d_1,*) \circ 0_1 \circ \Delta^{-1}$ and $d_2(x, y) = (d_2,*) \circ 0_1 \circ \Delta^{-1}$. Since $(d_1,*) \sim (0, d_1,\tau_1)$ and $(d_2,*) \sim (d_2,\tau_2)$, for $\Upsilon$ to commute here we need $d_1,\tau_1 = d_2,\tau_2$. In fact, we claim that $d_1,\tau_1 = d_2,\tau_2$ and $\tau_1 d_1,\tau_1 = d_2,\tau_2$, which gives us the result. The proof of this claim comes from carefully examining the smoothings corresponding to the spaces the maps act on.

$\tau_1 d_1,\tau_1 = d_2,\tau_2$:

\[
\begin{array}{c}
\begin{array}{ccc}
\varnothing_{/v,=0} & \xrightarrow{\Delta^{-1}} & \varnothing \\
\cong d_1,\tau_1 & \xrightarrow{\tau_1 d_1,\tau_1} & \varnothing \\
\mapstomap{\varnothing} & \xrightarrow{id} & \varnothing \\
\mapstomap{\varnothing} & \xrightarrow{d_2,\tau_2} & \varnothing \\
\end{array}
\end{array}
\]

$d_1,\tau_1 = \tau_2 d_2,\tau_2$:
The fact that the maps to and from \([ \nabla C ]_{v_+ = 0} \) are isomorphisms is explained in Section 3.3.2. Examining \( d_{1,1} \ast 0 \) and \( d_{2,1} \ast 0 \), the corresponding cobordisms are either a split or merge between the same strands (see figure 3.5). Since the differentials are determined by the cobordism and the spaces are isomorphic, the maps are the same. So the above diagrams commute.

![Diagram](image)

(a) \( d_{1,1} \ast 0 = d_{2,1} \ast 0 \)

(b) \( d_{2,1} \ast 0 = d_{1,1} \ast 0 \)

Figure 3.5: Differential comparison

With this final equivalence, we get that \( (C/C')/C'' \) and \( (D/D')/D'' \) are isomorphic chain complexes, and therefore have the same homology. The cancellation principle
passes this isomorphism up the sequence of sub- and quotient complexes, so \( H_*(\mathcal{C}) = H_*(\mathcal{D}) \).

\[ \square \]

### 3.4 Results from Khovanov Homology

One major result is that Khovanov homology detects the unknot.

**Theorem 3.4.1** (Kronheimer and Mrowka [7]). *A knot is the unknot if and only if its Khovanov homology is rank 1.*

This unassuming result is in fact quite the achievement. In particular, it is not known whether or not the Jones polynomial is an unknot detector. The extra structure of Khovanov homology is enough to do so. It has also been shown to detect the left and right trefoils [2] and the figure eight knot [1], as well as a few more knots. However, it is not powerful enough to detect everything: it is possible to construct different knots with the same Khovanov homology [9].


