ON THE NUMERICAL RANGE OF COMPACT OPERATORS

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Montserrat Dabkowski

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COMMITTEE MEMBERSHIP

TITLE:	On The Numerical Range of Compact Operators
AUTHOR:	Montserrat Dabkowski
DATE SUBMITTED:	June 2022
COMMITTEE CHAIR:	Linda Patton, Ph.D. Professor of Mathematics
COMMITTEE MEMBER:	Dylan Retsek, Ph.D. Professor of Mathematics
COMMITTEE MEMBER:	Emily Hamilton, Ph.D. Professor of Mathematics

ABSTRACT

On The Numerical Range of Compact Operators

Montserrat Dabkowski

One of the many characterizations of compact operators is as linear operators which can be closely approximated by bounded finite rank operators (theorem 25). It is well known that the numerical range of a bounded operator on a finite dimensional Hilbert space is closed (theorem 54). In this thesis we explore how close to being closed the numerical range of a compact operator is (theorem 56). We also describe how limited the difference between the closure and the numerical range of a compact operator can be (theorem 58). To aid in our exploration of the numerical range of a compact operator we spend some time examining its spectra, as the spectrum of a bounded operator is closely tied to its numerical range (theorem 45). Throughout, we use the forward shift operator and the diagonal operator (example 1) to illustrate the exceptional behavior of compact operators.

DEDICATION

To my parents, Ron and Rachael.

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Chapter 1

BACKGROUND

We begin with some definitions and useful results we wish to remind the reader of. We will omit the proofs, which can be found in the provided references.

Axiom 1 (Axiom of Completeness)**.** *Every nonempty set of real numbers that is bounded has a supremum and an infimum.*

Theorem 1 (Archimedean Property. [1], Theorem 1.4.2)**.**

1. *If $x \in \mathbb{R}$, then there exists some $n \in \mathbb{N}$ such that $n > x$*
2. *If $y \in \mathbb{R}$ is such that $y > 0$, then there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < y$*

Theorem 2 (Cayley-Hamilton Theorem. [3], Theorem 8.37)**.** *Let V be a complex vector space and $T \in \mathcal{L}(V)$. If p is the characteristic polynomial of T, then $p(T) = 0$.*

Theorem 3 ([2], Theorem 6.11)**.** *If V, W are metric spaces and $T : V \longrightarrow W$ is a function, then the following are equivalent:*

1. *T is continuous*
2. *$\lim_{n\to\infty} f_n = f$ in V implies $\lim_{n\to\infty} T(f_n) = Tf$ in W*
3. *$T^{-1}(G)$ is an open subset of V for every open set $G \subseteq W$*

4. $T^{-1}(F)$ *is a closed subset of* V *for every closed set* $F \subseteq W$

Theorem 4 (Cauchy-Schwarz Inequality. [2], Theorem 8.11)**.** *Let* V *be an inner product space and* $f, g \in V$*. Then,* $|\langle f, g\rangle| \leq \|f\| \, \|g\|$*. Equality holds if and only if* f, g *are scalar multiples of one another.*

Def: A **Banach space** is a complete normed vector space.

Theorem 5 (Open Mapping Theorem. [4], Theorem 12.1)**.** *If* V*,* W *are Banach spaces,* G *is open in* V*, and* $T : V \longrightarrow W$ *is continuous and onto, then* $T(G)$ *is open in* W*.*

Def: Let V, W be normed vector spaces and $T \in \mathcal{L}(V, W)$. The **norm** of T, denoted $\|T\|$, is defined by $\|T\| = \sup\{\|Tf\| : f \in V \text{ and } \|f\| = 1\}$. We say that T is **bounded** if $\|T\| < \infty$. We use $\mathcal{B}(V, W)$ to denote the set of bounded linear maps from V to W.

Theorem 6 ([2], Theorem 6.48)**.** *Let* V*,* W *be normed vector spaces and* $T \in \mathcal{L}(V, W)$*. Then,* T *is continuous if and only if* T *is bounded.*

Def: A **Hilbert space**, H, is an inner product space, such that under the norm induced by the inner product H is a Banach space.

Theorem 7 ([2], Theorem 6.16)**.** *Let* H *be a complete metric space. If* U *is a closed subspace of* H*, then* U *is complete.*

Note. *A subspace of a Hilbert space is not necessarily Hilbert. However, since Hilbert spaces are complete metric spaces, theorem 7 tells us that a closed subspace of a Hilbert space is Hilbert.*

Def: Let V be an inner product space and $U \subseteq V$. The **orthogonal complement** of U, denoted $U^{\perp}$, is the set $U^{\perp} = \{v \in V : \langle u, v \rangle \text{ for all } u \in U\}$.

Theorem 8 ([2], Theorem 8.40a)**.** *Let V be an inner product space and $U \subseteq V$, then $U^{\perp}$ is a subspace of V.*

Note. *We will be using the notation $\overline{U}$ to denote the closure of U.*

Theorem 9 ([2], Theorem 8.42)**.** *Let H be a Hilbert space and U be a subspace of H. Then, $\overline{U} = H$ if and only if $U^{\perp} = \{0\}$.*

Def: Let H be a Hilbert space and $\{h_n : n \in \mathbb{N}\} \subseteq H$. We say $\{h_n : n \in \mathbb{N}\}$ is a **basis** for H, if for each $h \in H$ there exist $h_{n_1}, \ldots, h_{n_k} \in \{h_n : n \in \mathbb{N}\}$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$, such that $h = \alpha_1 h_{n_1} + \cdots + \alpha_k h_{n_k}$.

Def: Let H be a Hilbert space and let

$$S = \{h_n : n \in \mathbb{N}, \langle h_n, h_m \rangle = 0 \text{ if } n \neq m \text{ and } \langle h_n, h_m \rangle = 1 \text{ otherwise}\} \subseteq H$$

We say S is an **orthonormal basis** for H, if for each $h \in H$ there exist $\alpha_n \in \mathbb{C}$ such that $h = \sum_{n=1}^{\infty} \alpha_n h_n$.

Theorem 10 (Tychonoff-Alaoglu Theorem. [8], Problem 17)**.** *Let H be a Hilbert space and B be the unit ball in H. Then, $\overline{B}$ is weakly compact.*

Theorem 11 (Heine-Borel Theorem. [13], Theorem 241)**.** *Let $K \subseteq \mathbb{R}^n$. Then, K is compact if and only if K is closed and bounded.*

Def: Let V, W be inner product spaces and $T \in \mathcal{L}(V, W)$. The **adjoint** of T is the function $T^* \in \mathcal{L}(W, V)$ defined by $\langle Tf, g\rangle = \langle f, T^*g\rangle$ for every $f \in V$ and every $g \in W$. We say T is **self-adjoint** if $T^* = T$. We say T is **normal** if $T^*T = TT^*$.

Theorem 12 ([2], Theorem 10.11)**.** *Let V, W be Hilbert spaces and $T \in \mathcal{B}(V, W)$. Then, $T^* \in \mathcal{B}(W, V)$, $(T^*)^* = T$, and $\|T^*\| = \|T\|$.*

Theorem 13 ([2], Theorem 10.12)**.** *If V, W, and U are Hilbert spaces, then:*

1. $(S + T)^* = S^* + T^*$ *for all* $S, T \in \mathcal{B}(V, W)$
2. $(\alpha T)^* = \overline{\alpha} T^*$ *for all* $\alpha \in \mathbb{F}$ *and all* $T \in \mathcal{B}(V, W)$
3. $I^* = I$
4. $(S \circ T)^* = T^* \circ S^*$ *for all* $T \in \mathcal{B}(V, W)$ *and all* $S \in \mathcal{B}(W, U)$

Theorem 14 ([2], Theorem 10.13)**.** *If V, W are Hilbert spaces and $T \in \mathcal{B}(V, W)$, then:*

1. $\mathit{null}\, T^* = (\mathit{range}\, T)^\perp$
2. $\overline{\mathit{range}\, T^*} = (\mathit{null}\, T)^\perp$
3. $\mathit{null}\, T = (\mathit{range}\, T^*)^\perp$
4. $\overline{\mathit{range}\, T} = (\mathit{null}\, T^*)^\perp$

Theorem 15 ([2], Theorem 10.14)**.** *Let V, W be Hilbert spaces and $T \in \mathcal{B}(V, W)$. Then, T has dense range if and only if T^* is one-to-one.*

Theorem 16 ([2], Theorem 10.19)**.** *Let H be a Hilbert space. $T \in \mathcal{B}(H)$ is invertible if and only if T^* is invertible. Futhermore, if T is invertible, then $(T^*)^{-1} = (T^{-1})^*$.*

Theorem 17 ([2], Theorem 10.22)**.** *Let V be a Banach space. If $T \in \mathcal{B}(V)$ and $\|T\| < 1$, then $I - T$ is invertible and $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$.*

Theorem 18 ([2], Theorem 10.25)**.** *Let V be a Banach space. Then, $\{T \in \mathcal{B}(V) : T \text{ is invertible}\}$ is an open subset of $\mathcal{B}(V)$.*

Theorem 19 ([2], Theorem 10.38)**.** *Let H be a nonzero complex Hilbert space and $T \in \mathcal{B}(H)$. Then, $\{\alpha \in \mathbb{C} : T - \alpha I \text{ is not invertible}\}$ is a nonempty subset of $\mathbb{C}$.*

Theorem 20 ([2], Theorem 10.54)**.** *Let H be a complex Hilbert space and $T \in \mathcal{B}(H)$. Then,*

1. *There exist unique self-adjoint operators $A = \dfrac{T^* + T}{2}$ and $B = \dfrac{i(T^* - T)}{2}$, such that $T = A + iB$*

2. *T is normal if and only if $AB = BA$*

Theorem 21 ([2], Theorem 10.57)**.** *Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be normal. If α, β are distinct eigenvalues of T, then the corresponding eigenvectors are orthogonal.*

Chapter 2

COMPACT OPERATORS

In this chapter we introduce compact operators. We include just enough proofs so that someone wanting to read about numerical ranges of compact operators (typically not included in most standard introductions to compact operators) can obtain enough background on compact operators to follow the later work with their numerical ranges.

Def: An operator T defined on a Hilbert space H is said to be **compact** if for every bounded sequence (f_n) in H, the sequence (Tf_n) has a convergent subsequence. We use $\mathcal{C}(H)$ to denote the set of all compact operators on H.

Theorem 22 ([2], Theorem 10.67)**.** *Let H be a Hilbert space and $T \in \mathcal{B}(H)$. If range T is finite, then $T \in \mathcal{C}(H)$.*

The following proof is adapted from the proof given in [2].

Proof. Assume H is a Hilbert space, $T \in \mathcal{B}(H)$, and $\dim(\text{range } T) < \infty$. Since range T is finite dimensional, there exists a finite basis for range T. Applying the Gram-Schmidt procedure, we obtain a finite orthonormal basis $\{e_1, \ldots, e_m\}$ for range T. Let (f_n) be a bounded sequence in H. Since $\{e_1, \ldots, e_m\}$ is an orthonormal basis for range T, for each $n \in \mathbb{N}$, we can write $Tf_n = \langle Tf_n, e_1\rangle e_1 + \cdots + \langle Tf_n, e_m\rangle e_m$. For each $n \in \mathbb{N}$ and each $j \in \{1, \ldots, m\}$,

$$|\langle Tf_n, e_j\rangle| \leq \|Tf_n\| \, \|e_j\| \tag{2.1}$$

$$
\begin{aligned}
&= \|Tf_n\| \qquad (2.2)\\
&\leq \|T\|\,\|f_n\|\\
&\leq \|T\| \sup\{\|f_n\| : n \in \mathbb{N}\}
\end{aligned}
$$

Line (2.1) follows from the Cauchy-Schwarz inequality. Line (2.2) follows from the e_j's being of unit length. Thus, we have shown that for each $j \in \{1, \ldots, m\}$, $(\langle Tf_n, e_j \rangle)$ is a bounded sequence in $\mathbb{C}$. Thus, by the complex version of the Bolzano-Weierstrass theorem, for each $j \in \{1, \ldots, m\}$, there exists some convergent subsequence $(\langle Tf_{n_k}, e_j \rangle)$. Thus, by taking subsequences of subsequences, we can find a subsequence (Tf_{n_l}) such that $\lim_{l\to\infty} \langle Tf_{n_l}, e_j \rangle = \alpha_j \in \mathbb{C}$ for every $j \in \{1, \ldots, m\}$. Therefore, by the algebraic limit theorem, $\lim_{l\to\infty} Tf_{n_l} = \alpha_1 e_1 + \cdots + \alpha_m e_m \in H$. Thus, we have shown that $T \in \mathcal{C}(H)$, as desired. □

The following is a prototypical compact operator and will be used as a basic example in the following sections.

Example 1. *Let (b_n) be a sequence in $\mathbb{F}$ such that $\lim_{n\to\infty} b_n = 0$. Define $T \in \mathcal{B}(\ell^2)$ by $T(a_1, a_2, \ldots) = (a_1 b_1, a_2 b_2, \ldots)$. T is compact.*

Proof. For each $n \in \mathbb{N}$, define $T_n \in \mathcal{B}(\ell^2)$ by

$$T_n(a_1, a_2, \ldots) = (a_1 b_1, \ldots, a_n b_n, 0, \ldots)$$

(i.e. T_n is the same as T for the first n entries, and has 0's in all the other entries). For a fixed $n \in \mathbb{N}$, elements of range T_n have the form

$$(a_1 b_1, \ldots, a_n b_n, 0, \ldots) = a_1 b_1 (1, 0, \ldots) + a_2 b_2 (0, 1, 0, \ldots) + \cdots + a_n b_n (0, \ldots, 0, 1, 0, \ldots).$$

Thus, for a fixed $n \in \mathbb{N}$, each element of range T_n can be written as a linear combination of n elements. Since the standard basis vectors are clearly linearly independent, $\{(0, \ldots, 0, 1, 0, \ldots)_i : i \in \{1, \ldots, n\}$ and $b_i \neq 0\}$ is a basis for range T_n. Therefore, dim(range T_n)$< \infty$. Since bounded operators with finite range are compact (this is theorem 22), it follows that T_n is compact for each $n \in \mathbb{N}$.

Next, we show that $\lim\limits_{n\to\infty} T_n = T$. This is equivalent to showing $\lim\limits_{n\to\infty} ||T_n - T|| = 0$. Let $\epsilon > 0$. By definition,

$$||T - T_n|| = \sup\{||(T - T_n)(a_n)|| : (a_n) \in \ell^2, ||(a_n)|| = 1\}$$

So, it suffices to show $||(T - T_n)(a_n)|| < \epsilon$ for an arbitrary $(a_n) \in \ell^2$ with $||(a_n)|| = 1$.

Since $||(a_n)|| = 1$, we have that $\sqrt{\sum\limits_{n=0}^{\infty} |a_n|^2} = 1$, and so $\sum\limits_{n=0}^{\infty} |a_n|^2 = 1$. Hence, because $\epsilon^2 > 0$, there exists $N_1 \in \mathbb{N}$, such that whenever $n_1 \geq N_1$, $\sum\limits_{i=n_1}^{\infty} |a_i|^2 < \epsilon^2$. Since $\lim\limits_{n\to\infty} b_n = 0$, and $1 > 0$, there exists some $N_2 \in \mathbb{N}$, such that whenever $n_2 \geq N_2$, $|b_n| < 1$. Let $N = \max\{N_1, N_2\}$.

By definition,

$$\begin{aligned} ||(T - T_n)(a_n)|| &= ||(0, \ldots, 0, a_{n+1}b_{n+1}, a_{n+2}b_{n+2}, \ldots)|| \\ &= \sqrt{\sum_{i=n+1}^{\infty} |a_i b_i|^2} \end{aligned}$$

Thus, $||(T - T_n)(a_n)||^2 = \sum\limits_{i=n+1}^{\infty} |a_i b_i|^2$.

So, letting $n \geq N$, we have that

$$\begin{aligned} ||(T - T_n)(a_n)||^2 &= \sum_{i=n+1}^{\infty} |a_i b_i|^2 \\ &= \sum_{i=n+1}^{\infty} (|a_i||b_i|)^2 \\ &= \sum_{i=n+1}^{\infty} |a_i|^2 |b_i|^2 \\ &< \sum_{i=n+1}^{\infty} |a_i|^2 \cdot 1 \qquad (2.3) \\ &= \sum_{i=n+1}^{\infty} |a_i|^2 \\ &< \varepsilon^2 \qquad (2.4) \end{aligned}$$

Line (2.3) follows from $|b_i|^2 < 1$, because $i \geq n+1 \geq N+1 \geq N_2 + 1 > N_2$. Line (2.4) follows from $\sum_{i=n+1}^{\infty} |a_i|^2 < \epsilon^2$, because $i \geq n+1 \geq N+1 \geq N_1 + 1 > N_1$.

Since norms are always non-negative and $\varepsilon > 0$, we can take the square root of both sides of the above inequality (without worrying about it changing directions), to find $||(T - T_n)(a_n)|| < \varepsilon$ whenever $n \geq N$. Thus, by the definition of supremum, $||T - T_n|| < \varepsilon$, whenever $n \geq N$. Thus, $\lim_{n\to\infty} ||T_n - T|| = 0$, and so $\lim_{n\to\infty} T_n = T$.

What we have shown is that (T_n) has a limit point T. Since $\mathcal{C}(\ell^2)$ is a closed subset of $\mathcal{B}(\ell^2)$ and (T_n) is a sequence in $\mathcal{C}(\ell^2)$, it follows that $T \in \mathcal{C}(\ell^2)$. □

Theorem 23 ([2], Theorem 10.68)**.** *Let H be a Hilbert space. If $T \in \mathcal{C}(H)$, then $T \in \mathcal{B}(H)$.*

The following proof is adapted from the proof given in [2].

Proof. Assume H is a Hilbert space and $T \in \mathcal{H}$. Seeking a contradiction, assume that T is not bounded. Then, there exists some bounded sequence (f_n) in H, such that $\lim_{n\to\infty} \|Tf_n\| = \infty$. Since T is compact, there exists some convergent subsequence (Tf_{n_k}) converging to some $y \in H$. Since H is a Hilbert space, it follows that $\|y\| < \infty$. If we consider only $(\|Tf_{n_k}\|)$ and $(\|Tf_n\|)$, we cannot have one converging to something with finite norm and the other not. Thus, no subsequence of (Tf_n) can converge, contradicting that $T \in \mathcal{C}(H)$. Therefore, we must conclude that $T \in \mathcal{B}(H)$, as desired. □

Note. *It seems intuitive that $\mathcal{C}(H)$ is a subspace of $\mathcal{B}(H)$. This is true and can be easily checked, so we omit its proof. More is true, and is summarized in theorem 24.*

Theorem 24 ([2], Theorem 10.69a)**.** *If H is a Hilbert space, then $\mathcal{C}(H)$ is a closed subspace of $\mathcal{B}(H)$.*

The proof of theorem 24 is omitted and can be found in Axler's book ([2]).

Note. *Compact operators are considered nice because they behave similarly to finite rank operators. This is summarized in the following theorem.*

Theorem 25. *Let H be an infinite dimensional Hilbert space and $T \in \mathcal{B}(H)$. $T \in \mathcal{C}(H)$ if and only if T is the limit of a sequence of operators in $\mathcal{B}(H)$ with finite dimensional range.*

Lemma 26. *Let H be a Hilbert space and $T \in \mathcal{C}(H)$. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for $\overline{\text{range}\, T}$ and P_n be the orthogonal projection of H onto span$\{e_1, \ldots, e_n\}$. Then,* $\lim_{n\to\infty} \|P_n T - T\| = 0$.

Proof of lemma 26. Let H be a Hilbert space and $T \in \mathcal{C}(H)$. Also, let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for $\overline{\text{range}\, T}$ and P_n be the orthogonal projection of H onto span$\{e_1, \ldots, e_n\}$. Seeking a contradiction, suppose $\lim_{n\to\infty} \|P_n T - T\| \neq 0$. Then, there exists some $\delta > 0$ and a sequence (n_j) in $\mathbb{N}$, such that $\left\|P_{n_j} T - T\right\| \geq \delta$. For ease of notation, we will refer to the sequence (P_{n_j}) as simply (P_n). We claim that there exists some sequence (x_n) in H with $\|x_n\| = 1$ such that $\|(P_n T - T)x_n\| \geq \dfrac{\delta}{2}$. To prove this claim, we suppose not. Then, for every $x \in H$ with $\|x\| = 1$, $\|(P_n T - T)x\| < \dfrac{\delta}{2}$. This implies that $\frac{\delta}{2}$ is an upper bound on $\{\|(P_n T - T)x\| : x \in H \text{ with } \|x\| = 1\}$. Since $\|P_n T - T\| = \sup\{\|(P_n T - T)x\| : x \in H \text{ with } \|x\| = 1\}$, it follows by the definition of supremum that $\|P_n T - T\| \leq \dfrac{\delta}{2}$. This contradicts that $\|P_n T - T\| \geq \delta$. Thus, we must conclude that there exists some sequence (x_n) contained in H with $\|x_n\| = 1$, such that $\|(P_n T - T)x_n\| \geq \dfrac{\delta}{2}$ as previously claimed.

Since $T \in \mathcal{C}(H)$ and (x_n) is bounded, (Tx_n) must have some convergent subsequence. Let (Tx_{n_k}) denote this convergent subsequence, and let $y \in H$ be the element it converges to. Let $\varepsilon > 0$. Then, there exists some $K_1 \in \mathbb{N}$ such that whenever $k \geq K_1$, $\|Tx_{n_k} - y\| < \dfrac{\varepsilon}{4}$.

Since $(Tx_{n_k}) \longrightarrow y$, we know $y \in \overline{\text{range}\, T}$. Thus, there exist some $\alpha_i \in \mathbb{C}$, such that $y = \sum_{i=1}^{\infty} \alpha_i e_i$. That is, there exists some $N \in \mathbb{N}$, such that whenever $n \geq N$,

$\left\| \sum_{i=1}^{n} \alpha_i e_i - y \right\| < \varepsilon$. Note that for $n \geq N$,

$$\begin{aligned}
\|P_n(y) - y\| &= \left\| P_n \left(\sum_{i=1}^{\infty} \alpha_i e_i \right) - y \right\| \\
&= \left\| \sum_{i=1}^{\infty} \alpha_i P_n(e_i) - y \right\| && (2.5) \\
&= \left\| \sum_{i=1}^{n} \alpha_i e_i - y \right\| && (2.6) \\
&< \varepsilon && (2.7)
\end{aligned}$$

Line (2.5) follows from the continuity of P_n, since $P_n \in \mathcal{B}(H)$. Line (2.6) follows from the definition of P_n. Line (2.7) follows from the fact that $n \geq N$. Thus, we have shown that $\lim_{n\to\infty} P_n y = y$. Therefore, the subsequence $(P_{n_k} y)$ must also converge to y. Thus, there exists some $K_2 \in \mathbb{N}$ such that whenever $k \geq K_2$, $\|P_{n_k} y - y\| < \frac{\varepsilon}{2}$.

Let $K = \max\{K_1, K_2\}$. Suppose $k \geq K$. Now,

$$\begin{aligned}
\|(P_{n_k}T - T)x_{n_k}\| &= \|(P_{n_k} - I)Tx_{n_k}\| \\
&= \|(P_{n_k} - I)Tx_{n_k} + (P_{n_k} - I)y - (P_{n_k} - I)y\| \\
&= \|(P_{n_k} - I)(Tx_{n_k} - y) + (P_{n_k} - I)y\| \\
&\leq \|(P_{n_k} - I)(Tx_{n_k} - y)\| + \|(P_{n_k} - I)y\| && (2.8) \\
&\leq \|(P_{n_k} - I)\| \, \|(Tx_{n_k} - y)\| + \|(P_{n_k} - I)y\| \\
&\leq (\|P_{n_k}\| + \|I\|) \, \|(Tx_{n_k} - y)\| + \|(P_{n_k} - I)y\| \\
&= 2 \, \|(Tx_{n_k} - y)\| + \|(P_{n_k} - I)y\| && (2.9) \\
&< 2 \left(\frac{\varepsilon}{4} \right) + \|P_{n_k} y - y\| && (2.10) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && (2.11)
\end{aligned}$$

$$= \varepsilon$$

Line (2.8) follows from the triangle inequality. Line (2.9) follows from the fact that orthogonal projection operators have norm 1. Line (2.10) follows from the fact that $k \geq K \geq K_1$. Line (2.11) follows from the fact that $k \geq K \geq K_2$. Thus, we have shown that $\|(P_{n_k}T - T)x_{n_k}\| < \varepsilon$. This contradicts that $\|(P_nT - T)x_n\| \geq \dfrac{\delta}{2}$ for every $n \in \mathbb{N}$, since we have just shown we can make a subsequence arbitrarily small. Thus, we must conclude that $\lim_{n\to\infty} \|P_nT - T\| = 0$. □

Proof of theorem 25. ($\Longrightarrow$) Suppose $T \in \mathcal{C}(H)$. Since H is a Hilbert space and $\overline{\text{range}\, T}$ is closed, it follows by theorem 7 that $\overline{\text{range}\, T}$ is a Hilbert space. Thus, there exists an orthonormal basis for $\overline{\text{range}\, T}$, which we will denote $\{e_n : n \in \mathbb{N}\}$. Let P_n denote the orthogonal projection of H onto $\text{span}\{e_1, \ldots, e_n\}$. By lemma 26, we know $\lim_{n\to\infty} P_nT = T$. Since $T \in \mathcal{C}(H)$, by theorem 23 $T \in \mathcal{B}(H)$. And since $P_n \in \mathcal{B}(H)$ for each $n \in \mathbb{N}$, it follows that $P_nT \in \mathcal{B}(H)$ for each $n \in \mathbb{N}$. Also note that $\dim(\text{range } P_nT) \leq \dim(\text{span}\{e_1, \ldots, e_n\}) = n$. So, P_nT has finite dimensional range for each $n \in \mathbb{N}$. Thus, by lemma 26 we have shown that T is the limit of a sequence of operators in $\mathcal{B}(H)$ with finite dimensional range.

($\Longleftarrow$) Assume $T \in \mathcal{B}(H)$ is such that $\lim_{n\to\infty} T_n = T$, where $T_n \in \mathcal{B}(H)$ and $\dim(\text{range } T_n) < \infty$ for each $n \in \mathbb{N}$. Then, by theorem 22, $T_n \in \mathcal{C}(H)$ for each $n \in \mathbb{N}$. Since $\mathcal{C}(H)$ is closed (this is theorem 24), it follows that $T \in \mathcal{C}(H)$.

Therefore, $T \in \mathcal{C}(H)$ if and only if it is the limit of a sequence of operators in $\mathcal{B}(H)$ with finite dimensional range. □

Theorem 27 ([2], Theorem 10.69b)**.** *Let H be a Hilbert space. If $T \in \mathcal{C}(H)$ and $S \in \mathcal{B}(H)$, then ST, $TS \in \mathcal{C}(H)$.*

The following proof is adapted from the proof given in [2].

Proof. Let H be a Hilbert space. Assume $T \in \mathcal{C}(H)$ and $S \in \mathcal{B}(H)$. Let (f_n) be some bounded sequence in H. Since $T \in \mathcal{C}(H)$, there exists some convergent subsequence (Tf_{n_k}) of (Tf_n). Let us call the element (Tf_{n_k}) converges to y. Then, $\lim_{k\to\infty} Tf_{n_k} = y$. Since S is a bounded linear operator, it is continuous. Thus, $\lim_{k\to\infty} S(Tf_{n_k}) = S(\lim_{k\to\infty} Tf_{n_k}) = Sy$. Thus, we have shown that (STf_n) has a convergent subsequence (STf_{n_k}). Thus, by definition we have $ST \in \mathcal{C}(H)$.

Now, we turn to showing that TS is compact. Since (f_n) is bounded there exists some $M > 0$ such that $\|f_n\| \leq M$ for every $n \in \mathbb{N}$. Since $S \in \mathcal{B}(H)$, $\|S\| < \infty$. Now,

$$\begin{aligned} \|Sf_n\| &\leq \|S\| \, \|f_n\| \\ &< \|S\| \, M \qquad (2.12) \\ &< \infty \qquad (2.13) \end{aligned}$$

Line (2.12) follows from (f_n) being bounded. Line (2.13) follows from S being bounded. Thus, we have shown that (Sf_n) is a bounded sequence. Since T is compact, (TSf_n) must have a convergent subsequence. Thus, $TS \in \mathcal{C}(H)$. □

Theorem 28 ([2], Theorem 10.73)**.** *Let H be a Hilbert space. $T \in \mathcal{C}(H)$ if and only if $T^* \in \mathcal{C}(H)$.*

The following proof is adapted from the proof given in [2].

Proof. ($\Longrightarrow$) Assume $T \in \mathcal{C}(H)$. By theorem 23, $T \in \mathcal{B}(H)$, hence $T^* \in \mathcal{B}(H)$, since $\|T\| = \|T^*\|$ (this is theorem 12). Since $\mathcal{C}(H)$ is a two-sided ideal, it follows that $TT^* \in \mathcal{C}(H)$. Let (f_n) be a bounded sequence in H. Thus, there exists some $M > 0$, such that $\|f_n\| < M$ for each $n \in \mathbb{N}$. Let $\varepsilon > 0$. Since $T \in \mathcal{C}(H)$, there exists some convergent subsequence of $(TT^* f_n)$. We will call this subsequence $(TT^* f_{n_j})$. Since $(TT^* f_{n_j})$ convergent and H is a Hilbert space, it follows that $(TT^* f_{n_j})$ is Cauchy. Thus, there exists some $N \in \mathbb{N}$, such that whenever $n_j, n_k \geq N$, $\left\|TT^* f_{n_j} - TT^* f_{n_k}\right\| < \dfrac{\varepsilon^2}{2M}$. Now,

$$\begin{aligned}
\left\|T^* f_{n_j} - T^* f_{n_k}\right\|^2 &= \langle T^* f_{n_j} - T^* f_{n_k}, T^* f_{n_j} - T^* f_{n_k}\rangle \\
&= \langle T^*(f_{n_j} - f_{n_k}), T^*(f_{n_j} - f_{n_k})\rangle \\
&= \langle f_{n_j} - f_{n_k}, TT^*(f_{n_j} - f_{n_k})\rangle && (2.14) \\
&\leq \left\|f_{n_j} - f_{n_k}\right\| \left\|TT^*(f_{n_j} - f_{n_k})\right\| && (2.15) \\
&\leq (\left\|f_{n_j}\right\| + \|f_{n_k}\|) \left\|TT^*(f_{n_j} - f_{n_k})\right\| && (2.16) \\
&< (M + M)\frac{\varepsilon^2}{2M} && (2.17) \\
&= \varepsilon^2
\end{aligned}$$

Line (2.14) follows from the definition of the adjoint. Line (2.15) follows from the Cauchy-Schwarz inequality. Line (2.16) follows from the triangle inequality and the fact that $\|h\| = \|-h\|$ for every $h \in H$. Line (2.17) follows from the fact that (f_n) is bounded and that $n_j, n_k \geq N$. Since norms are nonnegative taking the square root of both sides of the inequality yields $\left\|T^* f_{n_j} - T^* f_{n_k}\right\| < \varepsilon$. Thus, we have shown that $(T^* f_{n_j})$ is a Cauchy sequence. Therefore, $(T^* f_{n_j})$ is a convergent subsequence of $(T^* f_n)$. Thus, T^* is compact.

($\Longleftarrow$) Assume $T^* \in \mathcal{C}(H)$. Then, by the forward direction $(T^*)^* = T$ is compact.

Therefore, $T \in \mathcal{C}(H)$ if and only if $T^* \in \mathcal{C}(H)$, as desired. $\square$

Theorem 29 ([2], Theorem 10.74)**.** *Let H be a Hilbert space. If $T \in \mathcal{C}(H)$, then range T contains no infinite-dimensional closed subspaces.*

The following proof is adapted from the proof given in [2].

Proof. Assume $T \in \mathcal{C}(H)$, where H is a Hilbert space. Seeking a contradiction, suppose U is an infinite-dimensional closed subspace of range T. By theorem 23, T is bounded. Thus, T is continuous. Hence, $T^{-1}(U)$ is a closed subspace of H by theorem 3. Since $U, T^{-1}(U) \subseteq H$ are closed, it follows by theorem 7, that U and $T^{-1}(U)$ are complete subspaces of H. Since H is a Hilbert space, U and $T^{-1}(U)$ must be inner product spaces. Thus, under the norm induced by the inner product U and $T^{-1}(U)$ are metric spaces. Since U and $T^{-1}(U)$ are complete metric spaces and are subspaces of H, it follows that U and $T^{-1}(U)$ are Hilbert spaces in their own right.

Define $S = T\Big|_{T^{-1}(U)}$. S very clearly maps $T^{-1}(U)$ onto U. Thus, by the Open Mapping Theorem, S will map the open unit ball in $T^{-1}(U)$ to some open subset of U containing 0. Hence, there exists some $r > 0$ such that

$$\{g \in U : \|g\| < r\} \subseteq \{Tf : f \in T^{-1}(U) \text{ and } \|f\| < 1\}$$

Since U is a Hilbert space and $\dim(U) = \infty$, by applying the Gram-Schmidt procedure to some linearly independent sequence in U, we obtain an orthonormal sequence (e_n) contained in U. Since $\frac{r}{2} \in \mathbb{R}$ and $e_n \in U$ for each $n \in \mathbb{N}$, it follows that $\frac{re_n}{2} \in U$. Furthermore, $\left\|\frac{re_n}{2}\right\| = \frac{r}{2} < r$. So,

$$\frac{re_n}{2} \in \{g \in U : \|g\| < r\} \subseteq \{Tf : f \in T^{-1}(U) \text{ and } \|f\| < 1\}$$

Thus, for every $n \in \mathbb{N}$, there exists $f_n \in T^{-1}(U)$ with $\|f_n\| < 1$ such that $Tf_n = \frac{re_n}{2}$. In this way we construct the bounded sequence (f_n). Now, we examine the sequence (Tf_n):

$$\begin{aligned}
\|Tf_n - Tf_m\|^2 &= \left\|\frac{re_n}{2} - \frac{re_m}{2}\right\|^2 \\
&= \left\langle \frac{re_n}{2} - \frac{re_m}{2}, \frac{re_n}{2} - \frac{re_m}{2} \right\rangle \\
&= \left\langle \frac{r}{2}(e_n - e_m), \frac{r}{2}(e_n - e_m) \right\rangle \\
&= \frac{r}{2}\frac{\overline{r}}{2}\langle e_n - e_m, e_n - e_m\rangle \\
&= \frac{r^2}{4}\left(\langle e_n, e_n\rangle - \langle e_n, e_m\rangle - \langle e_m, e_n\rangle + \langle e_m, e_m\rangle\right) \\
&= \frac{r^2}{4}\left(1 - 0 - 0 + 1\right) \qquad (2.18) \\
&= \frac{r^2}{2}
\end{aligned}$$

Line (2.18) follows from the e_n's being orthonormal. Since $r > 0$ and norms are nonnegative, taking the square root gives $\|Tf_n - Tf_m\| = \frac{r}{\sqrt{2}} = \frac{\sqrt{2}r}{2}$. Since r is some fixed positive real number, we can never guarantee that $\|Tf_n - Tf_m\| < \varepsilon$ for every $\varepsilon > 0$ for n, m greater than or equal to some appropriate $N \in \mathbb{N}$. That is, (Tf_n) is not Cauchy, and more importantly, no subsequence of (Tf_n) is Cauchy. Thus, by the contrapositive, it follows that no subsequence of (Tf_n) is convergent. Therefore, T is not compact, contradicting the assumption that $T \in \mathcal{C}(H)$. Thus, we must conclude that range T cannot contain any closed infinite-dimensional subspaces. □

The following theorem is vital to proving the Fredholm Alternative Theorem. We omit its proof as it can be found in Axler's book ([2]).

Theorem 30 ([2], Theorem 10.77). *Let H be a Hilbert space. If $T \in \mathcal{C}(H)$, then range $(T - \alpha I)$ is closed for every $\alpha \in \mathbb{F}\backslash\{0\}$.*

Lemma 31 ([2], Lemma 10.83). *If $T \in \mathcal{L}(V)$ is one-to-one but not onto, then range $T \supsetneq$ range $T^2 \supsetneq$ range $T^3 \supsetneq \cdots$*

The following proof is adapted from the proof given in [2].

Proof. Assume $T \in \mathcal{L}(V)$ is one-to-one but not onto. Let $n \in \mathbb{N}$ and $f \in V$. Then, $T^n f = T^{n-1}(Tf) \in$ range T^{n-1}. Since $T^n f$ is a typical element of range T^n, it follows that range $T^n \subseteq$ range T^{n-1} for each $n \in \mathbb{N}$. Thus, we have the chain range $T \supseteq$ range $T^2 \supseteq$ range $T^3 \supseteq \cdots$. So, we need only show that none of these subsets can be equalities. Since T is not onto, there exists some $f \in V$ such that $f \notin$ range T. For any $n \in \mathbb{N}$, $T^n f \in$ range T^n. However, $T^n f \notin$ range T^{n+1}. If it were, then there would exist some $g \in V$ such that $T^n f = T^{n+1} g = T^n(Tg)$. Then, applying the fact that T is one-to-one n times, we would get $f = Tg \in$ range T. This would be a contradiction, so we must have $T^n f \notin$ range T^{n+1}. Thus, we have shown that for each $n \in \mathbb{N}$, there exists some element (namely $T^n f$) that is in range T^n but not in range T^{n+1}. Thus, range $T^{n+1} \subsetneq$ range T^n for each $n \in \mathbb{N}$. Therefore, range $T \supsetneq$ range $T^2 \supsetneq$ range $T^3 \supsetneq \cdots$, as desired. □

Theorem 32 (Fredholm Alternative Theorem)**.** *Let H be a Hilbert space and $T \in \mathcal{C}(H)$. Given $\alpha \in \mathbb{F}\backslash\{0\}$, the following are equivalent:*

1. *$T - \alpha I$ is not invertible*

2. *α is an eigenvalue of T*

3. *$T - \alpha I$ is not onto*

The following proof is adapted from Axler's proof of his theorem 10.85, which can be found in [2].

Proof. Let H be a Hilbert space, $T \in \mathcal{C}(H)$, and $\alpha \in \mathbb{F}\backslash\{0\}$.
($2 \Rightarrow 1$) Assume α is an eigenvalue of T. Then, $Tv = \alpha v$ for some $v \in H\backslash\{0\}$. Therefore, $(T - \alpha I)v = 0$. So, $v \in$ null $(T - \alpha I)$. Thus, $T - \alpha I$ is non-invertible.
($3 \Rightarrow 1$) Assume $T - \alpha I$ is not onto. Then, $T - \alpha I$ cannot be invertible.
($1 \Rightarrow 2$) Assume $T - \alpha I$ is not invertible. Seeking a contradiction, suppose α is not an eigenvalue of T. Then, $(T - \alpha I)v = 0$ has no nonzero solutions $v \in H$. Since it is always true that $(T - \alpha I)0 = 0$, null $(T - \alpha I) = \{0\}$. Thus, we see that $T - \alpha I$ is one-to-one. Since $T - \alpha I$ is non-invertible, it must follows that $T - \alpha I$ is not onto. Thus, we can apply lemma 31 to obtain the chain
range $(T - \alpha I) \supsetneq$ range $(T - \alpha I)^2 \supsetneq$ range $(T - \alpha I)^3 \supsetneq \cdots$. For each $n \in \mathbb{N}$,

$$\begin{aligned}(T - \alpha I)^n &= (T + (-\alpha I))^n \\ &= T^n(-\alpha I)^0 + nT^{n-1}(-\alpha I) + \cdots + nT(-\alpha I)^{n-1} + T^0(-\alpha I)^n \qquad (2.19) \\ &= T^n - \alpha n T^{n-1} + \cdots + (-\alpha)^{n-1} nT + (-\alpha)^n I \\ &= T^n - \alpha n T^{n-1} + \cdots + (-\alpha)^{n-1} nT - (-\alpha)^{n-1} I\end{aligned}$$

Line (2.19) follows from expansion using the binomial theorem. Since $\mathcal{C}(H)$ is a subspace of $\mathcal{B}(H)$ and an ideal in $\mathcal{B}(H)$, it follows that

$$T^n - \alpha n T^{n-1} + \cdots + (-\alpha)^{n-1} n T \in \mathcal{C}(H)$$

For ease, we define $S = T^n - \alpha n T^{n-1} + \cdots + (-\alpha)^{n-1} n T$. Thus, what we have shown is $(T - \alpha I)^n = S - (-\alpha)^{n-1} I$ for some $S \in \mathcal{C}(H)$. Since $\alpha \neq 0$, $(-\alpha)^{n-1} \neq 0$. Thus, by theorem 30, we have that range $(T - \alpha I)^n =$ range $(S - (-\alpha)^{n-1} I)$ is a closed subspace of H.

Since range $(T - \alpha I)^n \supsetneq$ range $(T - \alpha I)^{n+1}$, there exists some $f_n \in$ range $(T - \alpha I)^n$ such that $f_n \notin$ range $(T - \alpha I)^{n+1}$. Since 0 is in both range $(T - \alpha I)^n$ and range $(T - \alpha I)^{n+1}$, it must be that case that $f_n \neq 0$. Thus, we may assume without loss of generality that $\|f_n\| = 1$ (otherwise, we could just divide by $\|f_n\|$).

Since range $(T - \alpha I)^{n+1} \subsetneq$ range $(T - \alpha I)^n$, range $(T - \alpha I)^{n+1}$ is a proper subspace of range $(T - \alpha I)^n$. So, we can write

$$\text{range } (T - \alpha I)^n = \text{range } (T - \alpha I)^{n+1} \bigoplus (\text{range } (T - \alpha I)^{n+1})^{\perp}$$

Thus, because $f_n \in$ range $(T - \alpha I)^n$, we know that $f_n = h_n + g_n$, where $h_n \in$ range $(T - \alpha I)^{n+1}$ and $g_n \in$ (range $(T - \alpha I)^{n+1})^{\perp}$. Because we have $f_n \notin$ range $(T - \alpha I)^{n+1}$, it follows $g_n \neq 0$. If $h_n = 0$, then we automatically have $f_n = g_n$. If $h_n \neq 0$, we could simply replace f_n with g_n, since $g_n \in$ (range $(T - \alpha I)^{n+1})^{\perp}$ and $g_n \in$ range $(T - \alpha I)^{n+1}$. Therefore, we can assume without loss of generality that $f_n \in$ (range $(T - \alpha I)^{n+1})^{\perp}$.

So far, we have shown there exists some nonzero $f_n \in$ range $(T - \alpha I)^n \cap$ (range $(T - \alpha I)^{n+1})^{\perp}$ with $\|f_n\| = 1$ for each $n \in \mathbb{N}$. In this way we construct a sequence (f_n). Let $n, m \in \mathbb{N}$ such that $n < m$. Then,

$$\begin{aligned} Tf_n - Tf_m &= Tf_n - \alpha f_n + \alpha f_n - Tf_m + \alpha f_m - \alpha f_m \\ &= Tf_n - \alpha f_n - (Tf_m - \alpha f_m) - \alpha f_m + \alpha f_n \\ &= (T - \alpha I)f_n - (T - \alpha I)f_m - \alpha f_m + \alpha f_n \end{aligned} \tag{2.20}$$

Since $f_n \in$ range $(T - \alpha I)^n$, we know $(T - \alpha I)f_n \in$ range $(T - \alpha I)^{n+1}$. Similarly, because $f_m \in$ range $(T - \alpha I)^m$, we know $(T - \alpha I)f_m \in$ range $(T - \alpha I)^{m+1}$. Since $m > n$, $m + 1 > n + 1$. Thus, we know

$$\text{range } (T - \alpha I) \supsetneq \cdots \supsetneq \text{range } (T - \alpha I)^{n+1} \supsetneq \cdots \supsetneq \text{range } (T - \alpha I)^{m+1} \supsetneq \cdots$$

Hence, $(T - \alpha I)f_m \in$ range $(T - \alpha I)^{m+1}$ implies that $(T - \alpha I)f_m \in$ range $(T - \alpha I)^{n+1}$. Since $n < m$, $n \leq m$. Thus, $f_m \in$ range $(T - \alpha I)^m \subseteq$ range $(T - \alpha I)^{n+1}$. Notice that in this case the subset can be equal, because it is possible that $n + 1 = m$. Thus, since range $(T - \alpha I)^{n+1}$ is a subspace of H, it follows that $\alpha f_m \in$ range $(T - \alpha I)^{n+1}$. Thus, we have shown that each term in line (2.20), except αf_n, is in range $(T - \alpha I)^{n+1}$. And so, $(T - \alpha I)f_n - (T - \alpha I)f_m - \alpha f_m \in$ range $(T - \alpha I)^{n+1}$.

Since $f_n \in$ range $(T - \alpha I)^n \cap$ (range $(T - \alpha I)^{n+1})^\perp$, it follows that $\alpha f_n \in$ (range $(T - \alpha I)^{n+1})^\perp$. Now,

$$\begin{aligned} \|Tf_n - Tf_m\|^2 &= \langle Tf_n - Tf_m, Tf_n - Tf_m \rangle \\ &= \langle (T - \alpha I)f_n - (T - \alpha I)f_m - \alpha f_m + \alpha f_n, \\ &\quad (T - \alpha I)f_n - (T - \alpha I)f_m - \alpha f_m + \alpha f_n \rangle \\ &= \langle (T - \alpha I)f_n - (T - \alpha I)f_m - \alpha f_m, (T - \alpha I)f_n - (T - \alpha I)f_m - \alpha f_m \rangle \\ &\quad + \langle (T - \alpha I)f_n - (T - \alpha I)f_m - \alpha f_m, \alpha f_n \rangle \\ &\quad + \langle \alpha f_n, (T - \alpha I)f_n - (T - \alpha I)f_m - \alpha f_m \rangle + \langle \alpha f_n, \alpha f_n \rangle \end{aligned}$$

$$= \|(T-\alpha I)f_n - (T-\alpha I)f_m - \alpha f_m\|^2 + 0 + 0 + \|\alpha f_n\|^2 \tag{2.21}$$
$$\geq \|\alpha f_n\|^2$$

Line (2.21) follows from the fact that $(T-\alpha I)f_n - (T-\alpha I)f_m - \alpha f_m$ is perpendicular to αf_n. Since norms are always nonnegative, taking the square root of both sides of the inequality yields $\|Tf_n - Tf_m\| \geq \|\alpha f_n\|$. Thus we have that $\|Tf_n - Tf_m\| \geq \|\alpha f_n\| = |\alpha|\,\|f_n\| = |\alpha|$.

Since $\alpha \neq 0$, $|\alpha| > 0$. Thus, for $\varepsilon = |\alpha|$, for every $N \in \mathbb{N}$, whenever $n, m \geq N$, $\|Tf_n - Tf_m\| \geq |\alpha| = \varepsilon$. Thus, (Tf_n) cannot have any Cauchy subsequences. Consequently, (Tf_n) cannot have any convergent subsequences, contradicting the assumption that $T \in \mathcal{C}(H)$. Therefore, we must conclude that α is an eigenvalue of T.

$(1 \Rightarrow 3)$ Assume $T - \alpha I$ is not invertible. Thus, by theorem 16, $(T-\alpha I)^* = T^* - \overline{\alpha} I$ is not invertible. Since $1 \Rightarrow 2$, it follows $\overline{\alpha}$ is an eigenvalue of T^*. Thus, null $(T^* - \overline{\alpha} I) \neq \{0\}$. Hence, (null $(T^* - \overline{\alpha} I)^{\perp} \subsetneq H$. By theorem 14, $\overline{\text{range } (T-\alpha I)} = ((\text{null } (T-\alpha I)^*)^{\perp}$. Since $((\text{null } (T-\alpha I)^*)^{\perp} = (\text{null } (T^* - \overline{\alpha} I)^{\perp}$, it follows that $\overline{\text{range } (T-\alpha I)} \subsetneq H$. Hence, range $(T-\alpha I) \subsetneq H$. That is, $T - \alpha I$ is not onto, as desired.

Therefore, we have shown $1 \iff 2 \iff 3$. □

Note. *The above is not how we traditionally state the Fredholm Alternative Theorem. Traditionally it is stated as:*

If $T \in \mathcal{C}(H)$, where H is a Hilbert space, and $\alpha \in \mathbb{F}\backslash\{0\}$, then exactly one of the following holds:

1. *$Tf = \alpha f$ has no nonzero solution $f \in H$*
2. *$g = Tf - \alpha f$ has a solution $f \in H$ for every $g \in H$*

(1) is equivalent to the statement "α is an eigenvalue of T". (2) is equivalent to the statement "$T - \alpha I$ is onto H".

Lemma 33 is useful in proving the Spectral Theorem for Self-Adjoint Compact Operators. We omit its proof as it can be found in Axler's book ([2]).

Lemma 33 ([2], Lemma 10.93)**.** *Let H be a Hilbert space, $T \in \mathcal{C}(H)$, and $\sigma(T) = \{\alpha : T - \alpha I$ is not invetible$\}$. Then, for each $\delta > 0$, $\{\alpha \in \sigma(T) : |\alpha| \geq \delta\}$ is a finite set.*

The following lemma is useful in proving theorem 35. We omit its proof as it can be found in Axler's book ([2]).

Lemma 34 ([2], Lemma 10.96)**.** *Let H be a nonzero Hilbert space. If $T \in \mathcal{C}(H)$, then $T^*T - \|T\|^2 I$ is not invertible.*

Theorem 35 ([2], Theorem 10.99)**.** *Let H be a nonzero Hilbert space. If $T \in \mathcal{C}(H)$ is self-adjoint, then either $\|T\|$ or $-\|T\|$ is an eigenvalue of T.*

The following proof is adapted from the proof given in [2].

Proof. Let H be a nonzero Hilbert space. Assume $T \in \mathcal{C}(H)$ is self-adjoint. Then, $T^* = T$. So by lemma 34, $T^2 - \|T\|^2 I$ is not invertible. Since $T^2 - \|T\|^2 I = (T + \|T\| I)(T - \|T\| I)$, it must follow that $T + \|T\| I$ is not invertible or $T - \|T\| I$ is not invertible. Therefore, by the Fredholm Alternative Theorem, $\|T\|$ is an eigenvalue of T or $-\|T\|$ is an eigenvalue of T. □

Theorem 36 is necessary for proving the Spectral Theorem for Self-Adjoint Compact Operators. Since the proof can be found in Axler's book ([2]), we omit it here. Similarly, theorem 37 is necessary for proving the Spectral Theorem for Normal Compact Operators and we omit its proof for the same reason.

Theorem 36 ([2], Theorem 10.102). *If U is an invariant subspace for a self-adjoint operator T, then:*

1. *$U^{\perp}$ is an invariant subspace for T*
2. *$T\big|_{U^{\perp}}$ is a self-adjoint operator on $U^{\perp}$*

Theorem 37 ([2], Theorem 10.103). *Let H be a Hilbert space and $T \in \mathcal{B}(H)$. If there exists an orthonormal basis for H consisting of eigenvectors of T, then:*

1. *If $\mathbb{F} = \mathbb{R}$, then T is self-adjoint*
2. *If $\mathbb{F} = \mathbb{C}$, then T is normal*

Theorem 38 (Spectral Theorem for Self-Adjoint Compact Operators). *Let H be a Hilbert space. If $T \in \mathcal{C}(H)$ is self-adjoint, then:*

1. *There is an orthonormal basis for H consisting of eigenvectors of T*

2. *There is a countable set Ω, an orthonormal family $\{e_n : n \in \Omega\} \subseteq H$, and a family $\{\alpha_n : n \in \Omega\} \subseteq \mathbb{R}\backslash\{0\}$, such that $Tf = \sum_{n\in\Omega} \alpha_n \langle f, e_n\rangle e_n$ for every $f \in H$*

The following proof is adapted from Axler's proof of his theorem 10.106, which can be found in [2].

Proof. Assume H is a Hilbert space and $T \in \mathcal{C}(H)$ is self-adjoint.
(1) Define U to be the span of the eigenvectors of T. Since self-adjoint operators have real eigenvalues, it follows $U = \{v : Tv = \alpha v$ for some nonzero $v \in H$ and $\alpha \in \mathbb{R}\}$. It is easily verifiable that U is an invariant subspace for T. Thus, by theorem 36, $U^\perp$ is also an invariant subspace for T, and $T\Big|_{U^\perp}$ is a self-adjoint operator on $U^\perp$.

We claim that $T\Big|_{U^\perp}$ has no eigenvalues. Seeking a contradiction, suppose α is an eigenvalue of $T\Big|_{U^\perp}$. Then, there exists some nonzero $v \in U^\perp \subseteq H$, such that $T\Big|_{U^\perp} v = \alpha v$. This implies that v is an eigenvector of T. Thus, by definition of U, $v \in U$. Since $U \cap U^\perp = \{0\}$, it follows $v = 0$, contradicting that $v \neq 0$. Thus, we must conclude that $T\Big|_{U^\perp}$ has no eigenvalues as previously claimed.

By theorem 8 $U^\perp$ is a closed subspace of a Hilbert space, thus $U^\perp$ is a Hilbert space. Since $T\Big|_{U^\perp}$ is a self-adjoint operator on $U^\perp$ and it has no eigenvalues (as just shown), it follows by theorem 35 that $U^\perp = \{0\}$. So by theorem 9, $\overline{U} = H$.

For each eigenvalue α of T, we know there exists an orthonormal basis for null $(T - \alpha I)$ that consists of eigenvectors of T correspoding to the eigenvalue α, because every

Hilbert space has an orthonormal basis. We will refer to this basis as U_α. We claim that $\bigcup_{\alpha\in A} U_\alpha$ is an orthonormal family in H, where A is the set of all eigenvalues of T. Let $v \in \bigcup_{\alpha\in A} U_\alpha$. Then $v \in U_\alpha$ for some $\alpha \in A$. Since U_α is an orthonormal basis, it follows $\|v\| = 1$. Let $v, u \in \bigcup_{\alpha\in A} U_\alpha$. If $v = u$, then $\langle v, u\rangle = \langle v, v\rangle = \|v\|^2 = 1$. If $v \neq u$, then either $v, u \in U_\alpha$ for some $\alpha \in A$ or $v \in U_\alpha$ and $u \in U_\beta$ for some distinct $\alpha, \beta \in A$. In the first case, $\langle v, u\rangle = 0$, because U_α is an orthonormal basis. In the second case, because α, β are distinct eigenvalues of a self-adjoint operator, the corresponding eigenvectors u, v must be orthogonal by theorem 21. Thus, in the second case, we have $\langle v, u\rangle = 0$ as well. Therefore, $\bigcup_{\alpha\in A} U_\alpha$ is an orthonormal family in H, as claimed.

Since $\bigcup_{\alpha\in A} U_\alpha$ is the set of all the unit length eigenvectors of T, it follows that $\overline{\text{span}\left\{\bigcup_{\alpha\in A} U_\alpha\right\}} = \overline{U}$. As we showed previously, $\overline{U} = H$. Thus, $\overline{\text{span}\left\{\bigcup_{\alpha\in A} U_\alpha\right\}} = H$. Therefore, by definition $\bigcup_{\alpha\in A} U_\alpha$ is an orthonormal basis for H.

(2) Part (1) tells us there is an orthonormal basis of H consisting of eigenvectors of T. For ease in proving this second part, we will use $\{e_n : n \in \Gamma\}$ to denote this orthonormal basis of eigenvectors. Let $\{\alpha_n : n \in \Gamma\}$ be the set of corresponding eigenvalues. Since T is self-adjoint theorem 37 gives us that $\{\alpha_n : n \in \Gamma\} \subseteq \mathbb{R}$. Let $f \in H$. Since $\{e_n : n \in \Gamma\}$ is a basis for H, $f = \sum_{n\in\Gamma} \langle f, e_n\rangle e_n$. Now,

$$\begin{aligned} Tf &= T\left(\sum_{n\in\Gamma} \langle f, e_n\rangle e_n\right) \\ &= \sum_{n\in\Gamma} T(\langle f, e_n\rangle e_n) \qquad (2.22) \\ &= \sum_{n\in\Gamma} \langle f, e_n\rangle Te_n \end{aligned}$$

$$= \sum_{n \in \Gamma} \langle f, e_n \rangle \alpha_n e_n$$

Line (2.22) follows from the fact that $T \in \mathcal{C}(H) \subseteq \mathcal{B}(H)$ and that bounded linear operators are continuous. Define $\Omega = \{n \in \Gamma : \alpha_n \neq 0\}$. Then we can write $Tf = \sum_{n \in \Omega} \langle f, e_n \rangle \alpha_n e_n$. Thus, it only remains to show that Ω is countable. Lemma 33 says that if $\sigma(T)$ is infinite, then $\sigma(T)$ consists of 0 and a sequence in $\mathbb{R}$ converging to 0 ($\mathbb{F}$ in general, we use $\mathbb{R}$ because T is self-adjoint). We can index this sequence by $\mathbb{N}$. Thus, if $\sigma(T)$ were infinite, it would be countably infinite. Thus, we know $\sigma(T)$ is always countable as it is either finite or countably infinite. Since

$$\Omega \subseteq \{\alpha_n : n \in \Gamma\} \subseteq \{\alpha : \alpha \text{ an eigenvalue of } T\} \subseteq \sigma(T)$$

it must follow that Ω is countable. □

Theorem 39 (Spectral Theorem for Normal Compact Operators)**.** *Let H be a complex Hilbert space and $T \in \mathcal{C}(H)$. Then, there exists an orthonormal basis for H consisting of eigenvectors of T if and only if T is normal.*

The following proof is adapted from Axler's proof of his theorem 10.107, which can be found in [2].

Proof. Let H be a complex Hilbert space and $T \in \mathcal{C}(H)$.
($\Longrightarrow$) Assume there exists an orthonormal basis for H consisting of eigenvectors of T. Since $\mathbb{F} = \mathbb{C}$, by theorem 37, T is normal.

($\Longleftarrow$) Assume T is normal. By theorem 20, there exist self-adjoint operators $A = \dfrac{T^* + T}{2}$ and $B = \dfrac{i(T^* - T)}{2}$ on H, such that $T = A + iB$. Since T is normal, theorem 20 gives us that $AB = BA$. Since $T \in \mathcal{C}(H)$, theorem 28 yields $T^* \in \mathcal{C}(H)$. Since $\mathcal{C}(H)$ is a subspace, it follows that $A, B \in \mathcal{C}(H)$.

Let $\alpha \in \mathbb{R}$. We claim that null $(A - \alpha I)$ is an invariant subspace B. It is well known that the null space forms a subspace, so we need only show that null $(A - \alpha I)$ is invariant for B. Let $f \in$ null $(A - \alpha I)$. Now,

$$\begin{aligned}
(A - \alpha I)Bf &= ABf - \alpha Bf \\
&= BAf - B(\alpha f) \qquad (2.23) \\
&= B(Af - \alpha f) \\
&= B((A - \alpha I)f) \\
&= B0 \qquad (2.24) \\
&= 0
\end{aligned}$$

Line (2.23) follows from the fact that $AB = BA$. Line (2.24) follows from the assumption that $f \in$ null $(A - \alpha I)$. Thus $Bf \in$ null $(A - \alpha I)$. Therefore, null $(A - \alpha I)$ is an invariant subspace for B as claimed.

Since $B \in \mathcal{C}(H)$ and null $(A - \alpha I)$ is invariant for B, we have that $B\big|_{\text{null }(A-\alpha I)} \in \mathcal{C}(\text{null }(A - \alpha I))$. Since B is self-adjoint, $B\big|_{\text{null }(A-\alpha I)}$ is self-adjoint too. Thus, by the Spectral Theorem for Compact Self-Adjoint Operators, there exists an orthonormal basis for null $(A - \alpha I)$ consisting of eigenvectors of B (technically, this basis consists of eigenvectors of $B\big|_{\text{null }(A-\alpha I)}$, which are necessarily also eigenvectors of B).

If α is not an eigenvalue of A, then null $(A - \alpha I) = \{0\}$. It follows that $B\big|_{\text{null }(A-\alpha I)} = B\big|_{\{0\}} = 0$, which is not very interesting. However, the above tells us that if α is an eigenvalue of A, there exists an orthonormal basis for null $(A - \alpha I)$ consisting of eigenvectors of B. Let this orthonormal basis be denoted U_α. Define $\mathcal{A}$ to be the set of eigenvalues of A. By the exact same argument as in the proof of the Spectral Theorem for Self-Adjoint Compact Operators, we can conclude that $\bigcup_{\alpha\in\mathcal{A}} U_\alpha$ is an orthonormal family in H.

Since A is self-adjoint, by the proof of the Spectral Theorem for Self-Adjoint Compact Operators we can conclude that $\overline{\text{span}\left\{\bigcup_{\alpha\in\mathcal{A}} V_\alpha\right\}} = H$, where V_α is an orthonormal basis for null $(A - \alpha I)$ consisting of eigenvectors of A. Since eigenvectors of B form an orthonormal basis for null $(A - \alpha I)$, it follows that they must also be eigenvectors of A. Thus, we have $\bigcup_{\alpha\in\mathcal{A}} U_\alpha = \bigcup_{\alpha\in\mathcal{A}} V_\alpha$. Thus $\overline{\text{span}\left\{\bigcup_{\alpha\in\mathcal{A}} U_\alpha\right\}} = \overline{\text{span}\left\{\bigcup_{\alpha\in\mathcal{A}} V_\alpha\right\}}$. Therefore, $\overline{\text{span}\left\{\bigcup_{\alpha\in\mathcal{A}} U_\alpha\right\}} = H$. Thus, by definition $\bigcup_{\alpha\in\mathcal{A}} U_\alpha$ is an orthonormal basis for H.

So, it only remains to show that the elements of this orthonormal basis are eigenvectors of T. Let $v \in \bigcup_{\alpha\in\mathcal{A}} U_\alpha$. Then, $v \in U_\alpha$ for some eigenvalue α of A. Since U_α consists of eigenvectors of B, there must exist some corresponding eigenvalue β of B. Thus, $Bv = \beta v$. Since U_α is an orthonormal basis for null $(A - \alpha I)$, it follows that $v \in$ null $(A - \alpha I)$. Thus, $(A - \alpha I)v = 0$. It follows $Av = \alpha v$. Now,

$$
\begin{aligned}
Tv &= (A + iB)v \\
&= Av + iBv \\
&= \alpha v + i\beta v \\
&= (\alpha + i\beta)v
\end{aligned}
$$

Since $v \neq 0$ (as it is an eigenvector of B) and $\alpha + i\beta \in \mathbb{C}$, it follows that v is an eigenvector of T. Therefore, $\bigcup_{\alpha \in \mathcal{A}} U_\alpha$ is an orthonormal basis for H consisting of eigenvectors of T.

Thus, we have shown that T is normal if and only if there exists an orthonormal basis for H consisting of eigenvectors of T. $\square$

Chapter 3

SPECTRA

Theorem 40 (Inverse Mapping Theorem)**.** *If H and K are Banach spaces and $A \in \mathcal{B}(H, K)$ is bijective, then $A^{-1} \in \mathcal{B}(K, H)$ (that is, A^{-1} is bounded.)*

Proof. Assume H, K are Banach spaces and $A \in \mathcal{B}(H, K)$ is bijective. It immediately follows that $A^{-1} : K \longrightarrow H$. So, it only remains to show that A^{-1} is bounded. Since A is bounded, it is continuous by theorem 6. Since A is bijective, it is also onto. Let G be an arbitrary open set in H. Then, by the Open Mapping Theorem, $A(G)$ is open in K. By theorem 3, it follows that A^{-1} is continuous. Therefore, A^{-1} is bounded, as desired. □

Recall by theorem 6, for a linear map from one normed vector space to another, boundedness is equivalent to continuity. Thus, the Inverse Mapping Theorem lets us immediately conclude that the inverse of a continuous map is also continuous.

Note. *On a general Hilbert space (one that is possibly infinite-dimensional), it is not the case that an operator is one-to-one if and only if it is onto.*

Example 2. *The forward shift operator, S, is one-to-one but not onto.*

Proof. Suppose $(a_n), (b_n) \in \ell^2(\mathbb{N})$ are such that $S(a_n) = S(b_n)$. Evaluating, we have that $(0, a_1, a_2, \ldots) = (0, b_1, b_2, \ldots)$, which is true if and only if $0 = 0$ and $a_n = b_n$ for all $n \in \mathbb{N}$. Thus, we see that S is one-to-one.

To see that S is not onto, let $(a_n) \in \ell^2(\mathbb{N})$ be such that $a_1 \neq 0$. Therefore, there exists no $(b_n) \in \ell^2(\mathbb{N})$ such that $S(b_n) = (a_n)$, because $S(b_n) = (0, b_1, b_2, \ldots)$. Thus, S is not onto. □

Example 3. *The backward shift operator, S^*, is onto but not one-to-one.*

Proof. Let $(a_n) \in \ell^2(\mathbb{N})$. This implies $S^*(0, a_1, a_2, \ldots) = (a_1, a_2, \ldots) = (a_n)$. Thus, S^* is onto.

To see that S is not one-to-one, let $(a_n), (b_n) \in \ell^2(\mathbb{N})$ be such that $a_1 \neq b_1$ but $a_n = b_n$ for all $n \in \mathbb{N}\backslash\{1\}$. Thus, $(a_n) \neq (b_n)$, however, $S^*(a_n) = S^*(b_n)$, because $(a_2, a_3, \ldots) = (b_2, b_3, \ldots)$. Thus, S^* is not one-to-one. □

From the above examples with the shift operators, we see that an operator $A \in \mathcal{B}(H)$ can fail to be invertible for one of two reasons: it is not one-to-one or it is not onto. However, this is not the only way to break non-invertibility down into pieces.

Def: Let $A \in \mathcal{B}(H)$. The **spectrum** of A, denoted $\sigma(A)$, is the set $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}$.

Theorem 41 ([2], Theorem 10.76)**.** *Let H be an infinite-dimensional Hilbert space. If $T \in \mathcal{C}(H)$, then $0 \in \sigma(T)$.*

The following proof is adapted from the proof given in [2].

Proof. Let H be an infinite-dimensional Hilbert space. Assume $T \in \mathcal{C}(H)$. Seeking a contradiction, suppose $0 \notin \sigma(T)$. Thus, T is invertible. Hence, T is onto. That is, range $T = H$, and so dim(range T) $= \infty$. We will next show that range T is closed. To see this, let $f \in H$ be a limit point of range T. By definition, there exists a sequence (f_n) in H with $f_n \neq f$ for each $n \in \mathbb{N}$, such that $\lim_{n\to\infty} f_n = f$. Since (f_n) is convergent it must be Cauchy. Since H is a Hilbert space, it is complete. Thus, (f_n) converges to an element in $H =$ range T. Since limits are unique, and (f_n) converges to f, it follows that $f \in$ range T. Thus, range T contains all its limit points, and so is closed. Since range (T) is infinite-dimensional and is closed, this contradicts theorem 29. Therefore, we must conclude that $0 \in \sigma(T)$, as desired. □

Def: Let $A \in \mathcal{B}(H)$. The **point spectrum** of A, denoted $\sigma_p(A)$ or $\Pi_0(A)$, is the set of all eigenvalues of A; that is, $\sigma_p(A) = \{\lambda \in \mathbb{C} : Av = \lambda v \text{ for some } v \in H\backslash\{0\}\}$.

From the previous definition we see that eigenvalues work in a similar way on finite or infinite dimensional space.

Claim 1. *If $A \in \mathcal{B}(H)$, then $\sigma_p(A) = \{\lambda \in \mathbb{C} : A - \lambda I$ is not one-to-one$\}$.*

Proof. Assume $A \in \mathcal{B}(H)$.

($\subseteq$) Let $\lambda \in \sigma_p(A)$. Then, $Av = \lambda v$ for some $v \in H\backslash\{0\}$. Hence, we have

$0 = Av - \lambda v = (A - \lambda I)v$. And so, $v \in$ null $(A - \lambda I)$. Since $v \neq 0$, it follows that null $(A - \lambda I) \neq \{0\}$, and so $A - \lambda I$ is not one-to-one. Therefore, we have

$\sigma_p(A) \subseteq \{\lambda \in \mathbb{C} : A - \lambda I$ is not one-to-one$\}$.

($\supseteq$) Let $\lambda \in \mathbb{C}$ be such that $A - \lambda I$ is not one-to-one. Then, there exist $u, v \in H$ such that $u \neq v$ and $(A - \lambda I)v = (A - \lambda I)u$. Thus,

$$\begin{aligned} 0 &= (A - \lambda I)v - (A - \lambda I)u \\ &= (A - \lambda I)(v - u) \\ &= A(v - u) - \lambda(v - u) \end{aligned}$$

Hence, $A(v-u) = \lambda(v-u)$. Since $v \neq u$, $v - u \neq 0$, and so λ is an eigenvalue of A with corresponding eigenvector $v - u$. Hence, $\lambda \in \sigma_p(A)$. It follows that $\{\lambda \in \mathbb{C} : A - \lambda I \text{ is not one-to-one}\} \subseteq \sigma_p(A)$.
Therefore, $\sigma_p(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not one-to-one}\}$, as desired. □

Def: Let V be a metric space and $U \subseteq V$. We say U is **dense** in V, if $\overline{U} = V$, where $\overline{U}$ denotes the closure of U.

Note. *We don't divide $\sigma(A)$ into $\{\lambda \in \mathbb{C} : A - \lambda I$ is not one-to-one$\}$ and $\{\lambda \in \mathbb{C} : A - \lambda I$ is not onto$\}$, because $A - \lambda I$ can fail to be onto in two ways:*

1. *range $(A - \lambda I)$ is not closed*
2. *range $(A - \lambda I)$ is not dense in H*

These two ways of failure are not disjoint since it is possible that range $(A - \lambda I) \neq \overline{\text{range}\,(A - \lambda I)}$ and $\overline{\text{range}\,(A - \lambda I)} \neq H$.

Def: An operator A is **bounded below**, if there exists some $\delta > 0$, such that $||Af|| \geq \delta ||f||$ for all $f \in H$ with $f \neq 0$.

Claim 2. *If $B \in \mathcal{B}(H)$, then B is bounded below if and only if B is one-to-one and range B is closed.*

Proof. ($\Longrightarrow$) Assume B is bounded below, therefore, there exists some $\delta > 0$, such that $||Bf|| \geq \delta||f||$ for all $f \in H\backslash\{0\}$. To show B is one-to-one, let $f \in$ null (B). Consequently, $Bf = 0$, so $||Bf|| = ||0|| = 0$. Since B is bounded below, we have that $0 \geq \delta||f||$. Since norms are always non-negative, this inequality only holds if $||f|| = 0$, that is if $f = 0$. Thus, we've shown null $B \subseteq \{0\}$. Since the reverse containment is automatic, we have that null $B = \{0\}$, and so B is one-to-one.

It remains to show that range B is closed. We will do this using the limit point characterization since H is a Hilbert space; note that this works on the less restrictive Banach space as well. Let $y \in H$ be a limit point of range B. Thus, there exists a sequence (y_n) in range B such that $\lim_{n\to\infty} y_n = y$ and $y_n \neq y$ for each $n \in \mathbb{N}$. Since (y_n) is in range B, there exists some sequence (v_n) in H such that $(Bv_n) = (y_n)$. Since $\lim_{n\to\infty} y_n = y$ in H, (y_n) is a Cauchy sequence. Let $\epsilon > 0$. Consequently, $\epsilon\delta > 0$. Thus, because (y_n) is a Cauchy sequence, there exists some $N \in \mathbb{N}$ such that whenever $n, m \geq N$,

$$\begin{aligned} \epsilon\delta &> ||y_n - y_m|| \\ &= ||Bv_n - Bv_m|| && (3.1) \\ &= ||B(v_n - v_m)|| \\ &\geq \delta||v_n - v_m|| && (3.2) \end{aligned}$$

Line (3.1) follows from $(Bv_n) = (y_n)$, and line (3.2) follows from B being bounded below.

Since $\delta > 0$, we can divide both sides of the above inequality to obtain $\epsilon > ||v_n - v_m||$. Thus, we have shown that (v_n) is a Cauchy sequence in H. Since H is a Hilbert space it follows that (v_n) converges to some $v \in H$. Since B is bounded, it is continuous, and so $Bv = B\left(\lim_{n\to\infty} v_n\right) = \lim_{n\to\infty} Bv_n = \lim_{n\to\infty} y_n = y$. Thus, by definition, $y \in$ range B. Therefore, range B contains all its limit points, and so is closed.

($\Longleftarrow$) Assume range$B$ is closed and B is one-to-one. Define the linear map $\widetilde{B} : H \longrightarrow$ range B by $\widetilde{B}v = Bv$ for each $v \in H$. Since B is one-to-one, $\widetilde{B}$ is also one-to-one. $\widetilde{B}$ is clearly onto by definition and so $\widetilde{B}$ is invertible. We also note that H is a Banach space since it is a Hilbert space and that range B is a Banach space too as it is a subspace of a Banach space. Thus, by the Inverse Mapping Theorem, we conclude that $\widetilde{B}^{-1}$ is bounded since $\widetilde{B}$ is bounded (note $\widetilde{B}$ is bounded because B is bounded).

Since $\widetilde{B}$ is invertible and the zero map is not invertible we know $\widetilde{B}^{-1} \neq 0$, and so $||\widetilde{B}^{-1}|| \neq 0$. Thus, we can define $\delta = \dfrac{1}{||\widetilde{B}^{-1}||}$, which we will show bounds B below. Let $v \in H\backslash\{0\}$. Since $v \neq 0$, $Bv \neq 0$ because null $B = \{0\}$ (this is because B is one-to-one). Now,

$$\begin{aligned}
||v|| &= ||Iv|| \\
&= ||\widetilde{B}^{-1}(\widetilde{B}v)|| \\
&= ||\widetilde{B}^{-1}(Bv)|| \\
&\leq ||\widetilde{B}^{-1}|| \cdot ||Bv|| \\
&= \frac{1}{\delta}||Bv||
\end{aligned}$$

Thus, $\delta||v|| \leq ||Bv||$. Therefore, B is bounded below as desired. □

Claim 3. *An operator $A \in \mathcal{B}(H)$ is invertible if and only if A is bounded below and A has dense range in H.*

Proof. ($\Longrightarrow$) Assume $A \in \mathcal{B}(H)$ is invertible. Consequently, A must be onto, and so range $A = H$. Since range $A \subseteq \overline{\text{range } A} \subseteq H$, it must be the case that $\overline{\text{range } A} = H$. Thus, it remains only to show that A is bounded below. Since A is invertible, it cannot be the zero operator, and so $A^{-1} \neq 0$. Hence, $||A^{-1}|| \neq 0$. Thus, we can define $\delta = \dfrac{1}{||A^{-1}||}$. Now, for $v \in H \backslash \{0\}$,

$$\begin{aligned} ||v|| &= ||A^{-1}(Av)|| \\ &\leq ||A^{-1}|| \cdot ||Av|| \\ &= \frac{1}{\delta}||Av|| \end{aligned}$$

Thus, $\delta||v|| \leq ||Av||$; that is, A is bounded below, as desired.

($\Longleftarrow$) Assume $A \in \mathcal{B}(H)$ is bounded below and A has dense range in H. Since A is bounded below it follows by Claim 2 that A is one-to-one and range A is closed. Since range A is closed, range $A = \overline{\text{range } A}$. Since A has dense range in H, by definition $\overline{\text{range } A} = H$. Thus, range $A = H$, and so A is onto. Since A is both one-to-one and onto, it is invertible. □

Claim 3 suggests a way to divide the spectrum:

Def: Let $A \in \mathcal{B}(H)$. The **approximate point spectrum** of A, denoted $\sigma_{ap}(A)$, is the set $\sigma_{ap}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not bounded below}\}$.

Def: Let $A \in \mathcal{B}(H)$. The **compression spectrum** of A, denoted $\Gamma(A)$, is the set $\Gamma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ does not have dense range}\}$.

Note. *Claim 3 tells us that* $\sigma(A) = \sigma_{ap}(A) \cup \Gamma(A)$.

Def: Let A be a bounded linear operator. The **spectral radius** of A, denoted $r(A)$, is $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$.

Proposition 1 ([9], Theorem 5.6.12)**.** *Assume A is a square matrix with complex entries. Then,*

1. $r(A) < 1$ *if and only if* $\lim_{n\to\infty} A^n = 0$
2. $r(A) > 1$ *if and only if* $\lim_{n\to\infty} \|A^n\| = \infty$

Theorem 42 ([12] Theorem 1.2.4)**.** *Assume A is a bounded linear operator, then:*

1. *If* $||I - A|| < 1$*, then A is invertible*
2. $\sigma(A)$ *is a nonempty compact subset of* $\mathbb{C}$
3. *If A is invertible, then* $\sigma(A^{-1}) = \{\frac{1}{\lambda} : \lambda \in \sigma(A)\}$
4. $\sigma(A^*) = \{\overline{\lambda} : \lambda \in \sigma(A)\}$
5. *The spectral radius formula holds; that is,* $r(A) = \lim_{n\to\infty} ||A^n||^{1/n}$*. In particular,* $r(A) \leq ||A||$
6. *If A is an operator on a finite-dimensional space then* $\sigma(A) = \sigma_p(A)$

Proof. (1) Since $I - A$ is a bounded linear operator with $||I - A|| < 1$, it follows by theorem 17 that $I - (I - A) = A$ is invertible.

(2) By theorem 19, $\sigma(A)$ is nonempty.

Next, we show that $\sigma(A)$ is compact. First, we show it is closed by showing the complement is open. By theorem 18, we know that $\mathcal{I} = \{T \in \mathcal{B}(H) : T \text{ is invertible}\}$ is open in $\mathbb{C}$. Now, define the operator-valued function $\varphi : \mathbb{C} \longrightarrow \mathcal{B}(H)$ by $\varphi(\lambda) = A - \lambda I$. We claim that φ is continuous at each $\lambda_0 \in \mathbb{C}$. Let $\varepsilon > 0$. Define $\delta = \varepsilon$ and suppose $|\lambda - \lambda_0| < \delta$. Now,

$$\begin{aligned} ||\varphi(\lambda) - \varphi(\lambda_0)|| &= ||A - \lambda I - (A - \lambda_0 I)|| \\ &= ||(\lambda_0 - \lambda)I|| \\ &= |\lambda_0 - \lambda| \cdot ||I|| \\ &= |\lambda_0 - \lambda| \\ &< \delta \\ &= \varepsilon \end{aligned}$$

Therefore, φ is continuous at each $\lambda_0 \in \mathbb{C}$. Since φ is continuous and $\mathcal{I}$ is open in $\mathbb{C}$, it follows that $\varphi^{-1}(\mathcal{I}) = \mathbb{C}\backslash\sigma(A)$ is open in $\mathbb{C}$. Thus, the complement, $\sigma(A)$, must be closed in $\mathbb{C}$.

Lastly, we show that $\sigma(A)$ is bounded. Since $\sigma(A)$ is nonempty, there exists some $\lambda \in \sigma(A)$. Note we can demand that $\lambda \neq 0$ (the only scenario in which we cannot is when $\sigma(A) = \{0\}$, in which case the spectrum is clearly bounded). Since $\lambda \in \sigma(A)$, by definition, $A - \lambda I$ is not invertible. Thus, $\frac{-1}{\lambda}(A - \lambda I) = \frac{-1}{\lambda}A + I$ is not invertible. Thus, by the contrapositive of (1), $1 \leq ||I - (\frac{-1}{\lambda}A + I)|| = ||\frac{1}{\lambda}A|| = \frac{1}{|\lambda|}||A||$. Thus, $|\lambda| \leq ||A||$. Since A is bounded by assumption, $|\lambda| \leq ||A|| < \infty$, and so $\sigma(A)$ is bounded. Therefore, $\sigma(A)$ is a compact subset of $\mathbb{C}$.

(3) Assume A is invertible.

($\subseteq$) Let $\alpha \in \sigma(A^{-1})$. Then, $A^{-1} - \alpha I$ is not invertible. Note that $\alpha \neq 0$, since $A^{-1} - 0I = A^{-1}$ is invertible. Now,

$$\begin{aligned} A^{-1} - \alpha I &= A^{-1}(I - \alpha A) \\ &= -\alpha A^{-1}\left(\frac{-1}{\alpha}I + A\right) \\ &= -\alpha A^{-1}\left(A - \frac{1}{\alpha}I\right) \end{aligned}$$

Since A^{-1} is invertible, $-\alpha A^{-1}$ is invertible. Since $A^{-1} - \alpha I$ is not invertible, it must follow that $A - \frac{1}{\alpha}I$ is not invertible. Thus, $\frac{1}{\alpha} \in \sigma(A)$. Hence, $\alpha \in \{\frac{1}{\lambda} : \lambda \in \sigma(A)\}$, and so $\sigma(A^{-1}) \subseteq \{\frac{1}{\lambda} : \lambda \in \sigma(A)\}$.

($\supseteq$) Let $\gamma \in \{\frac{1}{\lambda} : \lambda \in \sigma(A)\}$. Then, $\gamma = \frac{1}{\lambda}$ for some $\lambda \in \sigma(A)$. Since $\lambda \in \sigma(A)$, $A - \lambda I$ is not invertible. Note that $\lambda \neq 0$, because $A - 0I = A$ is invertible. Now,

$$\begin{aligned} A - \lambda I &= A(I - \lambda A^{-1}) \\ &= -\lambda A\left(\frac{-1}{\lambda}I + A^{-1}\right) \\ &= -\lambda A(A^{-1} - \gamma I) \end{aligned}$$

Since A is invertible, $-\lambda A$ is invertible. Since $A - \lambda I$ is not invertible, it must follow that $A^{-1} - \gamma I$ is not invertible, and so $\gamma \in \sigma(A^{-1})$. Thus, $\{\frac{1}{\lambda} : \lambda \in \sigma(A)\} \subseteq \sigma(A^{-1})$. Therefore, $\sigma(A^{-1}) = \{\frac{1}{\lambda} : \lambda \in \sigma(A)\}$.

(4) In order to prove (4), we claim that if A is not invertible, then A^* is not invertible. Seeking a contradiction, suppose A^* is invertible. Then, $I = (A^*)^{-1}A^*$ and

$I = A^*(A^*)^{-1}$. Taking the adjoint of both sides of both equations gives us:

$$\begin{aligned} I^* &= ((A^*)^{-1}A^*)^* \\ &= A^{**}((A^*)^{-1})^* \\ &= A((A^*)^{-1})^* \end{aligned}$$

and

$$\begin{aligned} I^* &= (A^*(A^*)^{-1})^* \\ &= ((A^*)^{-1})^*A^{**} \\ &= A((A^*)^{-1})^*A \end{aligned}$$

Since $I^* = I$, we have $I = A((A^*)^{-1})^*$ and $I = ((A^*)^{-1})^*A$. That is, $A^{-1} = ((A^*)^{-1})^*$, contradicting the assumption that A is not invertible. Therefore, we must conclude that A^* is not invertible as we claimed.

($\subseteq$) Let $\gamma \in \sigma(A^*)$. Then, $A^* - \gamma I$ is not invertible. So, by our claim we have that $(A^* - \gamma I)^* = A^{**} - \overline{\gamma} I = A - \overline{\gamma} I$ is not invertible. Hence, $\overline{\gamma} \in \sigma(A)$, and so $\gamma = \overline{\overline{\gamma}} \in \{\overline{\lambda} \in \mathbb{C} : \lambda \in \sigma(A)\}$. Thus, $\sigma(A^*) \subseteq \{\overline{\lambda} : \lambda \in \sigma(A)\}$.

($\supseteq$) Let $\alpha \in \{\overline{\lambda} : \lambda \in \sigma(A)\}$. Consequently, $\alpha = \overline{\lambda}$ for some $\lambda \in \sigma(A)$. Since $\lambda \in \sigma(A)$, $A - \lambda I$ is not invertible. So, by our claim, $(A - \lambda I)^* = A^* - \overline{\lambda} I$ is not invertible. Thus, $\alpha = \overline{\lambda} \in \sigma(A^*)$, and so $\{\overline{\lambda} : \lambda \in \sigma(A)\} \subseteq \sigma(A^*)$.

Therefore, $\sigma(A^*) = \{\overline{\lambda} : \lambda \in \sigma(A)\}$.

(5) To stay within the scope of this thesis, we will prove only the finite-dimensional case, where A is a complex-valued m x m matrix. Case II of this proof is adapted from the proof given in [7].

Case I: If $r(A) = 0$, then by definition of the spectral radius we must have $\sigma(A) = \{0\}$. Thus, 0 is the only eigenvalue of A. Hence, the characteristic polynomial for A is $p(z) = z^m$. By the Cayley-Hamilton Theorem, $p(A) = A^m = 0$. Clearly, for all $n \in \mathbb{N}$ such that $n \geq m$, $A^n = 0$. Thus, for $n \geq m$, $\|A^n\|^{1/n} = \|0\|^{1/n} = 0^{1/n} = 0$. Therefore, $\lim_{n\to\infty} \|A^n\|^{1/n} = 0 = r(A)$.

Case II: Suppose $r(A) \neq 0$. Consider $\frac{1}{n} > 0$ for $n \in \mathbb{N}$ and define $A^+ = \dfrac{1}{r(A) + \frac{1}{n}} A$ and $A^- = \dfrac{1}{r(A) - \frac{1}{n}} A$. Since $r(A) \geq 0$, $r(A) + \frac{1}{n}$ is always positive. By picking n sufficiently large we can ensure that $r(A) - \frac{1}{n}$ is positive as well, since $r(A) \neq 0$. Thus, we have that $\dfrac{1}{r(A) + \frac{1}{n}}$ and $\dfrac{1}{r(A) - \frac{1}{n}}$ are both positive. Since the spectral radius is defined by $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$ and non-negative constants "factor out" of the supremum, it follows that $r(A^+) = \dfrac{1}{r(A) + \frac{1}{n}} r(A)$ and $r(A^-) = \dfrac{1}{r(A) - \frac{1}{n}} r(A)$. Since $\frac{1}{n} > 0$, $r(A) + \frac{1}{n} > r(A)$ and $r(A) - \frac{1}{n} < r(A)$. Thus, $r(A^+) = \dfrac{1}{r(A) + \frac{1}{n}} r(A) < 1$ and $r(A^-) = \dfrac{1}{r(A) - \frac{1}{n}} r(A) > 1$. Thus, by Proposition 1, $\lim_{n\to\infty} (A^+)^n = 0$. Hence, for $\varepsilon = 1$, there exists some $N_1 \in \mathbb{N}$ such that whenever $n \geq N_1$, $||(A^+)^n|| < 1$. Thus, by definition of $(A^+)^n$, $\left|\left|\left(\dfrac{1}{r(A) + \frac{1}{n}} A\right)^n\right|\right| < 1$. So, $||A^n|| < ||(r(A) + \frac{1}{n})^n|| = |(r(A) + \frac{1}{n})|^n$. Thus, $||A^n||^{1/n} < |r(A) + \frac{1}{n}|$. Also by Proposition 1, $r(A^{-1}) > 1$, $\lim_{n\to\infty} \|(A^-)^n\| = \infty$. So, for the specific $\varepsilon = 1$, there exists $N_2 \in \mathbb{N}$ such that $\|(A^-)\| > 1$. That is, by the definition of A^-, $\left\|\left(\dfrac{1}{r(A) - \frac{1}{n}} A\right)^n\right\| > 1$. So,
$\|A^n\| > \left\|(r(A) - \frac{1}{n})^n\right\| = |r(A) - \frac{1}{n}|^n$. Thus, $\|A^n\|^{1/n} > |r(A) - \frac{1}{n}|$. Thus, we have shown $|r(A) - \frac{1}{n}| < \|A^n\|^{1/n} < |r(A) + \frac{1}{n}|$. Therefore, by the squeeze theorem, it follows that $\lim_{n\to\infty} \|A^n\|^{1/n} = r(A)$, as desired.

(6) Assume $A \in \mathcal{B}(V)$ and $\dim(V) < \infty$. We claim that $\sigma(A)^C = \sigma_{ap}(A)^C$.

($\subseteq$) Let $\lambda \in \sigma(A)^C$. Since A is bounded, we know $A - \lambda I$ is bounded. By definition of the spectrum, $A - \lambda I$ is invertible. Thus, by claim 3, it follows that $A - \lambda I$ is bounded below, and so $\lambda \in \sigma_{ap}(A)^C$ by definition. Hence, $\sigma(A)^C \subseteq \sigma_{ap}(A)^C$.

($\supseteq$) Let $\lambda \in \sigma_{ap}(A)^C$. Then, $A - \lambda I$ is bounded below, and so by claim 2, $A - \lambda I$ is one-to-one. Since $\dim(V) < \infty$, one-to-one is equivalent to onto. Thus, $A - \lambda I$ is bijective, and so $\lambda \in \sigma(A)^C$ by definition. Thus, $\sigma_{ap}(A)^C \subseteq \sigma(A)^C$.

Therefore, $\sigma(A)^C = \sigma_{ap}(A)^C$. Taking the complement of both sides yields $\sigma(A) = \sigma_{ap}(A)$, as desired. □

Theorem 43 ([12] Theorem 1.2.7)**.** *For each $A \in \mathcal{B}(H)$, if $\sigma(A)$ is bounded, then the boundary of $\sigma(A)$ is contained in $\sigma_{ap}(A)$. In particular, $\sigma_{ap}(A) \neq \varnothing$.*

Lemma 44. *If x is a boundary point of S, then there exists a sequence (x_n) not contained in S such that $\lim_{n\to\infty} x_n = x$.*

Proof of Lemma 44. Let x be a boundary point of S. For each $n \in \mathbb{N}$, $\frac{1}{n} > 0$. So, the open ball $B(x, \frac{1}{n})$ around x of radius $\frac{1}{n}$ contains at least one point in S and at least one point in S^C. Thus, $B(x, \frac{1}{n}) \cap S^C \neq \varnothing$. Define this intersection to be B_n. Thus, for each $n \in \mathbb{N}$, we know there exists some $x_n \in B_n$. Thus, we construct a sequence (x_n) contained in S^C. Now, we argue that $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. By the Archimedean Property, there exists some $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Let $n \geq N$. Then, $\frac{1}{n} \leq \frac{1}{N} < \varepsilon$. By construction of the B_n's we have $|x_n - x| < \frac{1}{n}$, and so $|x_n - x| < \varepsilon$. Therefore, $\lim_{n\to\infty} x_n = x$. □

The following proof is adapted from the proof given in [12].

Proof of Theorem 43. Seeking a contradiction, let λ be in the boundary of $\sigma(A)$ and $\lambda \notin \sigma_{ap}(A)$. Since λ is in the boundary of $\sigma(A)$, by lemma 4 there exists a sequence (λ_n) not contained in $\sigma(A)$ such that $\lim_{n\to\infty} \lambda_n = \lambda$. We claim that there exists some

$k > 0$ and $M \in \mathbb{N}$, such that whenever $n \geq M$, $\|(A - \lambda_n I)f\| \geq k\|f\|$ for every $f \in H$. To prove this claim, suppose not. Then, for every $k > 0$ and every $M \in \mathbb{N}$, there exists some $n \geq M$ and some f_n such that $\|(A - \lambda_n I)f_n\| < k\|f_n\|$. We will assume without loss of generality that $\|f_n\| = 1$, otherwise we could simply replace f_n with $\dfrac{f_n}{\|f_n\|}$ (as the above inequality does not even hold for $f_n = 0$.) Let $\varepsilon > 0$, then there exists some $M \in \mathbb{N}$ such that whenever $n \geq M$, $|\lambda_n - \lambda| < \frac{\varepsilon}{2}$. With $\frac{\varepsilon}{2} > 0$ and $M \in \mathbb{N}$ fixed, we know by the above that there exists some $n \geq M$ and some f_n with $\|f_n\| = 1$ such that $\|(A - \lambda_n I)f_n\| < \frac{\varepsilon}{2}$. Thus,

$$\begin{aligned}
\|(A - \lambda I)f_n\| &= \|(A - \lambda_n I)f_n + (\lambda_n I - \lambda I)f_n\| \\
&\leq \|(A - \lambda_n I)f_n\| + \|(\lambda_n I - \lambda I)f_n\| \\
&= \|(A - \lambda_n I)f_n\| + |\lambda_n - \lambda|\,\|f_n\| \\
&= \|(A - \lambda_n I)f_n\| + |\lambda_n - \lambda| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon \\
&= \varepsilon\|f_n\|
\end{aligned}$$

Thus, we have shown for every $\varepsilon > 0$, there exists some $f_n \in H \backslash \{0\}$ such that $\|(A - \lambda I)f_n\| < \varepsilon\|f_n\|$, which is precisely the negation of the definition of bounded below. So, $A - \lambda I$ is not bounded below, contradicting the assumption that $\lambda \notin \sigma_{ap}(A)$. Therefore, there exists some $k > 0$ and $M \in \mathbb{N}$, such that whenever $n \geq M$, $\|(A - \lambda_n I)f\| \geq k\|f\|$ for every $f \in H$, proving our claim.

Recall $\sigma(A)$ is compact by theorem 42 part (2), and so it is closed. Since λ is a boundary point of $\sigma(A)$, it follows $\lambda \in \sigma(A)$. Since $\lambda \in \sigma(A) = \sigma_{ap}(A) \cup \Gamma(A)$ (by claim 3), if we show $\lambda \notin \Gamma(A)$ we will be forced to conclude $\lambda \in \sigma_{ap}(A)$, contradicting the assumption that $\lambda \notin \sigma_{ap}(A)$.

Let $g \in H \backslash \{0\}$. We will show that $g \in \overline{\text{range}\ (A - \lambda I)}$, which will let us conclude that $H = \overline{\text{range }(A - \lambda I)}$ because $0 \in \text{range }(A - \lambda I)$ automatically. Let $\varepsilon > 0$, then $\dfrac{k}{\|g\|}\varepsilon > 0$, so there exists some $N \in \mathbb{N}$ such that whenever $n \geq N$, $|\lambda_n - \lambda| < \dfrac{k}{\|g\|}\varepsilon$. Since $\lambda_n \notin \sigma(A)$, $A - \lambda_n I$ is invertible. Thus $(A - \lambda_n I)^{-1}g$ is some element in $H \backslash \{0\}$, which we will call f_n. We know $f_n \neq 0$, because $g \neq 0$ and $\ker(A - \lambda_n I) = \{0\}$). So, $(A - \lambda_n I)f_n = g$. By our claim, if $n \geq K \geq M$, then $\|(A - \lambda_n I)f_n\| \geq k\,\|f_n\|$; that is, $\|g\| \geq k\,\|f_n\|$.

Now,

$$
\begin{aligned}
\|(A - \lambda I)f_n - g\| &= \|(A - \lambda_n I)f_n - g + (\lambda_n I - \lambda I)f_n\| \\
&\leq \|(A - \lambda_n I)f_n - g\| + \|(\lambda_n I - \lambda I)f_n\| \\
&= \|g - g\| + |\lambda_n - \lambda|\,\|f_n\| \\
&= |\lambda_n - \lambda|\,\|f_n\| \\
&\leq |\lambda_n - \lambda|\frac{\|g\|}{k} \qquad (3.3) \\
&< \left(\frac{k}{\|g\|}\varepsilon\right)\frac{\|g\|}{k} \qquad (3.4) \\
&= \varepsilon
\end{aligned}
$$

Line (3.3) follows from the fact that $k\,\|f_n\| \leq \|g\|$ whenever $n \geq K$ and line (3.4) follows from the fact that $|\lambda_n - \lambda| < \dfrac{k}{\|g\|}\varepsilon$ whenever $n \geq K$.

Thus, we have shown that whenever $n \geq K$, $\|(A - \lambda I)f_n - g\| < \varepsilon$; that is, $\lim_{n\to\infty}(A - \lambda_n)f_n = g$. Hence, $g \in \overline{\text{range }(A - \lambda I)}$. Thus, $H \subseteq \overline{\text{range }(A - \lambda I)}$, and so $H = \overline{\text{range }(A - \lambda I)}$. Hence, $A - \lambda I$ has dense range, and so by definition $\lambda \notin \Gamma(A)$. Thus, since $\lambda \in \sigma(A) = \sigma_{ap}(A) \cup \Gamma(A)$, it must follow that $\lambda \in \sigma_{ap}(A)$, contradicting the assumption that $\lambda \notin \sigma_{ap}(A)$. Therefore, we must conclude that $\lambda \in \sigma_{ap}(A)$, and so $\partial\sigma(A) \subseteq \sigma_{ap}(A)$. Since $\sigma(A)$ is compact and $\sigma(A) \neq \varnothing$, the previous containment gives us that $\sigma_{ap}(A) \neq \varnothing$ as well. $\square$

We now make the leap from the spectrum to the numerical range with the following theorem. We will discuss the numerical range in more depth in chapter 4.

Theorem 45 ([12] Theorem 1.2.11)**.** *For each $A \in \mathcal{B}(H)$, $\sigma(A) \subseteq \overline{W(A)}$.*

The following proof draws from the proof given in [12].

Proof. Assume $A \in \mathcal{B}(H)$. Let $\lambda \in \sigma(A)$. By Claim 3, $\sigma(A) = \sigma_{ap}(A) \cup \Gamma(A)$. Thus, $\lambda \in \sigma_{ap}(A)$ or $\lambda \in \Gamma(A)$.

Case I: If $\lambda \in \sigma_{ap}(A)$, then $A - \lambda I$ is not bounded below. That is, for every $\delta > 0$, there exists $f \in H \backslash \{0\}$ such that $\|(A-\lambda I)f\| < \delta \|f\|$. Thus, for each $n \in \mathbb{N}$, there exists some $f_n \in H\backslash\{0\}$ such that $\|(A-\lambda I)f_n\| < \frac{1}{n}\|f_n\|$. Since each $f_n \neq 0$, $\|f_n\| \neq 0$. Thus, $\dfrac{\|(A-\lambda I)f_n\|}{\|f_n\|} < \dfrac{1}{n}$. Thus by linearity, we have that $\left\|(A-\lambda I)\left(\dfrac{f_n}{\|f_n\|}\right)\right\| < \dfrac{1}{n}$.

Define the sequence (g_n) by $g_n = \dfrac{f_n}{\|f_n\|}$ for each $n \in \mathbb{N}$. We note that $\|g_n\| = \left\|\dfrac{f_n}{\|f_n\|}\right\| = \dfrac{\|f_n\|}{\|f_n\|} = 1$ for every $n \in \mathbb{N}$. Let $\varepsilon > 0$ and $N = 1$. Then, by the Archimedean Property there exists some $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Thus, by the above work, whenever $n \geq N$,

$$\|(A-\lambda I)g_n\| = \left\|(A-\lambda I)\left(\frac{f_n}{\|f_n\|}\right)\right\| < \frac{1}{N} \leq \frac{1}{n} < \varepsilon$$

Thus, we have constructed a sequence (g_n) contained in H, such that $\|g_n\| = 1$ and $\lim_{n\to\infty} \|(A-\lambda I)(g_n)\| = 0$. Note that because $\|g_n\| = 1$ for each $n \in \mathbb{N}$, $(\langle Ag_n, g_n\rangle)$ is a sequence contained in W(A).

We claim that $\lim_{n\to\infty}\langle Ag_n, g_n\rangle = \lambda$. Let $\varepsilon > 0$. Since $\lim_{n\to\infty}\|(A-\lambda I)(g_n)\| = 0$, there exists some $N \in \mathbb{N}$, such that whenever $n \geq N$, $\|(A-\lambda I)g_n\| < \varepsilon$. Thus, letting $n \geq N$, we have

$$\begin{aligned}
|\langle Ag_n, g_n\rangle - \lambda| &= |\langle Ag_n, g_n\rangle - \lambda \left\|g_n\right\|^2| && (3.5)\\
&= |\langle Ag_n, g_n\rangle - \lambda\langle g_n, g_n\rangle|\\
&= |\langle Ag_n, g_n\rangle + \langle -\lambda g_n, g_n\rangle|\\
&= |\langle Ag_n - \lambda g_n, g_n\rangle|\\
&= |\langle (A-\lambda I)g_n, g_n\rangle|\\
&\leq \|(A-\lambda I)g_n\| \, \|g_n\| && (3.6)\\
&= \|(A-\lambda I)g_n\| && (3.7)\\
&< \varepsilon && (3.8)
\end{aligned}$$

Line (3.5) and (3.7) follow from the fact that $\|g_n\| = 1$. Line (3.6) follows from the Cauchy-Schwarz inequality. Line (3.8) follows from the assumption that $n \geq N$. Therefore, $\lim_{n\to\infty} \langle Ag_n, g_n\rangle = \lambda$, as we claimed.

Since $\lim_{n\to\infty} \langle Ag_n, g_n\rangle = \lambda$ and $(\langle Ag_n, g_n\rangle)$ is contained in $\mathrm{W}(A)$, it follows that either $\lambda = \langle Ag_n, g_n\rangle$ for some $n \in \mathbb{N}$ or $\lambda \neq \langle Ag_n, g_n\rangle$ for every $n \in \mathbb{N}$. In the former case, $\lambda \in \mathrm{W}(A) \subseteq \overline{\mathrm{W}(A)}$. In the latter case, λ is by definition a limit point of $\mathrm{W}(A)$, and so $\lambda \in \overline{\mathrm{W}(A)}$.

Case II: Assume $\lambda \in \Gamma(A)$. Then, $A - \lambda I$ does not have dense range in H; that is, $\overline{\text{range } (A-\lambda I)} \neq H$. Since $\overline{\text{range } (A-\lambda I)} \subseteq H$, there exists some $h \in H$ such that $h \notin \overline{\text{range } (A-\lambda I)}$. Since $\text{range } (A-\lambda I) \subseteq \overline{\text{range } (A-\lambda I)}$, it follows that $h \notin \text{range } (A-\lambda I)$. Since $H = \text{range } (A-\lambda I) \bigoplus (\text{range } (A-\lambda I))^{\perp}$, it follows that $h = f + g$, where $f \in \text{range } (A-\lambda I)$ and $g \in (\text{range } (A-\lambda I))^{\perp}$. Since $h \notin \text{range } (A-\lambda I)$, it must follow that $g \neq 0$. If $f = 0$, then $h = g$. If $f \neq 0$, then we could simply replace h with g, since $g \in H$ and necessarily

$g \notin$ range $(A-\lambda I)$ (this is because $g \neq 0$ and the intersection of a space with its orthogonal complement is $\{0\}$). Thus, by replacing h with g, we may assume without loss of generality that $h \in (\text{range } (A-\lambda I))^{\perp}$. We may also assume without loss of generality that $\|h\| = 1$; otherwise, we could simply divide by $\|h\|$ as $h \neq 0$, since $h \notin \overline{\text{range } (A-\lambda I)}$ and $0 \in \overline{\text{range } (A-\lambda I)}$. So, we have some $h \in H$ such that $\|h\| = 1$ and $h \in (\text{range } (A-\lambda I))^{\perp}$.

By definition, for every $f \in H$, $(A-\lambda I)f \in$ range $(A-\lambda I)$. Thus, $\langle (A-\lambda I)f, h\rangle = 0$ for every $f \in H$. In particular, this holds for $f = h$. So,

$$\begin{aligned} 0 &= \langle (A-\lambda I)h, h\rangle \\ &= \langle Af - \lambda h, h\rangle \\ &= \langle Ah, h\rangle + \langle -\lambda h, h\rangle \\ &= \langle Ah, h\rangle - \lambda \langle h, h\rangle \\ &= \langle Ah, h\rangle - \lambda \|h\|^2 \qquad (3.9) \\ &= \langle Ah, h\rangle - \lambda \end{aligned}$$

Line (3.9) follows from the fact that $\|h\| = 1$. Therefore, $\lambda = \langle Ah, h\rangle \in \mathrm{W}(A)$. Thus, $\Gamma(A) \subseteq \mathrm{W}(A) \subseteq \overline{\mathrm{W}(A)}$.

Therefore, by cases I and II, we have that $\sigma(A) = \sigma_{ap}(A) \cup \Gamma(A) \subseteq \overline{\mathrm{W}(A)}$, as desired.

□

Next, we look at the spectra of a non-compact operator and a prototypical compact operator. To do so we need the following two technical lemmas.

Lemma 46. $(1, \lambda, \lambda^2, \lambda^3, \ldots) \in \ell^2$ *if and only if* $\lambda \in \mathbb{D}$. *Furthermore,* $(1, \lambda, \lambda^2, \lambda^3, \ldots)$ *is an eigenvector of* S^* *with corresponding eigenvalue* λ *if and only if* $\lambda \in \mathbb{D}$.

Proof. Note $\sum_{n=0}^{\infty} |\lambda^n|^2 = \sum_{n=0}^{\infty} (|\lambda|^2)^n$ is a geometric series. Thus, it converges if and only if $|\lambda|^2 < 1$. Therefore, $(1, \lambda, \lambda^2, \lambda^3, \ldots) \in \ell^2$ if and only if $\lambda \in \mathbb{D}$.

Note that $(1, \lambda, \lambda^2, \lambda^3, \ldots)$ is not the zero vector due to the first entry. Also observe that,

$$\begin{aligned} S^*(1, \lambda, \lambda^2, \lambda^3, \ldots) &= (\lambda, \lambda^2, \lambda^3, \lambda^4, \ldots) \\ &= \lambda(1, \lambda, \lambda^2, \lambda^3, \ldots) \end{aligned}$$

Therefore, $(1, \lambda, \lambda^2, \lambda^3, \ldots)$ is an eigenvector of S^* with corresponding eigenvalue λ. We note it is only an eigenvector if $(1, \lambda, \lambda^2, \lambda^3, \ldots) \in \ell^2$, which is true if and only if $\lambda \in \mathbb{D}$. □

Def: For a set $\Omega \subseteq \mathbb{C}$, define $\Omega^* = \{\overline{\gamma} : \gamma \in \Omega\}$.

Lemma 47 ([8], problem 85)**.** *If $A \in \mathcal{B}(H)$, then $\sigma_p(A^*) = (\Gamma(A))^*$.*

Proof. ($\subseteq$) Let $\lambda \in \sigma_p(A^*)$. Then, λ is an eigenvalue of A^*, and so $A^* - \lambda I$ is not one-to-one. By 15 it follows that $(A^* - \lambda I)^* = A - \overline{\lambda} I$ does not have dense range in H. That is, $\overline{\lambda} \in \Gamma(A)$. Thus, $\lambda \in (\Gamma(A))^*$. Therefore, $\sigma_p(A^*) \subseteq (\Gamma(A))^*$.

($\supseteq$) Let $\lambda \in (\Gamma(A))^*$. Then, $\overline{\lambda} \in \Gamma(A)$. Thus, $A - \overline{\lambda} I$ does not have dense range in H. Hence, by theorem 15, $(A - \overline{\lambda} I)^* = A^* - \lambda I$ is not one-to-one. Thus, λ is an eigenvalue of A^*, and so $\lambda \in \sigma_p(A^*)$. Therefore, $(\Gamma(A))^* \subseteq \sigma_p(A^*)$.

Thus, $\sigma_p(A^*) = (\Gamma(A))^*$, as desired. □

Example 4. *For the forward shift operator $S \in \mathcal{L}(\ell^2)$, the spectrum decomposes as follows:*

1. $\sigma_p(S) = \mathbb{D}$

2. $\sigma(S) = \overline{\mathbb{D}}$

3. $\sigma_{ap}(S) = \partial\mathbb{D}$

4. $\Gamma(S) = \mathbb{D}$

Proof. (1) By lemma 46, λ is an eigenvalue of S^* if and only if $\lambda \in \mathbb{D}$. Therefore, $\sigma_p(S) = \mathbb{D}$.

(2) First, we will show that $\|S\| = 1$ using the equivalent definition, $\|S\| = \sup\{\|S(a_n)\| : (a_n) \in \ell^2 \text{ with } \|(a_n)\| = 1\}$. Let $(a_n) \in \ell^2$ with $\|(a_n)\|$. Then,

$$\begin{aligned}
\|S(a_n)\|^2 &= \|(0, a_0, a_1, \ldots)\|^2 \\
&= |0|^2 + |a_0|^2 + |a_1|^2 + \cdots && (3.10) \\
&= \sum_{n=0}^{\infty} |a_n|^2 \\
&= \|(a_n)\|^2 \\
&= 1 && (3.11)
\end{aligned}$$

Line (3.10) follows from the definition of the ℓ^2 norm. Line (3.11) follows from the assumption that $\|(a_n)\| = 1$. Since norms are always nonnegative, taking the square root of both sides of the equation yields $\|S(a_n)\| = 1$. Therefore, $\|S\| = \sup\{1\} = 1$, as desired. By theorem 42 part (6), $r(S) \leq \|S\| = 1$. By definition $r(S) = \sup\{|\lambda| : \lambda \in \sigma(S)\}$. Thus, $|\lambda| \leq 1$ for every $\lambda \in \sigma(S)$. Therefore, $\sigma(S) \subseteq \overline{\mathbb{D}}$. Next, we show that $\mathbb{D} \subseteq \sigma(S)$. Since $\mathbb{D} \subseteq \sigma(S)$, it follows $\overline{\mathbb{D}} \subseteq \overline{\sigma(S)}$. Since $\sigma(S)$ is compact, we have that $\overline{\sigma(S)} = \sigma(S)$. Thus, $\overline{\mathbb{D}} \subseteq \sigma(S)$. Therefore, $\sigma(S) = \overline{\mathbb{D}}$, as desired.

(3) Since $\sigma(S) = \overline{\mathbb{D}}$, it follows by theorem 43 that $\partial\overline{\mathbb{D}} \subseteq \sigma_{ap}(S)$. Since $\partial\mathbb{D} = \partial\overline{\mathbb{D}}$, we have $\partial\mathbb{D} \subseteq \sigma_{ap}(S)$. We claim that if $\lambda \in \mathbb{D}$, then $\|S(a_n) - \lambda(a_n)\| \geq (1 - |\lambda|)\,\|(a_n)\|$ for all $(a_n) \in \ell^2$. To prove this claim, let $\lambda \in \mathbb{D}$ and $(a_n) \in \ell^2$. Thus, $|\lambda| < 1$, and so $1 - |\lambda| > 0$. Hence, $|1 - |\lambda|| = 1 - |\lambda|$. By the reverse triangle inequality,

$$\begin{aligned} \|S(a_n) - \lambda(a_n)\| &\geq |\,\|S(a_n)\| - \|\lambda(a_n)\|\,| && (3.12)\\ &= |\,\|(a_n)\| - |\lambda|\,\|(a_n)\|\,| && (3.13)\\ &= |(1 - |\lambda|)\,\|(a_n)\|\,| \\ &= (1 - |\lambda|)\,\|(a_n)\| \end{aligned}$$

Line (3.12) is by the triangle inequality, as mentioned previously. Line (3.13) follows from the fact that $\|S(a_n)\| = \|(0, a_0, a_1, \ldots)\| = \sqrt{\sum_{n=0}^{\infty} |a_n|^2} = \|(a_n)\|$. Thus, if $\lambda \in \mathbb{D}$, then $S - \lambda I$ is bounded below as claimed. The contrapositive of this claim is that if $S - \lambda I$ is not bounded below, then $\lambda \in \mathbb{D}^C$. That is,

$$\sigma_{ap}(S) \subseteq \mathbb{D}^C = \{\lambda \in \mathbb{D} : |\lambda| \geq 1\} = \partial\mathbb{D} \cup \overline{\mathbb{D}}^C$$

Since $\sigma_{ap}(S) \subseteq \sigma(S) = \overline{\mathbb{D}}$, it must follow that $\sigma_{ap}(S) \subseteq \partial\mathbb{D}$, since $\sigma_{ap}(S) \cap \overline{\mathbb{D}}^C = \varnothing$. Therefore, $\sigma_{ap}(S) = \partial\mathbb{D}$, as desired.

(4) First, we show that $\sigma_p(S^*) = \mathbb{D}$. By lemma 46, if $\lambda \in \mathbb{D}$ then λ is an eigenvalue of S^*. Thus, we know $\mathbb{D} \subseteq \sigma_p(S^*)$. To show the reverse containment, let $\gamma \in \sigma_p(S^*)$. By definition, γ is an eigenvalue of S^*. Let $(a_n) \in \ell^2$ be the corresponding eigenvector. Then, $S(a_n) = \gamma(a_n)$; that is, $(a_1, a_2, \ldots) = (\gamma a_0, \gamma a_1, \ldots)$. Comparing entries, we have $a_{n+1} = \gamma a_n$ for every $n \in \mathbb{N} \cup \{0\}$. It follows that $a_n = \gamma^n a_0$ for every $n \in \mathbb{N}$. Thus, the eigenvector is $(a_0, \gamma a_0, \gamma^2 a_0, \ldots)$. We note that $a_0 \neq 0$, since if $a_0 = 0$, then $(a_n) = 0$, contradicting the definition of eigenvector. Since $(a_n) \in \ell^2$, $S^*(a_n) \in \ell^2$,

and so $\sum_{n=0}^{\infty} |\gamma^n a_0|^2$ converges. Note

$$\begin{aligned}\sum_{n=0}^{\infty} |\gamma^n a_0|^2 &= \sum_{n=0}^{\infty} |\gamma^n|^2 |a_0|^2 \\ &= |a_0|^2 \sum_{n=0}^{\infty} |\gamma^n|^2 \\ &= |a_0|^2 \sum_{n=0}^{\infty} (|\gamma|^2)^n\end{aligned}$$

Since this is a geometric series it is convergent if and only if $|\gamma|^2 < 1$. Thus, it is convergent if and only if $|\gamma| < 1$. Thus, $\gamma \in \mathbb{D}$. Thus, we have shown that $\sigma_p(S^*) \subseteq \mathbb{D}$. Therefore, $\sigma_p(S^*) = \mathbb{D}$.

By lemma 47, $\mathbb{D} = (\Gamma(S))^*$. Therefore, $\mathbb{D}^* = \Gamma(S)$. Since $\mathbb{D}^* = \mathbb{D}$, it follows $\mathbb{D} = \Gamma(S)$, as desired.

□

Example 5. *Let (b_n) be a sequence in $\mathbb{C}$ such that $\lim_{n\to\infty} b_n = 0$. Define the diagonal operator $T \in \mathcal{C}(\ell^2)$ by $T(a_n) = (a_n b_n)$. The spectrum of T decomposes as follows:*

1. $\sigma_p(T) = \{b_n : n \in \mathbb{N}\}$
2. $\sigma(T) = \{b_n : n \in \mathbb{N}\} \cup \{0\}$
3. $\sigma_{ap}(T) = \{b_n : n \in \mathbb{N}\} \cup \{0\}$
4. $\Gamma(T) = \{b_n : n \in \mathbb{N}\} \cup \{0\}$

Proof. (1)($\subseteq$) Let $\lambda \in \sigma_p(T)$. Let $(a_n) \in \ell^2$ be the corresponding eigenvector. Then, $T(a_n) = \lambda(a_n)$; that is, $(a_1b_1, a_2b_2, \ldots) = (\lambda a_1, \lambda a_2, \ldots)$. Comparing entries, we have that for each $n \in \mathbb{N}$, $a_nb_n = \lambda a_n$. Thus, for each $n \in \mathbb{N}$, $a_n = 0$ or $\lambda = b_n$. Since (a_n) is an eigenvector we cannot have $a_n = 0$ for every $n \in \mathbb{N}$. Thus, there must be some

$n \in \mathbb{N}$ such that $\lambda = b_n$. Hence, $\lambda \in \{b_n : n \in \mathbb{N}\}$. Thus, $\sigma_p(T) \subseteq \{b_n : n \in \mathbb{N}\}$.

($\supseteq$) Let $b_k \in \{b_n : n \in \mathbb{N}\}$. Define (c_n) by $c_n = 0$ for each $n \in \mathbb{N}\backslash\{k\}$ and $c_n = 1$ for $n = k$. Since $\|(c_n)\| = 1, (c_n) \in \ell^2$. Observe that

$$\begin{aligned} T(c_n) &= (0b_1, 0b_2, \ldots, 0b_{k-1}, 1b_k, 0b_{k+1}, \ldots) \\ &= (0, \ldots, 0, b_k, 0, \ldots) \\ &= b_k(0, \ldots, 0, 1, 0, \ldots) \\ &= b_k(c_n) \end{aligned}$$

Thus, b_k is an eigenvalue of T. Thus, by definition $b_k \in \sigma_p(T)$. Hence, $\{b_n : n \in \mathbb{N}\} \subseteq \sigma_p(T)$.

Therefore, $\{b_n : n \in \mathbb{N}\} = \sigma_p(T)$, as desired.

(2) By part (1), we know for each $n \in \mathbb{N}$, b_n is an eigenvalue of T. Thus, $T - b_nI$ is not one-to-one, and hence not invertible. Thus, $\{b_n : n \in \mathbb{N} \subseteq \sigma(T)$. By theorem 41, $0 \in \sigma(T)$. Thus, $\{b_n : n \in \mathbb{N}\} \cup \{0\} \subseteq \sigma(T)$. To show the reverse containment, let $\lambda \in \sigma(T)$. Either $\lambda = 0$ or $\lambda \neq 0$. If $\lambda = 0$, then $\lambda \in \{b_n : n \in \mathbb{N}\} \cup \{0\}$. If $\lambda \neq 0$, by the Fredholm Alternative Theorem, λ is an eigenvalue of T. Thus, $\lambda \in \sigma_p(T) = \{b_n : n \in \mathbb{N}\} \subseteq \{b_n : n \in \mathbb{N}\} \cup \{0\}$. Thus, we have shown that every $\lambda \in \sigma(T)$ is also in $\{b_n : n \in \mathbb{N}\} \cup \{0\}$; that is, $\sigma(T) \subseteq \{b_n : n \in \mathbb{N}\} \cup \{0\}$. Therefore, $\sigma(T) = \{b_n : n \in \mathbb{N}\} \cup \{0\}$.

(3) By claim 2, $\sigma_{ap}(T)^C = \{\lambda \in \mathbb{C} : T - \lambda I$ is one-to-one and range $(T - \lambda I)$ is closed$\}$. Since $T - \lambda I$ is one-to-one if and only if λ is not an eigenvalue of T, we have

$$\{\lambda \in \mathbb{C} : T - \lambda I \text{ is one-to-one}\} = \sigma_p(T)^C$$

$$= \{b_n : n \in \mathbb{N}\}^C \tag{3.14}$$

$$= \{\lambda \in \mathbb{C} : \lambda \neq b_n \text{ for any } n \in \mathbb{N}\}$$

Line (3.14) follows by part (1) of this example. By theorem 30, range $(T - \lambda I)$ is closed for every $\lambda \neq 0$. Thus, $\{\lambda \in \mathbb{C} : \text{range } (T - \lambda I) \text{ is closed}\} = \{0\}^C$. Hence,

$$\sigma_{ap}(T)^C = \sigma_p(T)^C \cap \{0\}^C \tag{3.15}$$

$$= (\sigma_p(C) \cup \{0\})^C$$

$$= (\{b_n : n \in \mathbb{N}\} \cup \{0\})^C \tag{3.16}$$

Line (3.15) follows from claim 2. Line (3.16) follows from part (1) of this example. By taking the complement of both sides of the above equation, we obtain $\sigma_{ap}(T) = \{b_n : n \in \mathbb{N}\} \cup \{0\}$.

(4) Since $T^*(a_n) = (a_n\overline{b_n})$, it follows by the same argument as in part (1) that $\sigma_p(T^*) = \{\overline{b_n} : n \in \mathbb{N}\}$. Thus, $\sigma_p(T^*)^C = \{\overline{b_n} : n \in \mathbb{N}\}^C$. Since T is compact, T^* is compact. Thus, by theorem 30, $\{\lambda \in \mathbb{C} : \text{range } (T^* - \lambda I) \text{ is closed}\} = \{0\}^C$. Thus,

$$\sigma_{ap}(T^*)^C = \sigma_p(T^*)^C \cap \{0\}^C \tag{3.17}$$

$$= (\sigma_p(T^*) \cup \{0\})^C$$

$$= (\{\overline{b_n} : n \in \mathbb{N}\} \cup \{0\})^C$$

Line (3.17) follows from claim 2. By taking the complement of both sides of the above equation, we have $\sigma_{ap}(T^*) = \{\overline{b_n} : n \in \mathbb{N}\} \cup \{0\}$. By lemma 47, $\Gamma(T) = (\sigma_{ap}(T^*))^*$. Therefore, $\Gamma(T) = \{b_n : n \in \mathbb{N}\} \cup \{0\}$. □

Chapter 4

NUMERICAL RANGE

Def: Let $A \in \mathcal{B}(H)$. The **numerical range** of A, denoted $\mathrm{W}(A)$, is the set $\mathrm{W}(A) = \{\langle Av, v\rangle : v \in H, \|v\| = 1\}$.

The following result that the numerical range of a bounded operator is always convex, known as the Toeplitz-Hausdorff Theorem, is one of the most famous results about the numerical range of a general bounded operator. We will make much use of this result throughout the rest of this thesis.

Theorem 48 (Toeplitz-Hausdorff Theorem)**.** *If $A \in \mathcal{B}(H)$, then $W(A)$ is a convex subset of $\mathbb{C}$.*

The following proof is adapted from [11].

Proof. Assume $A \in \mathcal{B}(H)$, where H is a Hilbert space. Let $\lambda, \mu \in \mathrm{W}(H)$ be distinct. We will show that $t\lambda + (1-t)\mu \in \mathrm{W}(A)$ for $t \in [0, 1]$. Since $\lambda \neq \mu$, we can define $\alpha = \dfrac{-\mu}{\lambda - \mu}$ and $\beta = \dfrac{1}{\lambda - \mu}$. We claim that $t\lambda + (1-t)\mu \in \mathrm{W}(A)$ if and only if $t \in \mathrm{W}(\alpha I + \beta A)$.

($\Longrightarrow$) Let $t\lambda + (1-t)\mu \in \mathrm{W}(A)$. Then, there exists some $v \in H$ with $\|v\| = 1$ such that $t\lambda + (1-t)\mu = \langle Av, v\rangle$. Since $\lambda - \mu \neq 0$, we can solve for t:

$$\begin{aligned} t &= \frac{1}{\lambda - \mu}\langle Av, v\rangle - \frac{\mu}{\lambda - \mu} \\ &= \beta\langle Av, v\rangle + \alpha \end{aligned}$$

$$
\begin{aligned}
&= \langle \beta A v, v\rangle + \alpha \|v\|^2 \\
&= \langle \beta A v, v\rangle + \alpha\langle v, v\rangle \\
&= \langle \beta A v, v\rangle + \langle \alpha v, v\rangle \\
&= \langle \beta A v + \alpha v, v\rangle \\
&= \langle (\alpha I + \beta A) v, v\rangle \\
&\in \mathrm{W}(\alpha I + \beta A) \qquad (4.1)
\end{aligned}
$$

Line (4.1) follows from $\|v\| = 1$.

($\Longleftarrow$) Let $t \in \mathrm{W}(\alpha I + \beta A)$. Then, there exists some $v \in H$ with $\|v\| = 1$ such that $t = \langle (\alpha I + \beta A)v, v\rangle$. By properties of inner products, it follows

$$
\begin{aligned}
t &= \langle (\alpha I + \beta A) v, v\rangle \\
&= \langle \alpha v + \beta A v, v\rangle \\
&= \langle \alpha v, v\rangle + \langle A v, v\rangle \\
&= \alpha\langle v, v\rangle + \beta\langle A v, v\rangle \\
&= \alpha + \beta\langle A v, v\rangle
\end{aligned}
$$

Thus, since $\beta \neq 0$, $\dfrac{t-\alpha}{\beta} = \langle Av, v\rangle \in \mathrm{W}(A)$. By the definition of α and β defined earlier, we have that

$$
\begin{aligned}
\frac{t-\alpha}{\beta} &= \frac{t + \frac{\mu}{\lambda-\mu}}{\frac{1}{\lambda-\mu}} \\
&= \left(t + \frac{\mu}{\lambda-\mu}\right)(\lambda-\mu) \\
&= t(\lambda-\mu) + \mu \\
&= \lambda t + \mu(1-t)
\end{aligned}
$$

Thus, $\lambda t + \mu(1-t) \in \mathrm{W}(A)$. Therefore, $t\lambda + (1-t)\mu \in \mathrm{W}(A)$ if and only if $t \in \mathrm{W}(\alpha I + \beta A)$ as previously claimed.

Since $\lambda, \mu \in \mathrm{W}(A)$, there exist $u, v \in H$ with $\|u\| = 1$ and $\|v\| = 1$, such that $\lambda = \langle Au, u\rangle$ and $\mu = \langle Av, v\rangle$. Now,

$$\begin{aligned}
\langle(\alpha I + \beta A)u, u\rangle &= \langle \alpha u, u\rangle + \beta\langle Au, u\rangle \\
&= \alpha\langle u, u\rangle + \beta\langle Au, u\rangle \\
&= \alpha + \beta\lambda \\
&= \frac{-\mu}{\lambda - \mu} + \frac{\lambda}{\lambda - \mu} \qquad (4.2) \\
&= \frac{\lambda - \mu}{\lambda - \mu} \\
&= 1
\end{aligned}$$

Line (4.2) follows from the definition of α and β. Similarly, we also have

$$\begin{aligned}
\langle(\alpha I + \beta A)v, v\rangle &= \alpha + \beta\mu \qquad (4.3) \\
&= \frac{-\mu}{\lambda - \mu} + \frac{\mu}{\lambda - \mu} \\
&= 0
\end{aligned}$$

Line (4.3) follows from the definition of α and β.

Define $g : \mathbb{R} \longrightarrow \mathbb{C}$ by $g(\theta) = \langle Sv, u\rangle e^{-i\theta} + \langle Su, v\rangle e^{i\theta}$, where u, v are defined above, and $S = \alpha I + \beta A$. Note that since $u, v \in H$ are fixed, $\langle Sv, u\rangle, \langle Su, v\rangle \in \mathbb{C}$ are also fixed. Thus, because $e^{-i\theta}, e^{i\theta}$ are continuous, it follows that g is continuous. Thus, the imaginary part of g, denoted Img, is a continuous real-valued function. Note that

$$\begin{aligned} g(\theta+\pi) &= \langle Sv, u\rangle e^{-i(\theta+\pi)} + \langle Su, v\rangle e^{i(\theta+\pi)} \\ &= \langle Sv, u\rangle e^{-i\theta}e^{-i\pi} + \langle Su, v\rangle e^{i\theta}e^{i\pi} \\ &= -\langle Sv, u\rangle e^{-i\theta} - \langle Su, v\rangle e^{i\theta} \\ &= -g(\theta) \end{aligned}$$

From the above equality and the fact that $g(0) = g(2\pi)$, it follows that $\text{Im}(g(0)) = -\text{Im}(g(\pi))$. Thus, we see that $\text{Im}(g(0))$ and $\text{Im}(g(\pi))$ have opposite signs. Since Img is a continuous function and $\text{Im}(g(0))$ and $\text{Im}(g(\pi))$ have opposite signs, it follows by the Intermediate Value Theorem that there exists some $\theta_0 \in (0, \pi)$ such that $\text{Im}(g(\theta_0)) = 0$.

We claim that v and $e^{i\theta_0}u$ are linearly independent. We proceed by contradiction. Assuming v and $e^{i\theta_0}u$ are linearly dependent, there exists some $\gamma \in \mathbb{C}$ such that $v = \gamma e^{i\theta_0}u$, unless $e^{i\theta_0}u = 0v$, in which case we would contradict that $\|u\| = 1$. Now,

$$\begin{aligned} 0 &= \langle(\alpha I + \beta A)v, v\rangle && (4.4) \\ &= \langle(\alpha I + \beta A)\gamma e^{i\theta_0}u, \gamma e^{i\theta_0}u\rangle \\ &= (\gamma e^{i\theta_0})(\overline{\gamma}e^{-i\theta_0})\langle(\alpha I + \beta A)u, u\rangle \\ &= (\gamma e^{i\theta_0})(\overline{\gamma}e^{-i\theta_0}) && (4.5) \\ &= |\gamma|^2 \end{aligned}$$

Line (4.4) follows from the previously proven fact that $\langle(\alpha I+\beta A)v,v\rangle=0$. Line (4.5) follows from the similar previously proven fact that $\langle(\alpha I+\beta A)u,u\rangle=1$. Since $|\gamma|^2=0$, it must follow $|\gamma|=0$, and so $\gamma=0$. This implies that $\|v\|=\|0\|=0$, which contradicts the assumption that $\|v\|=1$. Therefore, we must conclude that v and $e^{i\theta_0}u$ are linearly independent as previously claimed.

Since v and $e^{i\theta_0}u$ are linearly independent, it follows that $(1-t)v+te^{i\theta_0}u\neq 0$ for every $t\in[0,1]$. Thus, $\left\|(1-t)v+te^{i\theta_0}u\right\|\neq 0$. Hence, for each $t\in[0,1]$, we can define the unit vector $z_t=\dfrac{(1-t)v+te^{i\theta_0}u}{\|(1-t)v+te^{i\theta_0}u\|}$. We note that the function mapping $t\mapsto z_t$ is continuous, because t and $1-t$ are continuous functions and $v,e^{i\theta_0}u\in H$ are fixed. Also note that

$$
\begin{aligned}
\langle(\alpha I+\beta A)z_0,z_0\rangle &= \left\langle(\alpha I+\beta A)\frac{v}{\|v\|},\frac{v}{\|v\|}\right\rangle \\
&= \langle(\alpha I+\beta A)v,v\rangle \qquad (4.6)\\
&= 0 \qquad (4.7)
\end{aligned}
$$

Line (4.6) follows from the fact that $\|v\|=1$. Line (4.7) follows from the definition of v. Thus, we have shown $0\in \mathrm{W}(\alpha I+\beta A)$. Similarly, we also have

$$
\begin{aligned}
\langle(\alpha I+\beta A)z_1,z_1\rangle &= \left\langle(\alpha I+\beta A)\frac{e^{i\theta_0}u}{\|e^{i\theta_0}u\|},\frac{e^{i\theta_0}u}{\|e^{i\theta_0}u\|}\right\rangle \\
&= \frac{e^{i\theta_0}e^{-i\theta_0}}{\|e^{i\theta_0}\|^2}\left\langle(\alpha I+\beta A)\frac{u}{\|u\|},\frac{u}{\|u\|}\right\rangle \\
&= \frac{e^{i\theta_0}e^{-i\theta_0}}{\|e^{i\theta_0}\|^2}\langle(\alpha I+\beta A)u,u\rangle \qquad (4.8)\\
&= \frac{e^{i\theta_0}e^{-i\theta_0}}{\|e^{i\theta_0}\|^2} \qquad (4.9)\\
&= \frac{|e^{i\theta_0}|^2}{|e^{i\theta_0}|^2}
\end{aligned}
$$

$$= 1$$

Line (4.8) follows from the fact that $\|u\| = 1$. Line (4.9) follows from the definition of u. Thus, we have shown $1 \in \mathrm{W}(\alpha I + \beta A)$.

Lastly, define $f : [0,1] \longrightarrow \mathbb{C}$ by $f(t) = \langle(\alpha I + \beta A)z_t, z_t\rangle$. Since the function that maps $t \mapsto z_t$ is continuous and the inner product is continuous, it follows that f is a continuous function as well. By the work above, we have that
$f(0) = \langle(\alpha I + \beta A)z_0, z_0\rangle = 0$ and $f(1) = \langle(\alpha I + \beta A)z_1, z_1\rangle = 1$. Therefore, by the Intermediate Value Theorem, it must follow that $[0,1] \subseteq \mathrm{W}(\alpha I + \beta A)$. Therefore, by the very first claim (i.e. $t\lambda + (1-t)\mu \in \mathrm{W}(A) \Longleftrightarrow t \in \mathrm{W}(\alpha I + \beta A)$), it follows that $t\lambda + (1-t)\mu \in \mathrm{W}(A)$ for every $t \in [0,1]$; that is, $\mathrm{W}(A)$ is convex. □

Theorem 49. *Let H_1 and H_2 be inner product spaces. If $T_1 \in \mathcal{B}(H_1)$ and $T_2 \in \mathcal{B}(H_2)$, then $W(T_1 \bigoplus T_2) = conv\{W(T_1) \cup W(T_2)\}$.*

Proof. Assume H_1, H_2 are inner product spaces and $T_1 \in \mathcal{B}(H_1)$, $T_2 \in \mathcal{B}(H_2)$.

($\subseteq$) Let $\lambda \in \mathrm{W}(T_1 \bigoplus T_2)$. Thus, $\lambda = \left\langle (T_1 \bigoplus T_2)h, h\right\rangle_{H_1 \bigoplus H_2}$ for some $h \in H_1 \bigoplus H_2$ with $\|h\|_{H_1 \bigoplus H_2} = 1$. Since $h \in H_1 \bigoplus H_2$, there exist $h_1 \in H_1$ and $h_2 \in H_2$ such that $h = h_1 \oplus h_2$. Note

$$\begin{aligned}
\lambda &= \left\langle (T_1 \bigoplus T_2)h, h\right\rangle_{H_1 \bigoplus H_2} \\
&= \left\langle (T_1 \bigoplus T_2)(h_1 \oplus h_2), h_1 \oplus h_2\right\rangle_{H_1 \bigoplus H_2} \\
&= \langle T_1 h_1 \oplus T_2 h_2, h_1 \oplus h_2\rangle_{H_1 \bigoplus H_2} \qquad (4.10)
\end{aligned}$$

$$= \langle T_1 h_1, h_1 \rangle_{H_1} + \langle T_2 h_2, h_2 \rangle_{H_2} \tag{4.11}$$

$$= \|h_1\|_{H_1}^2 \left\langle T_1 \frac{h_1}{\|h_1\|}, \frac{h_1}{\|h_1\|} \right\rangle_{H_1} + \|h_2\|_{H_2}^2 \left\langle T_2 \frac{h_2}{\|h_2\|}, \frac{h_2}{\|h_2\|} \right\rangle_{H_2}$$

Line (4.10) follows from the definition of the direct sum of two operators. Line (4.11) follows from the definition of the inner product in the direct sum of two inner product spaces. By definition, $\left\langle T_1 \frac{h_1}{\|h_1\|}, \frac{h_1}{\|h_1\|} \right\rangle_{H_1} \in \mathrm{W}(T_1)$ and $\left\langle T_2 \frac{h_2}{\|h_2\|}, \frac{h_2}{\|h_2\|} \right\rangle_{H_2} \in \mathrm{W}(T_2)$. By definition of the inner product in $H_1 \bigoplus H_2$, $\|h\|_{H_1 \bigoplus H_2}^2 = \|h_1\|_{H_1}^2 + \|h_2\|_{H_2}^2$. Therefore, $\|h_1\|_{H_1}^2 + \|h_2\|_{H_2}^2 = 1$. Thus, by the definition of the convex hull, we have $\lambda = \|h_1\|_{H_1}^2 \left\langle T_1 \frac{h_1}{\|h_1\|}, \frac{h_1}{\|h_1\|} \right\rangle_{H_1} + \|h_2\|_{H_2}^2 \left\langle T_2 \frac{h_2}{\|h_2\|}, \frac{h_2}{\|h_2\|} \right\rangle_{H_2} \in \mathrm{conv}\{\mathrm{W}(T_1) \cup \mathrm{W}(T_2)\}$. Hence, we have shown that $\mathrm{W}(T) \subseteq \mathrm{conv}\{\mathrm{W}(T_1) \cup \mathrm{W}(T_2)\}$.

($\supseteq$) Let $\lambda \in \mathrm{conv}\{\mathrm{W}(T_1) \cup \mathrm{W}(T_2)\}$. Then $\lambda = \sum_{i=1}^{n} t_i \alpha_i$, where $\alpha_i \in \mathrm{W}(T_1) \cup \mathrm{W}(T_2)$, $\sum_{i=1}^{n} t_i = 1$, and $0 \leq t_i \leq 1$. Since $\alpha_i \in \mathrm{W}(T_1) \cup \mathrm{W}(T_2)$, $\alpha_i \in \mathrm{W}(T_1)$ or $\alpha_i \in \mathrm{W}(T_2)$. Assume without loss of generality that $\alpha_i \in \mathrm{W}(T_1)$. Then, there exists some $u_i \in H_1$ with $\|u_i\|_{H_1} = 1$, such that $\langle T_1 u_i, u_i \rangle_{H_1} = \alpha_i$. Since H_2 is a vectorspace, $0 \in H_2$. Thus, $u_i \oplus 0 \in H_1 \bigoplus H_2$. Note that

$$\begin{aligned}
\left\langle (T_1 \bigoplus T_2)(u_i \oplus 0), u_i \oplus 0 \right\rangle_{H_1 \bigoplus H_2} &= \langle T_1 u_i \oplus T_2 0, u_i \oplus 0 \rangle_{H_1 \bigoplus H_2} \\
&= \langle T_1 u_i, u_i \rangle_{H_1} + \langle T_2 0, 0 \rangle_{H_2} \\
&= \alpha_i + 0 \\
&= \alpha_i
\end{aligned}$$

By definition of the inner product in $H_1 \bigoplus H_2$, $\|u_i \oplus 0\|_{H_1 \bigoplus H_2}^2 = \|u_i\|_{H_1}^2 + \|0\|_{H_2}^2 = 1$. Thus, $\|u_i \oplus 0\|_{H_1 \bigoplus H_2} = 1$. Thus, by definition we have that $\alpha_i = \langle (T_1 \bigoplus T_2)(u_i \oplus 0), u_i \oplus 0 \rangle_{H_1 \bigoplus H_2} \in \mathrm{W}(T_1 \bigoplus T_2)$. Since $\mathrm{W}(T_1 \bigoplus T_2)$ is convex (this is the Toeplitz-Hausdorff theorem), it follows that $\lambda \in \mathrm{W}(T_1 \bigoplus T_2)$. Thus,

$\text{conv}\{\text{W}(T_1) \cup \text{W}(T_2)\} \subseteq \text{W}(T_1 \bigoplus T_2)$.

Therefore, $\text{W}(T_1 \bigoplus T_2) = \text{conv}\{\text{W}(T_1) \cup \text{W}(T_2)\}$, as desired. □

Proposition 2. *If $A \in \mathcal{B}(H)$, then $\sigma_p(A) \subseteq W(A)$.*

Proof. Assume $A \in \mathcal{B}(H)$. Let $\lambda \in \sigma_p(A)$. It follows by definition that λ is an eigenvalue of A. Thus, there exists some nonzero $v \in H$ such that $Av = \lambda v$. Since $v \neq 0$, $\|v\| \neq 0$. Hence, $A\frac{v}{\|v\|} = \lambda \frac{v}{\|v\|}$ and $\left\| \frac{v}{\|v\|} \right\| = 1$. Hence, $\left\langle A\frac{v}{\|v\|}, \frac{v}{\|v\|} \right\rangle = \left\langle \lambda\frac{v}{\|v\|}, \frac{v}{\|v\|} \right\rangle = \lambda \left\langle \frac{v}{\|v\|}, \frac{v}{\|v\|} \right\rangle = \lambda$. Thus, $\lambda \in \text{W}(A)$. Therefore, $\sigma_p(A) \subseteq \text{W}(A)$, as desired. □

Proposition 3. *If $A \in \mathcal{B}(H)$, then $W(A)$ lies in the closed disk of radius $\|A\|$ centered at 0.*

Proof. Assume $A \in \mathcal{B}(H)$. Let $a \in \text{W}(A)$. Then, $a = \langle Av, v\rangle$ for some $v \in H$ with $\|v\| = 1$. Now,

$$\begin{aligned} |a| &= |\langle Av, v\rangle| \\ &\leq \|Av\| \, \|v\| && (4.12) \\ &= \|Av\| \\ &\leq \|A\| && (4.13) \end{aligned}$$

Line (4.12) follows by the Cauchy-Schwarz inequality. Line (4.13) follows from the definition of the operator norm. Therefore, we have shown that $\text{W}(A)$ lies inside the closed disk of radius $\|A\|$, centered at 0. □

Claim 4. *If $A \in \mathcal{B}(H)$, then $W(A^*) = \{\overline{\lambda} : \lambda \in W(A)\}$*

Proof. Assume $A \in \mathcal{B}(H)$.

($\subseteq$) Let $\mu \in \mathrm{W}(A^*)$. Then there exists $v \in H$ with $\|v\| = 1$ such that $\mu = \langle A^*v, v\rangle$. By the definition of the adjoint $\mu = \langle A^*v, v\rangle = \langle v, A^{**}v\rangle = \langle v, Av\rangle$. Thus, $\overline{\mu} = \overline{\langle v, Av\rangle} = \langle Av, v\rangle$. Since $v \in H$ and $\|v\| = 1$, $\langle Av, v\rangle \in \mathrm{W}(A)$. Thus, $\overline{\mu} \in \mathrm{W}(A)$. Since $\mu = \overline{\overline{\mu}}$, it follows that $\mathrm{W}(A^*) \subseteq \{\overline{\lambda} : \lambda \in \mathrm{W}(A)\}$.

($\supseteq$) Let $\overline{\gamma} \in \{\overline{\lambda} : \lambda \in \mathrm{W}(A)\}$. Then, $\gamma \in \mathrm{W}(A)$. Hence, there exists some $v \in H$ with $\|v\| = 1$ such that $\overline{\gamma} = \langle Av, v\rangle$. By definition of the adjoint, $\overline{\gamma} = \langle v, A^*v\rangle$. Therefore, $\gamma = \overline{\overline{\gamma}} = \overline{\langle v, A^*v\rangle} = \langle A^*v, v\rangle$. By definition $\langle A^*v, v\rangle \in \mathrm{W}(A^*)$. Hence, $\{\overline{\lambda} : \lambda \in \mathrm{W}(A)\} \subseteq \mathrm{W}(A^*)$.

Therefore, $\mathrm{W}(A^*) = \{\overline{\lambda} : \lambda \in \mathrm{W}(A)\}$. □

Example 6. *Let $S \in \mathcal{B}(\ell^2)$ denote the forward shift operator. Then, $W(S) = \mathbb{D}$ (the open unit disk.)*

Proof. We will prove that $\mathrm{W}(S) = \mathbb{D}$ by showing $\mathrm{W}(S^*) = \mathbb{D}$, where S^* is the adjoint of S. Note that S^* is the backward shift operator.

($\subseteq$) In the proof of $\sigma_p(S) = \mathbb{D}$, we showed $\|S\| = 1$, hence, $\|S^*\| = 1$ since an operator and its adjoint have the same norm. Thus, by proposition 3, $\mathrm{W}(S^*)$ lies inside $\overline{\mathbb{D}}$; that is, $\mathrm{W}(S^*) \subseteq \overline{\mathbb{D}}$. Thus, we only need to show that $\mathrm{W}(S^*)$ contains no points in $\partial\mathbb{D}$ to complete the forwards containment. Let $\lambda \in \partial\mathbb{D}$. Consequently, $|\lambda| = 1$. Seeking a contradiction, suppose $\lambda \in \mathrm{W}(S^*)$. Then, there exists some $(a_n) \in \ell^2$ with $\|(a_n)\| = 1$, such that $\lambda = \langle S^*(a_n), (a_n)\rangle$. Thus,

$$\begin{aligned} 1 &= |\lambda| \\ &= |\langle S^*(a_n), (a_n)\rangle| \end{aligned} \tag{4.14}$$

$$
\begin{aligned}
&\leq \|S^*(a_n)\| \, \|(a_n)\| \qquad (4.15) \\
&= \|S^*(a_n)\| \\
&\leq \|S^*\| \, \|(a_n)\| \\
&= 1
\end{aligned}
$$

Line (4.15) follows by the Cauchy-Schwarz inequality. From the above we obtain the inequality $1 \leq \|S^*(a_n)\| \leq 1$. Thus, we have $\|S^*(a_n)\| = 1$. Hence, $\|S^*(a_n)\|^2 = 1$. Since $\|(a_n)\| = 1$, we have $\|(a_n)\|^2 = 1$, as well. Hence, $\|S^*(a_n)\|^2 = \|(a_n)\|^2$. It follows, by definition of the ℓ^2-norm, that $\sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=0}^{\infty} |a_n|^2$. Therefore, it must be the case that $|a_0|^2 = 0$, and so $a_0 = 0$. Also by the above, we must have equality between lines (4.14) and (4.15): $|\langle S^*(a_n), (a_n) \rangle| = \|S^*(a_n)\| \, \|(a_n)\|$. Thus, $S^*(a_n)$ is simply a scalar multiple of (a_n). In particular, the scalar multiple must be λ because $\lambda = \langle S^*(a_n), (a_n) \rangle$. Therefore, we have $S^*(a_n) = \lambda(a_n)$. So, by definition of the backward shift operator, S^*, we have $(a_1, a_2, a_3, \ldots) = (\lambda a_0, \lambda a_1, \lambda a_2, \ldots)$. Equating entries, we obtain $a_n = \lambda^n a_0$ for each $n \in \mathbb{N}$. Thus, we have $S^*(a_n) = a_0(1, \lambda, \lambda^2, \lambda^3, \ldots)$. Since $a_0 = 0$, we see $S^*(a_n) = (0)$. Thus, $(a_n) = (0)$. So, $\|(a_n)\| = 0$, which contradicts that $\|(a_n)\| = 1$. Therefore, we must conclude that $\mathrm{W}(S^*)$ contains no points in $\partial\mathbb{D}$. Therefore, $\mathrm{W}(S^*) \subseteq \mathbb{D}$.

($\supseteq$) Let $\lambda \in \mathbb{D}$. Then, $|\lambda| < 1$. Hence, by lemma 46, λ is an eigenvalue of S^*. Thus, by proposition 2, $\lambda \in \mathrm{W}(S^*)$. Thus, $\mathbb{D} \subseteq \mathrm{W}(S^*)$.

Therefore, $\mathrm{W}(S^*) = \mathbb{D}$. So, by claim 4, $\mathrm{W}(S) = \mathrm{W}(S^{**}) = \{\overline{\lambda} : \lambda \in \mathbb{D}\} = \mathbb{D}$, as desired. $\square$

Def: Let Λ be a set and $\lambda_{n_1}, \lambda_{n_2}, \ldots, \lambda_{n_k} \in \Lambda$. The **convex combination** of $\lambda_{n_1}, \lambda_{n_2}, \ldots, \lambda_{n_k}$ is $t_1\lambda_{n_1} + t_2\lambda_{n_2} + \cdots + t_k\lambda_{n_k}$, where $0 \leq t_1, t_2, \ldots, t_k \leq 1$ and $t_1 + t_2 + \cdots + t_k = 1$.

Def: Let Λ be a set. The **convex hull** of Λ, denoted conv(Λ), is the set of all finite convex combinations of Λ. The **infinite convex hull** of Λ, denoted $\text{conv}_\infty(\Lambda)$, is the set of all infinite convex combinations of Λ.

Lemma 50. *Let Λ be a set and $\alpha, \beta \in \mathbb{C}$. Then,*

1. $conv(\alpha\Lambda) = \alpha conv(\Lambda)$
2. $conv(\Lambda + \beta) = conv(\Lambda) + \beta$

Proof. Let Λ be a set and $\alpha, \beta \in \mathbb{C}$.

(1) By definition of conv(Λ), $x \in \alpha\text{conv}(\Lambda)$ if and only if

$$x = \alpha(t_1\lambda_{n_1} + t_2\lambda_{n_2} + \ldots + t_k\lambda_{n_k})$$

for some $\lambda_{n_1}, \lambda_{n_2}, \ldots, \lambda_{n_k} \in \Lambda$ and $0 \leq t_1, t_2, \ldots, t_k \leq 1$ such that $t_1 + t_2 + \cdots + t_k = 1$. Now,

$$\begin{aligned} x &= \alpha(t_1\lambda_{n_1} + t_2\lambda_{n_2} + \ldots + t_k\lambda_{n_k}) \\ &= t_1\alpha\lambda_{n_1} + t_2\alpha\lambda_{n_2} + \ldots + t_k\alpha\lambda_{n_k} \\ &\in \text{conv}(\alpha\Lambda) \end{aligned}$$

Therefore, $\text{conv}(\alpha\Lambda) = \alpha\text{conv}(\Lambda)$.

(2) Also by the definition of conv(Λ), $x \in$ conv$(\Lambda + \beta)$ if and only if there exist some $\lambda_{n_1}, \lambda_{n_2}, \ldots, \lambda_{n_k} \in \Lambda$ and $0 \leq t_1, t_2, \ldots, t_k \leq 1$ with $t_1 + t_2 + \cdots + t_k = 1$ such that $x = t_1(\lambda_{n_1} + \beta) + t_2(\lambda_{n_2} + \beta) + \cdots + t_k(\lambda_{n_k} + \beta)$. Now,

$$
\begin{aligned}
x &= t_1(\lambda_{n_1} + \beta) + t_2(\lambda_{n_2} + \beta) + \cdots + t_k(\lambda_{n_k} + \beta) \\
&= t_1\lambda_{n_1} + t_1\beta + t_2\lambda_{n_2} + t_2\beta + \cdots + t_k\lambda_{n_k} + t_k\beta \\
&= t_1\lambda_{n_1} + t_2\lambda_{n_2} + \cdots + t_k\lambda_{n_k} + (t_1 + t_2 + \cdots + t_k)\beta \\
&= t_1\lambda_{n_1} + t_2\lambda_{n_2} + \cdots + t_k\lambda_{n_k} + \beta \\
&\in \text{conv}(\Lambda) + \beta \qquad (4.16)
\end{aligned}
$$

Line (4.16) follows from the fact that $t_1 + t_2 + \cdots + t_k = 1$. Therefore, conv$(\Lambda + \beta)$ = conv$(\Lambda) + \beta$. □

Note. *By replacing the finite sums with infinite sums in the previous proof we find that* $conv_\infty(\alpha\Lambda) = \alpha conv_\infty(\Lambda)$ *and* $conv_\infty(\Lambda + \beta) = conv_\infty(\Lambda) + \beta$.

Proposition 4 (Shapiro, Prop 2.6)**.** *If* $\Lambda = \{\lambda_n \in \mathbb{C} : n \in \mathbb{N}\}$ *is a countable set of complex numbers, then* $conv_\infty(\Lambda) = conv(\Lambda)$.

The following proof is adapted from the proof given in [14].

Proof. Clearly, conv$(\Lambda) \subseteq$ conv$_\infty(\Lambda)$. To show the reverse containment, let $p \in$ conv$_\infty(\Lambda)$. Define $\Lambda_p = \Lambda - p$. Then, by lemma 50 and the following note, conv$_\infty(\Lambda) - p$ = conv$_\infty(\Lambda_p)$. Since $p \in$ conv$_\infty(\Lambda)$, it follows that $0 \in$ conv$_\infty(\Lambda_p)$. Seeking a contradiction, suppose that $p \notin$ conv(Λ). Then $0 \notin$ conv(Λ_p). Thus, there exists a line separating 0 from conv(Λ_p). We note that this separating line is not necessarily a strictly separating line, since if 0 were a limit point of Λ the convergence would violate the strict separation. By some rotation $e^{i\theta}$ about the origin and lemma 50, $\text{conv}(e^{i\theta}\Lambda_p) \subseteq \overline{\mathbb{H}^+}$, where $\overline{\mathbb{H}^+}$ denotes the closed upper half-plane. It follows that $\text{conv}_\infty(e^{i\theta}\Lambda_p) \subseteq \overline{\mathbb{H}^+}$. Since $e^{i\theta}\Lambda_p \subseteq \text{conv}(e^{i\theta}\Lambda_p)$, we also have $e^{i\theta}\Lambda_p \subseteq \overline{\mathbb{H}^+}$.

Since $0 \in \text{conv}_\infty(\Lambda_p)$, $0 \in e^{i\theta}\text{conv}_\infty(\Lambda)$. Thus, by lemma 50, $0 \in \text{conv}_\infty(e^{i\theta}\Lambda_p)$. Thus, we can write $0 = \sum_{n=1}^{\infty} a_n e^{i\theta}\lambda_n$, where $0 \leq a_n \leq 1$ for each $n \in \mathbb{N}$ with $\sum_{n=1}^{\infty} a_n = 1$, and $\lambda_n \in \Lambda_p$ for each $n \in \mathbb{N}$. If only finitely many a_n's are nonzero, then we automatically have $0 \in \text{conv}(e^{i\theta}\Lambda_p)$, which by lemma 50 gives $0 \in \text{conv}(\Lambda_p)$, contradicting that $0 \notin \text{conv}(\Lambda_p)$. Therefore, we must have that infinitely many a_n's must be nonzero. Let a_{n_j} denote these infinitely many nonzero scalars. Note that

$$\begin{aligned}
0 &= \sum_{n=1}^{\infty} a_n e^{i\theta}\lambda_n \\
&= \sum_{n=1}^{\infty} a_n(\text{Re}(e^{i\theta}\lambda_n) + i\text{Im}(e^{i\theta}\lambda_n)) \\
&= \sum_{n=1}^{\infty} a_n\text{Re}(e^{i\theta}\lambda_n) + i\sum_{n=1}^{\infty} a_n\text{Im}(e^{i\theta}\lambda_n)
\end{aligned}$$

Thus, it follows that $\sum_{n=1}^{\infty} a_n\text{Im}(e^{i\theta}\lambda_n) = 0$. Since $e^{i\theta}\Lambda_p \subseteq \overline{\mathbb{H}^+}$, it follows that $\text{Im}(e^{i\theta}\lambda_n) \geq 0$ for every $n \in \mathbb{N}$. By assumption $a_n \geq 0$ for every $n \in \mathbb{N}$. Thus, $a_n\text{Im}(e^{i\theta}\lambda_n) \geq 0$ for each $n \in \mathbb{N}$. Thus, the only way $\sum_{n=1}^{\infty} a_n\text{Im}(e^{i\theta}\lambda_n) = 0$ can hold is if $a_n\text{Im}(e^{i\theta}\lambda_n) = 0$ for each $n \in \mathbb{N}$. Therefore, we must have $\text{Im}(e^{i\theta}\lambda_n) = 0$ for each $a_n \neq 0$; that is, for each $n \in \mathbb{N}$ such that $a_n \neq 0$, $e^{i\theta}\lambda_n \in \mathbb{R}$. Note that for each $n \in \mathbb{N}$, $e^{i\theta}\lambda_n \neq 0$, because if $e^{i\theta}\lambda_n = 0$, then $\lambda_n = 0$, and since $\lambda_n \in \text{conv}(\Lambda_p)$, this would imply that $0 \in \text{conv}(\Lambda_p)$, contradicting that $0 \notin \text{conv}(\Lambda_p)$.

Recall that a_{n_j} denotes the infinitely many nonzero a_n's. Thus, $\sum_{j=1}^{\infty} a_{n_j} e^{i\theta} \lambda_{n_j} = 0$, where $0 < a_{n_j} \leq 1$ for each $n_j \in \mathbb{N}$, and $\sum_{j=1}^{\infty} a_{n_j} = 1$. Since each $\lambda_n \neq 0$, we know that each $\lambda_{n_j} \neq 0$. Since $a_{n_j} > 0$ and $e^{i\theta}\lambda_{n_j} \neq 0$, the only way $\sum_{j=1}^{\infty} a_{n_j} e^{i\theta} \lambda_{n_j} = 0$ can hold is if at least one $e^{i\theta}\lambda_{n_j}$ is negative and at least one is positive. So, assume $e^{i\theta}\lambda_{n_l} < 0$ and $e^{i\theta}\lambda_{n_m} > 0$. Then, $[e^{i\theta}\lambda_{n_l}, e^{i\theta}\lambda_{n_m}]$ is an interval on the real line containing the origin. Since $\lambda_{n_l}, \lambda_{n_m} \in \Lambda_p$, it follows that $[e^{i\theta}\lambda_{n_l}, e^{i\theta}\lambda_{n_m}] \subseteq e^{i\theta}\text{conv}(\Lambda_p)$ by lemma 50. In particular, $0 \in e^{i\theta}\text{conv}(\Lambda_p)$. Thus, $0 \in \text{conv}(\Lambda_p)$, contradicting that $0 \notin \text{conv}(\Lambda_p)$. Therefore, we must conclude that $p \in \text{conv}(\Lambda)$. Thus,
$\text{conv}_{\infty}(\Lambda) \subseteq \text{conv}(\Lambda)$. Therefore, $\text{conv}_{\infty}(\Lambda) = \text{conv}(\Lambda)$, as desired. □

Example 7. *Let (b_n) be a sequence in $\mathbb{C}$ such that $\lim_{n\to\infty} b_n = 0$. Define the diagonal operator $T \in \mathcal{C}(\ell^2)$ by $T(a_n) = (a_n b_n)$. Then, $W(T)$ is the convex hull of the eigenvalues of T.*

Proof. Let $B = \{b_n : n \in \mathbb{N}\}$. Note that by example 5, $B = \sigma_p(T)$, which by definition is the set of all eigenvalues of T.

($\subseteq$) Let $x \in \text{conv}(B)$. Then, $x = \sum_{i=1}^{k} t_i b_{n_i}$, where $0 \leq t_1, \ldots, t_k \leq 1$, $\sum_{i=1}^{k} t_i = 1$, and $b_{n_1}, \ldots, b_{n_k} \in B$. Since each b_{n_i} is an eigenvalue of T, let $u_{n_i} \in \ell^2$ be the corresponding unit eigenvectors. Since T is normal (this is obvious or can easily be shown), these eigenvectors must be orthogonal by theorem 21. Define $u = \sum_{i=1}^{k} \sqrt{t_i} u_{n_i}$. Note that

$$\begin{aligned}
\|u\|^2 &= \left\langle \sum_{i=1}^{k} \sqrt{t_i} u_{n_i}, \sum_{i=1}^{k} \sqrt{t_i} u_{n_i} \right\rangle \\
&= \sum_{i=1}^{k} \left\langle \sqrt{t_i} u_{n_i}, \sum_{i=1}^{k} \sqrt{t_i} u_{n_i} \right\rangle \\
&= \sum_{i=1}^{k} \left(\sum_{j=1}^{k} \langle \sqrt{t_i} u_{n_i}, \sqrt{t_j} u_{n_j} \rangle \right)
\end{aligned}$$

$$= \sum_{i=1}^{k} \left(\sum_{j=1}^{k} \sqrt{t_i} (\overline{\sqrt{t_j}}) \langle u_{n_i}, u_{n_j} \rangle \right)$$

$$= \sum_{i=1}^{k} t_i \langle u_{n_i}, u_{n_i} \rangle \tag{4.17}$$

$$= \sum_{i=1}^{k} t_i \tag{4.18}$$

$$= 1$$

Line (4.17) follows from the eigenvectors being orthogonal. Line (4.18) follows from the eigenvectors being of unit length. Since norms are always non-negative it follows that $\|u\| = 1$. Thus, $\langle Tu, u \rangle \in \mathrm{W}(T)$.

Also note that,

$$\begin{aligned}
\langle Tu, u \rangle &= \left\langle T \left(\sum_{i=1}^{k} \sqrt{t_i} u_{n_i} \right), u \right\rangle \\
&= \left\langle \sum_{i=1}^{k} T(\sqrt{t_i} u_{n_i}), u \right\rangle \\
&= \left\langle \sum_{i=1}^{k} \sqrt{t_i} b_{n_i} u_{n_i}, u \right\rangle \\
&= \sum_{i=1}^{k} \left\langle \sqrt{t_i} b_{n_i} u_{n_i}, u \right\rangle \\
&= \sum_{i=1}^{k} \left\langle \sqrt{t_i} b_{n_i} u_{n_i}, \sum_{j=1}^{k} \sqrt{t_j} u_{n_j} \right\rangle \\
&= \sum_{i=1}^{k} \left(\sum_{j=1}^{k} \langle \sqrt{t_i} b_{n_i} u_{n_i}, \sqrt{t_j} u_{n_j} \rangle \right) \\
&= \sum_{i=1}^{k} \langle \sqrt{t_i} b_{n_i} u_{n_i}, \sqrt{t_i} u_{n_i} \rangle \\
&= \sum_{i=1}^{k} \sqrt{t_i} (\overline{\sqrt{t_i}}) b_{n_i} \langle u_{n_i}, u_{n_i} \rangle
\end{aligned}$$

$$= \sum_{i=1}^{k} t_i b_{n_i} \langle u_{n_i}, u_{n_i} \rangle \tag{4.19}$$

$$= \sum_{i=1}^{k} t_i b_{n_i} \tag{4.20}$$

$$= x$$

Line (4.19) follows from the eigenvectors being orthogonal. Line (4.20) follows from the eigenvectors being of unit length. Thus, we have shown that $x = \langle Tu, u \rangle$. Thus, we have that $x = \langle Tu, u \rangle \in \mathrm{W}(T)$. Therefore, $\mathrm{conv}(B) \subseteq \mathrm{W}(T)$.

($\supseteq$) Let $x \in \mathrm{W}(T)$. Then, $x = \langle T(v_n), (v_n) \rangle$ for some $(v_n) \in \ell^2$ with $\|(v_n)\| = 1$. Thus, $\|(v_n)\|^2 = 1$. So, by definition of the ℓ^2-norm, $\sum_{n=1}^{\infty} |v_n|^2 = 1$. We note that $|v_n|^2 \geq 0$ for each $n \in \mathbb{N}$, and clearly $|v_n|^2 \leq 1$ for each $n \in \mathbb{N}$. Note that

$$\begin{aligned} x &= \langle Tv, v \rangle \\ &= \langle (v_n b_n), (v_n) \rangle \\ &= \sum_{n=1}^{\infty} v_n b_n \overline{v_n} \\ &= \sum_{n=1}^{\infty} |v_n|^2 b_n \end{aligned}$$

This last line is an infinite convex combination of b_n's. Thus, $x \in \mathrm{conv}_\infty(B)$. By proposition 4, $\mathrm{conv}_\infty(B) = \mathrm{conv}(B)$. Hence, $x \in \mathrm{conv}(B)$. And so, $\mathrm{W}(T) \subseteq \mathrm{conv}(B)$. Therefore, $\mathrm{W}(T) = \mathrm{conv}(B)$, as desired. $\square$

We now turn to generalizing the previous example. To do so, we first remind the reader of the difference between a basis and an orthonormal basis:

Def: Let H be a Hilbert space and $\{h_n : n \in \mathbb{N}\} \subseteq H$. We say $\{h_n : n \in \mathbb{N}\}$ is a **basis** for H, if for each $h \in H$ there exist $h_{n_1}, \ldots, h_{n_k} \in \{h_n : n \in \mathbb{N}\}$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$, such that $h = \alpha_1 h_{n_1} + \cdots + \alpha_k h_{n_k}$.

Def: Let H be a Hilbert space and let

$$S = \{h_n : n \in \mathbb{N}, \langle h_n, h_m \rangle = 0 \text{ if } n \neq m \text{ and } \langle h_n, h_m \rangle = 1 \text{ otherwise}\} \subseteq H$$

We say S is an **orthonormal basis** for H, if for each $h \in H$ there exist $\alpha_n \in \mathbb{C}$ such that $h = \sum_{n=1}^{\infty} \alpha_n h_n$.

Theorem 51. *If $T \in \mathcal{C}(H)$ is normal, then $W(T)$ is the convex hull of the eigenvalues of T.*

Proof. Assume $T \in \mathcal{C}(H)$ is normal. By the spectral theorem for normal compact operators, there exists an orthonormal basis for H composed of eigenvectors of T. Let $\{e_n\}_{n=1}^{\infty}$ denote these eigenvectors and $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ denote the corresponding eigenvalues.

($\subseteq$) Without any modification we use the proof of example 7 to conclude $\text{conv}(\Lambda) \subseteq \text{W}(T)$.

($\supseteq$) Let $x \in \text{W}(T)$. Then, $x = \langle Tf, f \rangle$ for some $f \in H$ with $\|f\| = 1$. Since $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis for H, there exist $\alpha_j \in \mathbb{C}$ such that $\sum_{n=1}^{\infty} \alpha_n e_n$. Since $\|f\| = 1$, $\|f\|^2 = 1$. Thus,

$$\begin{aligned} 1 &= \left\langle \sum_{n=1}^{\infty} \alpha_n e_n, \sum_{n=1}^{\infty} \alpha_n e_n \right\rangle \\ &= \sum_{n=1}^{\infty} \left\langle \alpha_n e_n, \sum_{n=1}^{\infty} \alpha_n e_n \right\rangle \end{aligned} \tag{4.21}$$

$$= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \langle \alpha_n e_n, \alpha_m e_m \rangle \right) \tag{4.22}$$

$$= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \alpha_n \overline{\alpha_m} \langle e_n, e_m \rangle \right)$$

$$= \sum_{n=1}^{\infty} |\alpha_n|^2 \tag{4.23}$$

Lines (4.21) and (4.22) follow from the inner product being continuous in one component. Line (4.23) follows from the e_n's being orthonormal. We note that $|\alpha_n|^2 \geq 0$ for each $n \in \mathbb{N}$. Clearly, it must also be the case that $|\alpha_n|^2 \leq 1$ for each $n \in \mathbb{N}$. Now,

$$x = \langle Tf, f \rangle$$

$$= \left\langle T \left(\sum_{n=1}^{\infty} \alpha_n e_n \right), \sum_{n=1}^{\infty} \alpha_n e_n \right\rangle$$

$$= \left\langle \sum_{n=1}^{\infty} T(\alpha_n e_n), \sum_{n=1}^{\infty} \alpha_n e_n \right\rangle \tag{4.24}$$

$$= \left\langle \sum_{n=1}^{\infty} \alpha_n \lambda_n e_n, \sum_{n=1}^{\infty} \alpha_n e_n \right\rangle$$

$$= \sum_{n=1}^{\infty} \left(\left\langle \alpha_n \lambda_n e_n, \sum_{n=1}^{\infty} \alpha_n e_n \right\rangle \right) \tag{4.25}$$

$$= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \langle \alpha_n \lambda_n e_n, \alpha_m e_m \rangle \right) \tag{4.26}$$

$$= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \alpha_n \lambda_n (\overline{\alpha_m}) \langle e_n, e_m \rangle \right)$$

$$= \sum_{n=1}^{\infty} |\alpha_n|^2 \lambda_n \tag{4.27}$$

$$\in \text{conv}_{\infty}(\Lambda) \tag{4.28}$$

Line (4.24) follows from the continuity of T. Lines (4.25) and (4.26) follow from the inner product being continuous in one component. Line (4.27) follows from the e_n's being orthonormal. Line (4.28) follows from $\sum_{n=1}^{\infty} |\alpha_n|^2 = 1$. By Proposition 4, we have

$x \in \text{conv}_\infty(\Lambda) \subseteq \text{conv}(\Lambda)$.

Therefore, $\text{W}(T) = \text{conv}(\Lambda)$. □

We now turn to some results on the numerical range of non-normal compact operators. We look at a few definitions and preliminary results vital to proving these theorems.

Def: Let B be a Banach space, (x_n) be a sequence contained in B, and $x \in B$. We say (x_n) **converges weakly** to x, if for each $b \in B$, $\lim_{n\to\infty} \langle x_n, b\rangle = \langle x, b\rangle$.

Lemma 52 ([6], Theorem 4.8.7)**.** *Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Then $T \in \mathcal{C}(H)$ if and only if (x_n) converging weakly to x implies (Tx_n) converges to x.*

Lemma 53. *Let H be an infinite-dimensional Hilbert space and $T \in \mathcal{C}(H)$. If $\lambda \neq 0$ is a limit point of $W(T)$, then there exists some $x \in H$ with $\|x\| \leq 1$, such that $\dfrac{\lambda}{\|x\|^2} \in W(T)$. Therefore, $\lambda \in \left(0, \dfrac{\lambda}{\|x\|^2}\right]$ (a line segment in the complex plane).*

The following proof is motivated by [5].

Proof. Assume H is an infinite-dimensional Hilbert space and $T \in \mathcal{C}(H)$. Let λ be a limit point of $\text{W}(T)$. By definition, there exists some sequence (λ_n) in $\text{W}(T)$, such that $\lim_{n\to\infty} \lambda_n = \lambda$ and $\lambda_n \neq \lambda$ for each $n \in \mathbb{N}$. Since each $\lambda_n \in \text{W}(T)$, there exist $x_n \in H$ with $\|x_n\| = 1$ such that $\lambda_n = \langle Tx_n, x_n\rangle$ for each $n \in \mathbb{N}$. Consider this new sequence (x_n) in H. Since $\|x_n\| = 1$ for each $n \in \mathbb{N}$, the sequence (x_n) is contained in $\overline{B}$, where B is the unit ball in H. By the Tychonoff-Alaoglu Theorem, $\overline{B}$ is weakly compact. In a metric space weakly compact is equivalent to weakly sequentially compact. Thus, there exists some subsequence (x_{n_k}) of (x_n) that converges weakly to some $x \in \overline{B} \subseteq H$. Thus, by lemma 52, $\lim_{k\to\infty} Tx_{n_k} = Tx$. Since the inner product is continuous in each component, it follows that $\lim_{k\to\infty} \langle Tx_{n_k}, x_{n_k}\rangle = \langle Tx, x\rangle$.

Since $\lim_{n\to\infty} \langle Tx_n, x_n\rangle = \lim_{n\to\infty} \lambda_n = \lambda$ any subsequence of $(\langle Tx_n, x_n\rangle)$ must also converge to λ. Since we just showed the subsequence $(\langle Tx_{n_k}, x_{n_k}\rangle)$ converges to $\langle Tx, x\rangle$, it must follow that $\langle Tx, x\rangle = \lambda$. Since $\lambda \neq 0$, it is not possible for $x = 0$, since $\langle Tx, x\rangle = \lambda$ and $\langle T0, 0\rangle = \langle 0, 0\rangle = 0$. Hence, we must have $x \neq 0$. Thus, $\|x\| \neq 0$ and $\|x\|^2 \neq 0$. Hence,

$$\begin{aligned}
\frac{\lambda}{\|x\|^2} &= \frac{1}{\|x\|^2}\langle Tx, x\rangle \\
&= \frac{1}{\|x\|}\left\langle \frac{1}{\|x\|}Tx, x\right\rangle \\
&= \frac{1}{\|x\|}\left\langle T\left(\frac{x}{\|x\|}\right), x\right\rangle \\
&= \left\langle T\left(\frac{x}{\|x\|}\right), \overline{\frac{x}{\|x\|}}\right\rangle \\
&= \left\langle T\left(\frac{x}{\|x\|}\right), \frac{x}{\|x\|}\right\rangle \qquad (4.29) \\
&\in \mathrm{W}(T) \qquad (4.30)
\end{aligned}$$

Line (4.29) follows from $\|x\|$ being real. Line (4.30) follows from $\left\|\frac{x}{\|x\|}\right\| = 1$. Since $x \in \overline{B}$, $\|x\| \leq 1$. Thus, λ lies in the line segment $\left(0, \frac{\lambda}{\|x\|^2}\right] \subseteq \mathbb{C}$. □

Def: Let S be a convex set. A point $x \in S$ is an **extreme point** of S if x does not lie in the interior any line segment joining two points of S.

Claim 5. *Let H be an infinite-dimensional Hilbert space, $T \in \mathcal{C}(H)$, and $x \in H$ be as in the conclusion of lemma 53. If $\lambda \neq 0$ is an extreme point of $\left[0, \frac{\lambda}{\|x\|^2}\right]$ (the intersection of a ray from 0 and $\overline{W(T)}$) and λ is a limit point of $W(T)$, then $\lambda \in W(T)$.*

Proof. Let H be an infinite-dimensional Hilbert space, $T \in \mathcal{C}(H)$, and $x \in H$ be defined as in lemma 53. Assume $\lambda \neq 0$ is an extreme point of $\left[0, \frac{\lambda}{\|x\|^2}\right]$ and is a limit point of $\mathrm{W}(T)$. Since λ is an extreme point of $\left[0, \frac{\lambda}{\|x\|^2}\right]$, either $\lambda = 0$ or $\lambda = \frac{\lambda}{\|x\|^2}$. By assumption $\lambda \neq 0$, so we must have $\lambda = \frac{\lambda}{\|x\|^2}$. Therefore, by lemma 53, $\lambda = \frac{\lambda}{\|x\|^2} \in \mathrm{W}(T)$. □

Recall that for a compact operator T, $0 \in \sigma(T) \subseteq \overline{\mathrm{W}(T)}$ (by theorems 41 and 45). Thus, claim 5 hints that $\overline{\mathrm{W}(T)}$ can be expressed as all rays from 0 intersected with $\overline{\mathrm{W}(T)}$. We formalize this in the following proposition, which will be of use later.

Proposition 5. *Let H be a Hilbert space. If $T \in \mathcal{C}(H)$, then $\overline{W(T)} = \{[0, b] : b \in \overline{W(T)}\}$.*

Proof. Let H be an infinite dimensional Hilbert space. Assume $T \in \mathcal{C}(H)$.
($\subseteq$) Let $x \in \overline{\mathrm{W}(T)}$. Clearly, $x \in [0, x]$. Note that $[0, x] \subseteq \{[0, b] : b \in \overline{\mathrm{W}(T)}\} \subseteq \overline{\mathrm{W}(T)}$. Thus, $\overline{\mathrm{W}(T)} \subseteq \{[0, b] : b \in \overline{\mathrm{W}(T)}\}$.

($\supseteq$) Consider $[0, b]$, where $b \in \overline{\mathrm{W}(T)}$. By theorem 41, $0 \in \sigma(T)$. By theorem 45, $\sigma(T) \subseteq \partial \mathrm{W}(T)$. Therefore, $0 \in \overline{\mathrm{W}(T)}$. Thus, by the convexity of $\overline{\mathrm{W}(T)}$, we know $[0, b] \subseteq \overline{\mathrm{W}(T)}$. Hence, $\{[0, b] : b \in \overline{\mathrm{W}(T)}\} \subseteq \overline{\mathrm{W}(T)}$.

Therefore, $\overline{\text{W}(T)} = \{[0, b] : b \in \overline{\text{W}(T)}\}$, as desired. □

Note. *It is well known that the numerical range of a bounded operator on a finite dimensional Hilbert space is closed (theorem 54). Since compact operators are closely related to finite dimensional operators (this is theorem 25), it seems plausible that its numerical range would be close to that of a finite dimensional operator. This is summarized in theorem 56.*

Theorem 54. *Let H be a Hilbert space and $T \in \mathcal{B}(H)$. If H is finite dimensional, then $W(T)$ is closed.*

Lemma 55. *Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Define $q : H \longrightarrow \mathbb{C}$ by $q(v) = \langle Tv, v\rangle$ for each $v \in H$. Then, q is continuous on H.*

Proof of lemma 55. Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Define $q : H \longrightarrow \mathbb{C}$ by $q(v) = \langle Tv, v\rangle$ for each $v \in H$. Let $\varepsilon > 0$ and $v_0 \in H$. If $T = 0$, we note $q = 0$ which is continuous. If $T \neq 0$, then $\|T\| \neq 0$. Define $\delta = \max\left\{1, \dfrac{\varepsilon}{3\|T\|(\|v_0\| + 1)}\right\}$. Suppose $v \in H$ is such that $\|v - v_0\| < \delta$. Now,

$$
\begin{aligned}
|q(v) - q(v_0)| &= |\langle Tv, v\rangle - \langle Tv_0, v_0\rangle| \\
&= |\langle Tv - Tv_0, v - v_0\rangle + \langle Tv - Tv_0, v_0\rangle + \langle Tv_0, v\rangle - \langle Tv_0, v_0\rangle| \\
&= |\langle Tv - Tv_0, v - v_0\rangle + \langle Tv - Tv_0, v_0\rangle + \langle Tv_0, v - v_0\rangle| \\
&\leq |\langle Tv - Tv_0, v - v_0\rangle| + |\langle Tv - Tv_0, v_0\rangle| + |\langle Tv_0, v - v_0\rangle| && (4.31) \\
&\leq \|T(v - v_0)\|\,\|v - v_0\| + \|T(v - v_0)\|\,\|v_0\| + \|Tv_0\|\,\|v - v_0\| && (4.32) \\
&\leq \|T\|\,\|v - v_0\|^2 + 2\|T\|\,\|v - v_0\|\,\|v_0\| && (4.33) \\
&< \|T\|\,\|v - v_0\| + 2\|T\|\,\|v - v_0\|\,\|v_0\| && (4.34)
\end{aligned}
$$

$$
\begin{aligned}
&< \|T\| \frac{\varepsilon}{3\|T\|(\|v_0\|+1)} + 2\|T\| \frac{\varepsilon}{3\|T\|(\|v_0\|+1)} \|v_0\| &(4.35)\\
&< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3}\\
&= \varepsilon
\end{aligned}
$$

Line (4.31) follows from the triangle inequality. Lines (4.32) and (4.33) follow from the Cauchy-Schwarz inequality. Line (4.34) follows from the fact that $\|v - v_0\| < \delta \leq 1$. Line (4.35) follows from the fact that $\|v - v_0\| < \delta \leq \frac{\varepsilon}{3\|T\|(\|v_0\|+1)}$. Therefore, q is continuous on H. □

Proof of theorem 54. Assume H is a finite dimensional Hilbert space and $T \in \mathcal{B}(H)$. Since H is a finite dimensional metric space, it is homeomorphic to $\mathbb{R}^n$, where $n = \dim(H)$. Since the closed unit ball $\overline{B}$ is closed and bounded, it follows by the Heine-Borel Theorem that $\overline{B}$ is compact. By lemma 55 q is continuous. Thus, $q(B) = \mathrm{W}(T)$ is compact. Again, by the Heine-Borel Theorem, $q(B) = \mathrm{W}(T)$ is closed and bounded. Therefore, $\mathrm{W}(T)$ is closed, as desired. □

Note. *Since convex sets are necessarily path connected, we will be using that the boundary points of the numerical range not contained in the numerical range are limit points of the numerical range.*

Theorem 56 (De Barra et al, Theorem 1)**.** *Let H be an infinite-dimensional Hilbert space and $T \in \mathcal{C}(H)$, then:*

1. $0 \in W(T)$ *if and only if* $W(T)$ *is closed.*
2. *If* $0 \notin W(T)$*, then* 0 *is an extreme point of* $\overline{W(T)}$*.*

The following proof is adapted from [5].

Proof. Let H be an infinite-dimensional Hilbert space and $T \in \mathcal{C}(H)$.

(1) ($\Longrightarrow$) Assume $0 \in \mathrm{W}(T)$. Let λ be a limit point of $\mathrm{W}(T)$. If $\lambda = 0$, we automatically have $\lambda \in \mathrm{W}(T)$ by assumption. If $\lambda \neq 0$, by lemma 53 $\dfrac{\lambda}{\|x\|^2} \in \mathrm{W}(T)$ for some $x \in H$ with $\|x\| \leq 1$. Since $\mathrm{W}(T)$ is convex, the line segment $\left[0, \dfrac{\lambda}{\|x\|^2}\right]$ is contained in $\mathrm{W}(T)$. Also by lemma 53, $\lambda \in \left(0, \dfrac{\lambda}{\|x\|^2}\right]$. Thus, $\lambda \in \mathrm{W}(T)$. Therefore, $\mathrm{W}(T)$ is closed.

($\Longleftarrow$) Assume $\mathrm{W}(T)$ is closed. Consequently, $\mathrm{W}(T) = \overline{\mathrm{W}(T)}$. By theorem 45, $\sigma(T) \subseteq \overline{\mathrm{W}(T)}$. Since T is compact, $0 \in \sigma(T)$, by theorem 41. Thus, we have the desired $0 \in \sigma(T) \subseteq \overline{\mathrm{W}(T)} = \mathrm{W}(T)$.

Therefore, $0 \in \mathrm{W}(T)$ if and only if $\mathrm{W}(T)$ is closed.

(2) Assume $0 \notin \mathrm{W}(T)$. Seeking a contradiction, suppose 0 is not an extreme point of $\overline{\mathrm{W}(T)}$. Then, 0 is contained on the interior of some line segment $[a, b]$ on $\partial \mathrm{W}(T)$. Since 0 is on the interior of $[a, b]$, we can break it into two line segments: $[a, 0]$ and $[0, b]$. Note that these line segments are the intersection of rays from 0 and $\overline{\mathrm{W}(T)}$, since $a, b \in \partial \mathrm{W}(T)$. Since 0 is on the interior of $[a, b]$, we know that $a, b \neq 0$. Since $a, b \in \partial \mathrm{W}(T)$, by the note a, b are limit points of $\mathrm{W}(T)$. Clearly, a is an extreme point of $[a, 0]$ and b is an extreme point of $[0, b]$. Thus, $a, b \in \mathrm{W}(T)$ by Claim 5. By the convexity of $\mathrm{W}(T)$, the line segment $[a, b]$ lies in $\mathrm{W}(T)$. Thus, we have

$0 \in [a,b] \subseteq \mathrm{W}(T)$, which contradicts the assumption that $0 \notin \mathrm{W}(T)$. Therefore, we must conclude 0 is an extreme point of $\overline{\mathrm{W}(T)}$. □

Lemma 57 and theorem 58 were formulated and proved while reading [5] (but do not appear in the paper).

Lemma 57. *Let H be an infinite dimensional Hilbert space, $T \in \mathcal{C}(H)$, and $0 \notin W(T)$. Suppose $[0,b]$ is the intersection of a ray from 0 and $\overline{W(T)}$. If $\lambda \in W(T) \cap [0,b]$, then $[\lambda, b] \subseteq W(T)$.*

Proof. Let H be an infinite dimensional Hilbert space, $T \in \mathcal{C}(H)$, and $0 \notin \mathrm{W}(T)$. Let $[0,b]$ denote the intersection of a ray from 0 and $\overline{\mathrm{W}(T)}$. Assume $\lambda \in \mathrm{W}(T) \cap [0,b]$. Since $[0,b] \subseteq \overline{\mathrm{W}(T)}$, $b \in \overline{\mathrm{W}(T)}$. If $b \in \mathrm{W}(T)$, then by convexity we automatically have $[\lambda, b] \subseteq \mathrm{W}(T)$. If $b \notin \mathrm{W}(T)$, then $b \in \overline{\mathrm{W}(T)} \backslash \mathrm{W}(T)$; that is, b is a limit point of $\mathrm{W}(T)$. If $b = 0$, then $[0,b] = \{0\} \subseteq \overline{\mathrm{W}(T)} \backslash \mathrm{W}(T)$. Thus, there can be no $\lambda \in \mathrm{W}(T) \cap [0,b]$, contradicting our assumption. Hence, we must have $b \neq 0$. Thus, by lemma 53, there exists some $x \in H$ with $\|x\| \leq 1$ such that $b \in \left(0, \dfrac{b}{\|x\|^2}\right]$ and $\dfrac{b}{\|x\|^2} \in \mathrm{W}(T)$. Since $0 < \|x\| \leq 1$, it follows that $(0,b] \subseteq \left(0, \dfrac{b}{\|x\|^2}\right]$; that is, $\dfrac{b}{\|x\|^2}$ sits on the same ray from 0 as b does, just potentially "farther out" than b on this ray. Since $[0,b]$ is the intersection of this ray from 0 with $\overline{\mathrm{W}(T)}$, it is not possible for $\dfrac{b}{\|x\|^2}$ to fall strictly after b on this ray. If $\dfrac{b}{\|x\|^2}$ did fall strictly after b on this ray, then $\dfrac{b}{\|x\|^2} \notin \overline{\mathrm{W}(T)}$. Thus, $\dfrac{b}{\|x\|^2} \notin \mathrm{W}(T)$, contradicting lemma 53. Since $\dfrac{b}{\|x\|^2}$ cannot fall strictly after b, we cannot have $\|x\|^2 < 1$. Hence, $\|x\|^2 = 1$. And so, $\|x\| = 1$. Thus, $b = \dfrac{b}{\|x\|^2} \in \mathrm{W}(T)$. Therefore, by convexity $[\lambda, b] \subseteq \mathrm{W}(T)$, as desired. □

The following figure illustrates the argument from the proof of lemma 57:

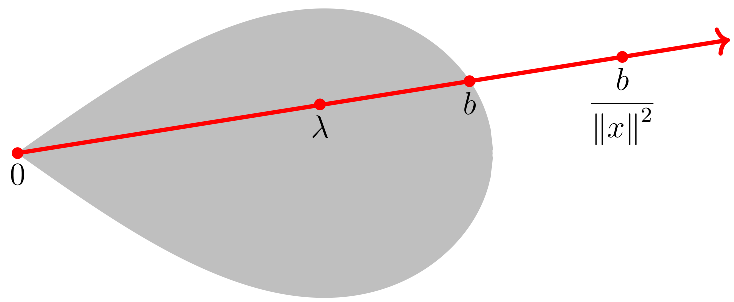

Figure 1.

Note. *If T is a bounded operator on a finite dimensional Hilbert space, then by theorem 54 $\overline{W(T)} \setminus W(T) = \varnothing$. Theorem 56 says that for a compact operator on an infinite dimensional Hilbert space, $\overline{W(T)} \setminus W(T) = \varnothing$ if and only if $0 \in W(T)$. The next theorem describes that for a compact operator T on an infinite dimensional Hilbert space and $0 \notin W(T)$, $\overline{W(T)} \setminus W(T)$ is small (either $\{0\}$ or the union of line segments).*

Theorem 58. *Let H be an infinite dimensional Hilbert space, $T \in \mathcal{C}(H)$, and $0 \notin W(T)$. Let $b \in \overline{W(T)} \setminus W(T)$ and $[0, c]$ be the intersection of $\overline{W(T)}$ with the ray from 0 that contains b. Then, either $b = 0$, or $[0, b] \subseteq \overline{W(T)} \setminus W(T)$ and there exists $\lambda \in \overline{W(T)}$ such that $b \in [0, \lambda]$, $[0, \lambda) \subseteq \overline{W(T)} \setminus W(T)$, and $(\lambda, c] \subseteq W(T)$.*

Proof. Let H be an infinite dimensional Hilbert space and $T \in \mathcal{C}(H)$. Assume $0 \notin \mathrm{W}(T)$. Suppose $b \in \overline{\mathrm{W}(T)} \setminus \mathrm{W}(T)$ and $b \neq 0$. Since $0 \in \overline{\mathrm{W}(T)}$, by convexity $[0, b] \subseteq \overline{\mathrm{W}(T)}$. So, we need only show that $[0, b] \cap \mathrm{W}(T) = \varnothing$. Seeking a contradiction, suppose there exists $\lambda \in [0, b] \cap \mathrm{W}(T)$. Then, by lemma 57,

$[\lambda, c] \subseteq \mathrm{W}(T)$. Since $b \in [\lambda, c]$, it follows that $b \in \mathrm{W}(T)$, contradicting the assumption that $b \in \overline{\mathrm{W}(T)} \backslash \mathrm{W}(T)$. Therefore, we must conclude that $[0, b] \cap \mathrm{W}(T) = \varnothing$. Hence, $[0, b] \subseteq \overline{\mathrm{W}(T)} \backslash \mathrm{W}(T)$.

By lemma 53, there exists $x \in H$ with $\|x\| \leq 1$ such that $b \in \left(0, \frac{b}{\|x\|^2}\right]$ and $\frac{b}{\|x\|^2} \in \mathrm{W}(T)$. Note we can write $\left(0, \frac{b}{\|x\|^2}\right] = \left\{tb : 0 < t \leq \frac{1}{\|x\|^2}\right\}$. Define $t_0 = \inf\left\{t : 0 < t \leq \frac{1}{\|x\|^2} \text{ and } tb \in \mathrm{W}(T)\right\}$. We note that the infimum exists by the Axiom of Completeness, since this set is bounded (below by 0 and above by $1/\|x\|^2$) and is nonempty, since $\frac{b}{\|x\|^2} \in \mathrm{W}(T)$.

Define $\lambda = t_0 b$. Since it's possible the above set does not achieve its infimum, all we can conclude is that $\lambda \in \overline{\mathrm{W}(T)}$. Since $\|x\| \leq 1$, it follows $1 \leq \frac{1}{\|x\|^2}$. Since $b \notin \mathrm{W}(T)$, $t_0 \geq 1$. Hence, b falls "before" $t_0 b$ on the line segment $[0, c]$ (or $b = t_0 b$). That is, $b \in [0, t_0 b]$.

By definition of infimum, $t_0 \leq t$ for each $0 < t \leq \frac{1}{\|x\|^2}$ such that $tb \in \mathrm{W}(T)$. Thus, $t_0 b$ falls "before" each $tb \in \mathrm{W}(T)$ that lies on $\left(0, \frac{b}{\|x\|^2}\right]$. This implies $(0, t_0 b)$ contains no points in $\mathrm{W}(T)$, otherwise we would contradict the definition of infimum. Thus, $(0, t_0 b) \subseteq \overline{\mathrm{W}(T)} \backslash \mathrm{W}(T)$. Since $0 \in \overline{\mathrm{W}(T)} \backslash \mathrm{W}(T)$, it follows $[0, t_0 b) \subseteq \overline{\mathrm{W}(T)} \backslash \mathrm{W}(T)$.

For every $\varepsilon > 0$, $[t_0 b + \varepsilon, c] \subseteq \mathrm{W}(T)$ by lemma 57. Thus, $(t_0 b, c] \subseteq \mathrm{W}(T)$. □

Example 8 ([5]). *Define* $T \in \mathcal{B}(\ell^2)$ *by*
$T(a_1, a_2, a_3, \ldots, a_n, \ldots) = \left(a_1, a_1 + a_2, a_3, \frac{1}{2}a_4, \ldots, \frac{1}{n-2}a_n, \ldots\right)$. $\overline{W(T)} \setminus W(T)$ *is the union of the two symmetric half-open line segments that contain* 0.

Solution. Note that we can write $T = T_1 \bigoplus T_2$, where $T_1 \in \mathcal{C}(\mathbb{F}^2)$ and $T_2 \in \mathcal{C}(\ell^2)$ are defined by $T_1(a_1, a_2) = (a_1, a_1 + a_2)$ and $T_2(a_n) = \left(\frac{1}{n}a_n\right)$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, T_2 is of the form described in example 1, and so is T_2 compact. By example 5, $\sigma_p(T_2) = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. By example 7, we know $\mathrm{W}(T_2) = \mathrm{conv}\{\sigma_p(T)\}$. Thus, $\mathrm{W}(T_2) = (0, 1]$.

On the other hand it can be shown that $\mathrm{W}(T_1) = \{z \in \mathbb{C} : |z - 1| \leq \frac{1}{2}\}$.

Thus, by theorem 49, $\mathrm{W}(T) = \mathrm{conv}\{\ \{z \in \mathbb{C} : |z - 1| \leq \frac{1}{2}\} \cup (0, 1]\}$, which is depicted in the figure below:

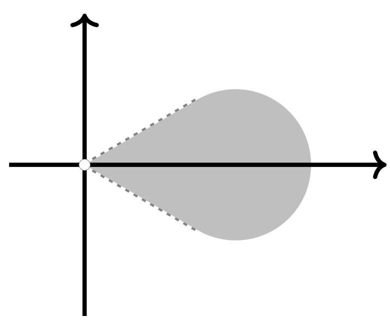

Figure 2. $\mathrm{W}(T)$

Therefore, we see that $\overline{\mathrm{W}(T)}$ is:

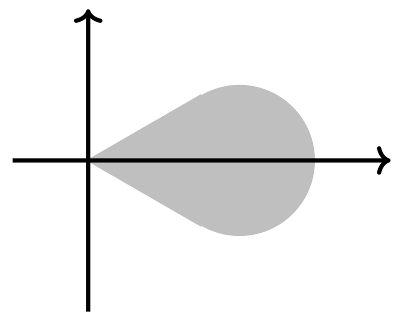

Figure 3. $\overline{\mathrm{W}(T)}$

Thus, $\overline{\mathrm{W}(T)} \backslash \mathrm{W}(T)$ is:

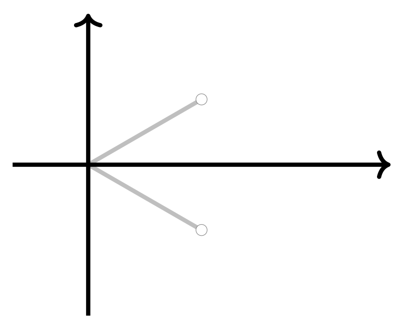

Figure 4. $\overline{\mathrm{W}(T)} \backslash \mathrm{W}(T)$

Which is the union of two half open line segments that contain 0, lie tangent to the circle $\{z \in \mathbb{C} : |z-1| = \frac{1}{2}\}$, and are symmetric about the real axis.

Note. *From the proof of theorem 58, it is not clear whether the endpoint λ of the line segment making up $\overline{W(T)} \backslash W(T)$ is in $W(T)$ or not. From the provided example (as well as the other examples in [5]), it seems plausible that $\lambda \in W(T)$ always. This is in fact the case ([10]), however, proving it requires the essential numerical range which is beyond the scope of this thesis. Lemma 5 provides a sufficient condition: If λ is an extreme point of $\overline{W(T)}$, then $\lambda \in W(T)$.*

Chapter 5

FUTURE WORK

In this thesis we explored the numerical range of a compact operator. We showed how a compact operator on a Hilbert space can be approximated by bounded finite rank operators (theorem 25). To illustrate, we introduced a prototypical compact operator (example 1). We showed the numerical range of this compact operator, and in fact all normal compact operators, is the convex hull of its eigenvalues (example 7 and theorem 51). We suspected that the numerical range of a compact operator would be similar to an operator on a finite dimensional Hilbert space (which is closed by theorem 54). We showed they share some similarity: the numerical range of a compact operator being closed depends on whether 0 is contained in the numerical range or not (theorem 56). We showed that the difference between the closure and the numerical range of a compact operator is a union of line segments that contain 0 (theorem 58). Lastly, we pointed out that a sufficient condition for these line segments to be open is for the nonzero endpoint to be an extreme point of the closure of the numerical range (an application of lemma 5).

Some questions that arose during the course of our research which we were unable to answer are:

- $\overline{\mathrm{W}(T)} \setminus \mathrm{W}(T)$ is $\{0\}$, one line segment containing 0, or the union of two line segments containing 0 ([10]). It seems plausible to prove $\overline{\mathrm{W}(T)} \setminus \mathrm{W}(T)$ consists of at most two line segments containing 0 using convexity.

- The line segments that make up $\overline{\mathrm{W}(T)} \backslash \mathrm{W}(T)$ (described in theorem 58) are always half open ([10]). Is it possible to prove this without resorting to the essential numerical range?

- When $\overline{\mathrm{W}(T)} \backslash \mathrm{W}(T)$ is symmetric (such as in example 8), what is revealed by such symmetry?

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