AN INTRODUCTION TO FRÖBERG'S CONJECTURE

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ABSTRACT

An Introduction to Fröberg's Conjecture

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The goal of this thesis is to make Fröberg's conjecture more accessible to the average math graduate student by building up the necessary background material to understand specific examples where Fröberg's conjecture is true.

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INTRODUCTION

The goal of this thesis is to make Fröberg's conjecture more accessible to the average math graduate student. Building up the necessary background material to understand specific examples where Fröberg's conjecture is true is the bulk of this thesis.

Let k be an infinite field and let $R = k[x_1, \dots, x_n]$ be the polynomial ring in n variables. A homogeneous ideal I in R is said to be of type $(n; d_1, \dots, d_r)$ if I is generated by generic forms f_i of degree d_i for $i = 1, \dots, r$. In other words we can write $I = (f_1, \dots, f_r)$ where each f_i is in some sense "random." While this is not quite correct, it can be helpful to think of each f_i as a homogeneous polynomial in n variable with the coefficients chosen randomly.

In loose terms, an ideal generated by a sequence of f_i's of degrees d_i are chosen "at random." We can identify each form with a point in $\prod_{i=1}^r R_{d_i}$ by sending the polynomials to the coordinates given by its coefficients . We say that a property P of such sequences is **generic** if it holds on a nonempty Zariski-open set $U \subseteq \prod_{i=1}^r R_{d_i}$. Loosely, such a property holds "most of the time," since open Zariski sets are known to be dense this property ought to hold for a randomly chosen sequence because the set of coefficients where our forms are not generic are closed in the Zariski topology. So the probability of picking those coefficients should be small.

The set of coefficients where our d_i-forms are generic is open in the Zariski topology, but it may be empty. Since nonempty open sets are dense, it it is true in one example it is true for many such ideals. We will not become preoccupied with the Zariski topology happening in the background, but will move forward thinking of our choices as "random".

Given degrees d_i for $i = 1, \ldots, r$ we can produce a generating function for the dimension of the k vector space of forms of each degree. We use absolute value notation to indicate that for a given series $\sum_{i=0}^{\infty} a_i t^i$, $a_i \in \mathbb{Z}$ for all i, let $|\sum_{i=0}^{\infty} a_i t^i|$ be the series $\sum_{i=0}^{\infty} b_i t^i$ where

$$b_i = \begin{cases} a_i, & \text{if } a_i > 0 \text{ for all } 0 \leq j \leq i \\ 0, & \text{otherwise} \end{cases}$$

So in the absolute value of a series, one a term becomes nonpositive, it and every term after it is set equal to 0.

In 1985, Fröberg conjectured that ideals generated by generic forms exhibit minimal Hilbert behavior. Note that the Hilbert series is another invariant that measures "size" of an ideal. Fröberg's conjecture states that

Conjecture 1.0.1.(Fröberg's Conjecture)

If k is an infinite field and I is generated by a generic sequence of polynomials of degrees $d_1, \ldots, d_r$, then

$$H_{R/I}(t) = \left| \frac{\prod_{i=1}^{r} \left(1 - t^{d_i}\right)}{(1-t)^n} \right|$$

[3]

where H is the Hilbert function; and in fact, there are many such ideals.

To generate such an ideal, we consider indeterminate d_i-forms—i.e. d-forms with indeterminate coefficients- then attempt to choose field elements for each coefficient so that the resulting ideal has the desired Hilbert function. The desired Hilbert function will place constraints on our choices. In particular, there is a homogeneous system of linear equations in our choices for coefficients whose solution set must be avoided.

Our Hilbert functions corresponding to underlying graded algebra that we want to understand. We need a vector space basis to be a certain size. We set the coefficients to indeterminates and solve the underlying system to determine potential forms. In our later examples, we will see how our underlying linear algebra affects the corresponding free resolutions and Betti tables. These connections are the main theme explored in this thesis.

Fröberg's conjecture is equivalent to the following conjecture:

Conjecture 1.0.2.

If k is an infinite field and $R = k[x_1, \ldots, x_n]$, and $d_1, \ldots, d_r$ are non-negative integers, then a generic sequence of polynomials of degrees $d_1, \ldots, d_r$ is semi-regular [3].

The benefit of this formulations is that semi-regular polynomials are more intuitive to work with. We are able to learn about the structure of our solution in terms of the generators themselves.

In general, we to show that the Zariski open set for which Fröberg's conjecture is nonempty. If so, since it is dense, "most" ideals will be such that Fröberg's conjecture is true. Thus for a particular small set of $\{d_1, \ldots, d_r\}$, the problem devolves into a simple case; it is enough to show there exists a homogeneous ideal with the given Hilbert series because then our Zariski set is non-empty. Finding such an ideal should be easy with a computer algebra system since if there is one, there should be many.

If we pass to a finite field and specify any n for the number of variables and a list of forms with degrees d_i, there is a computationally finite time for the computer to check all possible ideals with fixed coefficients. For every specific case tried an ideal can be produced given enough time over a finite field, but to prove Fröberg's conjecture in general has proven quite difficult. In the list below n is the number of variables in the polynomial ring and r is the number of forms. Fröberg's conjecture is known to be true for $r \leq n$; $n = 2$; $n = 3$; $r = n+1$ with char $k = 0$; $d_1, \cdots = d_r = 3$ and $n \leq 8$ [5]. This conjecture is interesting because it is wide open even though any particular case of small integers is immediately knowable.

Chapter 2

PRELIMINARIES

Throughout this paper $R = k[x_1, \ldots, x_n]$, with the natural grading by degree, k denotes the base field of R. The number r always denotes the number of forms in a sequence of interest in R.

Definition 2.0.1. (*Monomial*)
Let $R = k[x_1, \ldots, x_n]$. We say an element $p \in R$ is a **monomial** of degree d if $p = \prod_{i=1}^{n} x_i^{d_i}$ for $d_i \in \mathbb{N} \cup \{0\}$ where $\sum_{i=1}^{n} d_i = d$. Additionally we allow 0 to be a monomial of degree -1.

An example of a set of monomials of degree 2 in $R = \mathbb{R}[x, y, z]$ is

$$\{x^2, xy, xz, y^2, yz, z^2\}$$

The number of monomials of n variables in degree d is $\binom{d+n-1}{n-1}$. A **polynomial** in R are sums of monomials with coefficients in k.

Definition 2.0.2. (*Homogeneous polynomial*)
We say an element of degree d of R is **homogeneous** if it can be uniquely written by a sum of monomials of degree d with coefficients in k where not all of the coefficients are 0. As above the zero monomial has degree -1.

For example, if $R = \mathbb{R}[x, y, z]$ then a homogeneous element p of degree 2 would be

$$p = a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2$$

where $a_1, \ldots, a_6 \in \mathbb{R}$ and at least one $a_i \neq 0$. If all the $a_i = 0$ then we would have the zero polynomial. We often use the word form to denote a homogeneous polynomial.

Definition 2.0.3. (*Homogeneous Ideal*)

A **homogeneous ideal** $I \leq R$ is an ideal generated by homogeneous polynomials.

Since $R = k[x_1, \ldots, x_n]$ is know to be Noetherian, then every ideal of R can be finitely generated. For any homogeneous ideal I there exists $f_1, \ldots, f_r$ such that $I = (f_1, \ldots, f_r)$. For our purposes, if we state $I = (f_1, \ldots, f_r)$ we shall assume that I has already been reduced to a set of minimum generators.

2.1 Building up Graded Free Resolutions

In this section we will compile a list of definitions and theorems that will culminate in our definition of a graded free resolution.

Definition 2.1.1. (M_i)

Let M be an R-Module. Then we write $\mathbf{M_i}$ to be the k-vector space generated by the i^{th} degree parts of M.

In other words, for the polynomial ring $R = k[x_1, \ldots, x_n]$, R_i is the k-vector space of the homogeneous polynomials of degree i. So we can express R by

$$R = \bigoplus_{i=1}^{\infty} R_i$$

The $\dim_k R_i$ is the dimension of the i^{th} graded piece of R as a k-vector space.

Definition 2.1.2. (*Exact*)

If A, B, and C are R-Modules, and $\alpha : A \to B, \beta : B \to C$ are homomorphisms, then a pair of homomorphisms

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is **exact** if the image of α is equal to $\ker \beta$. In general, a sequence of maps between modules of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E$$

is exact if each pair of consecutive maps is exact.

Definition 2.1.3. (*Short exact sequence of complexes*)

A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

where α is an injection, β is a surjection, and the image of α is the kernel of β.

Definition 2.1.4. (*Complex of R-Modules*)

A **complex of R-Modules** is a sequence of modules F_i and maps $F_i \to F_{i-1}$ such that the compositions $F_{i+1} \to F_i \to F_{i-1}$ are all zero. The homology of this complex at F_i is the R-module

$$\ker(F_i \to F_{i-1}) \big/ \operatorname{im}(F_{i+1} \to F_i)$$

[2]

A free resolution is an exact complex.

Definition 2.1.5. (*Free resolutions*)

A **free resolution** of an R-Module M is a complex

$$\mathscr{F} : \ \ldots \longrightarrow F_n \xrightarrow{\varphi_n} \ldots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0$$

of free R modules such that $\mathscr{F}$ is exact. [2]

The image of the map φ_i is called the i^{th} syzygy module of M.

Definition 2.1.6. (*Graded Module*)

If $R = R_0 \oplus R_1 \oplus \ldots$ is a graded ring then a **graded module** over R is a module M with decomposition

$$M = \bigoplus_{-\infty}^{\infty} M_i$$

as abelian groups such that $R_i M_j \subseteq M_{i+j}$ for all i, j. [2]

A graded module allows us to keep track of elements degree-wise.

Definition 2.1.7. (*Graded Free Resolution*)

A resolution $\mathscr{F}$ is a **graded free resolution** if R is a graded ring, the F_i are graded free modules, and the maps are homogeneous maps of degree 0. [2]

If for some $n < \infty$ we have $F_{n+1} = 0$, but $F_i \neq 0$ for $1 \leq i \leq n$, then we shall say that $\mathscr{F}$ is a finite resolution of length n.

Example 2.1.8. Let $R = k[x, y]$ and let $I = (x, y)$. Then if we have the following mapping

$$R^2 \xrightarrow[\begin{bmatrix} x \\ y \end{bmatrix}]{} R \longrightarrow R/I$$

Then we are mapping degree 0 elements in R to degree 1 elements. Notice this sequence is exact, but it is not homogeneous because a degree 0 element gets mapped to a degree 1 element. We will need to fix this to give us a homogeneous sequence as well. Such a alteration is called:

Definition 2.1.9. (*dth Twist of M*)

Define $M(d)$ to b the altered graded module M shifted in its grading d steps. Then $M(d) \simeq M$ as a module and having grading defined by $M(d)_e = M_{d+e}$. Note that $M(d)$ is sometimes called the **dth Twist of M**.

So in order to preserve our degrees, we need to grade our left module by 1. So our free resolution becomes

$$R^2(-1) \xrightarrow[\begin{bmatrix} x \\ y \end{bmatrix}]{} R \longrightarrow R/I$$

This grading takes the degree of our map and brings it down by 1. Thus $1 \mapsto x, 1 \mapsto y$ maps a degree 1 element to a degree 1 element as desired. Our mapping here is an example of a zero map as stated in definition 2.1.7.

Note that the goal of such a dth twist is to take homogeneous elements to homogeneous elements. In our later examples, we more closely examine free resolutions of graded modules.

2.2 Hilbert Series

In this section, we develop our understanding of Hilbert series and learn how they relate to Fröberg's Conjecture.

Definition 2.2.1. (*Hilbert Function of M*)
Let M be a finitely generated graded module over $k[x_1, \ldots, x_r]$ with grading generated in positive degrees. The numerical function

$$H_M(s) := \dim_k M_s$$

is called the **Hilbert Function of M**. [2]

Note that the Hilbert function can be defined for negative degrees, but for simplicity we will just work with cases where it is positive.

Definition 2.2.2. (*Hilbert series*)

The **Hilbert series** of R/I is

$$H_{R/I}(t) = \sum_{i=0}^{\infty} \dim_k (R/I)_i t^i$$

Recall that $(R/I)_i$ is the k vector space of i-forms, thus $\dim_k (R/I)_i$ is the dimension of the i^{it} graded piece as a k-vector space.

Definition 2.2.3. (*Absolute value of a series*)

Given a series $\sum_{i=0}^{\infty} a_i t^i$, $a_i \in Z$ for all i, let $|\sum_{i=0}^{\infty} a_i t^i|$ be the series $\sum_{i=0}^{\infty} b_i t^i$ where

$$b_i = \begin{cases} a_i, & \text{if } a_j > 0 \text{ for all } 0 \leq j \leq i \\ 0, & \text{otherwise} \end{cases}$$

[4]

Recall from Linear Algebra,

Theorem 2.2.4. (Rank-Nullity Theorem)

Let $T : V \to W$ be a linear transformation between two vector spaces where T's domain V is finite dimensional. Then

$$\dim(\text{im}\,(T)) + \dim(\ker(T)) = \dim(V)$$

This will be useful in later examples, so is listed to refresh if needed.

Fröberg's conjecture is thought of in terms of semi-regular ideals. In order to build up to the definition of semi-regular, we first must understand regular sequences.

Definition 2.2.5. (*Regular sequence*)
A sequence of elements $f_1, \ldots, f_r$ in a ring R is a **regular sequence** on R if the ideal $(f_1, \ldots, f_r)$ is proper and for each i, the image of f_{i+1} is a non-zero divisor in $R/(f_1, \ldots, f_i)$.

Definition 2.2.6. (*Semi-regular, semi-regular sequence*)
Let $R = k[x_1, \ldots, x_n]$ and let I be a homogeneous ideal. A nonzero form $f \in R_d$ is called **semi-regular** on R/I if the multiplication maps $(R/I)_{a-d} \xrightarrow{f} (R/I)_a$ are linear maps of maximal rank for all a.

A sequence of forms $f_1, \ldots, f_r$ in R with degrees $d_1, \ldots, d_r$ is called a **semi-regular sequence** if f_i is semi-regular on $R/(f_1, \ldots, f_{i-1})$ for all $i = 1, \ldots, r$. [4]

Note that every regular sequence is automatically semi-regular, since all multiplication maps are injective, and thus of maximal rank.

Example 2.2.7. Let $R = \mathbb{R}[x, y, z]$ and $I = (x^2, y^2, z^2)$. I is generated by a regular sequence and therefore by a semi-regular sequence. A basis of elements for each

$(R/I)_i$ of our quotient ring are

$(R/I)_0:$	1	Which has dimension 1
$(R/I)_1:$	$x \quad y \quad z$	Which has dimension 3
$(R/I)_2:$	$xy \quad xz \quad yz$	Which has dimension 3
$(R/I)_3:$	xyz	Which has dimension 1

It follows that $H_{R/I} = 1 + 3t + 3t^2 + t^3$. Note that $(R/I)_4$ has dimension 0.

Now, if we wish to extend this semi-regular sequence we need our maps from $(R/I)_{a-d} \to (R/I)_a$ to be maximal. This means that our map f to be either injective or surjective from the Rank Nullity Theorem.

Let's show that $(x^2, y^2, z^2, x + 2y)$ is not a semi-regular sequence. Let $f : (R/I)_1 \to (R/I)_2$ be multiplication by $x + 2y$. If we apply f to each element in a basis for $(R/I)_1$, in order for $(x^2, y^2, z^2, x+2y)$ to be a semi-regular sequence we will need f to be injective or surjective, but if we apply f to a non-zero linear combination of basis elements in $(R/I)_1$ observe:

$$\begin{aligned} f(x - 2y) &= (x + 2y)(x - 2y) \\ &= x^2 + 2xy - 2xy - 4y^2 \\ &= 0 \in (R/I)_1 \end{aligned}$$

So f is not injective, as $x - 2y \neq 0 \in (R/I)_1$. Also note the dimension of $(R/I)_1$ is 3 and the dimension of (R/I_2) is 3. So for f to be surjective the kernel must be trivial, we have just shown that it is not. It follows that f is not surjective either. Therefore $(x^2, y^2, z^2, x + 2y)$ is not a semi-regular sequence.

Now we shall show that there does exists an order 1 term we can add to our sequence so that (x^2, y^2, z^2, f) is a semi-regular sequence. Let $f = ax + by + cz$. So

$$fx = ax^2 + bxy + cxz = bxy + cxz$$
$$fy = axy + by^2 + cyz = axy + cyz$$
$$fz = axz + byz + cy^2 = axz + byz$$

Now we need to show that these are linearly independent. Assume

$$\lambda_1(bxy + cxz) + \lambda_2(axy + cyz) + \lambda_3(axz + byz) = 0$$

Our goal is to show that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. This leads to the following system:

$$\lambda_1 b + \lambda_2 a = 0 \qquad \text{(from } xy \text{ term)}$$
$$\lambda_1 c + \lambda_3 a = 0 \qquad \text{(from } xz \text{ term)}$$
$$\lambda_2 c + \lambda_3 b = 0 \qquad \text{(from } yz \text{ term)}$$

So we can set up:

$$\begin{array}{c} \\ xy \\ xz \\ yz \end{array} \begin{array}{c} \begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_3 \end{array} \\ \left[\begin{array}{ccc|c} b & a & 0 & 0 \\ c & 0 & a & 0 \\ 0 & c & b & 0 \end{array}\right] \end{array}$$

Row reducing and assuming $b \neq 0 \neq c$ we arrive at:

$$\begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & a/b & 0 & 0 \\ 0 & 0 & 2a & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Therefore if $a \neq 0$ our forms are linearly independent. Therefore (x^2, y^2, z^2, f) is a semi-regular sequence if $f = ax + by + cz$ where $a, b, c \neq 0$. Specifically, if we let $f = \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z$ we have the following sequence where $\dim(R/I)_1 = 3$, $\dim(R/I)_2 = 3$, $\dim(R/I)_3 = 1$, and $\dim(R/I)_4 = 0$.

$$(R/I)_1 \longrightarrow (R/I)_2 \longrightarrow (R/I)_3 \longrightarrow (R/I)_4 = 0$$

With $f : (R/I)_1 \to (R/I)_2$ maximal since $\frac{1}{2} \neq 0$ and with the argument above. Now to show that $f : (R/I)_2 \to (R/I)_3$ is maximal we just need to show that it is surjective. Recall that the only monomial remaining in degree 3 is xyz. So

$$\begin{aligned} f(x+y-z) &= (\frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z)(x+y-z) \\ &= \frac{1}{2}x^2 + \frac{1}{2}xy - \frac{1}{2}xz + \frac{1}{2}xy + \frac{1}{2}y^2 - \frac{1}{2}yz + \frac{1}{2}xz + \frac{1}{2}yz - \frac{1}{2}z^2 \\ &= \frac{1}{2}xy - \frac{1}{2}xz + \frac{1}{2}xy - \frac{1}{2}yz + \frac{1}{2}xz + \frac{1}{2}yz \\ &= xy \end{aligned}$$

So we have found a linear term in $(R/I)_1$ such that if we apply f we get xy. Note that $(x+y-z)(z) \in (R/I)_2$. Then since $(xy)z = xyz$ it follows that $f(x+y-z)(z) = xyz$ and thus $f : (R/I)_2 \to (R/I)_3$ is surjective. Lastly, since the zero map is automatically surjective we are done. We have shown that $(x^2, y^2, z^2, \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z)$ is a semi-regular sequence.

More generally, we have that if R/I is Artinian (a ring that satisfies the descending chain condition on ideals; that is, there is no infinite descending sequence of ideals) and the highest degree in which R/I is nonzero is ρ, then any nonzero form of degree greater than ρ is semi-regular on R/I. This is because every multiplication map has maximal rank [4]. Note that the order of our sequence is important for semi-regularity.

Both regular and semi-regular elements can be characterized by Hilbert series [3]. We typically will demonstrate that a sequence is semi-regular via the Hilbert function. If we take see that the appropriate number of elements are killed off at each degree step we can determine that each element in our sequence is semi-regular.

An ideal being semi-regular leads to a nice generating function for its Hilbert series. A main take away of why this property is so attractive, is that if we have a semi-regular sequence for our ideal I then we can systematically compute the Hilbert series for R/I. Regular sequences are nice because they are composed of the torsion free elements of our ring. Semi-regular sequences are composed on non-zero divisors for the ring until possibly the "last step." Semi-regular sequences have a closely related Hilbert series to regular sequences which is as close as we can get to being Koszul (seen in a later section).

The Hilbert series of R/I where I is generated by a semi-regular sequence of forms of degrees $d_1, \ldots, d_r$ is precisely

$$H_{R/I}(t) = \left| \frac{\prod_{i=1}^{r} \left(1 - t^{d_i}\right)}{(1-t)^n} \right|$$

2.3 Betti Tables and Their Uses

Definition 2.3.1. *(i, jth graded Betti number)*

If I is an ideal in R, then R/I has a minimal graded free resolution

$$\cdots \to F_1 \to F_0 \to R/I$$

where the F_i are free R-modules. The **i, jth graded Betti number** of R/I is $\beta_{i,j}(R/I)$ which is equal to the number of free copies of R in F_i generated in degree j. [4]

So we have $\beta_{i,j}(R/I)$ equals the number of degree j generators in any set of minimal generators of the R-module F_i. Moreover, i represents the place in our free resolution while j represents the grading on each copy of our ring that is present at the ith place in our resolution.

Theorem 2.3.2 (Hilbert Syzygy Theorem). *If $R = k[x_1, \dots, x_n]$, then every finitely generated graded R-module has a finite graded free resolution of length $\leq n$, by finitely generated free modules.*

It follows from Hilbert's Syzygy theorem, that $\beta_{i,j}(R/I) = 0$ for $i > n$. Note that $F_0 = R$ and so $\beta_{0,0}(R/I) = 1$ since R is generated by $1 \in R$ as an R-module. Therefore $\beta_{0,j}(R/I) = 0$ for all $j \neq 0$. Since our free resolution has minimal grading, it follows that $B_{i,j}(R/I) = 0$ whenever $i > j$. The **Castelnuovo-Mumford regularity** $\rho(R/I)$, or simply ρ when context is clear, is the maximum value of j such that $\beta_{i,i+j}(S/I) \neq 0$ for some i. [4]. The **Poincaré series** $P_{R/I}(s,t) = \sum_{i=0}^{n} \sum_{j=0}^{\infty} \beta_{i,j} s^i t^j$ is the generating series of the graded Betti numbers. The **Betti Table** of R/I is a table with $\rho+1$ rows and $n+1$ columns where the i,jth entry, counting from zero, is $\beta_{i,i+j}(R/I)$.

For example, consider the Betti table

total:	1	3	3	1
0:	1	.	.	.
1:	.	1	.	.
2:	.	1	.	.
3:	.	.	1	.
4:	.	1	.	.
5:	.	.	1	.
6:	.	.	1	.
7:	.	.	.	1

For Betti tables we use a zero index for the rows and columns. Then starting at the top left corner with 0, every step down we add one to j and every step to the right we add one to i and j. This is sensible because of the fact that $B_{i,j}(R/I) = 0$ whenever $i > j$. We eliminate a whole region of our Betti table that would automatically be 0 to help streamline our information. While this notation can be difficult to grasp at first, with time it leads to an efficient way to examine our data.

Thus the way we place our Betti numbers in our table is as follows:

	0	1	2	3
0:	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	$\beta_{3,3}$
1:	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\beta_{3,4}$
2:	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\beta_{3,5}$

In our example, in the third row second column the entry $\beta_{1,2+1} = 1$ corresponds to a 3rd degree basis element. In other words, there is a $R(-3)$ graded copy of our ring at the second step of our free resolution. Each column represents a step of our resolution. For our example, our free resolution would be:

$$\begin{array}{ccccccccccc} R(-10) & \longrightarrow & R(-5) & \longrightarrow & R(-2) & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0 \\ & & \oplus & & \oplus & & & & & & \\ & & R(-7) & & R(-3) & & & & & & \\ & & \oplus & & \oplus & & & & & & \\ & & R(-8) & & R(-5) & & & & & & \end{array}$$

The Hilbert series and the Poincaré series are related by

$$(1-t)^n \, H_{R/I}(t) = P_{R/I}(-1,t)$$

where $P_{R/I}$ represents the Poincaré series. Recall that the Hilbert series of R/I where I is generated by a semi-regular sequence of forms of degrees $d_1, \ldots, d_r$ is

$$H_{R/I}(t) = \left| \frac{\prod_{i=1}^{r} (1-t^{d_i})}{(1-t)^n} \right|$$

So

$$P_{R/I}(-1,t) = (1-t)^n \left| \frac{\prod_{i=1}^{r} (1-t^{d_i})}{(1-t)^n} \right|$$

2.4 Koszul Complexes and Special Cases

Let $f_1, \ldots, f_r$ be a sequence of homogeneous polynomials of degrees $d_1, \ldots, d_r$. For each integer $i \geq 0$, let K_i be the free R-module with basis κ_σ indexed by the order i subsets $\sigma \in \{1, \ldots, r\}$. Let the degree of σ be $\sum_{h \in \sigma} d_h$.

For example, let $f_1, \dots, f_5$ be a sequence of homogeneous polynomials of degrees $d_1 = 2, d_2 = 3, d_3 = 3, d_4 = 3, d_5 = 4$. If $i = 3$ and $\sigma = \{1, 4, 5\}$ then $\deg \sigma = \sum_{h \in \sigma} d_h = d_1 + d_4 + d_5 = 2 + 3 + 4 = 9$.

Definition 2.4.1. (*Koszul Complex*)

The **Koszul complex** is defined by

$$\cdots \to K_2 \to K_1 \to K_0$$

where if $\sigma = \{\sigma_1 < \sigma_2 < \cdots < \sigma_i\}$ has order $i > 0$ then the image of κ_σ in K_{i-1} is $\sum_{h=1}^{i} (-1)^{i+h} f_{\sigma_h} \kappa_{\sigma - \sigma_h}$. [4]

Koszul complexes have a nice relationship with regular sequences. And recall that all regular sequences are semi-regular.

Theorem 2.4.2. *Let $f_1, \dots, f_r$ be a sequence of degrees $d_1, \dots, d_r$ and let K_i be defined as above. Then the Koszul complex is a minimal free resolution of R/I if and only if $f_1, \dots, f_r$ is a regular sequence.[4]*

In fact, we can look at the Poincaré series or the Betti table to see if I is generated by a regular sequence. Thus the following fact is true:

Theorem 2.4.3. *Let $K(d_1, \dots, d_r) = \prod_{i=1}^{r} (1 + st^{d_i})$. Then I is generated by a regular sequence of degrees $d_1, \dots, d_r$ if and only if $P_{R/I}(s, t) = K(d_1, \dots, d_r)$.*

If we implement an additional property on our sequence of homogeneous polynomials then there is a way to characterize ideals of degrees $d_1, \dots, d_r$ by their Betti numbers.

Definition 2.4.4. (*Special Numerical Condition*)

The nonnegative integers $n, d_1, \dots, d_r$ satisfy the **special numerical condition** if the first nonpositive coefficients of the series

$$\frac{\prod_{i=1}^{r}\left(1-t^{d_i}\right)}{(1-t)^n}$$

is equal to 0. [4]

With the special numerical condition we have the following theorem:

Theorem 2.4.5. *Let $r \geq n$ and let ρ be the degree of the polynomial*

$$\left|\frac{\prod_{i=1}^{r}\left(1-t^{d_i}\right)}{(1-t)^n}\right|$$

If $n, d_1, \dots, d_r$ satisfy the special numerical condition, then all ideals generated by semi-regular sequences of degrees $d_1, \dots, d_r$ have the same Betti numbers. [4]

This theorem gives us a way to characterise ideals that satisfy the special numerical criterion. This means that the Betti table for such ideals are identical. When $r = n+1$ the special numerical condition turns out to be particularly easy to compute.

Theorem 2.4.6. *Let I be an ideal in $R = k[x_1, \dots, x_n]$ generated by a semi-regular sequence $f_1, \dots, f_{n+1}$ of degrees $d_1 \leq \dots \leq d_{n+1}$ where $d_{n+1} \leq \sum_{i=1}^{n}(d_i - 1)$. Let ρ be the Castelnuovo-Mumford regularity of S/I. Then the coefficient of $t^{\rho+1}$ in*

$$\frac{\prod_{i=1}^{n+1}\left(1-t^{d_i}\right)}{(1-t)^n}$$

is 0 if and only if $\Delta = \sum_{i=1}^{n+1}(d_i - 1)$ is odd. When Δ is odd the graded Betti numbers of I are completely determined by the degree sequence $d_1 \leq \dots \leq d_{n+1}$. [4]

Theorem 2.4.7. *Let $K(d_1, \ldots, d_r) = \prod_{i=1}^{r}(1+st^{d_i})$. Then I is generated by a regular sequence of degrees $d_1, \ldots. d_r$ if and only if $P_{R/I}(s,t) = K(d_1, \ldots, d_r)$. [4]*

In other words, an ideal generated by a regular sequence is minimally resolved by a Koszul complex.

Example 2.4.8. Suppose we let $R = k(x_1, x_2, x_3, x_4)$ and let $I = (f_1, \ldots, f_5)$ have degrees $d_1 = d_2 = d_3 = 2$ and $d_4 = d_5 = 3$. Note that $d_1 \leq \cdots \leq d_5 \leq \sum_{i=1}^{4}(d_i - 1) = 5$. Also note that $\Delta = \sum_{i=1}^{5}(d_i - 1) = 7$ is odd. Therefore the graded Betti Numbers of I are completely determined by the degree sequence $d_1, \ldots, d_5$.

Then to show that a particular ideal I is generated by a semi-regular sequence it is enough to show that

$$H_{R/I}(t) = \left|\left(1 - t^{d_i}\right) H_{R/I}(t)\right|$$

for all $i = 1, \ldots, 5$. Via Mathematica we have

$$\begin{aligned} H_{R/I}(t) &= \left|\prod_{i=1}^{5} \frac{\left(1 - t^{d_i}\right)}{(1-t)^4}\right| \\ &= \left|1 + 4t + 7t^2 + 6t^3 + 0t^4 - 6t^5 - 7t^6 - \ldots\right| \\ &= 1 + 4t + 7t^2 + 6t^3 \end{aligned}$$

Note that the Castelnuovo-Mumford regularity is $\rho = 3$. Recall that $(1-t)^n H_{R/I}(t) = P_{R/I}(-1,t)$. The entries of our Betti table are determined by our Hilbert function.

$$(1 + 4t + 7t^2 + 6t^3)(1-t)^4 = 1 - 3t^2 - 2t^3 + 3t^4 + 12t^5 - 17t^6 + 6t^7$$

The first $\rho - 1$ rows are placed according to the Koszul complex. The rest (i.e. the bottom row) can be red off of the polynomial above. Placing these coefficients into our table we have

total:	1	5	15	17	6
0:	1	.	.	.	.
1:	.	3	.	.	.
2:	.	2	3	.	.
3:	.	.	12	17	6

So by Theorem 2.4.6 all semi-regular sequences of the form $(4; 2, 2, 2, 3, 3)$ will have the Betti table above. This characterizes these forms nicely.

Chapter 3

ILLUSTRATIVE EXAMPLES

With these examples, it is our goal to demonstrate the main themes present in the first half of this paper. Our first example, we focus on being able to generate the linear algebra inherent in our free resolution by hand and show how it could be translated to a computer for later more complex computations. We also want to illustrate the connection between the graded free resolution and the Betti table.

Example 3.0.1. Suppose we let $R = \mathbb{R}[x, y]$ and let $I = (x^2, xy, y^2)$. We want to construct the Betti table and the underlying free resolution corresponding to our quotient ring as well as the underlying linear algebra.

First we have $\varphi_0 : R \to R/I$ where $\ker \varphi_0$ is generated by (x^2, xy, y^2) and $1 \mapsto 1$.

$$R \xrightarrow{\varphi_0} R/I$$

Then we begin to construct φ_1. We will need $\operatorname{im} \varphi_1 = \ker \varphi_0$ for our sequence to be exact. To do this we will need three copies of our ring as there are three generators of $\ker \varphi_0$. Since we are taking a degree 0 element to degree 2 elements $1 \mapsto x^2, 1 \mapsto xy, 1 \mapsto y^2$ we will need to twist each piece by -2. So for this step in our free resolution we have three copies of $R(-2)$, which we express as $R(-2) \oplus R(-2) \oplus R(-2) = R^3(-2)$.

$$R^3(-2) \xrightarrow{\varphi_1} R \xrightarrow{\varphi_0} R/I$$

We can represent φ_1 with the following matrix.

$$\begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}$$

To construct φ_2 we need $\operatorname{im}\varphi_2 = \ker\varphi_1$. To see if we have a degree 0 element we take an arbitrary element $(a, b, c) \in R(-2) \oplus R(-2) \oplus R(-2)$ and see what values of a, b, c $\varphi_1(a, b, c) = 0$. Our map acts on a vector by matrix multiplication on the right.

$$\begin{aligned} \varphi_1(a, b, c) &= \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \\ &= ax^2 + bxy + cy^2 \end{aligned}$$

So $\varphi_1([a, b, c]) = 0$ if and only it $a = b = c = 0$.

For degree 1, similarly take a general 1-form and look for conditions which force the image to be zero.

$$\begin{aligned} \varphi_1(ax + by, cx + dy, ex + fy) &= \begin{bmatrix} ax + by & cx + dy & ex + fy \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \\ &= ax^3 + bx^2y + cx^2y + dxy^2 + exy^2 + fy^3 \\ &= ax^3 + (b + c)x^2y + (d + e)xy^2 + fy^3 \end{aligned}$$

So $\varphi_1(ax+by, cx+dy, ex+fy) = 0$ when $ax^3 = 0x^3$, $(b+c)x^2y = 0x^2y$, $(d+e)xy^2 = 0$, and $fy^3 = 0y^3$ which leads to the following system:

$$\begin{aligned} a &= 0 \\ b + c &= 0 \\ d + e &= 0 \\ f &= 0 \end{aligned}$$

compared to the system where labels on columns correspond to our variables and the rows correspond to the specific monomial in our quotient ring for degree 3.

$$\begin{array}{c} \\ x^3 \\ x^2y \\ xy^2 \\ y^3 \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

We can see that e and c are free and $d = -e, \quad b = -c$. Then we have $[ax + by, cx + dy, ex + fy] = [-cy, cx - ey, ex]$ and a basis is $\{(-y, x, 0), (0, -y, x)\}$.

Now we need to show that degree 2 adds no new information:

$$\begin{aligned} &\varphi_1(a_1x^2 + a_2xy + a_3y^2, b_1x^2 + b_2xy + b_3y^2, c_1x^2 + c_2xy + c_3y^2) \\ &= \begin{bmatrix} a_1x^2 + a_2xy + a_3y^2 & b_1x^2 + b_2xy + b_3y^2 & c_1x^2 + c_2xy + c_3y^2 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \\ &= a_1x^4 + (a_2 + b_1)x^3y + (a_3 + b_2 + c_1)x^2y^2 + (b_3 + c_2)xy^3 + c_3y^4 \end{aligned}$$

So $(a_1x^2 + a_2xy + a_3y^2, b_1x^2 + b_2xy + b_3y^2, c_1x^2 + c_2xy + c_3y^2) \in \ker\varphi_1$ when the following system is satisfied:

$$\begin{aligned} a_1 &= 0 \\ a_2 + b_1 &= 0 \\ a_3 + b_2 + c_1 &= 0 \\ b_3 + c_2 &= 0 \\ c_3 &= 0 \end{aligned}$$

given augmented matrix

$$\begin{array}{c} \\ x^4 \\ x^3y \\ x^2y^2 \\ xy^3 \\ y^4 \end{array} \begin{array}{c} \begin{array}{ccccccccc} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

it follows that $a_1 = c_3 = 0$ and b_1, b_2, c_1 and c_2 are free. So we have $a_2 = -b_1, a_3 = -b_2 - c_1, b_3 = -c_2$. So we have

$$[a_1x^2 + a_2xy + a_3y^2, b_1x^2 + b_2xy + b_3y^2, c_1x^2 + c_2xy + c_3y^2]$$

$$= [-b_1xy + (-b_2 - c_1)y^2, b_1x^2 + b_2xy + (-c_2)y^2, c_1x^2 + c_2xy]$$

which has a basis $\{(-xy, x^2, 0), (-y^2, xy, 0), (-y^2, 0, x^2), (0, -y^2, xy)\}$. Observe that

$$(-xy, x^2, 0) = x(-y, x, 0), (-y^2, xy, 0) = y(-y, x, 0), \text{ and } (0, -y^2, xy) = y(0, -y, x)$$

which are all boosted forms of our original basis from the degree 1 case. Our remaining vector $(-y^2, 0, x^2) = y(-y, x, 0) + x(0, -y, x)$ which is a combination of generators (as an R-module) of degree 1. This means that degree 2 offered no new information.

So we can represent φ_2 with the following matrix:

$$\begin{bmatrix} -y & x & 0 \\ 0 & -y & x \end{bmatrix}$$

Observe that:

$$\begin{aligned} \varphi_2(1,0) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -y & x & 0 \\ 0 & -y & x \end{bmatrix} \\ &= \begin{bmatrix} -y & x & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \varphi_2(0,1) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -y & x & 0 \\ 0 & -y & x \end{bmatrix} \\ &= \begin{bmatrix} 0 & -y & x \end{bmatrix} \end{aligned}$$

So the image of φ_2 is generated by $([-y, x, 0], [0, -y, x])$. Notice that these vectors are in the span of $\ker \varphi_1 = \{(-y, x, 0), (0, -y, x)\}$.

Note that the degree of x in our twisted ring $R(-2)$ is 3. So we need 2 copies of our ring graded by -3. Thus far we see:

$$R^2(-3) \xrightarrow{\varphi_2} R^3(-2) \xrightarrow{\varphi_1} R \xrightarrow{\varphi_0} R/I$$

By the Hilbert Syzygy Theorem we know we are done at this step. From this graded free resolution we can construct our Betti Table:

total:	1	3	2
0:	1	.	.
1:	.	3	2

Which produces the alternating sum $1-3t^2+2t^3$ (which is also our Poincaré series $P(-1,t)$). So our corresponding Hilbert series is given by

$$\frac{1-3t^2+2t^3}{(1-t)^2} = 1+2t.$$

Notice that the entries in our Betti table corresponds to the number of copies of our ring that we have at each step. And their placement corresponds to the grading.

In our next example, we illustrate with a picture. We can, in special cases, visualize these generators of our sub-modules in our free resolution.

Example 3.0.2. Suppose we let $R = \mathbb{R}[x,y,z]$ and let $I = (f_1, f_2, f_3, f_4)$ with $d_1 = 2, d_2 = 3, d_3 = 3, d_4 = 4$. We want to show that we can find specific f_1, f_2, f_3, f_4 such that, eventually as we move to higher degrees, everything is incorporated into the quotient ring. For example, if we let $f_1 = x^2, f_2 = y^3, f_3 = z^3, f_4 = y^2z^2$ then the following pictures illustrate that incorporation into the quotient ring as the degrees increase. The monomials enclosed in the following triangles are identically zero, so the remaining monomials give a k-vector space basis for the forms at each homogeneous degree.

1

Figure 3.1: Degree 0

x

y z

Figure 3.2: Degree 1

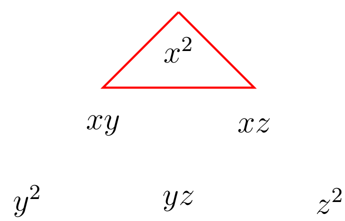

Figure 3.3: Degree 2

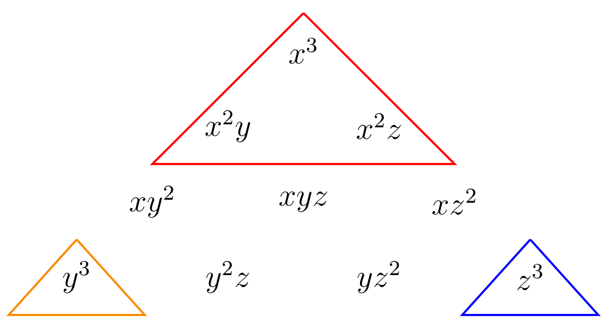

Figure 3.4: Degree 3

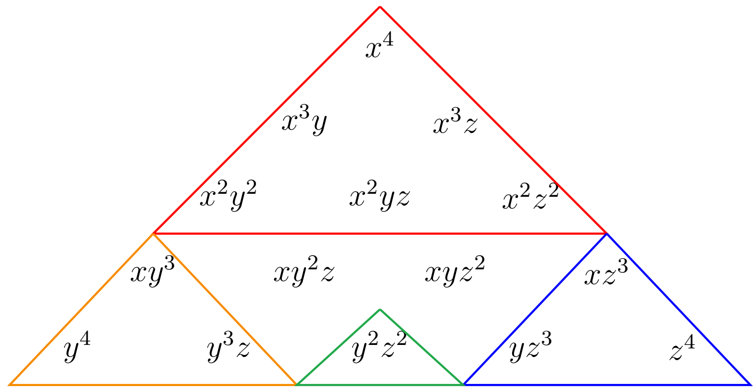

Figure 3.5: Degree 4

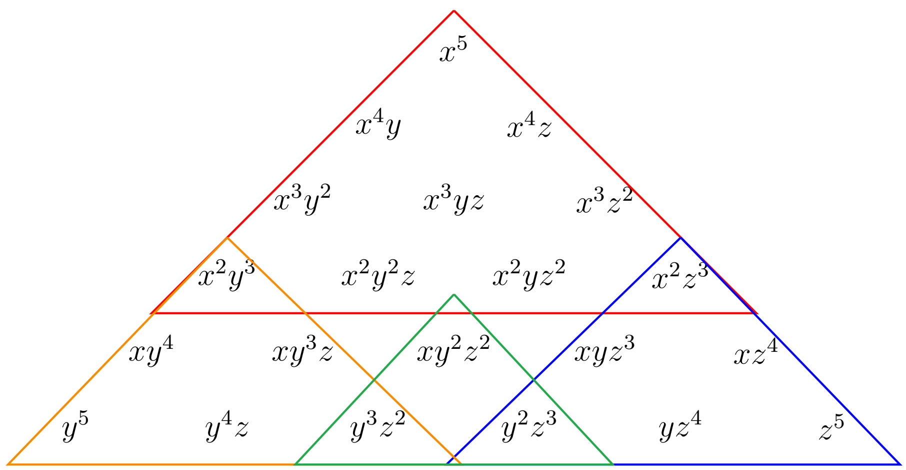

Figure 3.6: Degree 5

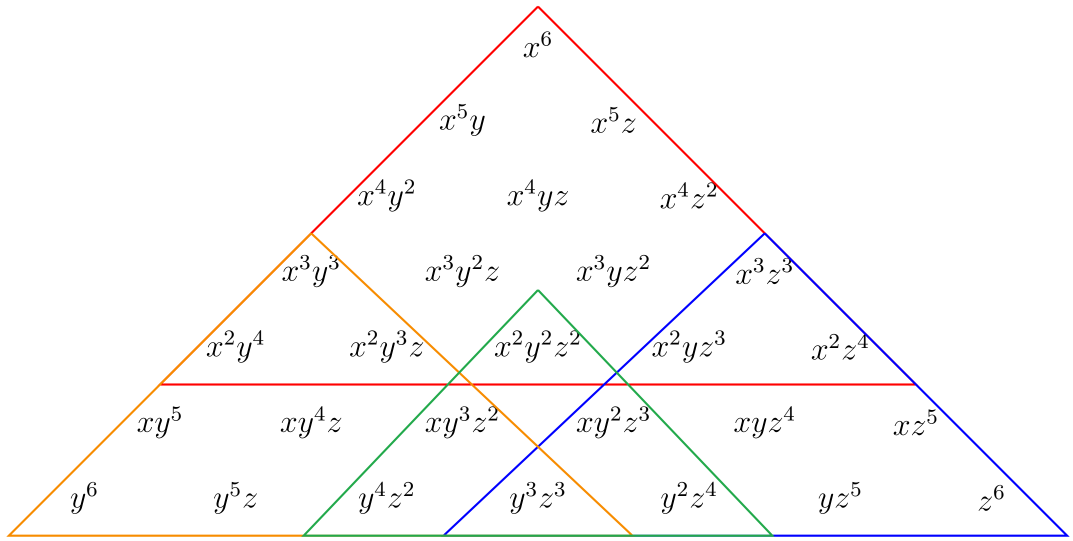

Figure 3.7: Degree 6

Notice that some monomials in the ideal are divisible by a unique minimal generator; for example, x^6 is divisible by x^2 while other elements are divisible by two minimal generators. For example y^3z^3 is divisible by y^3 and z^3. We can see the number of generators that divide an element in any given degree based on how many triangles it lies in.

A graded free resolution corresponding to this ideal is:

$$\begin{array}{ccccccccccc} R^2(-7) & \xrightarrow{C} & R^4(-5) & \xrightarrow{B} & R(-2) & \xrightarrow{A} & R & \longrightarrow & R/I & \longrightarrow & 0 \\ & & \oplus & & \oplus & & & & & & \\ & & R(-6) & & R^2(-3) & & & & & & \\ & & & & \oplus & & & & & & \\ & & & & R(-4) & & & & & & \end{array}$$

With maps

$$B = \begin{bmatrix} y^3 & -x^2 & 0 & 0 \\ 0 & z^2 & 0 & -y \\ z^3 & 0 & -x^2 & 0 \\ 0 & 0 & y^2 & -z \\ y^2z^2 & 0 & 0 & -x^2 \end{bmatrix} \qquad A = \begin{bmatrix} x^2 \\ y^3 \\ z^3 \\ y^2z^2 \end{bmatrix}$$

$$C = \begin{bmatrix} -z^2 & x^2 & 0 & 0 & y \\ 0 & 0 & -y^2 & -x^2 & z \end{bmatrix}$$

Notice that $\operatorname{im} B = \ker A$. This is the desired result. A consists of the generators of I since they are exactly the generators of the kernel of the map from $R \to R/I$.

Observe that A maps

$$R(-2) \oplus R^2(-3) \oplus R(-4) \to R$$

and the grading of $-2, -3, -3, -4$ comes from the degrees of the generators of our ideal. Then B maps

$$R^4(-5) \oplus R(-6) \to R(-2) \oplus R^2(-3) \oplus R(-4).$$

The grading here matches to the first time our triangles above overlap. Orange and Red overlap for the first time at x^2y^3 which is degree 5. A similar process is repeated to find all the terms that are the first to be in two triangles: x^2y^3, x^2z^3, y^3z^2, y^2z^3, $x^2y^2z^2$. This visual representation of our grading helps us develop an intuition about how our free resolutions are constructed.

For C we use Macaulay 2 to compute the kernel of B and place the generators in the rows. As we saw in the previous example, this can be computed by solving for the coefficient of each monomial in each degree. We know that this will terminate since we are in a Noetherian ring and thus ideals are finitely generated. This will be the final step in our free resolution by Hilbert's syzygy theorem because $\{(-z^2, x^2, 0, 0, y), (0, 0, -y^2 - x^2, z)\}$ are independent as elements of the R-module $R^4(-5) \oplus R(-6)$ and thus kernel is trivial.

Our free resolution leads to the following Betti Table:

total:	1	4	5	2
0:	1	.	.	.
1:	.	1	.	.
2:	.	2	.	.
3:	.	1	4	.
4:	.	.	1	2

This produces the alternating sum:

$$1 - t^2 - 2t^3 - t^4 + 4t^5 + t^6 - 2t^7$$

To see that our sequence is semi-regular observe that if we kill the generators one by one Hilbert function dies off at the appropriate pace. The first line represents the number of forms in 3 variables and each line beneath it represents to forms generated by each term in our ideal. We subtract all forms generated by our generators and if our end results are killed off at the appropriate rate, i.e. maximal until they have all died off. You can see this by looking back at the triangles.

$$\begin{aligned} 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + \dots \\ -t^2 - 3t^3 - 6t^4 - 10t^5 - \dots \\ -2t^3 - 6t^4 - 12t^5 - \dots \\ -t^4 - 3t^5 - \dots \end{aligned}$$

Which gives us $1 + 3t + 5t^2 + 5t^3 + 2t^4$.

Our Hilbert series for R/I with $I = (x^2, y^3, z^3, y^2z^2)$ is given by

$$H_{R/I}(t) = \left| \frac{(1-t^2)(1-t^3)^2(1-t^4)}{(1-t)^3} \right| = 1 + 3t + 5t^2 + 5t^3 + 2t^4$$

This matches our above result. Therefore our sequence is semi-regular.

Chapter 4

CONCLUSION

Fröberg's Conjecture is a relatively new idea in terms of the total scope of mathematics. This problem has been of interest in the field of commutative algebra for the past few decades. While great steps have been taken in certain cases, proving the conjecture in true generality has alluded mathematicians. By re-framing the problem in terms of semi-regular sequences and their generators has helped gain a new perspective on the problem.

A challenge in approaching Fröberg's conjecture is the depth of knowledge needed to even understand the hypotheses. One first must have an incredibly deep understanding of algebra in order to make headway on a problem like this. This conjecture is relevant because it is conjectured that most most sequences of polynomials are semi-regular [4]. So while ideals generated by semi-regular sequences are common, it can be difficult to prove this conjecture. Understanding the properties of semi-regular sequences may help provide an avenue of attack to the problem in general.

As stated previously, there are several cases where Fröberg's conjecture has been proven true: $r \leq n$; $n = 2$; $n = 3$; $r = n + 1$ with char $k = 0$; $d_1, \cdots = d_r = 3$ and $n \leq 8$ [5].

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