AN INVESTIGATION INTO CROUZEIX’S CONJECTURE

A Thesis
presented to
the Faculty of California Polytechnic State University,
San Luis Obispo

In Partial Fulfillment
of the Requirements for the Degree
Master of Science in Mathematics

by
Timothy Royston
June 2022
COMMITTEE MEMBERSHIP

TITLE: An Investigation Into Crouzeix’s Conjecture

AUTHOR: Timothy Royston

DATE SUBMITTED: June 2022

COMMITTEE CHAIR: Linda Patton, Ph.D.
Professor of Mathematics

COMMITTEE MEMBER: Erin Pearse, Ph.D.
Professor of Mathematics

COMMITTEE MEMBER: Caixing Gu, Ph.D.
Professor of Mathematics
We will explore Crouzeix’s Conjecture, an upper bound on the norm of a matrix after the application of a polynomial involving the numerical range. More formally, Crouzeix’s Conjecture states that for any $n \times n$ matrix $A$ and any polynomial $p$ from $\mathbb{C} \rightarrow \mathbb{C}$,

$$\|p(A)\| \leq 2 \sup_{z \in W(A)} |p(z)|.$$ 

Where $W(A)$ is a set in $\mathbb{C}$ related to $A$, and $\|\cdot\|$ is the matrix norm. We first discuss the conjecture, and prove the simple case when the matrix is normal. We then explore a proof for a class of matrices given by Daeshik Choi. We expand upon the proof where details are left out in the original. We also find and fix a small flaw in one section of the original paper.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of Figures</td>
<td>vi</td>
</tr>
<tr>
<td><strong>CHAPTER</strong></td>
<td></td>
</tr>
<tr>
<td>1 Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Definitions</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Crouzeix's Conjecture</td>
<td>2</td>
</tr>
<tr>
<td>1.3 Numerical Range and Matrix Norm Properties</td>
<td>3</td>
</tr>
<tr>
<td>1.4 Conformal Maps and Similarity</td>
<td>5</td>
</tr>
<tr>
<td>1.5 Relevant Facts and Inequalities</td>
<td>6</td>
</tr>
<tr>
<td>2 Choi's Proof</td>
<td>8</td>
</tr>
<tr>
<td>2.1 Simplification</td>
<td>9</td>
</tr>
<tr>
<td>2.2 Equivalent Statements</td>
<td>12</td>
</tr>
<tr>
<td>2.3 Proving Crouzeix's Conjecture</td>
<td>19</td>
</tr>
<tr>
<td>2.4 The n-1 Case</td>
<td>23</td>
</tr>
<tr>
<td>References</td>
<td>37</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Conformal Mapping of $W(J_\alpha)$</td>
<td>14</td>
</tr>
<tr>
<td>2.2</td>
<td>The Map $g$</td>
<td>22</td>
</tr>
<tr>
<td>2.3</td>
<td>$f(\theta)$, a Counterexample</td>
<td>29</td>
</tr>
</tbody>
</table>
1 Introduction

1.1 Definitions

Matrix Norm: Let $A$ be an $n \times n$ square matrix. The matrix norm of $A$ is defined to be

$$\|A\| := \sup_{v \in \mathbb{C}^n, |v| = 1} |Av|$$

where $| \cdot |$ is the usual Euclidean norm of a vector in $\mathbb{C}^n$.

Numerical Range: The numerical range of $A$ is a set $W(A) \subset \mathbb{C}$ defined as

$$W(A) := \{ (Av, v) : v \in \mathbb{C}^n, |v| = 1 \}.$$

Numerical Radius: The numerical radius of $A$ is defined as

$$\omega(A) = \sup_{z \in W(A)} |z|.$$

Spectrum: The spectrum of $A$, denoted $\sigma(A)$, is the set of all eigenvalues of $A$.

Spectral Radius: The spectral radius of $A$ is defined as

$$\rho(A) = \sup_{\lambda \in \sigma(A)} |\lambda|$$

Normal Matrix: A $n \times n$ matrix $A$ is unitary if $AA^T = A^T A$.

Unitary Matrix: A $n \times n$ matrix $U$ is unitary if $U^* = U^{-1}$ where $U^*$ is the conjugate transpose of $U$.

Unitary Similarity: Two $n \times n$ matrices $A$ and $B$ are unitarily similar if there exists a unitary matrix $U$ such that

$$A = UBU^*.$$

Condition Number: Given a similarity $A = CBC^{-1}$, not necessarily unitary, the condition number $\kappa$ is $\kappa = \|C\| \|C^{-1}\|$.
1.2 Crouzeix’s Conjecture

Michel Crouzeix’s conjecture, first seen in [4], states that for any matrix $A$, and any polynomial $p : \mathbb{C} \rightarrow \mathbb{C}$, we have the inequality

$$\|p(A)\| \leq 2 \sup_{z \in W(A)} |p(z)|$$  \hspace{1cm} (1)

Crouzeix’s conjecture first appeared in 2004 in [4], when it was proven for $2 \times 2$ matrices. In 2006, Crouzeix [5] proved that, for any matrix $A$, and polynomial $p$, then inequality (1) holds with $11.08$ in place of $2$. Crouzeix’s conjecture has been proven for matrices of the form

$$
\begin{pmatrix}
\lambda & \alpha_1 \\
\vdots & \ddots \\
\alpha_{n-1} & & \lambda
\end{pmatrix}
$$

where $\lambda, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$, and all nonspecified entries are $0$ by Choi [2]. In [6] Crouzeix offers a variety of open questions and partial results about the conjecture. And finally, in 2017, Crouzeix and César Palencia [7] prove the inequality (1) with a value of $1 + \sqrt{2}$ replacing $2$, dropping it from $11.08$ previously. In this paper we will be exploring the proof from Choi [2].

Some simpler cases of Crouzeix’s Conjecture have already been proven; namely that it holds for any normal matrix, and that it holds for any matrix $A$ where $W(A)$ is a disk. Badea et al [1] prove it whenever $W(A)$ is a disk. The proof for any normal matrix is fairly simple.

**Theorem 1.** Crouzeix’s Conjecture holds whenever $A$ is normal

**Proof.** Suppose $A$ is normal. Then $A = UDU^*$ for some unitary matrix $U$ and diagonal matrix $D$ whose entries are eigenvalues of $A$ by the spectral theorem. Using properties which are proven shortly, for any polynomial $p(x)$, we have $p(A) = Up(D)U^*$. Since $U$ and $U^*$ are unitary, $\|Up(D)U^*\| = \|p(D)\|$. Since $D$ is diagonal, say $D = \text{diag}\{d_1, d_2, \ldots, d_n\}$, we have $p(D) = \text{diag}\{p(d_1), p(d_2), \ldots, p(d_n)\}$. Thus $\|p(D)\| = \max\{ |p(d_1)|, |p(d_2)|, \ldots, |p(d_n)| \}$. By numerical range properties every eigenvalue of $A$ is in $W(A)$, so $\{p(d_1), p(d_2), \ldots, p(d_n)\} \subseteq \{p(z) : z \in W(A)\}$. Thus, $\|p(A)\| = \|p(D)\| \leq \sup_{z \in W(A)} |z|$. \hfill $\Box$

The matrices in Choi [2] are one of the next simplest matrices to work with after normal matrices and ones whose numerical range is a circle, which is why we will explore that proof in this thesis.
1.3 Numerical Range and Matrix Norm Properties

This section includes some basic proofs about numerical ranges and the matrix norm.

**Theorem 2.** If \( A \) and \( B \) are unitarily similar matrices, then \( W(A) = W(B) \)

*Proof.* Suppose \( A = U^*BU \) for some unitary matrix \( U \). We will show \( W(A) \subseteq W(B) \). Note that the subset equality \( W(B) \subseteq W(A) \) comes immediately from the same proof with \( B \) and \( A \) swapped, so we only need to show it once.

Claim 1: If \( v, w \) are complex vectors, and \( U \) is unitary, then \( \langle Uv, w \rangle = \langle v, U^*w \rangle \).

*Proof:* Note that we can write \( \langle u, v \rangle = v^*u \). Then we have that \( \langle Uv, w \rangle = w^*Uv \). Using the fact that \( (AB)^* = B^*A^* \) for any matrices \( A, B \), we get

\[
\langle Uv, w \rangle = w^*(U^*)^*v = w^*Uv = \langle v, U^*Uv \rangle = \langle v, v \rangle
\]

Completing the proof of the claim.

Claim 2: \( \|v\| = \|Uv\| \) if \( U \) is unitary.

*Proof:* \( \|v\| = \|Uv\| \) if and only if \( \|v\|^2 = \|Uv\|^2 \), which is equivalent to \( \langle v, v \rangle = \langle Uv, Uv \rangle \). And by the previous claim we have that \( \langle Uv, Uv \rangle = \langle v, U^*Uv \rangle = \langle v, v \rangle \). This completes the proof of the claim.

Now, let \( w \in W(A) \), so \( w = \langle Av, v \rangle \) for some \( v \in \mathbb{C}^n \) with \( \|v\| = 1 \). We have that

\[
\langle Av, v \rangle = \langle U^*BUv, v \rangle = \langle BUv, Uv \rangle,
\]

and \( \|Uv\| = \|v\| = 1 \). Therefore \( w \) is also an element of \( W(B) \), so \( W(A) \subseteq W(B) \), completing the proof.

**Theorem 3.** If \( A \) is any \( n \times n \) matrix, \( I_n \) is the \( n \times n \) identity matrix and \( \alpha, \delta \in \mathbb{C} \), then \( W(\alpha A + \delta I_n) \) is \( \alpha W(A) + \delta \).
Proof. Let $A$ be an $n \times n$ matrix, and let $I_n$ be the $n \times n$ identity matrix, and let $\alpha$ and $\delta$ be complex numbers. Then we have,

\[
W(\alpha A + \delta I_n) = \{(\alpha A + \delta I_n)v, v : \|v\| = 1}\}
\]
\[
= \{(\alpha Av + \delta I_nv, v) : \|v\| = 1\} \text{ by IP properties}
\]
\[
= \{\alpha \langle Av, v \rangle + \delta \langle v, v \rangle : \|v\| = 1\} \text{ by IP properties and } I_nv = v.
\]
\[
= \{\alpha \langle Av, v \rangle + \delta \|v\|^2 : \|v\| = 1\} \text{ since } \|v\|^2 = \langle v, v \rangle
\]
\[
= \{\alpha \langle Av, v \rangle + \delta : \|v\| = 1\} \text{ since } \|v\| = 1
\]
\[
= \{\alpha w + \delta : w \in W(A)\}
\]
\[
= \alpha W(A) + \delta
\]

\[\Box\]

Theorem 4. The matrix norm is sub-multiplicative. That is, if $A$ and $B$ are matrices, $\|AB\| \leq \|A\| \|B\|$

Proof. We start with the definition of matrix norm, and after some manipulation, will get the result we desire.

\[
\|AB\| = \sup_{\|v\| = 1} |(AB)v|
\]
\[
= \sup_{\|v\| = 1} |A(Bv)|
\]
\[
= \sup_{\|v\| = 1} |A(Bv)| |Bv|
\]
\[
= \sup_{\|v\| = 1} A \left( \frac{Bv}{|Bv|} \right) |Bv|
\]
\[
\leq \sup_{\|v\| = 1} \|A\| \|Bv\|
\]
\[
= \|A\| \sup_{\|v\| = 1} |Bv|
\]
\[
\leq \|A\| \|B\|
\]

Thus matrix norm is sub-multiplicative. \[\Box\]
Theorem 5. If a matrix \( M \) consists of at most one nonzero entry in each row and column, then \( \| M \| \) is the maximum entry. Further, \( |Mv| \) will attain its maximum when \( v \) is the appropriate standard unit basis vector.

Proof. Suppose \( M \) consists of at most one nonzero entry in each row and column. Then \( M = UDU^* \) is unitarily equivalent to a diagonal matrix \( D = \text{diag}\{d_1, \ldots, d_n\} \), where the unitary matrix \( U \) is a permutation matrix. Thus \( \| M \| = \| D \| \), and for a diagonal matrix, every standard unit basis vector \( e_j \) is an eigenvector with eigenvalue \( d_j \). Assume without loss of generality, that \( |d_k| = \max_{1 \leq j \leq n} |d_j| \).

Thus we have for any \( v \in \mathbb{C}^n \),

\[
|Dv| = |D(c_1e_1 + \cdots + c_ne_n)| \\
= |c_1d_1e_1 + c_2d_2e_2 + \cdots + c_nd_ne_n| \\
\leq |c_1d_ke_1 + c_2d_ke_2 + \cdots + c_nd_ne_n| \\
= |d_k(c_1e_1 + \cdots + c_ne_n)| \\
= |d_kv| \\
= |d_k||v|
\]

Therefore if \( |v| = 1 \), \( |Dv| \leq |d_k| \) and equality is attained when \( v = e_k \), and so \( |d_k| = \| D \| = \| M \| \). \( \square \)

1.4 Conformal Maps and Similarity

Theorem 6. If \( p(z) \) is a polynomial over \( \mathbb{C} \), then matrix similarity holds through the application of \( p \). That is, for any \( m \times m \) matrices \( A \) and \( B \), \( p(BAB^{-1}) = Bp(A)B^{-1} \).

Proof. Since \( p(x) \) is a polynomial, \( p(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0 \) where \( c_i \in \mathbb{C} \). Note that for any \( k \in \mathbb{Z} \),

\[
(BAB^{-1})^k = BAB^{-1}BAB^{-1} \cdots BAB^{-1} = BA^kB^{-1}
\]

Therefore, we have

\[
p(BAB^{-1}) = c_n(BAB^{-1})^n + c_{n-1}(BAB^{-1})^{n-1} + \cdots + c_1(BAB^{-1}) + c_0I \\
= c_nBA^nB^{-1} + c_{n-1}BA^{n-1}B^{-1} + \cdots + c_1BAB^{-1} + c_0BB^{-1} \\
= B(c_nA^n + C_{n-1}A^{n-1} + \cdots + c_1A + c_0)B^{-1} \\
= Bp(A)B^{-1}
\]
Thus matrix similarity holds through polynomial application.

Since conformal mappings in \( \mathbb{C} \) are complex analytic functions, we can say that any conformal map \( \phi(z) \) can be written as a convergent power series, \( \phi(z) = \sum_{n=0}^{\infty} c_n z^n \), and so by the above proof and a limiting argument, matrix similarity holds for conformal maps as well, \( \phi(BAB^{-1}) = B\phi(A)B^{-1} \).

### 1.5 Relevant Facts and Inequalities

The following facts and inequalities are vital in certain proofs throughout the paper.

Schwartz’ lemma, seen in Gamelin’s *Complex Analysis* \([8]\) Chapter 9.1, gives an upper bound for the magnitude of a complex function from the unit disk \( \mathbb{D} \) to the complex plane. Let \( f : \mathbb{D} \to \mathbb{C} \) be an analytic function such that \( f(0) = 0 \) and \( |f(z)| \leq 1 \) on \( \mathbb{D} \). Then \( |f(z)| \leq |z| \) for all \( z \in \mathbb{D} \).

Gelfand’s formula, seen in MacCluer’s *Elementary Functional Analysis* \([10]\) Theorem 5.15, gives the spectral radius as a limit of matrix norms. For an \( n \times n \) matrix \( A \), and any matrix norm \( \| \cdot \| \)

\[
\rho(A) = \lim_{n \to \infty} \| A^n \|^{1/n}. \tag{2}
\]

The inequality of arithmetic and geometric means, seen in Cloud et al. \([3]\), states that the arithmetic mean is always larger than the geometric mean. For \( n \) variables, it states

\[
\frac{1}{n} \sum_{k=1}^{n} x_k \geq \left( \prod_{k=1}^{n} x_k \right)^{1/n}. \tag{3}
\]

The last needed fact is that for any \( a, b > 0 \) and \( 0 \leq t \leq 1 \),

\[
a^t b^{1-t} \leq \max(a, b). \tag{4}
\]

**Proof.** We will show that \( f(t) = a^t b^{1-t} \) is monotonic on \( 0 \leq t \leq 1 \). Let’s look at the derivative,

\[
\frac{d}{dt} a^t b^{1-t} = \frac{d}{dt} e^{t \ln a} e^{(1-t) \ln b} \\
= \frac{d}{dt} e^{t (\ln a - \ln b)} + \ln b \\
= (\ln a - \ln b) e^{t (\ln a - \ln b)} + \ln b
\]
Since $e^x$ is always positive, the sign of $f'$ is the same as the sign of $\ln a - \ln b$. Thus, if $a \geq b$, then $f$ is increasing, and if $a \leq b$, then $f$ is decreasing. So we may say that on $[0, 1]$, $f(t) \leq \max(f(0), f(1))$, and $f(0) = b$, $f(1) = a$. Therefore, $a^t b^{1-t} \leq \max(a, b)$ for $0 \leq t \leq 1$. \qed
2 Choi’s Proof

In [2], Choi proves Crouzeix’s conjecture for the class of matrices

\[
\begin{pmatrix}
\lambda & \alpha_1 \\
& \ddots & \ddots \\
& & \ddots & \alpha_{n-1} \\
& & & \alpha_n & \lambda
\end{pmatrix}
\]

where \(\lambda, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}\), and all nonspecified entries are 0. Throughout the paper, we will be identifying \(\alpha_{n+j}\) with \(\alpha_j\), so \(\alpha_{n+1} = \alpha_1, \alpha_{n+2} = \alpha_2, \) etc.

The following proposition states that Crouzeix’s conjecture will follow from the right conformal map from \(W(A)\) to the unit disk.

**Proposition 1.** For any matrix \(A\), if there exists a conformal bijection \(\phi\) from \(W(A)\) to the unit disk, such that \(\phi(A)\) is similar to a contraction where the condition number of the similarity is at most 2, then Crouzeix’s conjecture holds.

**Proof.** Suppose we have a \(\phi\) as described above. Then \(\phi(A) = XCX^{-1}\) with \(\|X\|\|X^{-1}\| \leq 2\) and we have

\[
\|p(A)\| = \|(p \circ \phi^{-1})(\phi(A))\|
\]

\[
= \|(p \circ \phi^{-1})(XCX^{-1})\|
\]

\[
= \|X(p \circ \phi^{-1})(C)X^{-1}\|
\]

\[
\leq \|X\| \|p \circ \phi^{-1}(C)\| \|X^{-1}\|
\]

\[
\leq 2\|(p \circ \phi^{-1})(C)\|
\]

\[
\leq 2\sup_{z \in \mathbb{D}} |(p \circ \phi^{-1})(z)|
\]

\[
= 2\sup_{z \in W(A)} |p(z)|
\]

Where the last inequality holds from von Neumann’s Inequality, which can be found in [1] chapter 1. Thus Crouzeix’s conjecture holds.
2.1 Simplification

To simplify the problem, we will prove some properties of Crouzeix’s conjecture that will allow us
to work with a simpler matrix.

**Lemma 1.** If $A = \mu B + \lambda I$ or if $A$ is unitarily similar to $B$, then Crouzeix’s Conjecture holds for $A$ if and only if it holds for $B$.

**Proof.** Suppose $A = \mu B + \lambda I$. For any polynomial $p(z)$, define $q(z) = p(\mu z + \lambda)$. It follows that $q(B) = q(\mu B + \lambda I) = p(A)$, and by properties of the numerical range, $W(A) = \mu W(B) + \lambda$.

Thus we have

$$\|q(B)\| \leq 2 \sup_{z \in W(B)} |q(z)|$$

if and only if

$$\|q(B)\| \leq 2 \sup_{z \in W(B)} |p(\mu z + \lambda)|$$

if and only if

$$\|q(B)\| \leq 2 \sup_{z \in W(A)} |p(z)|$$

Since $q(B) = p(A)$, we have that Crouzeix’s conjecture holds for $B$ if and only if it holds for $A$.

Since $A$ and $B$ are unitarily similar $A = UBU^*$ for some unitary $U$. Thus we have $W(A) = W(B)$, and $\|p(A)\| = \|p(UBU^*)\| = \|Up(B)U^*\| = \|B\|$. Therefore, we have that

$$\|p(A)\| \leq 2 \sup_{z \in W(A)} |p(z)|$$

and

$$\|p(B)\| \leq 2 \sup_{z \in W(B)} |p(z)|$$

are equivalent. \[\square\]

Now, we can take our matrix, and split it to make it easier to work with by noting

$$\begin{pmatrix} \lambda & \alpha_1 \\ \vdots & \ddots \\ \alpha_{n-1} & \alpha_n \end{pmatrix} = J_\alpha + \lambda I,$$
where

\[ J_\alpha = \begin{pmatrix}
0 & \alpha_1 \\
& \ddots & \ddots \\
& & \ddots & \alpha_{n-1} \\
\alpha_n & & & 0
\end{pmatrix}. \]

And now by Lemma 1 we can prove Crouzeix’s conjecture for \( J_\alpha \) and it will follow for our given matrix.
We can further simplify so that we only need to work with positive real numbers as our $\alpha_j$.

Since $\alpha_j \in \mathbb{C}$, we can write $\alpha_j = r_je^{i\theta_j}$ where $r_j \geq 0$ and $\theta_j \in \mathbb{R}$. If we let $\mu = e^{i(\theta_1+\theta_2+\cdots+\theta_n)/n}$ and $D = \text{diag}(z_1, z_2, \ldots, z_n)$, where $z_j = \mu^{-1}e^{-i(\theta_1+\theta_2+\cdots+\theta_j)}$, we get $J_\alpha = \mu DJ_rD^{-1}$ where $r = (r_1, r_2, \ldots, r_n)$.

$$\mu DJ_rD^{-1} = \mu \begin{pmatrix}
  \frac{1}{z_1} & 0 & \cdots & 0 \\
  0 & \frac{1}{z_2} & \cdots & 0 \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & \cdots & \cdots & \frac{1}{z_n}
\end{pmatrix} = \mu \begin{pmatrix}
  \frac{z_1 r_1}{z_1} & \frac{z_2 r_1}{z_2} & \cdots & \frac{z_n r_1}{z_n} \\
  \frac{z_1 r_2}{z_1} & \frac{z_2 r_2}{z_2} & \cdots & \frac{z_n r_2}{z_n} \\
  \vdots & \ddots & \ddots & \vdots \\
  \frac{z_1 r_n}{z_1} & \frac{z_2 r_n}{z_2} & \cdots & \frac{z_n r_n}{z_n}
\end{pmatrix} = \mu\begin{pmatrix}
  \frac{z_1 r_1}{z_1} & \frac{z_2 r_1}{z_2} & \cdots & \frac{z_n r_1}{z_n} \\
  \frac{z_1 r_2}{z_1} & \frac{z_2 r_2}{z_2} & \cdots & \frac{z_n r_2}{z_n} \\
  \vdots & \ddots & \ddots & \vdots \\
  \frac{z_1 r_n}{z_1} & \frac{z_2 r_n}{z_2} & \cdots & \frac{z_n r_n}{z_n}
\end{pmatrix}
$$

So we want to show that $\mu \frac{z_j r_j}{z_{j+1}} = \alpha_j$ for all $j = 1, \ldots, n$. Recall we identify $z_{n+1}$ with $z_1$.

$$\frac{\mu z_j r_j}{z_{j+1}} = \mu \mu^{-1} e^{-i(\theta_1+\theta_2+\cdots+\theta_j)} r_j e^{-i(\theta_1+\theta_2+\cdots+\theta_j-1)} = r_j e^{i\theta_j}$$

and so we have that $J_\alpha = \mu DJ_rD^{-1}$. The matrix $D$ is unitary, so by Lemma 1 if we prove Crouzeix’s conjecture for $J_r$ it will hold for $J_\alpha$ and thus for our original matrix.

We can go slightly further, and say that all $\alpha_j$ must be nonzero.

**Lemma 2.** If $\alpha_j = 0$ for some $j$, then $W(J_\alpha)$ is a disk.

**Proof.** For any $q = (q_1, q_2, \ldots, q_n) \in \mathbb{C}^n$ with $|q| = 1$, we can see that

$$\langle J_\alpha q, q \rangle = \langle (\alpha_1 q_2, \alpha_2 q_3, \ldots, \alpha_{n-1} q_n, \alpha_n q_1), (q_1, q_2, \ldots, q_n) \rangle = \alpha_1 q_2 \overline{q_1} + \alpha_2 q_3 \overline{q_2} + \cdots + \alpha_{n-1} q_n \overline{q_{n-1}} + \alpha_n q_1 \overline{q_n} = \sum_{j=1}^{n} \alpha_j q_{j+1} \overline{q_j}$$

Here we use the convention that $q_{n+1} = q_n$, so $q_{n+1} = q_1$. With this, we see that if we have $\beta$, a cyclic permutation of $\alpha$, then $\langle J_\beta v, v \rangle = \langle J_\alpha q, q \rangle$ where $v$ is the same cyclic permutation from $\alpha$ to
\( \beta \), applied to \( q \), and so we get that \( W(J_\alpha) = W(J_\beta) \) for any \( \beta \), a cyclic permutation of \( \alpha \). Therefore, we may assume without loss of generality that \( \alpha_n = 0 \).

Now, let \( \theta \in \mathbb{R} \), and \( q \in \mathbb{C}^n \). Let \( D_\theta = \text{diag}(1,e^{i\theta},e^{2i\theta},\ldots,e^{(n-1)i\theta}) \). Now let's consider \( \langle J_\alpha (D_\theta q), D_\theta q \rangle \).

\[
\langle J_\alpha (D_\theta q), D_\theta q \rangle = \langle (\alpha_1 e^{i\theta} q_2, \alpha_2 e^{2i\theta} q_3, \ldots, \alpha_{n-1} e^{(n-1)i\theta} q_n, 0), (q_1, e^{i\theta} q_2, \ldots, e^{(n-1)i\theta} q_n) \rangle \\
= \alpha_1 e^{i\theta} q_2 q_1 + \alpha_2 e^{2i\theta} q_3 q_2 + \cdots + \alpha_{n-1} e^{(n-1)i\theta} q_n q_{n-1} + 0 \\
= e^{i\theta} \sum_{j=1}^{n} \alpha_j q_{j+1} q_j \\
= e^{i\theta} \langle J_\alpha q, q \rangle 
\]

Thus, we have for any point \( c \) in \( W(J_\alpha) \), the entire circle with radius \(|c|\) is in \( W(J_\alpha) \). Let \( c \) be a point that attains the magnitude equal to the numerical radius \( r(J_\alpha) \). Since \( W(J_\alpha) \) is convex, it follows that \( W(J_\alpha) \) is a circular disk with radius \( r(J_\alpha) \).

Therefore, since Crouzeix’s conjecture has already been solved for any matrix with a circular numerical range, we can assume that \( \alpha_j > 0 \) for all \( j \).

To reiterate this section. We have shown that in order to solve Crouzeix’s conjecture for our original class of matrices, it suffices to show that it holds for the matrices \( J_\alpha \) where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) with \( \alpha_j > 0 \) for all \( 1 \leq j \leq n \).

### 2.2 Equivalent Statements

In this section we will derive some statements equivalent to Crouzeix’s conjecture for \( J_\alpha \). First, we will look at some properties of \( J_\alpha \).

**Lemma 3.** The following are properties of \( J_\alpha \)

1. \( \|J_\alpha^k\| = \max \{ \alpha_j \alpha_{j+1} \cdots \alpha_{j+k-1} : 1 \leq j \leq n \} \) for any \( k = 1, \ldots, n \). That is, \( \|J_\alpha^k\| \) is the largest product of \( k \) consecutive \( \alpha_j \).
2. \( \sigma(J_\alpha) = \{ (\alpha_1 \cdots \alpha_n)^{1/n} e^{2\pi ik/n} : k = 1, \ldots, n \} \)
3. \( W(J_\alpha) \) contains the origin.
Thus whenever

\[ \text{Let} \]

must be the characteristic polynomial for \( x \) of \( 1 \leq j \leq \text{that each row and column has exactly one nonzero element. Thus}, \]

\[ ||J^k_\alpha|| = \max \{ \alpha_j \alpha_{j+1} \ldots \alpha_{j+k-1} : 1 \leq j \leq n \}. \]

b) We can see that \( J^\alpha_n = \alpha_1 \alpha_2 \ldots \alpha_n I \), where \( I \) is the \( n \times n \) identity matrix. And so \( J_\alpha \) is a zero of \( x^n - \alpha_1 \alpha_2 \ldots \alpha_n \). Since \( n \) is the first integer where \( J^\alpha_n \) is a multiple of the identity matrix, this must be the characteristic polynomial for \( J_\alpha \), and so \( \sigma(J_\alpha) = \{(\alpha_1 \ldots \alpha_n)^{1/n} e^{2\pi jk/n} : k = 1, \ldots, n\} \).

c) Simply note that \( \langle J_\alpha e_1, e_1 \rangle = 0 \) where \( e_1 \) is the first standard unit vector.

d) Let \( \omega_k = e^{2\pi ki/n} \) and \( D_k \) be the diagonal matrix \( \text{diag}(1, \omega_k, \omega_k^2, \ldots, \omega_k^{n-1}) \) for \( k \in \{1, \ldots, n\} \).

Let \( v = (v_2, \ldots, v_n) \) be a unit vector. Note that \( D_k \) is unitary; thus \( D_k v \) is also a unit vector. Then \( \langle J_\alpha v, v \rangle \in W(J_\alpha) \) and \( \langle J_\alpha D_k v, D_k v \rangle \in W(J_\alpha) \). We will show that \( \langle J_\alpha D_k v, D_k v \rangle = \omega_k \langle J_\alpha v, v \rangle \).

Thus whenever \( z \in J_\alpha \), we will have \( e^{2\pi ki/n} z \in J_\alpha \) and so \( W(J_\alpha) \) consists of \( n \) identical regions.

\[
\langle J_\alpha D_k v, D_k v \rangle = \langle J_\alpha(v_1, \omega_k v_2, \ldots, \omega_k^{n-1} v_n), (v_1, \omega_k v_2, \ldots, \omega_k^{n-1} v_n) \rangle \\
= \langle (\alpha_1 \omega_k v_2, \alpha_2 \omega_k^2 v_3, \ldots, \alpha_{n-1} \omega_k^{n-1} v_n, \alpha_n v_1), (v_1, \omega_k v_2, \ldots, \omega_k^{n-1} v_n) \rangle \\
= \alpha_1 \omega_k v_2 v_1 + \alpha_2 \omega_k^2 v_3 v_2 + \cdots + \alpha_{n-1} \omega_k^{n-1} v_n v_1 + \alpha_n v_1 v_1 \\
= \omega_k \langle J_\alpha v, v \rangle.
\]

Thus, \( W(J_\alpha) \) consists of \( n \) identical regions. To show that each region is bisected into two identical pieces, we will show that \( W(J_\alpha) \) has conjugate symmetry, splitting the segment with argument from \(-\pi/n\) to \(\pi/n\) into two, and by symmetry splitting all segments into two identical pieces. Suppose \( z = \langle J_\alpha v, v \rangle \) for some unit vector \( v \). Note that all entries of \( J_\alpha \) are real.

\[
\overline{z} = \langle J_\alpha v, v \rangle \\
= \langle J_\alpha v, \overline{v} \rangle \\
= \langle J_\alpha \overline{v}, v \rangle
\]

Since \( v \) is a unit vector, \( \overline{v} \) is a unit vector, so \( \overline{z} \in W(J_\alpha) \). Thus, \( W(J_\alpha) \) consists of \( n \) identical regions, with each region consisting of two symmetric pieces.
Due to the symmetry we have from Lemma 2 d), when we conformally map $W(J_\alpha)$ to the unit disk, we can first focus on the region with angle ranging from 0 to $2\pi/n$, which we will call $Y$. We will map this region to the sector of the unit disc with the same angles, which we will call $Z$. To do this, we will use Theorem 5.10g from [9]. Which states that we can conformally map $Y$ to $Z$ by choosing 3 points on the boundary of $Y$ and mapping those to 3 points on the boundary of $Z$. If we take the points from $Y$ as the origin, and the endpoints of each line segment leading away from the origin, and the points of $Z$ similarly as the origin and the end of each line segment; we get that the region of $W(J_\alpha)$ gets mapped to the sector of the unit disk, with the line segments from the numerical range getting mapped to the same line segments on the unit disk. This map is conformal on the interiors, and extends continuously to the boundary, which is exactly the conditions necessary to apply the Schwarz Reflection Principle shortly. As an example, suppose the blue curve in (a) was the boundary of this region of the numerical range and the gray section was the unit disk. The pairs of points we map to each other are $(A, A)$, $(E, C)$, and $(D, B)$. The resulting map would send the line segment $AE$ to $AC$ and the segment $AD$ to $AB$.

![Figure 2.1: Conformal Mapping of $W(J_\alpha)$](image)

Now, we can use the Schwarz Reflection Principle. Since we mapped lines to themselves, we can analytically continue the map across them. We are extending the domain by the reflection over the line, and the corresponding range will be the reflection of the original range. Above, (b) is the map after extending via reflection over the real axis. The blue outline is the domain, and the gray area is the range.

Due to the symmetry of $W(J_\alpha)$, when the original region is reflected repeatedly, the domain will be exactly $W(J_\alpha)$, and the range will be exactly the unit disk. We will call this map from $W(J_\alpha)$ to
the unit disk $\phi$. We have that $\phi(0) = 0$, and $\phi$ maps the real subset of $W(J_\alpha)$ onto the real subset of the unit disk.

**Proposition 2.** $\phi(J_\alpha) = cJ_\alpha$ for some unknown $c \in \mathbb{R}$

**Proof.** When we map $W(J_\alpha)$ to the unit disk via $\phi$, the real eigenvalue of $J_\alpha$ will be sent to some real number in $(0, 1)$. So we may say that it was multiplied by some $c \in \mathbb{R}$. Since all eigenvalues lie on the same lines that were used during our construction of $\phi$, all eigenvalues of $J_\alpha$ will be multiplied by the same constant $c$. Note that $J_\alpha$ has $n$ distinct eigenvalues, so it can be diagonalized, for some matrix $P$,

$$J_\alpha = P \begin{pmatrix} \lambda_1 & 0 \\ \vdots & \ddots \\ 0 & \lambda_{n-1} & \lambda_n \end{pmatrix} P^{-1}$$

And so, since $\phi$ is a power series, we have that

$$\phi(J_\alpha) = P \phi(\text{diag}(\lambda_1 \ldots \lambda_n)) P^{-1} = P \begin{pmatrix} \phi(\lambda_1) & 0 \\ \vdots & \ddots \\ 0 & \phi(\lambda_{n-1}) & \phi(\lambda_n) \end{pmatrix} P^{-1} = P \begin{pmatrix} c\lambda_1 & 0 \\ \vdots & \ddots \\ 0 & c\lambda_{n-1} & c\lambda_n \end{pmatrix} P^{-1} = cP \begin{pmatrix} \lambda_1 & 0 \\ \vdots & \ddots \\ 0 & \lambda_{n-1} & \lambda_n \end{pmatrix} P^{-1} = cJ_\alpha$$

Therefore, $\phi(J_\alpha) = cJ_\alpha$ for some unknown $c$. 

\[\square\]
Now, if we were to assume Crouzeix’s conjecture holds for \( \phi_k(z) = \phi(z)^k \) where \( \phi(z) \) is our bijective conformal map from \( W(A) \) to the unit disk, we would get a number of inequalities \( \| \phi(J_\alpha)^k \| \leq 2 \sup_{z \in W(J_\alpha)} |\phi(z)^k| \). Since \( \phi \) maps \( W(J_\alpha) \) to the unit disk, \( |\phi(z)^k| \leq 1 \). Thus we would have

\[
\| \phi(J_\alpha)^k \| \leq 2 \sup_{z \in W(J_\alpha)} |\phi(z)^k| \leq 2 \tag{5}
\]

We don’t know if these inequalities are true yet, but we can actually use them to prove Crouzeix’s conjecture for \( J_\alpha \).

**Theorem 7.** Crouzeix’s Conjecture holds for \( J_\alpha \) if \( \text{(5)} \) holds.

**Proof.** Assume \( \text{(5)} \). We will show there exists a matrix \( X \) such that \( \kappa(X) \leq 2 \) and \( \| X \phi(J_\alpha)X^{-1} \| \leq 1 \). From which Crouzeix’s conjecture will follow from Prop \( \text{[1]} \). We will use the constant \( c \) with \( \phi(J_\alpha) = cJ_\alpha \) from Prop \( \text{[2]} \) We will define the matrix \( X \) by \( X = \text{diag}(x_1, x_2, \ldots, x_n) \) where

\[
x_1 = \max\{1, c\alpha_n, c^2\alpha_n\alpha_{n-1}, \ldots, c^{n-1}\alpha_n\alpha_{n-1} \ldots \alpha_2\}
\]

\[
x_2 = \max\{1, c\alpha_1, c^2\alpha_1\alpha_n, \ldots, c^{n-1}\alpha_1\alpha_n \ldots \alpha_3\}
\]

\[
\vdots
\]

\[
x_j = \max\{1, c\alpha_{j-1}, c^2\alpha_{j-1}\alpha_{j-2}, \ldots, c^{n-1}\alpha_{j-1}\alpha_{j-2} \ldots \alpha_{j+1}\}
\]

\[
\vdots
\]

\[
x_n = \max\{1, c\alpha_{n-1}, c^2\alpha_{n-1}\alpha_{n-2}, \ldots, c^{n-1}\alpha_{n-1}\alpha_{n-2} \ldots \alpha_1\}
\]

We have that \( \| \phi(J_\alpha)^k \| = \| c^k J_\alpha^k \| = c^k \| J_\alpha^k \| = c^k \max\{c^k \alpha_j \alpha_{j+1} \ldots \alpha_{j+k-1} : 1 \leq j \leq n\} \) by Lemma \( \text{[3]} \) For any \( j \) we see that the entry consisting of a product of \( k \) alphas, that is \( c^k \alpha_{j-1} \alpha_{j-2} \ldots \alpha_{j-k} \), is less than \( c^k \| J_\alpha^k \| = \| \phi(J_\alpha)^k \| \) by Lemma \( \text{[3]} \) which is less than 2 by \( \text{(5)} \). Since each \( x_j \) is the max of some set whose entries are all less than 2, we get that \( x_j \leq 2 \). Thus, since \( 1 \leq x_j \leq 2 \), it follows that \( 1 \leq \| X \| \leq 2 \), and \( \frac{1}{2} \leq \| X^{-1} \| \leq 1 \). Therefore \( \kappa(X) = \| X \| \| X^{-1} \| \leq 1 \cdot 2 = 2 \). Now we see
that

\[ X \phi(J_\alpha) X^{-1} = XcJ_\alpha X^{-1} \]

\[
= \begin{pmatrix}
  x_1 & 0 & & & \\
  & \ddots & & & \\
  & & x_{n-1} & & \\
  & & & x_n & \\
 0 & & & & 1
\end{pmatrix}
\begin{pmatrix}
  0 & c\alpha_1 & & & \\
  & \ddots & \ddots & & \\
  & & \ddots & c\alpha_{n-1} & \\
  & & & 0 & \\
 0 & & & & 1
\end{pmatrix}
\begin{pmatrix}
  x_1^{-1} & & & & \\
  & \ddots & & & \\
  & & x_{n-1}^{-1} & & \\
  & & & x_n^{-1} & \\
 0 & & & & 1
\end{pmatrix}
\]

And so we get that \( X \phi(J_\alpha) X^{-1} = J_\beta \) where \( \beta = (\frac{c\alpha_1 x_1}{x_2}, \frac{c\alpha_2 x_2}{x_3}, \ldots, \frac{c\alpha_n x_n}{x_1}) \). Therefore, we can show that \( cx_j \alpha_j \leq x_{j+1} \) for each \( j \) to get that \( \|X \phi(J_\alpha) X^{-1}\| \leq 1 \).

We know, by Gelfand’s Formula, that the spectral radius of \( \phi(J_\alpha) \) is

\[ \rho(\phi(J_\alpha)) = \lim_{k \to \infty} \|\phi(J_\alpha)^k\|^{1/k} \leq \lim_{k \to \infty} 2^{1/k} = 1 \]

so \( \rho(\phi(J_\alpha)) \leq 1 \), and we know that \( \phi(J_\alpha) = cJ_\alpha \), so \( \rho(\phi(J_\alpha)) = c \max |\sigma(J_\alpha)| = c(\alpha_1 \ldots \alpha_n)^{1/n} \) by Lemma 3. So we have that \( c^n \alpha_1 \alpha_2 \ldots \alpha_n \leq 1 \). Now, lets look at \( c\alpha_j x_j \) and \( x_{j+1} \):

\[ c\alpha_j x_j = \max\{c\alpha_j, c^2 \alpha_j \alpha_{j-1}, \ldots, c^{n-1} \alpha_j \alpha_{j-1} \ldots \alpha_{j+2}, c^n \alpha_j \alpha_{j-1} \ldots \alpha_{j+1}\} \]

\[ x_{j+1} = \max\{1, c\alpha_j, c^2 \alpha_j \alpha_{j-1}, \ldots, c^{n-1} \alpha_j \alpha_{j-1} \ldots \alpha_{j+2}\} \]

Note that their elements match except for 1 and \( c^n \alpha_1 \alpha_2 \ldots \alpha_n \), so since \( c^n \alpha_1 \alpha_2 \ldots \alpha_n \leq 1 \) we have that \( c\alpha_j x_j \leq x_{j+1} \) which completes the proof.

Now, we can restate (5) inequality in terms of our constant \( c \).
Lemma 4. \[c \cdot \max \left\{ \frac{\| J_\alpha \|}{2}, \left( \frac{\| J_\alpha^2 \|}{2} \right)^{1/2}, \ldots, \left( \frac{\| J_\alpha^{n-1} \|}{2} \right)^{1/(n-1)}, \| J_\alpha^n \|^{1/n} \right\} \leq 1 \] (6)

Proof. First, we will show that
\[
\sup_{k \in \mathbb{N}} \left( \frac{\| J_k \|}{2} \right)^{1/k} = \max \left\{ \frac{\| J_\alpha \|}{2}, \left( \frac{J_\alpha^2}{2} \right)^{1/2}, \ldots, \left( \frac{J_\alpha^{n-1}}{2} \right)^{1/(n-1)}, \| J_\alpha^n \|^{1/n} \right\}
\]

By Lemma 3 a) and b), \( \rho(J_\alpha) = (\alpha_1 \ldots \alpha_n)^{1/n} = \| J_\alpha^n \|^{1/n} \). By Gelfand’s Formula, we have that
\[
\lim_{k \to \infty} \left( \frac{\| J_k \|}{2} \right)^{1/k} = \rho(J_\alpha) = \| J_\alpha^n \|^{1/n},
\]
so every element on the right hand side is either an element of the left hand side, or the limit of the left hand side. Since we have a supremum on the left, that means that the left hand side is greater than or equal to the right hand side.

To show that the right hand side is not less than the left hand side, we need to show that no element \( \left( \frac{\| J_k \|}{2} \right)^{1/k} \) is greater than or equal to all elements of the right hand side. If \( k \leq n - 1 \), \( \left( \frac{\| J_k \|}{2} \right)^{1/k} \) is in the right hand side, so we are done. If \( k \geq n \), then \( k = m + (n - 1) \) for some \( m \in \mathbb{N} \). In this case, we have
\[
\left( \frac{\| J_k \|}{2} \right)^{1/k} = \left( \frac{\| J_\alpha^{m+(n-1)} \|}{2} \right)^{1/(m+(n-1))} \\
\leq \left( \frac{\| J_\alpha^m \| \| J_\alpha^{n-1} \|}{2} \right)^{1/(m+(n-1))} \\
= \left( \frac{\| J_\alpha^m \|}{2} \right)^{1/(m+(n-1))} \left( \frac{\| J_\alpha^{n-1} \|}{2} \right)^{1/(m+(n-1))} \\
= \left( \frac{\| J_\alpha^m \|}{2} \right)^{m/(m+(n-1))} \left( \frac{\| J_\alpha^{n-1} \|}{2} \right)^{(n-1)/(m+(n-1))} \\
\leq \max \left\{ \left( \frac{\| J_\alpha^m \|}{2} \right)^{1/m}, \left( \frac{\| J_\alpha^{n-1} \|}{2} \right)^{1/(n-1)} \right\}
\]

Where the last inequality comes from \[1\]. If \( m < n \), we are done as both elements of the maximum are from our right hand side. If not, either \( \left( \frac{\| J_k \|}{2} \right)^{1/k} \leq \left( \frac{\| J_\alpha^{n-1} \|}{2} \right)^{1/(n-1)} \), in which
case we are done, or \( \left( \frac{\|J^n\|}{2} \right)^{1/k} \leq \left( \frac{\|J^m\|}{2} \right)^{1/m} \). In the latter case, we use an induction argument of the same process repeated using \( m \) instead of \( k \). Since \( m < k \), every step will either terminate or decrease \( m \). Since the process terminates whenever \( m \leq n - 1 \), the algorithm must terminate at some point. In fact, since \( k = q(n - 1) + r \) where \( 0 \leq r < n - 1 \), it will terminate after at most \( q \) repetitions. Thus, we have shown the equality

\[
\sup_{k \in \mathbb{N}} \left( \frac{\|J^k\|}{2} \right)^{1/k} = \max \left\{ \frac{\|J_n\|}{2}, \left( \frac{\|J^2_n\|}{2} \right)^{1/2}, \ldots, \left( \frac{\|J_{n-1} - 1\|}{2} \right)^{1/(n-1)}, \|J_n\|^{1/n} \right\}
\]

Therefore we have,

\[
1 \geq c \cdot \max \left\{ \frac{\|J_n\|}{2}, \left( \frac{\|J^2_n\|}{2} \right)^{1/2}, \ldots, \left( \frac{\|J_{n-1} - 1\|}{2} \right)^{1/(n-1)}, \|J_n\|^{1/n} \right\}
\]

\[
= c \cdot \sup_{k \in \mathbb{N}} \left( \frac{\|J^k\|}{2} \right)^{1/k}
\]

\[
= \sup_{k \in \mathbb{N}} \left( \frac{|cJ^k_n|}{2} \right)^{1/k}
\]

\[
= \sup_{k \in \mathbb{N}} \left( \frac{\|cJ_n^k\|}{2} \right)^{1/k}
\]

Thus, \( \|\phi(J_n)^k\| \leq 2 \) for all \( k \in \mathbb{N} \), proving (5).

\[
2.3 \text{ Proving Crouzeix’s Conjecture}
\]

Now, we will go about proving (6), which will prove Crouzeix’s conjecture for our desired class of matrices by the chain of conditionals from the previous section.

First, let

\[
s = \max \left\{ \frac{\|J_n\|}{2}, \left( \frac{\|J^2_n\|}{2} \right)^{1/2}, \ldots, \left( \frac{\|J_{n-1} - 1\|}{2} \right)^{1/(n-1)}, \|J_n\|^{1/n} \right\}
\]

(7)

Lemma 5. The following properties of \( s \) are true.

(a) \( \rho(J_n) \leq s \)
(b) \( s \in W(J_\alpha) \)

(c) If \( s = \left( \frac{\|J_\alpha^n\|}{2} \right)^{1/k} \) for some \( k \in \{1, \ldots, n-2\} \), then \( W(J_\alpha) \) contains the disk centered at the origin of radius \( s \).

Proof. We know \( \rho(J_\alpha) = (\alpha_1 \alpha_2 \ldots \alpha_n)^{1/n} = \|J_\alpha^n\|^{1/n} \). So \( \rho(J_\alpha) \leq s \) is clearly true.

If \( s = \|J_\alpha^n\|^{1/n} \), then \( s \) is the positive eigenvalue of \( J_\alpha \) and so \( s \in W(J_\alpha) \).

Suppose \( s = \left( \frac{\|J_\alpha^n\|}{2} \right)^{1/k} \) for \( k \in \{1, \ldots, n-2\} \). Then, as we have shown before, if \( \beta \) is a cyclic permutation of \( \alpha \), \( W(J_\alpha) = W(J_\beta) \), so we may assume without loss of generality that \( s = (\frac{\alpha_1 \ldots \alpha_k}{2})^{1/k} \). Define the vector \( q_\theta \) in \( \mathbb{C}^n \) by

\[
q_\theta = \frac{1}{\sqrt{k}} \left( \frac{1}{\sqrt{2}} e^{i\theta}, e^{2i\theta}, \ldots, e^{(k-1)i\theta}, \frac{1}{\sqrt{2}} e^{ik\theta}, 0, \ldots, 0 \right)^T
\]

Then \( q_\theta \) is a unit vector, since

\[
q_\theta q_\theta^* = \frac{1}{k} \left( \frac{1}{2} + 1 + \cdots + 1 + \frac{1}{2} + 0 + \cdots + 0 \right) = \frac{k}{k} = 1.
\]

Since it is a unit vector, \( \langle J_\alpha q_\theta, q_\theta \rangle \) is a unit vector, \( \langle J_\alpha q_\theta, q_\theta \rangle \in W(J_\alpha) \). Also,

\[
\langle J_\alpha q_\theta, q_\theta \rangle = \frac{1}{\sqrt{k}} \left( \alpha_1 e^{i\theta}, \alpha_2 e^{2i\theta}, \ldots, \alpha_{k-1} e^{(k-1)i\theta}, \frac{\alpha_k}{\sqrt{2}} e^{ik\theta}, 0, \ldots, 0, \frac{\alpha_n}{\sqrt{2}} \right)^T
\]

And, by the inequality of arithmetic and geometric means, we have

\[
s = \left( \frac{\alpha_1 \ldots \alpha_k}{2} \right)^{1/k} \leq \frac{1}{k} \left( \frac{\alpha_1}{\sqrt{2}} + \alpha_2 + \cdots + \alpha_{k-1} + \frac{\alpha_k}{\sqrt{2}} \right)^k
\]

Since \( W(J_\alpha) \) is convex and contains 0, \( s \in W(J_\alpha) \). Since \( \theta \) was arbitrary, \( e^{i\theta} s \in W(J_\alpha) \), so if \( s = \left( \frac{\|J_\alpha^n\|}{2} \right)^{1/k} \) for \( k \in \{1, \ldots, n-2\} \), we have that \( W(J_\alpha) \) contains the circle centered at the origin of radius \( s \).

Now we are left with the case where \( s = \left( \frac{\|J_\alpha^n\|}{2} \right)^{1/(n-1)} \) for \( n \in \{1, \ldots, n \} \). Define the unit vector \( q = \frac{1}{\sqrt{n-1}} \left( \frac{1}{\sqrt{2}}, 1, \ldots, 1, \frac{1}{\sqrt{2}} \right)^T \in \mathbb{C}^n \).
Then we get
\[
\langle J_\alpha q, q \rangle = \frac{1}{\sqrt{n}-1} \left( \alpha_1, \alpha_2, \ldots, \alpha_{n-2}, \frac{\alpha_{n-1}}{\sqrt{2}}, \frac{\alpha_n}{\sqrt{2}} \right), q) \\
= \frac{1}{n-1} \left( \frac{\alpha_1}{\sqrt{2}} + \alpha_2 + \cdots + \alpha_{n-2} + \frac{\alpha_{n-1}}{\sqrt{2}} + \frac{\alpha_n}{2} \right) \\
\geq \left( \frac{\alpha_1 \cdots \alpha_{n-1}}{2} \right)^{1/(n-1)} + \frac{\alpha_n}{2(n-1)} \\
\geq s
\]
Where the first inequality comes from the inequality of arithmetic and geometric means on all but the last term, and the second is true since $\alpha_n > 0$.

Thus, since $W(J_\alpha)$ is convex and includes $0$ and $\langle J_\alpha q, q \rangle$, we get $s \in W(J_\alpha)$. □

Now, note that with this definition of $s$, [3] is equivalent to $cs \leq 1$. So if we can show that $cs \leq 1$, then we get that Equation [5] holds, and therefore Crouzeix’s conjecture holds for $J_\alpha$.

We will show that $cs \leq 1$ for all possible values of $s$. For almost all cases, this follows from Lemma 5 (c). The case where $s = \|J_\alpha^n\|^{1/n}$ is also simple. The problem comes with the case where $s = \left( \frac{\|J_\alpha^{n-1}\|}{2} \right)^{1/(n-1)}$. In this case we will have to devote much more work to showing $cs \leq 1$.

In the following proof we will prove $cs \leq 1$ for all cases except the last mentioned above. We will then outline what is necessary for that case and deal with it separately.

**Theorem 8.** $cs \leq 1$ whenever $s \neq \left( \frac{\|J_\alpha^{n-1}\|}{2} \right)^{1/(n-1)}$

**Proof.** First, suppose $s = \|J_\alpha^n\|^{1/n}$. Then $s = (\alpha_1 \cdots \alpha_n)^{1/n}$ is the positive eigenvalue of $J_\alpha$, and so $\phi(s) = cs$ by the properties of $\phi$. Since $\phi$ is a conformal map from $W(J_\alpha)$ to the unit disk, we immediately get $cs = \phi(s) \leq 1$.

Now, let $s = \left( \frac{\|J_\alpha^k\|^{1/k}}{2} \right)$ where $k \in \{1, 2, \ldots, n-2\}$. Then by Lemma 5 (c), we know that $W(J_\alpha)$ contains $D_s$, the disk of radius $s$ centered at the origin. Therefore, we can look at $\phi|D_s$, which is $\phi$ whose domain is restricted to $D_s$. Equivalently, we can look at the function $\phi(sz)$ where $z$ is in the unit disk. Note that $|\phi(sz)| \leq 1$ and $\phi(0) = 0$; therefore, we can then apply Schwarz’ Lemma to $g(z) = \phi(sz)$, to get that $|g(z)| = |\phi(sz)| \leq |z|$ for all $z$ in the unit disk. Now, we look at $x = sz$ and we get that $|\phi(x)| \leq \frac{2}{|z|}$ for $|x| \leq s$. Notably, since $s \geq (\alpha_1 \cdots \alpha_n)^{1/n}$, we can plug in an
eigenvalue, \( x = \lambda \), and we get

\[
|\phi(\lambda)| \leq \frac{\lambda}{s} \\
|c\lambda| \leq \frac{\lambda}{s} \\
|c||\lambda| \leq \frac{\lambda}{s} \\
|c||s| \leq 1 \\
cs \leq 1
\]

And so if \( s \neq \left( \frac{\|J^{-1}\alpha\|}{2} \right)^{1/(n-1)} \), we get that \( cs \leq 1 \).

In the case where \( s = \left( \frac{\|J^{-1}\alpha\|}{2} \right)^{1/(n-1)} \), we want to do the same thing as above, except we do not know that we have a disk of radius \( s \) contained in \( W(J\alpha) \) to use Schwarz’ Lemma. In the following section, we will show that there exists an open subset \( U \subset W(J\alpha) \) which contains \( \sigma(J\alpha) \), along with a conformal map \( \psi \) from \( U \) to \( V \) where \( \psi(0) = 0, \psi(\lambda) = s^{-1}\lambda \) for an eigenvalue \( \lambda \) of \( J\alpha \), and \( V \) contains the unit disk. If we have that, then we may create a holomorphic function \( g \) from \( V \) to the unit disk, defined by \( g = \phi \circ \psi^{-1} \). In the figure below, the green arrows are the already known map \( \phi \) from \( W(J\alpha) \) to \( \mathbb{D} \), and the purple arrows are the map \( \psi \) which we are looking for, \( \psi \), from \( U \) to a new space \( V \). The map \( g \) then follows the purple arrows backwards and the green arrows forward to create a map from \( V \) to \( \mathbb{D} \).

Figure 2.2: The Map \( g \)
The function $g$ then satisfies the properties for Schwarz’ Lemma to apply on $g$ restricted to $\mathbb{D}$; namely that $g(0) = \phi \circ \psi^{-1}(0) = \phi(0) = 0$, and $|g(z)| \leq 1$ for $z \in \mathbb{D}$, since $|\phi(z)| \leq 1$ for all $z$ in its domain. Thus, we can apply Schwarz’ lemma on $g$ to get that $|g(z)| \leq |z|$ for any $z \in \mathbb{D}$. In particular, $s^{-1} \lambda \in \mathbb{D}$ for any $\lambda$ since $s > (\alpha_1 \ldots \alpha_n)^{1/n} = \lambda$. Then we get that $g(s^{-1} \lambda) = \phi \circ \psi^{-1}(s^{-1} \lambda) = \phi(\lambda) = c\lambda$, so $|c\lambda| \leq |s^{-1} \lambda|$, and we get $cs \leq 1$.

Now, we simply need to show that such a $U$ and $\psi$ exist.

### 2.4 The n-1 Case

For this section, we will be exploring when $s = \|J_{n-1}^\alpha\|^{1/(n-1)}$, where $s$ is defined in (7).

Assume $s = \left(\frac{\|J_{n-1}^\alpha\|}{2}\right)^{1/(n-1)}$. We know $\|J_{n-1}^\alpha\| = \max\{\alpha_j\alpha_{j+1} \ldots \alpha_{j+(n-1)-1} : 1 \leq j \leq n\}$.

As we have done before, we can use cyclic permutations of $\alpha$ to assume without loss of generality, that $s = \left(\frac{\alpha_1\alpha_2 \ldots \alpha_{n-1}}{2}\right)^{1/(n-1)}$. Now, by Lemma 1, we know that Crouzeix’s conjecture holds for $J_{\alpha}$ if and only if it holds for $kJ_{\alpha}$ for some scalar $k$. Thus, we may scale $J_{\alpha}$ such that $s = \left(\frac{\alpha_1\alpha_2 \ldots \alpha_{n-1}}{2}\right)^{1/(n-1)} = 1$.

We can use this simplified $s$ to find properties of $\alpha$.

**Lemma 6.** The vector $\alpha$ satisfies both $s = \left(\frac{\alpha_1\alpha_2 \ldots \alpha_{n-1}}{2}\right)^{1/(n-1)}$ and $s = 1$ if and only if $\alpha$ satisfies (a), (b), and (c) below:

(a) $\alpha_1 \ldots \alpha_{n-1} = 2$
(b) $1 \leq \alpha_j \ldots \alpha_{n-1} \leq 2$ for any $j = 2, \ldots, n-1$
(c) $\alpha_n \leq \frac{1}{2}$

**Proof.** For the forward direction, assume $s = \left(\frac{\alpha_1\alpha_2 \ldots \alpha_{n-1}}{2}\right)^{1/(n-1)}$ and $s = 1$. The proof of (a) is simple:

$$\left(\frac{\alpha_1\alpha_2 \ldots \alpha_{n-1}}{2}\right)^{1/(n-1)} = 1$$

$$\frac{\alpha_1\alpha_2 \ldots \alpha_{n-1}}{2} = 1$$

$$\alpha_1\alpha_2 \ldots \alpha_{n-1} = 2$$

For (c), we know that $s \geq \|J_{\alpha}\|^{1/n}$ by the definition of $s$, so we have

$$s = 1 \geq \|J_{\alpha}\|^{1/n} = (\alpha_1 \ldots \alpha_n)^{1/n} = (2\alpha_n)^{1/n}$$

23
Thus, we get that $1 \geq 2\alpha_n$, so $\alpha_n \leq \frac{1}{2}$, and we get (c).

For (b), again by the definition of $s$, we know $\left(\frac{\|J_k\|}{2}\right)^{1/k} \leq 1$ for any $k \in \{1, \ldots, n\}$. Therefore, since $\|J_\alpha\| = \max\{\alpha_m \alpha_{m+1} \ldots \alpha_{m+k-1} : 1 \leq m \leq n\}$, we get that $\alpha_1 \ldots \alpha_j \leq 2$ (letting $m = 1$, and $k = j$), as well as $\alpha_j \ldots \alpha_{n-1} \leq 2$ (letting $m = j$, and $k = n - j$) for any $j$. So we get the 2nd inequality of (b), that $\alpha_j \ldots \alpha_{n-1} \leq 2$. As for the first inequality, we have, from (a),

$$\alpha_1 \ldots \alpha_{n-1} = 2$$
$$\alpha_1 \ldots \alpha_{j-1} \alpha_j \ldots \alpha_{n-1} = 2$$
$$\alpha_j \ldots \alpha_{n-1} = \frac{2}{\alpha_1 \ldots \alpha_{j-1}}$$
$$\alpha_j \ldots \alpha_{n-1} \geq \frac{2}{2}$$
$$\alpha_j \ldots \alpha_{n-1} \geq 1,$$

where the inequality comes from the fact that $\alpha_1 \ldots \alpha_j \leq 2$ for any $j$. Thus, if $s = \left(\frac{\alpha_1 \alpha_2 \ldots \alpha_{n-1}}{2}\right)^{1/(n-1)}$ and $\left(\frac{\alpha_1 \alpha_2 \ldots \alpha_{n-1}}{2}\right)^{1/(n-1)} = 1$, we satisfy the three conditions.

For the converse direction, suppose (a),(b), and (c) hold. We will show that $s = \left(\frac{\alpha_1 \alpha_2 \ldots \alpha_{n-1}}{2}\right)^{1/(n-1)}$ and $\left(\frac{\alpha_1 \alpha_2 \ldots \alpha_{n-1}}{2}\right)^{1/(n-1)} = 1$, starting from the definition of $s$ from (7). We will do this by showing all terms inside the maximum are less than or equal to 1, and that $\left(\frac{\|J_{\alpha}\|}{2}\right)^{1/(n-1)} = 1$. For the $\|J_{\alpha}\|^{1/n}$ term, since $\alpha_1 \ldots \alpha_{n-1} = 2$ and $\alpha_n \leq \frac{1}{2}$, we get that $\|J_{\alpha}\|^{1/n} = (\alpha_1 \ldots \alpha_n)^{1/n} \leq 1^{1/n} = 1$. Now we just need to show that for any of the $\left(\frac{\|J_{\alpha}\|}{2}\right)^{1/k}$ terms, that they are less than 1, and that $\left(\frac{\|J_{\alpha}\|}{2}\right)^{1/(n-1)} = 1$.

We will show that $\alpha_j \alpha_{j+1} \ldots \alpha_m \leq 2$ for all choices of $j$ and $m$ in $\{1, 2, \ldots, n\}$. That is, the products of $\alpha_i$ which start at $j$ and end at $m$ are bounded above by 2. Note that we must also check cases where $j > m$, in which case we use the notation that $\alpha_{n+j} = \alpha_j$. As an example, $\alpha_{n+1} \alpha_1 \alpha_2$ is the case where $j = n - 1$ and $m = 2$.

(Case 0 $j \leq n - 1$, $m = n - 1$): $\alpha_j \ldots \alpha_{n-1} \leq 2$ by (b)

(Case 1 $j = 1$, $1 \leq m \leq n - 2$): $\alpha_1 \ldots \alpha_m = \frac{\alpha_1 \ldots \alpha_{n-1}}{\alpha_{m+1} \ldots \alpha_{n-1}} = \frac{2}{\alpha_{m+1} \ldots \alpha_{n-1}} \in [1, 2]$ since $\alpha_1 \ldots \alpha_{n-1} = 2$ by (a), and $1 \leq \alpha_{m+1} \ldots \alpha_{n-1} \leq 2$ by (b)

(Case 2 $2 \leq j \leq m \leq n - 2$): $\alpha_j \ldots \alpha_m = \frac{\alpha_1 \ldots \alpha_{n-1}}{\alpha_{1} \ldots \alpha_{j-1} \alpha_{j+1} \ldots \alpha_{n-1}} = \frac{2}{\alpha_{1} \ldots \alpha_{j-1} \alpha_{j+1} \ldots \alpha_{n-1}} \in [\frac{1}{2}, 2]$ since $\alpha_1 \ldots \alpha_{j-1} = 2$ by (a), $1 \leq \alpha_1 \ldots \alpha_{j-1} \leq 2$ by Case 1, and $1 \leq \alpha_{m+1} \ldots \alpha_{n} \leq 2$ by (b).
(Case 3) $1 \leq j \leq n = m$: $\alpha_j \ldots \alpha_n \leq 2\alpha_n \leq 1$ Since $\alpha_j \ldots \alpha_{n-1} \leq 2$ by (b), and $\alpha_n \leq \frac{1}{2}$ by (c)

(Case 4) $m < j \leq n - 1$: $\alpha_j \ldots \alpha_n \alpha_1 \ldots \alpha_m \leq \alpha_1 \ldots \alpha_m \leq 2$ since $\alpha_j \ldots \alpha_n \leq 1$ by Case 3. And $\alpha_1 \ldots \alpha_m \leq 2$ by Case 1.

(Case 5) $j = n$, $1 \leq m \leq n - 1$: $\alpha_n \alpha_1 \ldots \alpha_m \leq \alpha_1 \ldots \alpha_m \leq 2$ since $\alpha_j \ldots \alpha_n \leq 1$ by Case 3. And $\alpha_1 \ldots \alpha_m \leq 2$ by Case 1.

And so, we have shown for every possible choice of $j$ and $m$, that $\alpha_j \ldots \alpha_m \leq 2$. Therefore, $\|J^k\| \leq 2$ for all $k = 1, 2, \ldots, n$, and so $\left(\frac{\|\alpha\|}{2}\right)^{1/k} \leq 1$ for all $k$. Additionally, $\left(\frac{\|\alpha\|}{2}\right)^{1/(n-1)} \geq (\frac{\alpha_1 \ldots \alpha_{n-1}}{2})^{1/(n-1)} = 1$. Therefore we get that $s = (\frac{\alpha_1 \ldots \alpha_{n-1}}{2})^{1/(n-1)} = 1$ by definition of $s$.

Thus, conditions (a) (b) and (c) are the necessary and sufficient conditions for $\alpha$ to have $s = 1$.

Now that we have some simplification, we will start working towards finding our open set $U$.

Let $q = \frac{1}{\sqrt{n-1}} \left(\frac{1}{\sqrt{2}}, 1, \ldots, 1, \frac{1}{\sqrt{2}}\right)^T \in \mathbb{C}^n$. Note that $|q|^2 = \frac{1}{2^2 + 1^2 + \ldots + 1^2} = \frac{n-1}{n-1} = 1$, so $q$ is a unit vector. We will define a function

$$w(\theta) = \langle J_\alpha q_\theta, q_\theta \rangle$$

where $q_\theta = \text{diag}\{1, e^{i\theta}, e^{2i\theta}, \ldots, e^{(n-1)i\theta}\}q$. The open set bounded by this function will be our desired $U$. First we need to simplify $w(\theta)$ to get an easier formula to work with.

We know $q_\theta = \frac{1}{\sqrt{n-1}} \left(\frac{1}{\sqrt{2}}, e^{i\theta}, e^{2i\theta}, \ldots, e^{(n-2)i\theta}, \frac{1}{\sqrt{2}}e^{(n-1)i\theta}\right)^T$, which gives us

$$J_\alpha q_\theta = \begin{pmatrix} 0 & \alpha_1 & & \\ & \ddots & \ddots & \\ & & \alpha_{n-1} & 1 \\ \alpha_n & 0 \\ \end{pmatrix} \frac{1}{\sqrt{n-1}} \left(\frac{1}{\sqrt{2}}, e^{i\theta}, e^{2i\theta}, \ldots, e^{(n-2)i\theta}, \frac{1}{\sqrt{2}}e^{(n-1)i\theta}\right)^T$$

$$= \frac{1}{\sqrt{n-1}} \left(\alpha_1 e^{i\theta}, \alpha_2 e^{2i\theta}, \ldots, \alpha_{n-2} e^{(n-2)i\theta}, \frac{\alpha_n - 1}{\sqrt{2}} e^{(n-1)i\theta}, \frac{\alpha_n}{\sqrt{2}}\right)^T$$
Therefore, we have

\[ w(\theta) = \langle J_\alpha q_\theta, q_\theta \rangle \]
\[ = \frac{1}{n-1} \left( \frac{\alpha_1}{\sqrt{2}} e^{i\theta} + \alpha_2 e^{i\theta} + \cdots + \alpha_{n-2} e^{i\theta} + \frac{\alpha_{n-1}}{\sqrt{2}} e^{i\theta} + \alpha_n e^{-(n-1)i\theta} \right) \]
\[ = \frac{e^{i\theta}}{n-1} \left( \frac{\alpha_1}{\sqrt{2}} + \alpha_2 + \cdots + \alpha_{n-2} + \frac{\alpha_{n-1}}{\sqrt{2}} + \alpha_n e^{-ni\theta} \right) \]
\[ = e^{i\theta} \left( a + e^{-in\theta} \frac{\alpha_n}{2(n-1)} \right) \]

Where \( a \) is the constant

\[ a = \frac{1}{n-1} \left( \frac{\alpha_1}{\sqrt{2}} + \alpha_2 + \cdots + \alpha_{n-2} + \frac{\alpha_{n-1}}{\sqrt{2}} \right) \tag{9} \]

We can now show the following properties of \( w(\theta) \).

**Lemma 7.** The following are properties of \( w(\theta) \)

1. \( |w(\theta)| > 0 \) for all \( \theta \)
2. \( w(\theta + \frac{2\pi k}{n}) = e^{\frac{2\pi ik}{n}} w(\theta) \) for any integer \( k \).
3. \( w(-\theta) = w(\theta) \)
4. \( \arg(w(\theta)) = \theta \) for \( \theta = k\pi, k = 1, \ldots, 2n \)
5. \( \arg(w(\theta)) \) is a one-to-one function on \([0, \frac{\pi}{n}]\)

**Remark 1.** I note that this statement differs from the original paper, which states: \( w(\theta + \frac{2\pi k}{n}) = w(\theta) \) for any integer \( k \), \( |w(\frac{2\pi k}{n} - \theta)| = |w(\theta)| \) for \( 0 \leq \theta \leq \frac{\pi}{n} \), as well as that \( w(\theta) \) is a one-to-one function.

The first appears to be a typo, as we want \( w(\theta) \) to have the same symmetric properties as \( W(J_\alpha) \).

For the second, I believe c) is a simpler way to see a slightly stronger symmetry. The last, however, is true, but does not seem to be a strong enough condition for a statement made later in the paper.

The proof for e) will be separate, and the section that requires the change in condition is remark 2.

**Proof.** a) \( |w(\theta)| > 0 \) for all \( \theta \):

First note that \( a \) as well as \( \alpha_n \) are strictly positive, since in the beginning of this paper we were able to assume that \( \alpha_j > 0 \) for all \( j \). Since \( a \) and \( \alpha_n \) are real, \( w(\theta) \) can only be 0 if

\[ a - \frac{\alpha_n}{2(n-1)} = 0 \]

We have that \( a \geq 1 \), since by the inequality of arithmetic and geometric means,

\[ a = \frac{1}{n-1} \left( \frac{\alpha_1}{\sqrt{2}} + \alpha_2 + \cdots + \alpha_{n-2} + \frac{\alpha_{n-1}}{\sqrt{2}} \right) \geq \frac{\alpha_1 \ldots \alpha_{n-1}}{2} = 1 \]
We also know that $\alpha_n \leq \frac{1}{2}$, so we get that $\frac{\alpha_n}{2(n-1)} \leq \frac{1}{4(n-1)} \leq \frac{1}{4}$. Therefore, we get that

$$a - \frac{\alpha_n}{2(n-1)} \geq 1 - \frac{1}{4} = \frac{3}{4} > 0$$

And so $|w(\theta)| > 0$ for all $\theta$.

b) $w(\theta + \frac{2\pi k}{n}) = e^{\frac{2\pi ik}{n}} w(\theta)$ for any integer $k$:

Let $k$ be any integer, then we have.

$$w(\theta + \frac{2\pi k}{n}) = e^{i(\theta + \frac{2\pi k}{n})} \left( a + e^{-in(\theta + \frac{2\pi k}{n})} \frac{\alpha_n}{2(n-1)} \right)$$

$$= e^{i\theta} e^{\frac{2\pi ik}{n}} \left( a + e^{-in\theta} e^{-2\pi ik} \frac{\alpha_n}{2(n-1)} \right)$$

$$= e^{\frac{2\pi ik}{n}} e^{i\theta} \left( a + e^{-in\theta} \frac{\alpha_n}{2(n-1)} \right)$$

$$= e^{\frac{2\pi ik}{n}} w(\theta)$$

c) $w(-\theta) = \overline{w(\theta)}$:

If we simply plug in $-\theta$, we see that

$$w(-\theta) = e^{-i\theta} \left( a + e^{in\theta} \frac{\alpha_n}{2(n-1)} \right)$$

$$= e^{i\theta} \left( a + e^{-in\theta} \frac{\alpha_n}{2(n-1)} \right)$$

$$= \overline{w(\theta)}$$

d) $\arg(w(\theta)) = \theta$ for $\theta = \frac{k\pi}{n}$, $k \in \{1, \ldots, 2n\}$:

Let $k \in \{1, \ldots, 2n\}$. Then we have

$$\arg \left( w \left( \frac{k\pi}{n} \right) \right) = \arg \left( e^{i\frac{k\pi}{n}} \left( a + e^{-ik\pi} \frac{\alpha_n}{2(n-1)} \right) \right)$$

$$= \arg \left( e^{i\frac{k\pi}{n}} \left( a + (-1)^k \frac{\alpha_n}{2(n-1)} \right) \right)$$

$$= \frac{k\pi}{n},$$

Since $a + (-1)^k \frac{\alpha_n}{2(n-1)}$ by the proof of (a).
Lemma 8. \( \arg(w(\theta)) \) is a one-to-one function on \([0, \frac{\pi}{n}]\)

Proof. Let \( b = \frac{\alpha n}{2(n-1)} \), and let \( \arg(z) \) be the argument of \( z \) on the principal branch, \((-\pi, \pi]\). We have that

\[
\arg(w(\theta)) = \arg(e^{i\theta}(a + e^{-in\theta}b)) = \theta + \arg(a + e^{-in\theta}b) = \theta + \arg(a + b\cos(-n\theta) + bi\sin(-n\theta)) = \theta + \tan^{-1}\left(\frac{b\sin(-n\theta)}{a + b\cos(-n\theta)}\right).
\]

The value \( \arg(z) = \tan^{-1}\left(\frac{\Im(z)}{\Re(z)}\right) \) is well defined for \( z \) with \( \Re(z) > 0 \), so for the above to be true, we need to show that \( a + b\cos(-n\theta) > 0 \). The function \( a + b\cos(-n\theta) \) is minimized at \( a - b \), when \( \theta = \pi/n \). Since \( a \geq 1 \) and \( b \leq \frac{1}{4(n-1)} < 1 \), it follows that \( a - b > 0 \), so we can use \( \tan^{-1} \).

We will show that this is a one-to-one function by showing it is monotonically increasing on the interval \([0, \frac{\pi}{n}]\). To this end we will show that the derivative is always positive.

\[
\frac{d}{dx} \arg(w(x)) = 1 - \frac{b^2n\sin^2(-nx) + \frac{bn\cos(-nx)}{a + b\cos(-nx)}}{(a + b\cos(-nx))^2 + 1} + \frac{b^2n\sin^2(-nx) + abn\cos(nx) + nb^2\cos^2(nx)}{a^2 + 2ab\cos(nx) + b^2\cos^2(nx) + b^2\sin^2(nx)}
\]

\[
= 1 - \frac{b^2n\sin^2(nx) + abn\cos(nx) + nb^2\cos(nx)}{a^2 + 2ab\cos(nx) + b^2\cos^2(nx)}
\]

\[
= \frac{a^2 - b^2(n-1) - ab(n-2)\cos(nx)}{a^2 + b^2 + 2ab\cos(nx)}
\]

Now, note that the denominator is always positive; at worst, it is \( a^2 + b^2 - 2ab = (a-b)^2 > 0 \) when \( \cos(nx) = -1 \). For the numerator, we will need to use the fact that \( a \geq 1 \) and \( 0 < b \leq \frac{1}{4(n-1)} \).
worst, the numerator is \( a^2 - b^2 (n-1) - ab (n-2) \) when \( \cos(nx) = 1 \).

\[
a^2 - b^2 (n-1) - ab (n-2) \geq a - \frac{1}{16(n-1)} - \frac{a(n-2)}{4(n-1)} \\
= a \left( 1 - \frac{n-2}{4(n-1)} \right) - \frac{1}{16(n-1)} \\
= a \frac{3n-2}{4(n-1)} - \frac{1}{16(n-1)} \\
\geq \frac{3n-2}{4(n-1)} - \frac{1}{16(n-1)} \\
= \frac{12n-9}{16(n-1)} \\
> 0.
\]

The first inequality holds since \( a \geq 1 \) and \( b \leq \frac{1}{4(n-1)} \). The 2nd holds since \( a \geq 1 \) and the last holds since \( n > 2 \). Therefore, the argument is strictly increasing and thus one-to-one on \([0, \pi/n]\).

**Remark 2.** The following proof is why we needed to change conditions in Remark 1. In the original paper, instead of \( \arg(w(\theta)) \), it simply stated \( w(\theta) \) was one-to-one on \([0, \pi/n]\). In this case, it is possible to create a curve that has the properties \( a - d \) and is one-to-one, that does not enclose a star shaped set, as a ray from the origin can intersect it twice. The curve \( f(\theta) \) below is such an example for \( n = 4 \). On the left is the segment for \( \theta \in [0, \pi/4] \), which has the property that \( \arg(f(\pi/4)) = \pi/4 \). This segment is then reflected so that it has the desired symmetries, shown on the right. This curve clearly is one-to-one, but is not star shaped with respect to the origin.

![Figure 2.3: f(\theta), a Counterexample](image)

Now, we will use the facts about \( w(\theta) \) to show that it encloses a star shaped set with respect to the origin.
Lemma 9. A curve $C$ who has properties a-e listed in Lemma 7 encloses a star shaped set with respect to the origin.

Proof. For a star shaped domain $D$, we need that for every point $d \in D$, the line $td$, $0 \leq t \leq 1$ lies entirely in $D$. Since we are working with the open set bounded by a curve, it suffices to show that the line $tc$, $0 \leq t < 1$ lies entirely in $D$ for every $c \in C$. By the symmetry we have from (b) and (c), we only need to check for lines emanating from the origin with arguments from $0$ to $\frac{\pi}{n}$. Since $|c| > 0$, we know that the curve does not cross over the origin, and since $\arg(c)$ is one-to-one, we know that a line from the origin will only ever hit $C$ once. Thus, we get that $D$ is star-shaped with respect to the origin.

Let $U$ be the open set enclosed by $w(\theta)$. Since $q_0$ is a unit vector for all $\theta$, $w(\theta) \in W(J_\alpha)$. $W(J_\alpha)$ is a convex set containing $0$, and $U$ is a star shaped set with respect to $0$, so $U \subseteq W(J_\alpha)$.

We know that $w(\frac{2k\pi}{n}) = e^{2k\pi/n}(a + e^{-2k\pi - \frac{\alpha_n}{2(n-1)}}) = e^{2k\pi/n}(a + \frac{\alpha_n}{2(n-1)})$ and $a \geq 1$. Since $U$ is a star shaped set with respect to the origin, we know that $re^{2k\pi/n} \subseteq U$ for all $r \leq a + \frac{\alpha_n}{2(n-1)}$.

Notably, since $s = 1$, we have $(\alpha_1\alpha_2 \cdots \alpha_n)^{1/n} \leq 1$. Thus, $(\alpha_1\alpha_2 \cdots \alpha_n)^{1/n} \leq 1 \leq a + \frac{\alpha_n}{2(n-1)}$ since $a \geq 1$ and $\alpha_n \geq 0$. So all eigenvalues of $J_\alpha$, $e^{2k\pi/n}(\alpha_1\alpha_2 \cdots \alpha_n)^{1/n} k \in \{1, \ldots, n\}$ are contained in $U$. Thus, $U$ is a subset of $W(J_\alpha)$ which contains the spectrum of $J_\alpha$.

We now want to find a bijective holomorphic function $\psi$ from $U$ to some set $V$ containing the unit disk. We need $\psi(0) = 0$, and $\psi(\lambda) = \lambda$ for some eigenvalue $\lambda$ of $J_\alpha$. The minimum polynomial of $J_\alpha$ is $z^n - \alpha_1 \cdots \alpha_n = z^n - 2\alpha_n$. To get these properties, we look for a function of the form

$$\psi(z) = z(1 - \xi(z^n - 2\alpha_n))$$

(10)

for some holomorphic $\xi$ with $\xi(0) = 0$. The multiplication by $z$ will give us $\psi(0) = 0$, and the $1 - \xi(z^n - 2\alpha_n)$, since $\xi(0) = 0$ and $\lambda$ will zero out the minimum polynomial, will give us $\psi(\lambda) = \lambda$.

We will choose the function $\xi$ to be constant, $\xi(z) = \frac{\alpha_n}{2(n-1)z^{n-1}}$ where $a$ is the constant from $[9]$, which was used in $w(\theta)$. So, now we want to show that $\psi(U)$ contains the unit disk, and that $\psi$ is one-to-one.

Lemma 10. Let $\psi$ be the polynomial defined above and let $U$ be the set defined above, then $\psi(U)$ contains the unit disk.

Proof. We will show that $\delta \psi(U) \subseteq \psi(\delta U)$, where $\delta$ denotes the boundary of a set. Let $|\delta \psi(U)| \geq 1$ denote $|z| \geq 1$ for all $z \in \delta \psi(U)$. Since $U$ contains 0 in its interior and $\psi(0) = 0$, we know that 0 is in the interior of $\delta \psi(U)$, thus for any angle, a ray originating from 0 with that angle will intersect
\( \delta \psi(U) \). It follows that \( \psi(U) \) contains the unit disk is equivalent to \( |\delta \psi(U)| \geq 1 \) and that will hold when \( |\delta(U)| \geq 1 \). Note that \( \delta U = w(\theta) \), and we will only have to show that \( |\psi(w(\theta))| \geq 1 \). \( \psi \) is a polynomial, and thus is continuous on \( x \). By the open mapping theorem, we know that \( \psi \) maps the open set \( U \) to the open set \( \psi(U) \), and the closed set \( U \cup \delta U \) to another closed set, \( \psi(U \cup \delta U) \).

If \( x \in \delta \psi(U) \), then since \( \psi(U \cup \delta U) \) is closed and contains \( \psi(U) \), \( x \in \psi(U \cup \delta U) \). Thus \( x = \psi(z) \) for some \( z \) either from \( U \) or \( \delta U \). If \( z \) came from \( U \), then \( \psi(z) = x \in \psi(U) \), which contradicts \( \psi(U) \) being open. Thus, it must be the case that \( z \in \delta U \), and so \( x \in \psi(\delta U) \). Therefore, we get that \( \delta \psi(U) \subseteq \psi(\delta U) \) and so we only need to show that \( |\psi(w(\theta))| \geq 1 \) to show that \( \psi(U) \) contains the unit disk.

Now, let’s show \( |\psi(w(\theta))| \geq 1 \). Note that for our choice of \( \xi \), \( w_{\theta} = w(\theta) = e^{i\theta} (a + e^{-in\theta}\xi a^{n+1}) \).

\[
\psi(w_{\theta}) = w_{\theta}(1 - \xi (w_{\theta}^2 - 2\alpha_n))
= w_{\theta}(1 + 2\xi\alpha_n) - \xi w_{\theta}^{n+1}
= w_{\theta}(1 + 2\xi\alpha_n) - \xi w_{\theta}^n - (1 + 2\xi\alpha_n)e^{i(n+1)\theta}\xi a^{n+1} + (1 + 2\xi\alpha_n)e^{i(n+1)\theta}\xi a^n
= (1 + 2\xi\alpha_n)(w_{\theta} - e^{i(n+1)\theta}\xi a^{n+1}) - \xi w_{\theta}^n + (1 + 2\xi\alpha_n)e^{i(n+1)\theta}\xi a^n
= (1 + 2\xi\alpha_n)(w_{\theta} - e^{i(n+1)\theta}\xi a^{n+1}) + 2\xi^2\alpha_n e^{i(n+1)\theta} a^{n+1} + \xi \left( e^{i(n+1)\theta} a^{n+1} - w_{\theta}^{n+1} \right).
\]

Thus, by the reverse triangle inequality, we get that

\[
|\psi(w_{\theta})| \geq |(1 + 2\xi\alpha_n)(w_{\theta} - e^{i(n+1)\theta}\xi a^{n+1})| - |2\xi^2\alpha_n e^{i(n+1)\theta} a^{n+1}| - \left| \xi \left( e^{i(n+1)\theta} a^{n+1} - w_{\theta}^{n+1} \right) \right|.
\]

We will start by simplifying each term. First we look at \(|(1 + 2\xi\alpha_n)(w_{\theta} - e^{i(n+1)\theta}\xi a^{n+1})|\).

\[
|(1 + 2\xi\alpha_n)(w_{\theta} - e^{i(n+1)\theta}\xi a^{n+1})| = (1 + 2\xi\alpha_n)|w_{\theta} - e^{i(n+1)\theta}\xi a^{n+1}|
= (1 + 2\xi\alpha_n)|e^{i\theta}(a + e^{-in\theta}\xi a^{n+1}) - e^{i(n+1)\theta}\xi a^{n+1}|
= (1 + 2\xi\alpha_n)|a + e^{-in\theta}\xi a^{n+1} - e^{in\theta}\xi a^{n+1}|
= (1 + 2\xi\alpha_n)|a + \xi a^{n+1}2i\sin(n\theta)|
\geq (1 + 2\xi\alpha_n)a
\]

31
Now, we will look at $|\xi \left( e^{i(n+1)\theta} a^{n+1} - w_{\theta}^{n+1} \right) |$.

\[
|\xi \left( e^{i(n+1)\theta} a^{n+1} - w_{\theta}^{n+1} \right) | = \xi \left| e^{i(n+1)\theta} a^{n+1} - w_{\theta}^{n+1} \right|
\]

\[
= \xi a^{n+1} \left| e^{i(n+1)\theta} \left( \frac{w_{\theta}}{a} \right)^{n+1} - 1 \right|
\]

\[
= \xi a^{n+1} \left| \left( \frac{e^{i\theta} (a + e^{-i\theta} \xi a^{n+1})}{ae^{i\theta}} \right)^{n+1} - 1 \right|
\]

\[
= \xi a^{n+1} \left| (1 + e^{-i\theta} \xi a^{n})^{n+1} - 1 \right|
\]

\[
= \xi a^{n+1} \left| -1 + \sum_{k=0}^{n+1} \binom{n+1}{k} (e^{-i\theta} \xi a^{n})^k \right|
\]

\[
\leq \xi a^{n+1} \sum_{k=1}^{n+1} \binom{n+1}{k} (e^{-i\theta} \xi a^{n})^k
\]

\[
= \xi a^{n+1} \sum_{k=1}^{n+1} \binom{n+1}{k} (\xi a^n)^k
\]

Combining these two, we get

\[
|\psi(w_{\theta})| \geq |(1 + 2\xi \alpha_n)(w_{\theta} - e^{i(n+1)\theta} \xi a^{n+1})| - |2\xi^2 \alpha_n e^{i(n+1)\theta} a^{n+1}| - \left| \xi \left( e^{i(n+1)\theta} a^{n+1} - w_{\theta}^{n+1} \right) \right|
\]

\[
\geq (1 + 2\xi \alpha_n)a - 2\xi^2 \alpha_n a^{n+1} - \xi a^{n+1} \sum_{k=1}^{n+1} \binom{n+1}{k} (\xi a^n)^k
\]

\[
= a + 2\alpha_n (a - \xi a)^{n+1} - \xi a^{n+1}((\xi a^n + 1)^{n+1} - 1)
\]

\[
= a - \alpha_n a^{n+1} \left( 2\xi \alpha_n - 2\alpha_n a^{-n} + (\xi a^n + 1)^{n+1} - 1 \right)
\]

\[
= a - \alpha_n \left( 2\alpha_n (\xi - a^{-n}) + (\xi a^n + 1)^{n+1} - 1 \right)
\]

\[
= a - \frac{\alpha_n}{2(n-1)} \left( 2\alpha_n \left( \frac{\alpha_n}{2(n-1)} - a^{-n} \right) + \left( \frac{\alpha_n}{2\alpha(n-1) + 1} \right)^{n+1} - 1 \right)
\]

Now, if we were to let $x = a^{-1}$, and $t = \frac{\alpha_n}{2(n-1)}$, which turns the our desired inequality to

\[
x^{-1} - t(4tx^n(n-1)(tx - 1) + (tx + 1)^{n+1} - 1) \geq 1
\]

32
And we define the function \( f(x; t) \) as

\[
f(x; t) = x + xt[(tx + 1)^{n+1} - 1 - 4tx^n(n - 1)(1 - tx)]
\]  

(11)

Then our desired inequality becomes \( f(x; t) \leq 1 \) for those specific \( x = a^{-1} \) and \( t = \frac{\alpha_n}{2(n-1)} \). Since \( \alpha \geq 1 \), and \( \alpha_n \leq 1/2 \), if \( f(x; t) \leq 1 \) for all \( 0 \leq x \leq 1 \), and \( 0 \leq t \leq \frac{1}{4(n-1)} \), then we will have our desired inequality hold.

To show \( f(x; t) \leq 1 \), let’s differentiate with respect to \( x \).

\[
f_x(x, t) = 1 + tl[(tx + 1)^{n+1} - 1 - 4(n - 1)(1 - tx)tx^n + x(t(n + 1)(tx + 1)^n - 4(n - 1)(1 - tx)tnx^{n-1} + 4(n - 1)t^2x^n)]
\]

\[
= 1 + tl[(tx + 1)^{n+1} - 1 - 4(n - 1)(1 - tx)tx^n + tx(n + 1)(tx + 1)^n - 4tx^n(n - 1)(1 - tx) + 4t^2x^{n+1}(n - 1)]
\]

\[
= 1 + tl[(tx + 1)^n(tx + 1 + tx(n + 1)) - 1 + 4t^2x^{n+1}(n - 1) - 4tx^n(n - 1)(1 - tx)(1 + n)]
\]

\[
\geq 1 + tl[-1 - 4tx^n(n^2 - 1)(1 - tx)]
\]

\[
\geq 1 + tl[-1 - 4t(n^2 - 1)]
\]

\[
\geq 1 + tl[-1 - 4t(n^2 - 1)]
\]

The first two inequalities are simply dropping positive terms (since both \( x \) and \( t \) are positive on \( x \in [0, 1] \) and \( t \in [0, \frac{1}{4(n-1)}] \)), and the last is true since \( 0 < x \leq 1 \). Now, we will have that \( f_x \geq 0 \) if \( t(1 + 4t(n^2 - 1)) \) is less than \( 1 \) for \( t \in [0, \frac{1}{4(n-1)}] \). This is a quadratic function in \( t \), and is increasing for positive \( t \), so it attains its maximum at \( t = \frac{1}{4(n-1)} \). The value of this max is \( \frac{n+2}{4(n-1)} \), which is less than one since \( n \geq 3 \).

Thus, \( f_x(x, t) \geq 0 \) for any \( x, t \) on \( x \in [0, 1] \) and \( t \in [0, \frac{1}{4(n-1)}] \), so for any fixed \( t \), \( f(x, t) \) is an increasing function of \( x \). So \( f(x, t) \leq 1 \) follows from \( f(1, t) \leq 1 \).

Note that:

\[
f(1, t) \leq 1
\]

if and only if

\[
1 + tl[(t + 1)^{n+1} - 1 - 4(n - 1)(1 - t)t] \leq 1
\]

if and only if

\[
(t + 1)^{n+1} - 1 - 4(n - 1)(1 - t)t \leq 0
\]

\[
(t + 1)^{n+1} - 4(n - 1)(1 - t)t \leq 1
\]
Let \( g(t) = (t+1)^{n+1} - 4(n-1)(1-t)t \). We will show this is a decreasing function for \( 0 \leq t \leq \frac{1}{4(n-1)} \) and \( n \geq 3 \). \( g'(t) = (n+1)(t+1)^n + 4(n-1)(2t-1) \) is clearly increasing for all \( t \). Therefore, the derivative \( g'(t) < 0 \) for \( 0 \leq t \leq \frac{1}{4(n-1)} \) is equivalent to \( g'(\frac{1}{4(n-1)}) < 0 \). Thus we need to show

\[
(n+1)\left(\frac{1}{4(n-1)} + 1\right)^n + 4(n-1)\left(\frac{2}{4(n-1)} - 1\right) < 0
\]

\[
(n+1)\left(\frac{1}{4(n-1)} + 1\right)^n + (6-4n) < 0
\]

\[
\left(\frac{1}{4(n-1)} + 1\right)^n < \frac{4n-6}{n+1}
\]

for all \( n \geq 3 \). The left hand side is strictly decreasing in \( n \), with a value of \((9/8)^3 \approx 1.423 \) at \( n = 3 \), while the right hand side is strictly increasing, with a value of \( 6/4 = 1.5 \). Thus, the inequality is true for all \( n \geq 3 \), so \( \psi(U) \) contains the unit disk.

**Lemma 11.** The polynomial \( \psi \) is one-to-one in \( U \).

**Proof.** Suppose \( \psi(z) = \psi(w) \) for some \( z, w \in U \) with \( z \neq w \). Then we have

\[
z(1 - \xi(z^n - 2\alpha_n)) = w(1 - \xi(w^n - 2\alpha_n))
\]

\[
z - \xi z^{n+1} + 2\alpha_n \xi z = w - \xi w^{n+1} + 2\alpha_n \xi w
\]

\[-\xi z^{n+1} + z(1 + 2\alpha_n \xi) = -\xi w^{n+1} + w(1 + 2\alpha_n \xi)
\]

\[
\xi(z^{n+1} - w^{n+1}) = z(1 + 2\alpha_n \xi) - w(1 + 2\alpha_n \xi)
\]

\[
\xi(z^{n+1} - w^{n+1}) = (z - w)(1 + 2\alpha_n \xi)
\]

\[
\frac{z^{n+1} - w^{n+1}}{z - w} = \xi^{-1} + 2\alpha_n
\]

\[
\frac{z^{n+1} - w^{n+1}}{z - w} = 2(n-1)a^{n+1}\alpha_n^{-1} + 2\alpha_n
\]

Now, we can find a lower bound for the left hand side, and an upper bound for the right, to show that equality is impossible.

For the left hand side, note that

\[
\frac{z^{n+1} - w^{n+1}}{z - w} = z^n + z^{n-1}w + z^{n-2}w^2 + \cdots + zw^{n-1} + w^n = \sum_{k=0}^{n} z^{n-k}w^k
\]
Thus, we can see that
\[
\left| \frac{z^{n+1} - w^{n+1}}{z - w} \right| \leq \sum_{k=0}^{n} \left| z^{n-k} w^k \right|
\]
\[
\leq \sum_{k=0}^{n} \max_{x \in U} |x|^{n-k} |x|^k
\]
\[
= \sum_{k=0}^{n} \max_{x \in U} |x|^n
\]
\[
= (n + 1) \max_{x \in U} |x|^n
\]
\[
= (n + 1) \max_{\theta \in [0, 2\pi]} |w(\theta)|^n
\]

Where the second line holds since \( z, w \in U \), and the last line holds since \( U \) is the open set bounded by \( w(\theta) \). Note \( w(\theta) = e^{i\theta} \left( a + e^{-in\theta} \frac{\alpha_n}{2(n-1)} \right) \) which has a maximum magnitude of \( a + \frac{\alpha_n}{2(n-1)} \). Thus, it suffices to show that
\[
\left( a + \frac{\alpha_n}{2(n-1)} \right)^n < 2 \frac{(n-1)a^{n+1}\alpha_n^{-1} + \alpha_n}{n + 1}
\]

Note that the constant \( a \), from (9) is the same as in \( w(\theta) \), so we know \( a \geq 1 \). Now, for \( n \geq 3 \), we have on the left hand side
\[
\left( a + \frac{\alpha_n}{2(n-1)} \right)^n = a^n \left( 1 + \frac{\alpha_n}{2a(n-1)} \right)^n
\]
\[
\leq a^n \left( 1 + \frac{1}{4a(n-1)} \right)^n \quad \text{Since } \alpha_n \leq 1/2
\]
\[
\leq a^n \left( 1 + \frac{1}{4(n-1)} \right)^n \quad \text{Since } a \geq 1
\]
\[
\leq a^n \left( 1 + \frac{1}{8} \right)^3 \quad \text{Since } n \geq 3
\]
\[
\leq 1.43 a^n
\]
We can do a similar bound on the right hand side.

\[
2 \frac{(n - 1)a^{n+1}a_n^{-1} + a_n}{n + 1} \geq 2 \frac{(n - 1)a^{n+1}a_n^{-1}}{n + 1} \quad \text{Since } \alpha_n \geq 0
\]

\[
\geq 4 \frac{(n - 1)a^{n+1}}{n + 1} \quad \text{Since } \alpha_n^{-1} \geq 2
\]

\[
\geq 4 \frac{(n - 1)a^n}{n + 1} \quad \text{Since } a > 1
\]

\[
\geq 2a^n \quad \text{Since } n \geq 3, \text{ so } \frac{n - 1}{n + 1} \geq 1/2
\]

Thus, the left hand side is less than 1.43a^n and the right hand side is larger than 2a^n, so equality is impossible. Therefore, \( \psi \) must be one-to-one. \( \square \)

This completes the proof of Crouzeix’s Conjecture for \( J_\alpha \). To reiterate the outline of the proof; by Theorem 7 and Lemma 4, we get that Crouzeix’s Conjecture holds for \( J_\alpha \) if \( cs \leq 1 \), where \( s \) is defined in equation (7). By Theorem 8, \( cs \leq 1 \) for all values of \( s \) except \( s = \left( \frac{J_n - 1}{2} \right)^{(1/(n-1))} \). For the case when \( s = \left( \frac{J_n - 1}{2} \right)^{(1/(n-1))} \), we found the function \( \psi \) and set \( U \) such that \( \psi(U) \) contains the unit disk \( \mathbb{D} \), and \( \psi \) is one-to-one from \( U \) to \( \psi(U) \). Thus, we may define the function \( g \) from \( \mathbb{D} \) to \( \mathbb{D} \) by \( g(z) = \phi \circ \psi^{-1}(z) \). See Figure 2.2 for a visual aid for \( g \). \( \phi \) and \( \psi \) have properties such that \( g(0) = 0 \) and \( |g(z)| \leq 1 \) for \( z \in \mathbb{D} \), so we may apply Schwarz’ lemma on \( g \). Therefore, \( |g(z)| \leq |z| \) for all \( z \in \mathbb{D} \). In particular, \( |g \left( \frac{\lambda}{2} \right) | \leq \left| \frac{\lambda}{2} \right| \). Since \( g \left( \frac{\lambda}{2} \right) = c\lambda \), we have that \( cs \leq 1 \). Therefore, \( cs \leq 1 \) for any choice of \( s \), and so Crouzeix’s Conjecture holds for \( J_\alpha \).
References


