OPTIMIZATION OF INTERPLANETARY TRANSFER TRAJECTORIES
USING THE INVARIANT MANIFOLDS OF HALO ORBITS

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ABSTRACT

Optimization of Interplanetary Transfer Trajectories Using the Invariant Manifolds of Halo Orbits

Ryan Yedinak

Traditionally, two-body dynamics have been used to design orbital trajectories for interplanetary missions using a series of Lambert’s transfers and gravity assists. Although these are reliable methods, they have extremely high fuel requirements, especially for missions to outer planets. From an orbital mechanics perspective, three ways of reducing fuel requirements for these types of missions are utilizing low energy transfer trajectories, applying low thrust engine parameters, and implementing orbit optimization techniques.

The goal of this thesis is to combine low energy transfers created from the dynamics of the Circular Restricted Three-Body Problem (CRTBP) with low-thrust orbit optimization techniques to develop interplanetary missions that require less fuel to reach their destinations when compared to transfers that implement traditional two-body dynamics. A Patched Conics with Manifolds method was used to maneuver a spacecraft from low Earth orbit to a halo orbit and then use exterior manifolds to link a spacecraft along a low-thrust interplanetary trajectory between the halo orbits of Earth and the destination planet. Indirect and direct optimization are applied to both an initial orbit raising maneuver and the low-thrust interplanetary transfer to attempt to reduce the fuel requirements needed for each mission phase. These transfers are tested for missions from Earth to Jupiter and Earth to Saturn and the fuel requirements and time of flight are analyzed.

Results from this research showed that applying low thrust optimization to portions of the interplanetary transfer using manifolds reduced the fuel required for each phase of
the transfer for both missions to Jupiter and Saturn. For the low thrust orbit raising phase, the fuel requirements were reduced by 13-17%, while for the interplanetary phase fuel requirements were reduced by 23-27% for the mission to Jupiter and 15-17% for the mission to Saturn when compared to previous methods. In addition, additional fuel savings are found by the elimination of the need for a second stage to arrive at the destination planet due to the low-thrust maneuver accounting for the reduction in speed due to the constant burn throughout the interplanetary phase.

Analysis of the flight times for each test case showed that the Patched Conics with Manifolds method combined with low-thrust maneuvers increased the amount of travel time by 6.5 years for a mission to Jupiter and 8.5 years for a mission to Saturn. Overall, the fuel and time of flight results show that there is a trade-off between significantly reducing the fuel needed for this type of transfer and significantly increasing the transfer time that must be considered for each particular mission application.
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Chapter 1

INTRODUCTION

1.1 Problem Statement

Throughout the history of spaceflight, interplanetary travel has been accomplished using the principles of two-body dynamics and gravity assists to reach the distant locations in our solar system. These techniques proved to be successful in the 1960s with the Voyager program, and have continued this success with recent missions such as New Horizons. However, with the continued interest in interplanetary exploration, orbital mechanics solutions must be developed to reduce the amount of fuel required to reach these destinations in favor of including more science equipment, traveling further, or visiting multiple locations in one mission. Three solutions for accomplishing this goal are utilizing low energy transfer trajectories, applying low thrust engine parameters, and implementing orbit optimization techniques.

N-body dynamical systems, which take into account the gravitational effects of more than two objects, can be used to produce low energy trajectories for traveling through space. In 2018, Rund applied the Circular Restricted Three Body Problem (CRTBP) to compare transfers using manifolds of halo orbits to reduce the $\Delta V$ necessary to connect a spacecraft to a hyperbolic escape trajectory on a course to an outer planet [12]. This method, called the Manifold to Hyperbolic Escape Trajectory method, was tested on transfers to both Jupiter and Saturn. When compared to traditional Hohmann transfers, the Manifold to Hyperbolic Escape Trajectory method reduced the $\Delta V$ by 1 km/s for a mission to Jupiter and 3.6 km/s for a mission to Saturn [12]. In addition, Topputo et al. [13] explored using an alternative interplanetary
transfer method using manifolds in which the exterior manifolds of halo orbits in two planetary systems are used as departure and arrival points for an interplanetary Lambert’s transfer. This method, deemed the Patched Conics with Manifolds method, was tested on missions to Mars and Venus. When compared to Hohmann transfers, the Patched Conics with Manifolds method reduced the $\Delta V$ by 0.7 km/s for a mission to Mars and 0.6 km/s for a mission to Venus [13].

Low-thrust engines with high specific impulses can be used to constantly accelerate spacecraft at low levels of thrust, decreasing the amount of fuel mass required to reach its destination. Combining these engine parameters with orbit optimization techniques is useful for finding the optimum trajectory to minimize the amount of fuel needed to complete a mission. In 2016, Pritchett compared the results of applying indirect and direct optimization methods to both circular-to-circular orbit transfers and transfers between manifolds of halo orbits [11]. Although minimal differences in fuel mass reduction were found for the applications of circular-to-circular and manifold-to-manifold orbit transfers, the principles of orbit optimization were proven to be successful in determining the minimum parameter required to complete the mission objective. Moore et al. [8] as well as Mingotti et al. [7] have applied optimal control theory and direct transcription optimization principles to transfers connecting Earth unstable manifolds and Lunar stable manifolds for missions between Earth and the Moon. In both cases, the $\Delta V$ needed to complete these missions when compared to methods using two-body dynamics were reduced by about 0.1 km/s.

The findings discussed above have shown that low-thrust trajectories and orbit optimization techniques are useful in reducing fuel requirements for a range of mission applications. By combining the principles of n-body dynamics and low-thrust orbit optimization, the $\Delta V$ required for interplanetary missions can be lowered significantly, making missions that may have once been impossible now seem feasible.
1.2 Thesis Objectives and Scope

The main objective of this thesis is to combine low energy transfer trajectories with low-thrust orbit optimization techniques to develop interplanetary missions that reduce the fuel necessary to reach their destination when compared to transfers that implement traditional two-body dynamics. Specifically, a variation of the Patched Conics with Manifolds method presented by Topputo [13] will be combined with optimized low-thrust trajectories created using indirect and direct optimization approaches, such as those presented by Pritchett [11], to reduce the amount of fuel needed to reach interplanetary destinations.

To accomplish this goal, a mission profile must be developed that incorporates both trajectories that result from the CRTBP and optimized low-thrust trajectories. Manifolds constructed from halo orbits about Lagrange points of Earth and the destination planet will be used as departure and arrival trajectories that link the interplanetary transfer between the two planets. One optimized low-thrust trajectory created using two-body dynamics will boost the spacecraft from a low Earth orbit to a target higher Earth orbit, while another will be used for the interplanetary phase of the transfer. The principles of the CRTBP and low-thrust transfer trajectories will be discussed in Chapter 3, while the full mission profile will be discussed in Chapter 4.

In order to minimize fuel requirements for these transfers, indirect and direct optimization methods will be applied and compared. Indirect optimization involves using Euler-Lagrange theory to convert the problem of optimizing a cost function into a two-point boundary value problem (TPBVP) [11]. Direct optimization, on the other hand, involves operating on the cost function directly by implicitly integrating on the equations of motion to transcribe an optimal control problem into a nonlinear programming problem [11]. For the purposes of this thesis, collocation, which is a
form of direct transcription, will be applied by approximating the state and control variables using polynomial functions. Each of these methods will be explained in Chapter 5.

The interplanetary transfer method and orbit optimization schemes developed in this thesis will be applied to theoretical missions to Jupiter and Saturn. The main goal will be to compare the amount of fuel needed, as a percentage of total spacecraft mass, to complete each mission using the methods outlined in this thesis to the methods used by Rund [12], as well as to Hohmann transfers developed from traditional two-body dynamics. Time of flight to complete each mission will also be compared. The results for both missions will be presented in Chapter 6 and final conclusions and future work will be discussed in Chapter 7.
Chapter 2

LITERATURE REVIEW

In this chapter a review of previous work in the areas of orbital transfers using three-body dynamics and orbit optimization methods will be presented along with how each topic relates to the work done in this thesis. The past missions and research done in the Sun-Earth and Earth-Moon three-body systems will be discussed along with research involving interplanetary transfers using three-body dynamical systems. In terms of orbit optimization, both indirect and direct methods will be discussed along with their application to both two-body and three-body systems for orbital transfers.

2.1 Orbital Transfers Using Three-Body Dynamics

The study of the gravitational effects that three masses have on each other was first formulated by Newton in the 1600s, and refined by Poincaré in the 1890s [11]. It was not until the late 1960s that research began on using the CRTBP to create orbital trajectories for space missions. Since this early research only a few missions have utilized these dynamics to create their trajectories, with all of them being Earth orbiting spacecraft. These dynamics have also been studied to create low-energy trajectories for missions between Earth and the Moon and for interplanetary missions.

The Genesis mission was designed by NASA’s Jet Propulsion Laboratory (JPL) to observe the solar wind from a point outside Earth’s atmosphere and magnetic field environment. To accomplish this, the spacecraft traveled to the Sun-Earth L₁ point via a stable manifold and then completed five orbits about the L₁ point to gather data. The spacecraft then left the L₁ point along an unstable manifold and connected
to an $L_2$ stable manifold to loop around the $L_2$ point in order to return to Earth in daylight. By using the invariant manifold trajectories, Genesis used much less fuel than it would have had it not taken advantage of three-body dynamics [9]. Figure 2.1 shows the trajectory that the Genesis spacecraft used to complete its mission [12].

Another mission designed using three-body dynamics is the James Webb Space Telescope (JWST) which is set to launch in late 2021. The JWST will use these dynamics to orbit the Sun-Earth $L_2$ point to observe distant objects in the universe and investigate the formations of galaxies [10]. The orbit created using three-body dynamics is advantageous for this type of mission because it allows the spacecraft’s position to remain relatively in-line with the Sun and Earth due to the rotation of the Lagrange point with the rotating frame. This means the spacecraft will have a constant line of sight with Earth for relaying data and always have the Sun behind the telescope to make better observations of objects in deep space.

Figure 2.1: Trajectory of the Genesis Mission, with the approach trajectory (purple), halo orbit (red), and return trajectory (blue) [12].
In addition to utilizing manifold connections to reach orbits about Lagrange points in the Sun-Earth system, research has been done using the same methods for creating manifold trajectories between the Earth and Moon. In 1993 Belbruno and Miller studied the connection of unstable manifolds, a collection of trajectories that depart a Lagrange point, from the Sun-Earth L₂ to stable manifolds, a collection of trajectories that arrive at a Lagrange point, from the Earth Moon L₂ to complete low energy transfers from the Earth to the Moon [1]. This is possible because if each set of manifolds is propagated for long enough, connections can be made from the departing Sun-Earth L₂ unstable manifolds to the Earth-Moon L₂ stable manifolds in the configuration space. This ultimately allows for the creation of a single orbital path with a small burn at the manifold-manifold overlap point to complete the transfer. An example of this type of transfer, where unstable manifolds departing Earth and stable manifolds approaching the Moon can be connected to create a single trajectory, is shown in Figure 2.1 [8].

![Figure 2.2: Trajectory an Earth to Moon transfer using manifolds (black-dotted), with Sun-Earth L₂ unstable (red) and stable (blue) manifolds, and Earth-Moon L₂ stable manifolds (green) [8].](image)
Researchers have also investigated the possibilities of using these dynamics to create interplanetary trajectories that can reduce the large amounts of fuel required to reach these destinations using traditional transfer methods. At JPL, researchers including Martin Lo have developed the idea of an interplanetary superhighway in which manifolds are used to travel around the solar system with low fuel requirements [12]. However, for interplanetary missions other than those between outer planets, the unstable manifolds of the departure planet and stable manifolds of the arrival planet will only ever connect in the configuration space after propagation of the orbits for decades or even hundreds of years due to the vast distances between planets [13].

The two methods that have been developed to overcome this problem are the Patched Conics with Manifolds method and Manifold to Hyperbolic Escape Trajectory method. The Patched Conics with Manifolds Method, explored by Topputo et al. [13], matches the unstable manifolds of the departure planet and stable manifolds of the arrival planet in the phase space rather than the configuration space. This allows for a spacecraft to use a transit orbit based on the unstable manifolds of the departure planet to connect to an interplanetary transfer in the two-body space, and then connect to a ballistic capture transit orbit via the stable manifolds at the arrival planet. This method was tested for missions to Venus and Mars and found that the \( \Delta V \) needed to complete the interplanetary transfer using manifolds was reduced by 0.6 km/s and 0.7 km/s, respectively, when compared to Hohmann transfers [13]. Figure 2.3 shows an example of the interplanetary phase of the Patched Conics with Manifolds transfer for a mission to Mars as developed in Topputo [13].
The Manifold to Hyperbolic Escape Trajectory method used by Rund [12], links a spacecraft from an unstable manifold at the departure planet to a hyperbolic escape trajectory to connect to an interplanetary trajectory. Once the spacecraft reaches the destination planet, it connects to a ballistic capture trajectory at the arrival planet that matches with a stable manifold to coast from the interplanetary transfer exit to a halo orbit in the arrival planet system. When this method was applied to missions to Jupiter and Saturn, it was determined that the $\Delta V$ savings were approximately 1 km/s and 3.6 km/s, respectively [12]. Figure 2.4 shows an example of a trajectory created using the Manifold to Hyperbolic Escape Trajectory method as applied by Rund [12].
Figure 2.4: Manifold to Hyperbolic Escape Trajectory Method trajectory showing the Sun-Earth L₁ halo orbit (blue), unstable manifold (purple), and hyperbolic escape trajectory (green) for the departure phase of a mission to Jupiter [12].

Although both methods were able to reduce the amount of fuel needed to reach the interplanetary destinations studied, further research could be done into optimizing these transfers to further reduce fuel requirements. This thesis intends to commit to that research by exploring the use of low-thrust orbit optimization techniques as applied to interplanetary transfers using manifolds. When comparing each method’s viability for the application of low-thrust transfers, the Patched Conics with Manifolds method is more suitable due to the transition phases between the Earth, interplanetary transfer, and destination planet along the exterior manifolds of the departure and arrival systems. However, the method of reaching the halo orbit discussed in Rund is utilized rather than creating a transit orbit that passes through the halo used by Topputo. This is because it is more difficult to find a transit orbit that passes close to Earth (within 50,000 km) than to find a stable manifold directly from a halo orbit that passes close to Earth [13]. In addition, the results from Rund were used as a point of comparison to provide more insight into how the use of manifolds for interplanetary transfers affect the fuel savings for missions to outer planets.
2.2 Trajectory Optimization Approaches

With the improvements in guidance, navigation, and control systems for space applications, trajectory optimization methods have been incorporated into mission designs in order to minimize specific parameters such as fuel consumption or travel time. Optimization methods are generally placed into one of three categories: indirect methods, direct methods, and evolutionary algorithms. As the goal of this thesis is to apply low-thrust trajectory optimization techniques to interplanetary transfers using manifolds to reduce fuel requirements, further research was done on each type of method. This thesis will focus on both indirect and direct optimization methods in the form of Euler-Lagrange optimization and collocation, respectively. Evolutionary algorithms were not considered as the approaches for each evolutionary method vary widely so it would be difficult to determine which approach is best suited for the transfers optimized in this work.

The indirect optimization approach is derived from the calculus of variations which uses changes in functions and functionals (a mapping of a space of functions into real numbers) to find the maximum or minimum values of the functionals. An example problem in which the calculus of variations is used is the brachistochrone problem raised by Bernoulli, the goal of which is to determine a path (function) that minimizes a scalar function of that path (functional) [11]. The calculus of variations was eventually refined by Euler and Lagrange into the Euler-Lagrange Theorem, which is a method of determining a function that minimizes a functional by using a system of second-order differential equations whose solutions are zero for the given functional [11]. The advantages of indirect methods are that they provide high accuracy for solutions, are lower in computational cost and time, and provide theoretical insight.
into the problem. On the other hand, these methods have difficulty in characterizing systems with complex equations of motion.

While indirect methods are reliable for solving continuous optimization problems by converting them into TPBVPs, direct methods address the drawbacks of indirect methods. Direct optimization works by converting the continuous optimal control problem into a set of discrete variables which can be reformulated into a nonlinear programming problem [4]. The process by which the optimal control problem is discretized is called direct transcription. By discretizing the system, the need for costate variables used for indirect methods, which can be difficult to solve for more complex problems, is removed. The scheme by which direct transcription is used is often denoted by the form of integration used to determine the optimal path or control states within the problem, with one of the more popular methods being collocation [4]. The advantages of direct methods are that they are more capable of dealing with complex problems and constraints are easier to structure, while the disadvantages are lower accuracy of solutions and higher computational cost and time.

2.3 Optimization as Applied to Orbital Mechanics Problems

Optimization algorithms have been used in both two-body systems and three-body systems for a variety of applications including low-thrust orbit raising trajectories, minimum fuel trajectories for interplanetary transfers, and determining the ideal locations for mid-course corrections throughout a mission, among others. This thesis intends to apply the indirect and direct optimization approaches discussed above to portions of an interplanetary transfer that utilizes manifolds. Therefore, the orbital mechanics applications of both methods were explored to determine the viability of applying them to this type of transfer.
Optimization algorithms are particularly useful in three-body systems when attempting to create connections between departure and arrival manifolds created from orbits about Lagrange points of planetary bodies. This has been studied thoroughly for transfers between the Earth and the Moon. By finding a link point between the unstable manifolds from an orbit about a Lagrange point in the Sun-Earth-Spacecraft three-body system to the stable manifolds created from an orbit about a Lagrange point in the Earth-Moon-Spacecraft three-body system.

The Discrete Mechanics and Optimal Control (DMOC) method developed by Moore et al. at JPL was used to create minimum fuel trajectories as applied to transfers that start in Earth orbit and connect Sun-Earth unstable manifolds to Earth-Moon stable manifolds to arrive at a lunar orbit. This method is based on a discretization of the Lagrange-d’Alembert principle that leads to forced discrete Euler-Lagrange equations that are used as constraints in an optimal control algorithm to solve a cost function [8]. The optimization can then be solved using a sequential quadratic programming (SQP) solver. The results of this analysis showed that the $\Delta V$ needed to travel from Earth to the Moon was reduced by about 0.3 km/s when compared to a traditional Hohmann transfer between the two bodies [8].

In addition, Mingotti et al. uses collocation along with attainable sets to determine a suitable initial guess to find optimal trajectories between the Earth and Moon by connecting manifolds. An attainable set is a collection of low-thrust trajectories that reach a defined surface perpendicular to the x-y plane at different times based on thrust profiles found by applying an optimal control algorithm [7]. A collocation scheme with multiple shooting is used to convert the optimal control problem into a nonlinear programming problem [7]. When applied to a transfer from Earth to a halo orbit about the Earth-Moon L1 point, the $\Delta V$ was found to be 3.6 km/s for the entire
transfer, which is slightly higher than the $\Delta V$ determined by the DMOC algorithm for a similar transfer [7].

This research intended to create low-thrust trajectories directly along the manifolds to link a spacecraft from a circular orbit about Earth to an interplanetary transfer like in the research discussed above. However, it proved to be difficult to apply the indirect optimization method to such a maneuver due to the large variability between the costates at the initial point in the Earth orbit and the final point in the halo orbit. Small differences in the guesses of the initial costates led to large differences in the final costates. Therefore, it was determined that applying low-thrust optimization methods to transfers that use two-body dynamics, such as an orbit raising or interplanetary maneuver, would be more suitable. Therefore, applications of optimized low-thrust transfers in two-body systems were explored.

To illustrate a simple application of indirect and direct optimization techniques using two-body dynamics, Pritchett tested such algorithms on low-thrust circular-to-circular orbit transfers [11]. Euler-Lagrange theory was used for the indirect approach, while a high-order Legendre-Gauss collocation collocation scheme was used for the direct approach. Variations of the low-thrust optimization principles presented by Pritchett were used as a basis for the low-thrust orbit raising and interplanetary transfers created in this thesis. By inserting these low-thrust trajectories into the mission profile for an interplanetary mission that utilizes the Patched Conics with Manifolds method, testing could be done in this work to determine if adding low-thrust optimization would reduce fuel requirements for interplanetary missions.
Two-body systems, those in which a larger primary body (a planet) exerts a gravitational force on a smaller secondary body (the spacecraft), are the standard model used for determining spacecraft trajectories. However, \( n \)-body systems, those that take into account the gravitational forces of \( n \) primary bodies acting on the spacecraft, have also been analyzed to create trajectories by taking advantage of the unique dynamics of the system. In this chapter, the principles of an \( n \)-body system, particularly the Circular Restricted Three-Body Problem (CRTBP), will be presented along with the different types of orbit maneuvers, impulse and low-thrust, to lay the groundwork for the mission profile discussed in Chapter 4. In addition, it will be assumed that no external forces other than the gravitational forces of the primary bodies will be acting on the spacecraft, when in reality, other forces such as solar radiation pressure, aerodynamic drag, and primary body oblateness will have an effect. These other external forces will be ignored as they are much smaller in magnitude than these gravitational forces.

### 3.1 Impulse vs. Low-Thrust Transfers

There are two types of orbital maneuvers: impulsive transfers and low-thrust transfers, and thus two types of engines needed to complete these transfers: high thrust engines and low-thrust engines. Impulsive maneuvers are those that occur instantaneously with respect to the time scale of the transfer and require high thrust chemical engines to produce a large amount of thrust. By producing this high thrust, the space-
craft’s velocity vector can be changed instantaneously without changing its position vector. An example of a spacecraft completing an impulsive transfer from a 2,000 km altitude circular orbit to a 20,000 km altitude circular orbit about Earth is shown in Figure 3.1.

![Figure 3.1: An example of an impulsive transfer.](image)

In Figure 3.1, the spacecraft completes a transfer from the red 2,000 km orbit to the blue 20,000 km orbit via the black transfer path. The spacecraft only needs to thrust to change its velocity at two points: once at point A where the red and black orbits intersect, and once at point B where the black and blue orbits intersect. By completing the first burn, the spacecraft will change its velocity and increase the semi-major axis of its orbit to begin its coast along the black orbit. Completing the
second burn then allows the spacecraft to slow down and decrease the semi-major axis of its orbit to arrive along the higher orbit.

On the other hand, low-thrust maneuvers are completed by electric propulsion engines that produce much less thrust than their chemical counterparts, but are continuously thrusting for most of the orbit, changing both the position and velocity vector continuously over time. Low-thrust engines have an advantage over chemical engines as they are much more fuel efficient, but result in much longer transfer times due to the low levels of thrust produced. An example of a spacecraft completing a low-thrust transfer from a 2,000 km altitude circular orbit to a 20,000 km altitude circular orbit about Earth is shown in Figure 3.2.

![Figure 3.2: An example of a low-thrust transfer.](image)
In Figure 3.2, the spacecraft completes a transfer from the red 2,000 km orbit to the blue 20,000 km orbit via the black transfer path. Along the black transfer path, the spacecraft is continuously thrusting as the thrust-pointing vector for the spacecraft changes over time. The thrust pointing vector is denoted by the orange arrows extending from the transfer path. Since a small $\Delta V$ is imparted throughout the transfer, the spacecraft spirals out to the final orbit over a much longer time period, unlike the impulsive transfer shown previously. In this work, both impulsive and low-thrust transfers will be used at different points of the interplanetary transfers modeled, and will be explained further in the mission profile in Chapter 4.

### 3.2 Circular Restricted Three-Body Problem

The Circular Restricted Three-Body Problem (CRTBP) is a dynamical system that takes into account the effects of the gravitational forces of two primary bodies acting on one smaller secondary body. For example, for a spacecraft traveling in the Sun-Earth system, the Sun and Earth would be the two primary bodies, while the spacecraft would be the secondary body. Within this system, the mass of the secondary body is small enough that the gravitational effects it has on the system can be ignored and it is assumed that the orbits of the primary bodies are circular [12].

These are valid assumptions for most Sun-Planet systems in the solar system due each planet’s orbit about the Sun having an eccentricity close to zero. In addition, when comparing the mass of the spacecraft (secondary body) to the masses of the celestial bodies (primaries), the order of magnitude is much smaller, making it essentially negligible. Through defining the system in this way, several unique trajectories can be created due to the gravitational relationships between the three bodies.
3.2.1 Synodic Frame

The most commonly used reference frame for the CRTBP is the synodic coordinate frame, where the origin of the system is set at the center of mass of the primary bodies and is known as the barycenter. The x-y plane of the frame makes up the orbital plane in line with the two primary bodies and the z-direction is out-of-plane with the primary bodies. The frame rotates about the fixed center of mass with an angular velocity of $\omega_s$. Figure 3.3 shows a diagram of the synodic coordinate frame [12]. In this diagram the unit vectors $\hat{x}_s$ and $\hat{y}_s$ define the x and y axes of the frame while the z axis points out of the page. The more massive primary body, for example the Sun, is represented by $m_1$ while the smaller of the primaries, for example the Earth, is represented by $m_2$. The distances $r_{B,1}$ and $r_{B,2}$ are the distances between the primary bodies $m_1$ and $m_2$ and the barycenter, respectively. The vectors $\vec{r}_1$ and $\vec{r}_2$ are the distances between the primary bodies and the spacecraft, while $\vec{r}_{B,sc}$ is the distance from the barycenter to the spacecraft.
3.2.2 Equations of Motion

When analyzing the dynamics of the CRTBP, canonical units are used to simplify the calculations and avoid the issue of the masses of and distances between objects not being well known. This is carried out by setting $m_1 = 1 - \mu^*$ and $m_2 = \mu^*$, where $\mu^*$ is defined as the mass ratio:

$$\mu^* = \frac{m_2}{m_1 + m_2} \quad (3.1)$$

The angular velocity of the rotation of the primaries about the barycenter $\omega_s$ is also set equal to one. The synodic coordinate frame used in this thesis will be the right-handed system, in which the smaller primary body is located along the positive x-axis,
so the distances between the primaries and the barycenter then become $r_{B,1} = -\mu^*$ and $r_{B,2} = 1 - \mu^*$. The magnitudes of the distances between the primary bodies and the spacecraft, $\vec{r}_1$ and $\vec{r}_2$ can be calculated using the following equations:

\[ r_1 = \sqrt{(x + \mu^*)^2 + y^2 + z^2} \quad (3.2) \]

\[ r_2 = \sqrt{(x + \mu^* - 1)^2 + y^2 + z^2} \quad (3.3) \]

The equations of motion for the acceleration of the secondary body in $x$, $y$, and $z$ for the CRTBP dynamics are shown in the equations below. A full derivation of the equations can be found in Vallado [15].

\[ \ddot{x} = 2\dot{y} + x - \frac{(1 - \mu^*)(x + \mu^*)}{r_1^3} - \frac{\mu^*(x + \mu^* - 1)}{r_2^3} \quad (3.4) \]

\[ \ddot{y} = -2\dot{x} + y - \frac{y(1 - \mu^*)}{r_1^3} - \frac{y\mu^*}{r_2^3} \quad (3.5) \]

\[ \ddot{z} = -\frac{z(1 - \mu^*)}{r_1^3} - \frac{z\mu^*}{r_2^3} \quad (3.6) \]

### 3.2.3 Lagrange Points

For each three-body system within the CRTBP, there are five equilibrium points in the synodic frame where the velocity and acceleration, due to gravitational fields and the momentum of the system, are zero. These points are known as Lagrange or Libration points. The $L_1$, $L_2$, and $L_3$ points are colinear and lie along the x-axis of the synodic frame, while the $L_4$ and $L_5$ points lie in the x-y plane where the distances between each primary body and the barycenter, $\vec{r}_1$ and $\vec{r}_2$, are equal. In addition, all Lagrange points within the three-body system have a z-component equal to zero, or oscillations would occur within the system.
The $L_1$ is defined as the point between the two primaries, the $L_2$ is the point outside the smaller of the primaries, and the $L_3$ is the point on the opposite side of the larger primary. The $x$ locations of these three points can be found using Equation 3.7 through Equation 3.9, while the $y$ coordinate of these points is zero as they lie along the $x$-axis [15].

\[
L_1 = x - \frac{(1 - \mu^*)}{(x + \mu^*)^2} + \frac{\mu^*}{(x + \mu^* - 1)^2} = 0 \tag{3.7}
\]

\[
L_2 = x - \frac{(1 - \mu^*)}{(x + \mu^*)^2} - \frac{\mu^*}{(x + \mu^* - 1)^2} = 0 \tag{3.8}
\]

\[
L_3 = x + \frac{(1 - \mu^*)}{(x + \mu^*)^2} + \frac{\mu^*}{(x + \mu^* + 1)^2} = 0 \tag{3.9}
\]

The $x$ and $y$ coordinates of the $L_4$ and $L_5$ points can then be found using Equation 3.10 and Equation 3.11 [15].

\[
L_4 = \left(\frac{1}{2} - \mu^*, \frac{\sqrt{3}}{2}\right) \tag{3.10}
\]

\[
L_5 = \left(\frac{1}{2} - \mu^*, -\frac{\sqrt{3}}{2}\right) \tag{3.11}
\]

An example of the Lagrange points for the Sun-Earth system is shown in Figure 3.4.
3.3 Halo Orbits

Halo orbits are characterized as perfectly periodic orbits about Lagrange points that retrace their path in space repeatedly rather than diverging onto a different trajectory. These types of orbits, when set about the $L_1$ or $L_2$ points, are advantageous for deep space telescopes, Sun observing satellites, or Earth relay satellites because the Sun or Earth remain in the same location relative to the spacecraft throughout its orbit.

There are two classes of halo orbits: northern halos and southern halos. For northern halos, the spacecraft spends the majority of the orbit above the x-y plane in the synodic frame, while for southern halos, the spacecraft spends the majority of the orbit below the x-y plane in the synodic frame. Examples of northern and southern halo orbits in the x-z plane of the synodic frame are shown in Figure 3.5.
Figure 3.5: Examples of an $L_2$ northern class halo (top) and $L_2$ southern class halo (bottom) about Sun-Earth L2 with a $z$-amplitude of 100,000 km.

Halo orbits are defined by their amplitude in the $x$, $y$, and $z$ direction within the synodic frame, and the size of a halo orbit can be changed by altering the amplitude of the halo in the $z$-direction. Figure 3.6 shows a family of halo orbits about the Sun-Earth $L_2$ with $z$-direction amplitudes ranging from 100,000 km to 1,000,000 km.
Figure 3.6: Family of halo orbits about Sun-Earth L₂ ranging from 100,000 km to 800,000 km in z-amplitude.

The most accurate approximation of a halo orbit is determined by first finding an initial estimate of a state at time zero using an analytical solution, and then using this initial state as an estimate to numerically integrate on to determine the correct initial state and then the rest of the orbit. The analytical solution is found using the Lindstedt-Poincare method as derived by Richardson and presented in Koon [5], while the numerical solution is developed by Howell [3] which relies on iteration to find a solution by altering the initial state until established tolerances are on the met changes in velocity needed to establish the entire orbit. The equations and calculation processes needed to solve for the numerical and analytical solutions are summarized in Rund [12] and were used in this work to establish the halo orbits needed to create the manifolds used for the interplanetary mission profile discussed in Chapter 4.
3.4 Invariant Manifolds

Manifolds are sets of trajectories that either approach or depart from a Lagrange point. Stable manifolds are paths that travel toward the Lagrange point, while unstable manifolds travel away from the point. Manifolds are calculated by finding the eigenvalues and eigenvectors of the Jacobian of the state vector at a Lagrange point. The eigenvector corresponding to the larger of the two real eigenvalues of the Jacobian indicates the direction of the stable manifold while the eigenvector of the smaller real eigenvalue indicates the direction of the unstable manifold [12]. By perturbing each eigenvector, the initial state of the manifold can be found. The full manifold trajectories can then be found by propagating the initial state using the equations of motion of the CRTBP [12]. The stable manifolds are found by propagating the initial states forward in time while the unstable manifolds are found by propagating the initial states backwards in time [12]. Unstable periodic orbits about Lagrange points also have stable and unstable manifolds. Since a manifold trajectory can be created for each point along a halo orbit, for example, this set of trajectories form tube-like structures known as invariant manifolds.

Once all points on a halo orbit have been found, invariant manifolds can be calculated using a numerical solution. First, the eigenvalues and eigenvectors of the monodromy matrix — the state transition matrix after one orbital period (at time $T$) — are found and used in place of the Jacobian at each point in the halo orbit [12]. This is because it is more computationally expensive to use the Jacobian, and the monodromy matrix, calculated using Equation 3.12, contains information about the stability of the entire orbit, so its eigenvalues are still usable [12].

$$M = \Phi(t_0 + T, t_0)$$

(3.12)
Of the six eigenvalues and eigenvectors of the monodromy matrix, only the eigenvectors for the maximum and the minimum real eigenvalues are needed to find the invariant manifolds. The eigenvector corresponding to the minimum real eigenvalue is used to calculate the stable manifolds, while the eigenvector corresponding to the maximum real eigenvalue is used to calculate the unstable manifolds [12]. The eigenvector for the stable manifold will be denoted as $V^S$ and the eigenvector for the unstable manifold will be denoted as $V^U$. These eigenvectors can then be found at every point along the halo orbit using the state transition matrix at each time step. This is shown in Equation 3.13 and Equation 3.14, where $i$ represents the point along the halo orbit at which the eigenvector is being evaluated [12].

$$V^S_i = \Phi(t_0 + t_i, t_0)V^S$$

$$V^U_i = \Phi(t_0 + t_i, t_0)V^U$$

In order to find the initial conditions for the stable and unstable manifold trajectories from each point along the halo ($X^S_i$ and $X^U_i$), the eigenvectors are used to perturb the state at each point on the halo using the following equations [12]:

$$X^S_i = X_i \pm \epsilon \frac{V^S_i}{|V^S_i|}$$

$$X^U_i = X_i \pm \epsilon \frac{V^U_i}{|V^U_i|}$$

In these equations, $X_i$ represents the state at the point along the halo and $\epsilon$ is a small number used to perturb the state. For this work, a value of $10^{-7}$ was used for $\epsilon$ in all cases as this is equivalent to a perturbation on the order of hundreds of kilometers in canonical units for the Sun-Earth system. In addition, adding the perturbation to the initial states of the halo points will produce the interior manifolds, while subtracting the perturbation from the initial states will produce the exterior manifolds.
The initial states for the manifolds can then be propagated using the equations of motion of the CRTBP. Again, the unstable manifolds are propagated forward in time as they depart the halo orbit, while the stable manifolds are propagated backwards in time since they approach the halo orbit. The interior manifolds are those that flow towards the planet, while the exterior manifolds are those that flow away from the planet. Examples of the interior and exterior unstable and stable manifolds from halo orbits about the Sun-Earth $L_1$ and Sun-Earth $L_2$ are shown in Figures 3.7 and Figure 3.8, respectively.

Figure 3.7: Example of the four manifolds types produced from a Sun-Earth $L_1$ halo orbit.
Figure 3.8: Example of the four manifolds types produced from a Sun-Earth $L_2$ halo orbit.
Chapter 4

MISSION PROFILE

4.1 Mission Overview

The two missions developed for testing optimized interplanetary transfers using the invariant manifolds of halo orbits are Earth to Jupiter and Earth to Saturn missions. The purpose of selecting these test cases is to compare to the results from Rund [12]. Each mission will have four phases that will require four different coordinate frames and sets of system parameters. Each coordinate frame will be described briefly in this chapter; however, more detailed information including visuals and the canonical unit conversions for each frame can be found in Appendix A. In addition, three orbit maneuvers will be needed throughout the mission to take the spacecraft from Earth to the destination planet. Table 4.1 summarizes the coordinate frame and maneuvers needed for each phase of the mission.

<table>
<thead>
<tr>
<th>Mission Phase</th>
<th>Orbit Maneuver</th>
<th>Coordinate Frame</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth to Halo Orbit</td>
<td>Low-Thrust Orbit Raising</td>
<td>Earth-Centered Inertial (ECI)</td>
</tr>
<tr>
<td></td>
<td>Manifold Merge</td>
<td>Sun-Earth Synodic (SES)</td>
</tr>
<tr>
<td>Earth Departure</td>
<td>− − −</td>
<td>Sun-Earth Synodic (SES)</td>
</tr>
<tr>
<td>Interplanetary</td>
<td>Low-Thrust Interplanetary</td>
<td>Heliocentric</td>
</tr>
<tr>
<td>Destination Planet</td>
<td>− − −</td>
<td>Sun-Jupiter Synodic (SJS)</td>
</tr>
<tr>
<td>Arrival</td>
<td></td>
<td>Sun-Saturn Synodic (SSS)</td>
</tr>
</tbody>
</table>

The first phase is a transfer from an Earth parking orbit to a halo orbit about the Sun-Earth L1 point. As part of this phase, the spacecraft will first raise its orbit using
an optimized low-thrust transfer to a target higher Earth orbit that intersects a stable manifold. At the intersection point between the higher Earth orbit and manifold, an impulse maneuver is used to merge the spacecraft onto the manifold to take it to the halo orbit. The second phase is the interplanetary departure phase in which the spacecraft leaves the halo orbit along an exterior unstable manifold to link to the interplanetary transfer trajectory. The third phase is a low-thrust interplanetary transfer from Earth to the selected destination planet using a Patched Conics solution. Finally, the fourth phase is the destination planet arrival phase in which the spacecraft arrives along an exterior stable manifold about the Sun-Destination Planet \( L_1 \) point and coasts to a halo orbit. From the halo orbit, the spacecraft can then enter into an orbit about the destination planet, perform a flyby, or continue on with another interplanetary transfer to another planet using another set of manifolds. However, the analysis for this thesis will end once the spacecraft reaches the halo orbit at the destination planet, as the possible mission extensions would all require vastly different additional mission requirements.

4.2 Mission Phases

The following sections will cover the four phases of the interplanetary mission using invariant manifolds and low-thrust maneuvers analyzed in this thesis. The general process for these transfers will be discussed in the following sections along with reasoning for the mission profile decisions. The actual results from the two mission test cases will be discussed in the Results section in Chapter 6.
4.2.1 Earth Orbit to Halo Orbit Transfer Phase

In order to use manifolds for an interplanetary transfer, the spacecraft must first reach a halo orbit about a Sun-Earth Lagrange point. To accomplish this, an intersection point between an interior stable manifold and a parking orbit around Earth must be found to allow the spacecraft to perform a single burn at the intersection point and connect to the manifold. Once on the manifold, the spacecraft will get a free ride to the halo orbit about the Lagrange point without the need for additional burns. This method was used in Rund [12], where the interior stable manifolds of a Sun-Earth Lagrange point were calculated and the manifold that passed closest to Earth was used as a basis for the parking orbit about Earth at which the spacecraft would start its mission. It was also found that the $\Delta V$ needed to complete this phase of the transfer decreased as the altitude of the parking orbit increased; however, additional $\Delta V$ would be needed for a Hohmann transfer from the low altitude orbit to the higher altitude manifold-intersecting orbit if the higher orbit were chosen to intersect the manifold [12].

This work intends to use higher altitude parking orbits for connections to the stable manifolds as discussed above because the amount of fuel needed to complete the impulse burn at the manifold merge point is much less than when completing the maneuver from LEO orbits. In addition, the amount of fuel mass needed to complete the transfer from the initial LEO orbit to the higher altitude orbit will be mitigated by applying low-thrust optimization to the transfer. Therefore, the first phase of the interplanetary transfer method used in this work involves the spacecraft raising its orbit from a circular LEO orbit to a higher circular orbit between a 2,000 km and 50,000 km altitude using low-thrust propulsion. From this higher altitude orbit, the spacecraft will then be able to connect to an exterior stable manifold to venture to the halo orbit using less fuel than determined for previous methods. Therefore, this phase
consists of two distinct parts: first a low-thrust transfer from a LEO circular orbit to a higher circular orbit, and second, an instantaneous burn to merge the spacecraft from the target orbit onto the stable manifold trajectory.

For all test cases, the spacecraft will start in a 500 km circular orbit and the fuel mass needed to launch the spacecraft to this orbit will be ignored, although it must be acknowledged that a large chemical rocket would be needed to get the spacecraft to the initial 500 km orbit. An altitude of 500 km was chosen for the initial orbit as the starting location just needed to be somewhere in the LEO orbit regime. Regardless of the starting altitude in LEO, results similar to those found with a 500 km starting altitude would be obtained. Increasing the starting orbit altitude would lead to a slight decrease in the amount of fuel needed to raise the orbit, while decreasing the starting orbit would lead to a slight increase in the amount of fuel needed. In addition, only the Sun-Earth L₁ was considered for the halo staging point due to the direction in which the exterior manifolds flow from each Lagrange point. This will be explained further in section 4.2.3 when the Interplanetary Transfer Phase is discussed.

The higher target orbit to which the spacecraft will travel is dependent on both the stable manifolds of the halo orbit that is selected for the mission and the change in altitude between the starting 500 km orbit and the target higher Earth orbit. To determine the optimal target orbit and manifold intersection point, between 250 and 300 interior stable manifolds of a chosen halo orbit were propagated (depending on the size of the halo) and the manifold with the closest point to a given target high Earth orbit was found. From this point, a target circular orbit could be created based on the COEs of the of that point in space. The initial 500 km orbit could then be created by matching its COEs (particularly inclination, RAAN, and argument of perigee) with the higher target orbit, thus creating two coplanar circular orbits about Earth, with the higher orbit intersecting the chosen stable manifold. Now the
fuel mass needed to complete a low-thrust transfer from the initial 500 km orbit and target orbit could be calculated along with the fuel mass needed to merge from the target orbit to the stable manifold at the manifold intersection point. This process was repeated for northern and southern halo orbits with amplitudes between 100,000 km and 500,000 km and for target higher Earth orbits between 2,000 km and 50,000 km. Figure 4.1 shows a plot comparing the fuel mass percentage needed to raise the spacecraft’s orbit and allow it to merge onto the stable manifold based on the target orbit for a 300,000 km amplitude southern halo orbit.

![Comparison of fuel mass percentage needed to raise the spacecraft’s orbit and merge onto a stable manifold for several target orbits.](image)

**Figure 4.1:** Comparison of fuel mass percentage needed to raise the spacecraft’s orbit and merge onto a stable manifold for several target orbits.

The plot in Figure 4.1 shows that as the target orbit increases in altitude, the fuel mass percentage needed for the impulse transfer onto the manifold decreases, while the fuel
mass percentage needed to complete the low-thrust orbit raising increases. Overall, the fuel mass percentage needed decreases as the target orbit altitude increases when both parameters are combined. Based on these results, target orbits between 30,000 km and 40,000 km were chosen for the mission test cases as they require between 50% and 52% of the spacecraft’s mass dedicated to fuel. Although higher orbits result in an additional 1-2% decrease in fuel mass percentage needed, they require an extra 10,000 to 20,000 km of low-thrust travel distance before merging onto the manifold, which would require more time.

![Figure 4.2: Comparison of fuel mass percentage needed to merge onto a stable manifold from different sized halo orbits.](image-url)
In addition, Figure 4.2 compares the fuel mass percentage needed to reach a target orbit between 30,000 km and 40,000 km for different sized halo orbits between 100,000 km and 500,000 km. Based on these results, Sun-Earth L₁ halo orbits between 200,000 km and 300,000 km were considered for the mission test cases, as these halo orbits require the lowest fuel mass percentage to merge from the target orbit to the stable manifold. Lower amplitude halo orbits are also important for the Earth Departure Phase, which will be discussed in the following section, due to the exterior manifolds being more coplanar with Earth’s orbital plane.

With parameters for selecting a halo orbit amplitude, intersecting stable manifold, and target higher Earth orbit defined, an example of the full Earth Orbit to Halo Orbit transfer phase trajectory is shown in Figure 4.3. Additionally, a closer view of the low-thrust propulsion part of the transfer is shown in Figure 4.4. Lastly, theoretically, once the spacecraft merges onto the manifold it will not require any further maneuvers to maintain its path to reach the halo. In reality, orbital perturbations and corrections would need to be considered to keep the spacecraft on the proper trajectory; however, these are not taken into account for this work.
Figure 4.3: Full Earth Orbit to Halo Orbit Transfer Phase trajectory with the spacecraft’s transfer path on green.
4.2.2 Earth Departure Phase

To complete an interplanetary transfer using the Patched Conics with Manifolds method, the spacecraft must depart the halo orbit along an exterior unstable manifold to connect with an interplanetary trajectory at an injection point. After reaching the arrival point along the halo from the Earth Orbit to Halo Orbit Transfer Phase, the spacecraft will orbit about the halo until the departure point is reached. Once it reaches this point, a small deflection in the spacecraft’s speed and position will allow it to coast along the exterior unstable manifold. An example of an exterior unstable manifold selected for the Earth Departure Phase is shown in Figure 4.5.

Figure 4.4: Close up view of the low-thrust orbit raising portion of the transfer with the spacecraft’s transfer path in green.
Figure 4.5: Example of trajectory the spacecraft would take along an exterior unstable manifold to depart a halo orbit about the Sun-Earth L1 to link to the interplanetary phase.

To determine the best candidate for the departure phase manifold, between 250 and 300 exterior unstable manifold trajectories (depending on the size of the halo orbit) were calculated based on evenly spaced initial points along the halo. The manifold selected for the departure phase is the one closest to parallel with Earth’s orbital plane. This is accomplished by creating a plane in the phase space at the end of the manifold propagation and determining which manifold most closely aligns with Earth’s orbital plane in the heliocentric frame. The Heliocentric frame is equivalent to the x-y plane in the Sun-Earth Synodic Frame. The purpose of this selection is
to allow for calculation of an approximated two-dimensional interplanetary transfer, rather than one in three dimensions, which is more difficult to complete using low-thrust orbit optimization. Figure 4.6 shows a plot of the endpoints of several exterior unstable manifolds from a Sun-Earth L$_1$ halo orbit and identifies the endpoint closest to the x-y plane.

![Figure 4.6: Exterior unstable manifold endpoints (blue) in relation to the synodic x-y plane, along with the closest point (red).](image)

The Patched Conics with Manifolds method is made possible by the fact that once the spacecraft travels far enough away from Earth, the gravitational effect of Earth is no longer present and the spacecraft’s motion is only governed by the Sun [13].
This can be shown by calculating the energy associated with the manifold as it is propagated in time from the halo orbit using the following equation, where the radius, \( r \), and velocity, \( v \), magnitudes are calculated in the heliocentric frame and \( \mu_s \) is the gravitational parameter of the Sun:

\[
E = \frac{1}{2} v^2 - \frac{\mu_s}{r} \tag{4.1}
\]

Figure 4.7 shows an example of the shift in energy of the manifold trajectory as it is propagated in time and as the spacecraft travels further away from Earth.

Figure 4.7: Two-body energy associated with the exterior unstable manifold along which the spacecraft departs from Earth.
After an initial increase in energy, when the spacecraft is close to Earth, the energy associated with the orbit begins to stabilize in an oscillatory fashion at which point it is only under the influence of the Sun’s gravitational effects. This allows for calculation of a two-body interplanetary transfer starting from the end of the propagated manifold, when the spacecraft transitions from being influenced by the gravity of both primaries, to being influenced only by the gravity of the larger primary, the Sun.

4.2.3 Interplanetary Transfer Phase

Once the spacecraft reaches the transition point along the manifold where the energy of the orbit shifts from being associated with both the Sun and Earth to only the Sun, a shift occurs from a three-body problem in the Sun-Earth Synodic Frame to a two-body problem in the Heliocentric Frame. The standard method for completing an interplanetary transfer between two planets would be to create a conic arc using a Hohmann transfer trajectory or Lambert’s transfer trajectory as described in Rund [12]. For this work, the interplanetary phase will be done using a low-thrust trajectory between the departure and arrival manifolds.

In order to simplify the optimization problem, the orbits of the planets and the transfer orbit will be assumed to be in the same plane. This is a good assumption for the planetary orbits, as the inclination of Earth’s orbit relative to the Sun is about five degrees different from those of Jupiter and Saturn. For the transfer orbit, in order for it to be completed in the two-dimensional frame, the departure and arrival manifolds that are selected for the Earth Departure Phase and Destination Planet Arrival Phase must align with the planetary orbit. To accomplish this, the departure and arrival manifolds that are closest to the synodic x-y plane in the specific coordinate systems are chosen. The full process for the selection of these manifolds is explained in the Earth Departure Phase and Destination Planet Arrival Phase sections.
Similar to the low-thrust orbit raising used in the Earth Orbit to Halo Orbit Phase, a circular orbit to circular orbit low-thrust transfer will be used to complete the interplanetary maneuver between Earth and the destination planet. In particular, the transfer will be from the transition point of the exterior unstable manifold from the Earth halo orbit to the transition point of the exterior stable manifold from the destination planet halo orbit. For this thesis, it is assumed that at the interplanetary injection point at the end of the exterior unstable manifold, the spacecraft will now have the speed of the planet at its current location relative to the Heliocentric Frame due to the shift in energy discussed in the previous section. This assumption is similar to that of a traditional patched conics solution that uses hyperbolic escape trajectories. For those transfers, once the spacecraft is on the escape trajectory, it is assumed that once it reaches the edge of Earth’s sphere of influence, it will now have Earth’s orbital speed relative to the Sun in the Heliocentric Frame. This assumption is also used at the destination planet where the spacecraft must change its speed to reach the speed of the destination planet during the interplanetary transfer. An example of an interplanetary transfer using this method is shown in Figure 4.8.
It should be noted from Figure 4.8 that along the red exterior unstable manifold trajectory, the Earth is moving closely with the spacecraft’s trajectory due to the rotational nature of the synodic frame. Similarly with the blue exterior stable manifold trajectory, Jupiter is moving along its orbital path closely with the manifold as the spacecraft travels from the interplanetary arrival point to the halo injection point. However, Earth and Jupiter are only plotted at their initial and final points, respectively, in Figure 4.8 to not obstruct the other important features of the transfer. Lastly, as both indirect and direct methods were used to create the low-thrust interplanetary transfers, two different methods of trajectory creation were needed. For the direct method, since the state and control information is needed along the en-
tire trajectory for the initial guess, two low-thrust trajectories needed to be patched together, one that starts at the end of the departure manifold and is propagated forward, and one that starts at the end of the arrival manifold and is propagated backwards. These trajectories then meet at the low-thrust patch point to form one trajectory. There is a small error at the patch point, as the two trajectories do not meet exactly; however, it is on the order of a few thousand kilometers, so it would be assumed that a small course correction could be used to align the trajectories. This patch point is not needed for the indirect method, since only the initial states and costates are guessed in order to create the entire trajectory, and a full trajectory can be created that ends at the end of the arrival manifold without the need for a patch point. The patch point will be discussed further in Chapter 5 and Chapter 6 when explaining the optimization of the interplanetary phase of the mission.

4.2.4 Destination Planet Arrival Phase

Upon exiting the interplanetary phase of the transfer, the spacecraft will arrive along an exterior stable manifold derived from a halo orbit about the Sun-Destination Planet L₁ point. Once the spacecraft arrives along the manifold, it will coast to the halo orbit without the need to complete any more burns. Figure 4.9 shows the route the spacecraft would take to reach the halo orbit from the exterior stable manifold, starting from the arrival transition point (red) from the low-thrust interplanetary transfer, traveling along the green exterior stable manifold trajectory (green), and arriving at the Sun-Jupiter L₁ halo orbit (black) at the injection point (blue).
Figure 4.9: Example of trajectory the spacecraft would take along an exterior stable manifold to reach a halo orbit about the Sun-Destination Planet L$_1$.

Similar to the Earth Departure Phase, the best candidate for the exterior stable manifold along which the spacecraft will travel to reach the halo orbit in the destination planet system is the one which aligns closest with the destination planet’s orbital plane, or the synodic x-y plane. To determine this manifold, between 250 and 300 exterior stable manifold trajectories (depending on the size of the halo orbit) were calculated based on evenly spaced initial points along the halo. By creating a plane in the phase space at the end of the propagation of the manifolds and determining which one most closely aligns with the orbital plane in the of the heliocentric frame, which is equivalent to the x-y plane in the Sun-Earth Synodic Frame, the required manifold was found. Again, the purpose of this selection is to allow for an approx-
imated two-dimensional interplanetary transfer to be calculated, rather than one in three-dimensions. Figure 4.10 shows a plot of the endpoints of several exterior stable manifolds from a Sun-Jupiter L$_1$ halo orbit and identifies the endpoint closest to the x-y plane.

![Figure 4.10: Exterior stable manifold endpoints (blue) in relation to the synodic x-y plane, along with the closest point (red).](image)

Again, similar to the Earth Departure Phase, once the spacecraft gets close enough to the smaller primary in the system, the destination planet in this case, along the exterior stable manifold it is now subject to the gravitational force of both the larger and smaller primaries, not just the Sun. Therefore, the reference frame and equations under which the spacecraft’s motion is governed shifts to the Sun-Destination Planet Synodic Frame and the CRTBP equations of motion. Figure 4.11 shows how the
two-body energy of the spacecraft’s orbit shrinks when close to the destination planet after being under the gravitational effects of the Sun.

Figure 4.11: Two-body energy associated with the exterior stable manifold along which the spacecraft arrives at the destination planet.

Similar to the Earth Departure Phase, Equation 4.1 was used to calculate the two-body energy of the spacecraft as it travels along the exterior stable manifold. If the plot in Figure 4.11 is observed from right to left, it can be seen that the energy of the trajectory is oscillatory when the spacecraft is far away from the destination planet, but decreases as it gets closer to the planet. This can be seen as the spacecraft’s two-body energy being associated with the Sun during the oscillatory portion and then hits a transition where the spacecraft falls under the influence of both the Sun and destination planet. Therefore, the Patched Conics method can be used to assume
that the spacecraft will be traveling with the speed at which the planet is orbiting the Sun as it approaches the halo along the exterior stable manifold.

From the halo orbit about the Sun-Destination Planet L1 point, the spacecraft may now enter into an orbit about the destination planet by merging onto a stable manifold, perform a flyby, or continue onto another interplanetary transfer to an additional planet using a different set of manifolds. However, the analysis for this thesis will end once the spacecraft reaches the halo orbit, as the three possible mission extensions would all require vastly different additional mission requirements in terms of fuel needed and transfer time.
Chapter 5

LOW-THRUST OPTIMIZATION

5.1 General Optimal Trajectory Design Problem

The goal of every optimal trajectory design problem is to minimize or maximize a cost function based on several constraints. The cost function includes two terms, a terminal cost $K$ that depends on the system boundaries, and a path cost $L$, that depends on the change in the system parameters over time [11]. The general cost function takes the form:

$$\text{Min. } J = K(t_f, x_f) + \int_{t_0}^{t_f} L(t, x, u) \, dt$$  \hspace{1cm} (5.1)$$

Problems with only a terminal cost are said to be in Mayer form, while problems with only a path cost are said to be in Lagrange form. Problems that include both terminal and cost terms are said to be in Bolza form [4]. For this work, since the focus is on maximizing final mass, only the Mayer form is needed. Optimization problems are subject to several constraints, the first of which is the set of governing differential equations that model the dynamics of the given system [4]:

$$\dot{x} = f(t, x(t), u(t))$$ \hspace{1cm} (5.2)$$

where the dynamics are a function of $n$ state variables in the state vector $x$, and $m$ control variables in the control vector $u$, both of which are a function of time $t$. Next are the path constraints that restrict motion along the trajectory at specific times, or throughout the entire trajectory [4].
Another set of constraints are the upper and lower bounds on the state and control variables to ensure that certain parameters stay within defined regions [4]. For example, limits on the angle at which a thruster can be gimbaled.

\[ x_{low} \leq x(t) \leq x_{upp} \]  
\[ u_{low} \leq u(t) \leq u_{upp} \]  

Lastly, it is important to define the limits on the initial and final state and time [4]. These ensure that the solution falls along a path that reaches a certain goal within a specified time to some chosen tolerance, or reaches a solution that falls at specific location.

\[ t_{low} \leq t_0 \leq t_f \leq t_{upp} \]  
\[ x_{0,low} \leq x(t_0) \leq x_{0,upp} \]  
\[ x_{f,low} \leq x(t_f) \leq x_{f,upp} \]  

5.2 Low-Thrust Equations of Motion

For both phases in the mission profile in which trajectory optimization is utilized, two-dimensional Cartesian coordinates were used to model the dynamics of the spacecraft. In addition, this work assumes that the spacecraft is equipped with a constant thrust, constant specific impulse (CSI) engine, which adds a mass state, \( m \), to the state vector of the spacecraft dynamics which already includes four states that define the position and velocity of the spacecraft:
\[
\begin{bmatrix}
x \\
y \\
\dot{x} \\
\dot{y} \\
m
\end{bmatrix}
\quad (5.9)
\]

where \(x\) and \(y\) are the position components of the spacecraft in DU, \(\dot{x}\) and \(\dot{y}\) are the velocity components of the spacecraft in DU/TU, and \(m\) is the spacecraft mass in MU. The control vector that determines the motion of the spacecraft deploying a low-thrust engine along with the five state variables. Two control variables govern the thrust pointing vector in the \(x\) and \(y\) directions of the Cartesian coordinate system, while the third determines whether the spacecraft thrust is turned on or off:

\[
\begin{bmatrix}
u_x \\
u_y \\
T
\end{bmatrix}
\quad (5.10)
\]

The equations of motion for a thrusting spacecraft that employs a CSI engine can then be introduced as follows:

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
-\frac{x}{r^3} + \frac{T u_x}{m} \\
-\frac{y}{r^3} + \frac{T u_y}{m} \\
-\frac{T}{I_{sp} g}
\end{bmatrix}
\quad (5.11)
\]

where \(r = \sqrt{x^2 + y^2}\) in DU, \(I_{sp}\) is the specific impulse of the engine in TU, and \(g\) is gravity in DU/TU\(^2\).
5.3 Cost Function

The goal of optimization for all test cases in this work is to minimize the amount of fuel consumed during the transfer, or in other words, maximize the final mass of the spacecraft. Therefore, the cost function can be stated using the following equation where the negative is used to represent finding the maximum of the cost function rather than the minimum:

\[ J = -m_f \]  

(5.12)

The cost function is the same for both the indirect and direct approaches that were analyzed in this thesis and the specifics of how the cost function is used for each will be discussed in the following sections.

5.4 Indirect Approach - Euler-Lagrange Method

Indirect optimization approaches are analytical techniques derived from the calculus of variations that rely on satisfying a set of necessary conditions to determine an optimal solution rather than operating directly on the cost function [11]. In this work, Euler-Lagrange (E-L) theory is implemented to convert the problem of optimizing the cost function into a two-point boundary value problem (TPBVP). In general, the E-L theorem produces a set of necessary conditions that must be satisfied to allow for the optimal control parameters to minimize the scalar cost function, \( J \). The E-L theorem is an indirect optimization method because it finds an optimal solution that satisfies a set of boundary conditions rather than one that minimizes or maximizes the cost function directly. The derivation of the E-L theory can be found in both Conway [2] and Longuski et al. [6] and is also summarized in Pritchett [11]. The E-L theory as
applied to the low-thrust circular-to-circular and interplanetary transfers analyzed in this work will be described here.

Three steps are needed to apply the E-L theorem to an optimal control problem, the first of which is to form the Hamiltonian. The Hamiltonian consists of the path cost and the vector multiplication of the costates, or Lagrange multipliers related to the states, and the equations of motion [11]. Equation 5.13 shows the Hamiltonian used for the system described in Section 5.2.

\[ H = \lambda_1 \dot{x} + \lambda_2 \dot{y} - \lambda_3 \left( \frac{x}{r^3} + \frac{T u_x}{m} \right) - \lambda_4 \left( \frac{y}{r^3} + \frac{T u_y}{m} \right) - \frac{\lambda_5 T}{I_{sp} g} \]  

The second step is to determine the partial derivatives of the Hamiltonian with respect to each state variable using Equation 5.14 and each control variable using Equation 5.15 [11].

\[ \dot{\lambda} = \frac{\partial H}{\partial x} = H_x \]  
\[ \frac{\partial H}{\partial u} = H_u = 0 \]  

These equations help form the conditions necessary to have a valid TPBVP, but it may still be ill-defined due to not enough boundary conditions being supplied. The third step is to supply those additional boundary conditions based on initial and final constraints on the problem. For a TPBVP to be well-defined, a total of \(2n+2\) boundary conditions are needed, where \(n\) is the number of state variables [11]. However, for some problems, the optimal solution can be found with fewer boundary conditions than what is deemed necessary to be well-defined. For the circular-to-circular orbit raising maneuver, only 9 conditions were needed, while for the interplanetary transfer, 13 conditions were supplied.
For the circular-to-circular orbit transfer, the first five constraints come from setting the initial position, velocity, and mass of the spacecraft. The next three come from the circular orbit constraints at the final state in the orbit shown in Equation 5.16 through Equation 5.18. These equations ensure that the spacecraft’s position, velocity, and momentum are defined such that the spacecraft is on a circular orbit at the end of the transfer. The last constraint comes from the transversality condition, shown in Equation 5.19, which constrains the costate variables at the terminal point on the transfer path [11].

\[
x^2_f + y^2_f = r^2_f \tag{5.16}
\]

\[
v^2_{xf} + v^2_{yf} = v^2_f \tag{5.17}
\]

\[
v_{yf}x_f - v_{xf}y_f = r_f v_f \tag{5.18}
\]

\[
H_f dt_f - \lambda_f^T d\mathbf{x}_f + dK = 0 \tag{5.19}
\]

The same constraints are used for the low-thrust interplanetary transfer, while four additional constraints are added constraining the final position and velocity of the spacecraft to ensure that it arrives at the injection point along the stable manifold at the destination planet.

Once the necessary conditions and boundary conditions have been solved for, an initial guess of the states and costates are supplied and an optimization tool can be used to solve the optimization problem. In this case fsolve in Matlab will be used to find the optimal solution as it forces all the constraints to zero based on a set tolerance given by the user. The program will continue to run through additional iterations of solving for each set of constraints and propagating the trajectory until an optimal solution is found. The guess of the initial costates is important because if it is too far away from the true initial guess that the optimization algorithm solves for, then the minimum fuel trajectory will not be found. Therefore, an iterative process was used in which
the five costates are guessed and the trajectory is propagated forward to determine how close the boundary conditions are to what they should be. Then based on how close the boundary conditions are, the costates are adjusted until a reasonable first guess is found. This was done as opposed to using an adjoint control transformation which would relate the costates to physical properties of the dynamics to assist in a guess of the initial costates.

For both low-thrust transfer cases, the goal is to maximize the final spacecraft, or to minimize the amount of fuel mass used. There is a difference in the optimization between the orbit raising and interplanetary transfer. For the orbit raising, the spacecraft can finish its trajectory at any point along the target orbit. For interplanetary transfers, the spacecraft must reach the specified interplanetary arrival point at which the interplanetary transfer merges with the exterior stable manifold at the destination planet.

5.5 Direct Approach - Collocation

Direct optimization methods are numerical techniques that convert a continuous optimal control problem into a set of discrete variables which can be reformulated into a nonlinear programming problem (NLP). This is accomplished by discretizing the continuous trajectory into a set of finite points with state and control information and then using a collocation scheme to interpolate between discrete points in time to construct an optimal solution that minimizes or maximizes a given cost function [4].

NLPs are called such because they are constrained parameter optimization problems that include non-linear terms in the cost and constraint functions [4]. The structure of an NLP is similar to that of the general optimal trajectory design problem in that the goal is to find the $n$ variable vector $x$ that minimizes the cost function subject
to $m$ constraints with a set of bounds on the state and control parameters. The Lagrangian is used to find a solution to the NLP and is determined using [11]:

$$L(x, \mu, \lambda) = J(x) + \mu^T g(x) + \lambda^T f(x)$$

(5.20)

where the Lagrange multipliers $\mu$ relate to the inequality constraints $g(x)$ and the Lagrange multipliers $\lambda$ relate to the equality constraints $f(x)$. The necessary conditions needed to minimize the cost function and solve the NLP are known as the Karush-Kuhn-Tucker (KKT) conditions. The conditions for systems where both equality and inequality constraints are as follows [11]:

$$\nabla J(x^*) + g(x^*)^T \mu + f(x^*)^T \lambda = 0$$

(5.21)

$$f(x^*) = 0$$

(5.22)

$$g(x^*) \leq 0$$

(5.23)

$$\mu \geq 0$$

(5.24)

$$\mu^T g(x^*) = 0$$

(5.25)

The first condition ensures stationarity, or that the state has a near constant variance over the time span of the trajectory. The second and third conditions ensure dual feasibility of the problem, while the fourth condition ensures primal feasibility. The last condition ensures that there is complementary slackness in the problem. These conditions, along with the Hessian, which is a matrix of the second derivatives of the equality and inequality constraints and shown in Equation 5.26, are used to form a system of equations that can be differentiated to find the changes in the state and Lagrangian variables that are used to solve the NLP [11].
In order to discretize the optimal trajectory optimization problem to use the NLP method that is created by the KKT equations to find the minimum or maximum of a given cost function, direct collocation is used. Collocation is a method of implicitly integrating differential equations by fitting piecewise polynomials to discrete points in a system that are governed by a set of equations of motion that are ordinary differential equations [4]. Collocation is useful for converting continuous optimal control problems, such as those described in section 5.1, into NLP problems. For orbital mechanics problems, the discretization is most commonly made up of discrete points in time over some trajectory, i.e.,

\[ M : t_1 < t_2 < ... < t_n = t_f \]  

(5.27)

where \( M \) is a mesh made comprised of \( n \) mesh points. Each mesh point, \( Z_i \), contains state and control information for that discrete point in time, i.e.,

\[ Z = (x_1, u_1, ..., x_n, u_n)^T \]  

(5.28)

Collocation schemes are defined by the implicit integration method that is used to approximate the solution to a set of differential equations. The accuracy of the collocation scheme is therefore dependent on the accuracy of the implicit integration method. A collocation problem is solved when all defects, or the difference between the states determined by Euler’s method and the actual states at \( i + 1 \), which are defined by the initial guess for the optimization scheme, are equal to zero within a defined tolerance. This ensures that the segments between adjacent points in the mesh are continuous. The two collocation schemes investigated in this paper were the

\[
H = \nabla_x^2 J + \sum_{i=1}^{m} \mu_i \nabla_x^2 g_i + \sum_{i=1}^{n} \lambda_i \nabla_x^2 f_i
\]  

(5.26)
Trapezoid and Hermite-Simpson schemes. Trapezoidal collocation, which implements the trapezoid implicit integration rule, is a second order accurate method. The defect constraints for this method are defined as [4]:

$$\xi_i = x_i - x_{i+1} + \frac{\delta t_i}{2} (f(t_i, x_i, u_i) + f(t_{i+1}, x_{i+1}, u_{i+1})) = 0$$  \hspace{1cm} (5.29)$$

In addition, Hermite-Simpson collocation is a third order accurate method that employs the Hermite-Simpson implicit integration rule. The defect constraints for this method are defined as [14]:

$$\xi_i = x_i - x_{i+1} + \frac{\delta t_i}{6} (f(t_i, x_i, u_i) + 4f(t_c, x_c, u_c) + f(t_{i+1}, x_{i+1}, u_{i+1})) = 0$$ \hspace{1cm} (5.30)$$

where the state and control variables at the collocation point $x_c$ are calculated as [14]:

$$x_c = \frac{1}{2}(x_i + x_{i+1}) + \frac{\delta t_i}{8} (f(t_i, x_i, u_i) - f(t_{i+1}, x_{i+1}, u_{i+1}))$$  \hspace{1cm} (5.31)$$

$$u_c = \frac{u_i + u_{i+1}}{2}$$ \hspace{1cm} (5.32)$$

Both the Trapezoid and Hermite-Simpson collocation schemes were investigated and tested first on a minimum time circular-to-circular orbit transfer problem for verification. Both methods were able to produce a minimum time solution within two minutes of the result found using the indirect method described in section 5.4. Because both methods produced similar results, the Trapezoid method was ultimately used for the minimum fuel transfers that are part of the mission profile for the interplanetary transfers using manifolds because the computation time is much lower when compared to the Hermite-Simpson method. The full results of the verification testing can be found in Appendix B.
The direct collocation method described here was applied to both the low-thrust transfers used for the orbit raising portion of the Earth Orbit to Halo Orbit Transfer Phase and the interplanetary transfer for the Interplanetary Transfer Phase. The same set of 2D Cartesian coordinate equations used for the indirect method were used for the collocation method.

Before a collocation scheme is employed, an initial guess must first be supplied that gives a value for each state and control variable at each point in the mesh. For a low-thrust orbital mechanics problem, this means simply supplying a guess orbit with position, velocity, mass, and thrust pointing parameters. In addition, bounds must be placed on the state and control variables. Once the initial guess and bounds are set, \textit{fmincon} in Matlab was used as the optimization tool to solve for the minimum fuel trajectory. After all the defects are found for each set of constraints throughout the entire trajectory, the optimization \textit{fmincon} will force these constraints to zero based on the set tolerance. The program will continue to run through additional iterations of solving for each set of constraints and propagating the trajectory until an optimal solution is found.

Similar to the indirect method, there is a difference between the orbit raising and interplanetary transfers. For orbit raising, the spacecraft can finish its trajectory at any point along the target orbit, while for the interplanetary transfers, the spacecraft must reach the specified interplanetary arrival point at which the transfer merges with the exterior stable manifold at the destination planet.

As was discussed in the Interplanetary Transfer Phase of the Mission Profile in Section 4.2.3, the low-thrust interplanetary transfer had to be split into two parts that meet at a patch point to provide a proper initial guess for the collocation scheme. Because the two trajectories do not align exactly at the patch point, two separate collocation problems were run: one that starts from the exterior unstable manifold endpoint at
the Earth system and ends at the patch point, and one that starts at the patch point and ends at the exterior stable manifold endpoint at the destination planet system. This ensured that the collocation scheme did not have to incorporate the inexact alignment of the two trajectories at the patch point.
Chapter 6

MISSION TEST CASES

The main goal of this work is to reduce the fuel mass needed for missions that implement the invariant manifolds of halo orbits for trajectory design by applying optimization algorithms to these transfers. The mission design discussed in Chapter 4 and optimization procedures outlined in Chapter 5 have the potential to reduce the amount of fuel required for these missions. This section will cover the test cases used to demonstrate the fuel mass reduction obtained with these methods introduced. Missions between Earth and Jupiter and Earth and Saturn will be investigated to determine the amount of fuel required to complete each phase of the mission, along with the time of flight required. These two missions were tested in Rund [12] using the Manifold to Hyperbolic Escape Trajectory method with no optimization applied to the transfers, so the results found there will be used as a point of comparison to the results found in this work in terms of fuel mass used and time of flight. The results found in this paper will also be compared to a standard Hohmann transfer. Although the results presented in Rund are for $\Delta V$ in km/s, the results for fuel in this paper will be presented in fuel mass percentage of the overall spacecraft and will be based on the thrust and specific impulse of the engines used for the impulse and low-thrust maneuvers presented. All mission considerations for each phase of the mission for both test cases will be discussed in the following sections as well. In addition, as these types of transfers have not been undertaken for real-life missions yet, mass, engine specific impulse, and thrust for a theoretical spacecraft will be used to gather data for the transfers outlined in this work.
6.1 Earth to Jupiter Transfer

As discussed throughout this work, Rund tested the method of using manifolds along with hyperbolic escape trajectories to reduce the $\Delta V$ needed to transfer between Earth and Jupiter and found that by using this method, the overall mission $\Delta V$ could be reduced by 1 km/s [12]. The goal of this work was to attempt to further reduce the fuel required by substituting in optimized low-thrust transfers for parts of the full interplanetary transfer. The following work describes the trajectories used to complete a mission using low-thrust maneuvers combined with manifolds between Earth and Jupiter along with the fuel mass and time of flight required for each portion of the transfer. In addition, the results found in this work will be compared to the results found in Rund for the Manifold to Hyperbolic Escape Trajectory method and for a Hohmann Transfer.

As the results for the transfers will be presented in fuel mass percentage of the spacecraft used rather than $\Delta V$, the spacecraft mass and engine parameters are needed for both the impulse transfer and the low-thrust transfers. The initial mass of the spacecraft was set at 500 kg, while Table 6.1 shows the thrust and specific impulse parameters used for each maneuver during the mission. Different levels of thrust were used for the indirect and direct optimized interplanetary transfers due to the differences in the way each trajectory is constructed as discussed in Chapter 4. In order for a constant thrust trajectory to be created that satisfied the boundary conditions of the direct method transfer, the thrust had to be slightly lower.
Table 6.1: Spacecraft and Engine Parameters for Impulse and Low-Thrust Maneuvers for Jupiter Mission.

<table>
<thead>
<tr>
<th>Description</th>
<th>Specific Impulse (sec)</th>
<th>Thrust (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Impulse</td>
<td>300</td>
<td>-</td>
</tr>
<tr>
<td>Low-Thrust (OR)</td>
<td>3000</td>
<td>10</td>
</tr>
<tr>
<td>Low-Thrust (IP)</td>
<td>3000</td>
<td>1 (Indirect)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.9 (Direct)</td>
</tr>
</tbody>
</table>

The mission begins with the spacecraft starting in a 500 km circular orbit about Earth. The halo orbit from which the stable manifolds are propagated was chosen to be 250,000 km as this lies within the regime discussed in Section 4.2.1, which leads to the lowest amounts of fuel needed to merge between the target orbit and the manifold for L₁ halo orbits in the Sun-Earth system. The target orbit selected for the spacecraft’s low-thrust orbit raising maneuver was chosen to be 30,000 km as this also lies within the regime of orbit altitudes that saves the most on fuel. When the manifolds were propagated from the halo orbit, the manifold that passed closest to the chosen target orbit was at an altitude of 31,174 km, making this the true target orbit to which the spacecraft will reach from the orbit raising maneuver. In addition, the manifold intersects the target orbit altitude with an inclination of 28°, RAAN of 13°, and argument of perigee of 25°. These parameters define the orbital plane of the two-dimensional circular-to-circular low-thrust orbit raising maneuver.

Both the indirect and direct optimization methods were used to create minimal fuel trajectories that reach the target orbit using a specific impulse of 3000 seconds and constant thrust of 10 N. Additionally, the target time span of the low-thrust transfers was chosen to be two days. Although a longer time span would reduce the necessary thrust needed to complete the transfer, more mass may be required due to the engine producing continuous thrust. Figure 6.1 shows the spacecraft’s transfer path along the green trajectory starting from the initial 500 km orbit, raising its altitude to
the target orbit using the low-thrust transfer, arriving at the target orbit at a true anomaly of 186°, coasting to the manifold intersection point, and performing the impulse maneuver to merge onto the manifold. A similar plot for the low-thrust trajectory created using the direct method can be found in Appendix C.

Figure 6.1: View of the spacecraft’s trajectory (green) during the orbit raising and manifold merge portions of the Earth Orbit to Halo Orbit Transfer Phase.

Table 6.2 shows the fuel mass percentage and time of flight results for the orbit raising maneuver for both optimization methods.
Table 6.2: Optimization Results for Spacecraft Orbit Raising (Jupiter).

<table>
<thead>
<tr>
<th>Method</th>
<th>Fuel Mass Percentage</th>
<th>Time of Flight (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Indirect</td>
<td>14%</td>
<td>2.34</td>
</tr>
<tr>
<td>Direct</td>
<td>13%</td>
<td>2.10</td>
</tr>
</tbody>
</table>

The results from the table show that the direct method was able to produce a trajectory that requires 1% less fuel than the trajectory produced by the indirect method. This is due to the transfer time for the direct method being slightly less than that of the indirect method and because the collocation algorithm required fewer time steps to determine a viable solution. Figure 6.2 and Figure 6.3 show the low-thrust orbit raising trajectories in two-dimensions for the indirect and direct methods, respectively.

Figure 6.2: Low-thrust orbit raising trajectory for the indirect method.
When comparing the low-thrust orbit raising trajectories created by the indirect and direct optimization methods in Figure 6.2 and Figure 6.3, respectively, the indirect trajectory spirals out more elliptically whereas the direct trajectory spirals out more circularly. This is mainly due to the direct method trajectory being very close to the optimized trajectory, which is inherent in the collocation approach. In addition, the direct trajectory has a much lower fidelity, especially closer to Earth, due to fewer points being used to approximate the solution.

To merge the spacecraft from the target orbit to the stable manifold to travel to the halo orbit, an impulse maneuver with a $\Delta V$ of 1.4 km/s is required, which is equivalent to 38% of the spacecraft’s total mass required for fuel. This is about half
the fuel required when compared to Rund’s results of a $\Delta V$ of 3.1 km/s, or 65% mass ratio, when a lower orbit was used to merge onto the stable manifold. With the additional fuel used to perform the low-thrust orbit raising included, the proposed method produces fuel savings of about 13% when compared to the previous method.

Figure 6.4 shows the spacecraft’s full trajectory after merging onto the stable manifold to reach the 250,000 km halo orbit. The time of flight from the manifold intersection point to the halo injection point is 387 days.

Figure 6.4: Transfer path from Earth to halo orbit along the stable manifold trajectory.

Once the spacecraft arrives at the 250,000 km halo orbit, the next maneuver is to traverse the halo to the halo departure point to begin the Earth Departure Phase.
To accomplish this, the exterior stable manifold must be chosen along with the propagation time before linking to the interplanetary trajectory. The process described in Section 4.2.2 was used to select the manifold that lies closest to the orbital plane at the time at which the manifold reaches the energy transition point. Plots of the manifold selection and energy transition point can be found in Appendix C. Based on these plots, it was determined that the spacecraft reached the energy transition point after traveling along the manifold for 147 days.

As the spacecraft reaches the halo departure point, a small deflection in its position and velocity cause it to merge onto the exterior unstable manifold and travel towards the interplanetary injection point. The spacecraft spends 147 days on this trajectory before it links to the interplanetary transfer. Figure 6.5 shows the route the spacecraft takes from the halo departure point to the interplanetary injection point to link to the low-thrust interplanetary trajectory.
Figure 6.5: Trajectory of the spacecraft along the chosen exterior unstable manifold (green) to connect to the interplanetary transfer to depart Earth.

Once the spacecraft reaches the interplanetary injection point, the reference frame shifts to the Heliocentric Frame and it is now assumed that the spacecraft has the speed of the Earth relative to its rotation about the Sun. To find an optimal trajectory that minimizes the fuel required to reach Jupiter, both the indirect and direct optimization methods were used. Before determining the low-thrust trajectory, the departure and arrival dates were selected to determine the locations of each planet within their orbital planes. The date at which the spacecraft departs from the halo along the exterior unstable manifold was selected to be September 2, 1977. This date matches with the work of Rund, and is close to the launch date of Voyager 1, to which
the results from that work were compared. This selection determines the location of
the Earth at the departure point and Jupiter at the arrival point as well as the lo-
cations of the manifold endpoints in the Heliocentric Frame. Figure 6.6 shows the
optimal trajectory found using the indirect method and Figure 6.7 shows the optimal
trajectory found using the direct method.

![Figure 6.6: Optimal Interplanetary trajectory for the indirect method.](image)

Figure 6.6: Optimal Interplanetary trajectory for the indirect method.
Both interplanetary trajectories show the spacecraft starting in the Earth system, traveling along the exterior manifold (the same trajectory shown in Figure 6.5 in Heliocentric Frame), transitioning to the low-thrust interplanetary trajectory, and arriving along the Jupiter stable manifold. As discussed in Section 4.2.3, the indirect trajectory was constructed by guessing the initial states and costates to find a trajectory that reaches the interplanetary arrival point at Jupiter, while the direct trajectory was created by propagating two different routes from the ends of each manifold and patching them together at an intermediate patch point.
When comparing the interplanetary trajectories created for the indirect and direct optimization methods, the indirect trajectory cycles twice within Earth’s orbit before traveling out to the endpoint of the Jupiter stable exterior manifold, while the direct trajectory cycles once within Earth’s orbit before taking a much wider angle to reach the manifold endpoint. In addition, the indirect trajectory takes a much sharper angle toward the manifold endpoint when compared to the direct trajectory. Table B.1 shows the fuel mass percentage and time of flight necessary for the trajectories found for both optimization methods.

**Table 6.3: Optimization Results for Interplanetary Transfer (Jupiter).**

<table>
<thead>
<tr>
<th></th>
<th>Fuel Mass Percentage</th>
<th>Time of Flight (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Indirect</td>
<td>56%</td>
<td>1500</td>
</tr>
<tr>
<td>Direct</td>
<td>52%</td>
<td>1573</td>
</tr>
</tbody>
</table>

The results from the table show that the direct method trajectory requires 73 more days to reach the Jupiter manifold, but needs 4% less fuel than for the indirect trajectory. Similar to the orbit raising optimization, fewer time steps are required for the collocation algorithm to solve for the optimal trajectory which means the spacecraft will need to thrust at fewer points along the orbit and less fuel used.

Upon arrival at the exterior stable manifold for the Jupiter halo orbit, the spacecraft reaches another energy transition point where the problem shifts from a two-body problem in the Heliocentric Frame to another three-body problem in the Sun-Jupiter Synodic Frame. At this point the spacecraft is deemed to have reached a velocity that is sufficient for it to link to the manifold without any additional maneuvers. The halo size chosen for the Sun-Jupiter L₁ halo orbit was 900,000 km. The size of this halo orbit is arbitrary for the Patched Conics with Manifolds method as any size halo orbit will have manifolds that align with the synodic x-y plane. Due to the mass of Jupiter, halo orbits on the order of hundreds of thousands of kilometers in amplitude
are almost parallel to the synodic x-y plane, unlike those of Earth, meaning that their manifolds will be more parallel to the x-y plane as well. Plots of the manifold selection and energy transition point selection can be found in Appendix C. Based on these plots, it was determined that the spacecraft reaches the halo injection point after traveling along the manifold for 1337 days.

Once the spacecraft reaches the interplanetary arrival point, it connects with the exterior stable manifold and travels towards halo orbit without the need for any additional maneuvers, arriving at the halo injection point after 1337 days. Figure 6.8 shows the route the spacecraft takes from the interplanetary arrival point to the halo injection point along the exterior stable manifold in the Sun-Jupiter Synodic Frame.
Figure 6.8: Trajectory of the spacecraft along the chosen exterior stable manifold (green) to leave the interplanetary transfer and arrive at the Sun-Jupiter L$_1$ halo.

Once the spacecraft reaches the halo at the halo injection point, the mission will be considered complete for the purposes of this work; however, the spacecraft could complete several other types of maneuvers from this location including a Jupiter flyby, orbit about the planet, or connect to another manifold to complete a further interplanetary transfer. The fuel mass percentages used and time of flight data for each stage of the transfer are compiled in Table 6.4 and Table 6.5 for missions using the indirect and direct optimization methods, respectively.
Table 6.4: Fuel Mass Used and Time of Flight (ToF) for Each Mission Phase of Jupiter Mission (Indirect).

<table>
<thead>
<tr>
<th>Transfer Phase</th>
<th>Remaining Mass (kg)</th>
<th>ToF (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbit Raising</td>
<td>430</td>
<td>2</td>
</tr>
<tr>
<td>Manifold Merge</td>
<td>267</td>
<td>—</td>
</tr>
<tr>
<td>Earth to Halo</td>
<td>267</td>
<td>387</td>
</tr>
<tr>
<td>Exterior Manifold</td>
<td>267</td>
<td>147</td>
</tr>
<tr>
<td>Interplanetary Phase</td>
<td>118</td>
<td>1500</td>
</tr>
<tr>
<td>Jupiter Arrival Phase</td>
<td>118</td>
<td>1337</td>
</tr>
</tbody>
</table>

Table 6.5: Fuel Mass Used and Time of Flight (ToF) for Each Mission Phase of Jupiter Mission (Direct).

<table>
<thead>
<tr>
<th>Transfer Phase</th>
<th>Remaining Mass (kg)</th>
<th>ToF (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbit Raising</td>
<td>435</td>
<td>2</td>
</tr>
<tr>
<td>Manifold Merge</td>
<td>270</td>
<td>—</td>
</tr>
<tr>
<td>Earth to Halo</td>
<td>270</td>
<td>387</td>
</tr>
<tr>
<td>Exterior Manifold</td>
<td>270</td>
<td>147</td>
</tr>
<tr>
<td>Interplanetary Phase</td>
<td>140</td>
<td>1573</td>
</tr>
<tr>
<td>Jupiter Arrival Phase</td>
<td>140</td>
<td>1337</td>
</tr>
</tbody>
</table>

Based on the results from Tables 6.4 and 6.5, the spacecraft finishes with a final mass of 140 kg for the direct approach mission, and 118 kg for the indirect approach mission. The time of flight for the direct mission is also 73 days longer due to the extra time required for the interplanetary transfer.

To compare the results found using the Patched Conics with Manifolds method for an interplanetary transfer to the Manifold to Hyperbolic Escape Trajectory method used in Rund and to traditional Hohmann transfers, Table 6.6 shows comparisons of fuel mass percentage used for each phase of the transfers and Table 6.7 shows the total time of flight required for each method.
Table 6.6: Fuel Mass Percentage Comparison between Methods (Jupiter).

<table>
<thead>
<tr>
<th></th>
<th>Patched Conics (Indirect)</th>
<th>Patched Conics (Direct)</th>
<th>Manifold to Hyperbolic</th>
<th>Hohmann Transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbit Raising</td>
<td>14%</td>
<td>13%</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Manifold Merge</td>
<td>38%</td>
<td>38%</td>
<td>65%</td>
<td>–</td>
</tr>
<tr>
<td>IP Phase (Stage 1)</td>
<td>56%</td>
<td>52%</td>
<td>79%</td>
<td>92%</td>
</tr>
<tr>
<td>IP Phase (Stage 2)</td>
<td>–</td>
<td>–</td>
<td>91%</td>
<td>94%</td>
</tr>
</tbody>
</table>

When the combination of the orbit raising and manifold merge phases of the Patched Conics with Manifolds method are compared to the manifold merge phase for the Manifold to Hyperbolic Escape Trajectory method, the latter requires about 13-14% less spacecraft mass dedicated to fuel over the former, depending on the optimization method used. When comparing the interplanetary phases of each method, the Patched Conics with manifolds method only requires one stage that also uses at least 23% less fuel than the other two methods. Both the Manifold to Hyperbolic Escape and Hohmann methods also require second stages than need over 90% of their spacecraft mass dedicated to fuel.

Table 6.7: Time of Flight Comparison between Methods (Jupiter).

<table>
<thead>
<tr>
<th></th>
<th>Patched Conics (Indirect)</th>
<th>Patched Conics (Direct)</th>
<th>Manifold to Hyperbolic</th>
<th>Hohmann Transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>ToF (days (yrs))</td>
<td>3373 (9.2)</td>
<td>3446 (9.4)</td>
<td>919 (2.5)</td>
<td>553 (1.5)</td>
</tr>
</tbody>
</table>

Although there are major savings on fuel using the Patched Conics with Manifolds method, there is a major increase in the time of flight. This method requires a total flight time of over nine years from Earth to Jupiter, while the Manifold to Hyperbolic Escape and Hohmann transfer methods require about 2.5 and 1.5 years, respectively.
Lastly, Table 6.8 shows the number of points used for each of the low-thrust optimized maneuvers used in the mission profile along with the run times of the optimization algorithms.

**Table 6.8: Number of Points and Run Times for Each Optimized Low-Thrust Maneuver.**

<table>
<thead>
<tr>
<th>Maneuver</th>
<th>Number of Points</th>
<th>Run Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbit Raising (Indirect)</td>
<td>6173</td>
<td>00:01:13</td>
</tr>
<tr>
<td>Orbit Raising (Direct)</td>
<td>400</td>
<td>04:21:46</td>
</tr>
<tr>
<td>Interplanetary (Indirect)</td>
<td>1089</td>
<td>00:00:10</td>
</tr>
<tr>
<td>Interplanetary (Direct - 1st Leg)</td>
<td>300</td>
<td>01:38:31</td>
</tr>
<tr>
<td>Interplanetary (Direct - 2nd Leg)</td>
<td>300</td>
<td>01:57:02</td>
</tr>
</tbody>
</table>

When comparing the run times for the indirect and direct approaches, the run times for the direct method are significantly longer for both the orbit raising and interplanetary maneuvers. The indirect problems are solved within a few minutes, while the direct problems take a few hours to solve. This is due to the indirect algorithm only needing to check the constraints at the boundaries of the problem, while the direct algorithm must check constraints at each point along the trajectory, making the solving process much longer. In addition, when comparing the run times and numbers of points used for the orbit raising to the interplanetary transfer, the interplanetary transfer requires fewer points to estimate the trajectory and shorter run times. This is because the orbit raising maneuver requires the spacecraft to complete several cycles around Earth to get to the target orbit, while the spacecraft only has to cycle around Earth once for the interplanetary transfer before maneuvering out towards the Jupiter exterior stable manifold.
6.2 Earth to Saturn Transfer

Rund also tested the method of using manifolds along with hyperbolic escape trajectories to reduce the $\Delta V$ needed to transfer between Earth and Saturn and found that by using this method, the overall mission $\Delta V$ could be reduced by 3.6 km/s when compared to a Hohmann transfer [12]. The goal of this work was to attempt to further reduce the fuel required by substituting in optimized low-thrust transfers for parts of the full interplanetary transfer. In the following sections, the trajectories used to complete a transfer using low-thrust trajectories combined with manifolds between Earth and Saturn along with the fuel mass and time of flight required for each portion of the transfer. In addition, the results found in this work will be compared to the results found in Rund for the Manifold to Hyperbolic Escape Trajectory method and for a Hohmann Transfer.

As the results for the transfers will be presented in fuel mass percentage of the spacecraft used rather than $\Delta V$, the spacecraft mass and engine parameters are needed for both the impulse transfers and the low-thrust transfers. Similar to the Jupiter mission, the initial mass of the spacecraft was set as 500 kg, while Table 6.9 shows these parameters used for the transfer between Earth and Jupiter for both the impulse and low-thrust maneuvers. Again, as with the Jupiter mission, the thrust used for creating the interplanetary trajectory with the direct method was slightly lower than for the indirect method in order for a constant thrust trajectory to be created that satisfied the boundary conditions.
Table 6.9: Spacecraft and Engine Parameters for Impulse and Low-Thrust Maneuvers for Saturn Mission.

<table>
<thead>
<tr>
<th>Description</th>
<th>Specific Impulse (sec)</th>
<th>Thrust (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Impulse</td>
<td>300</td>
<td>- - -</td>
</tr>
<tr>
<td>Low-Thrust (OR)</td>
<td>3000</td>
<td>10</td>
</tr>
<tr>
<td>Low-Thrust (IP)</td>
<td>3000</td>
<td>0.5 (Indirect)</td>
</tr>
</tbody>
</table>

The mission begins with the spacecraft starting in a 500 km altitude circular orbit about Earth. The halo orbit from which the stable manifolds are propagated was chosen to be 300,000 km as this lies within the regime discussed in Section 4.2.1, which leads to the lowest amounts of fuel needed to merge between the target orbit and the manifold for L₁ halo orbits in the Sun-Earth system. The target orbit selected for the spacecraft’s low-thrust orbit raising maneuver was chosen to be 35,000 km as this also lies within the regime of orbit altitudes that saves the most on fuel. A different sized halo orbit and target orbit altitude were chosen to obtain new results for the optimization methods used for the orbit raising transfer. When the manifolds were propagated from the halo orbit, the manifold that passed closest to the chosen target orbit was at an altitude of 36,869 km, making this the true target orbit to which the spacecraft will reach from the orbit raising maneuver. In addition, the manifold intersects the target orbit altitude with an inclination of 34°, RAAN of 17°, and argument of perigee of 25°. These parameters define the orbital plane of the two-dimensional circular-to-circular low-thrust orbit raising maneuver.

Again, both the indirect and direct optimization methods were used to create minimal fuel trajectories that reach the target orbit using a specific impulse of 3000 seconds and constant thrust of 10 N. Additionally, the target time span of the low-thrust transfers was chosen to be two days. Figure 6.9 shows the spacecraft’s transfer path along the green trajectory starting from the initial 500 km orbit, raising its altitude.
to the target orbit using the low-thrust transfer, arriving at the target orbit at a true anomaly of 230°, coasting to the manifold intersection point, and performing the impulse maneuver to merge onto the manifold. The low-thrust trajectory for this transfer was found using the direct method. A similar plot for the low-thrust trajectory created using the indirect method can be found in Appendix D.

![Diagram showing spacecraft trajectory](image)

**Figure 6.9:** View of the spacecraft’s trajectory (green) during the orbit raising and manifold merge portions of the Earth Orbit to Halo Orbit Transfer Phase.

Table 6.10 shows the fuel mass percentage and time of flight results for the orbit raising maneuver for both optimization methods.
Table 6.10: Optimization Results for Spacecraft Orbit Raising (Saturn).

<table>
<thead>
<tr>
<th></th>
<th>Fuel Mass Percentage</th>
<th>Time of Flight (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Indirect</td>
<td>15%</td>
<td>2.40</td>
</tr>
<tr>
<td>Direct</td>
<td>14%</td>
<td>2.20</td>
</tr>
</tbody>
</table>

Similar to the trajectory created for the Earth to Jupiter case, the direct method was able to produce a trajectory that requires 1% less fuel than the trajectory produced by the indirect method. This is also due to the transfer time for the direct method being slightly less that of the indirect method and because the collocation algorithm required fewer time steps to determine a viable solution. Figure 6.10 and Figure 6.11 show the low-thrust orbit raising trajectories in two-dimensions for the indirect and direct methods, respectively.

When comparing the low-thrust orbit raising trajectories created by the indirect and direct optimization methods in Figure 6.10 and Figure 6.11, respectively, the indirect trajectory spirals out slightly more elliptically whereas the direct trajectory spirals out more circularly. This is mainly due to the direct method trajectory being very close to the optimized trajectory, which is inherent in the collocation approach. In addition, the direct trajectory has a much lower fidelity, especially closer to Earth, due to fewer points being used to approximate the solution as was shown with the Jupiter mission trajectories.
Figure 6.10: Low-thrust orbit raising trajectory for the indirect method.

Figure 6.11: Low-thrust orbit raising trajectory for the direct method.
To merge the spacecraft from the target orbit to the stable manifold to travel to the halo orbit an impulse maneuver with a $\Delta V$ of 1.3 km/s is required, which is equivalent to 35% of the spacecraft’s total mass required for fuel. This is about half the fuel required when compared to the results found from Rund of a $\Delta V$ of 3.2 km/s, or 66% mass ratio, when a lower orbit was used to merge onto the stable manifold. With the additional fuel used to perform the low-thrust orbit raising added in, the proposed method produces fuel savings of about 16% when compared to the previous method.

Figure 6.12 shows the full trajectory that the spacecraft travels along after merging onto the stable manifold to reach the 300,000 km halo orbit. The time of flight from the manifold intersection point to the halo injection point is 388 days.

**Figure 6.12:** Transfer path from Earth to halo orbit along the stable manifold trajectory.
The spacecraft then arrives at the 300,000 km halo orbit and the next maneuver is to traverse the halo to the halo departure point to begin the Earth Departure Phase. To accomplish this, the exterior stable manifold must be chosen along with the propagation time before linking to the interplanetary trajectory. The process described in Section 4.3.2 was used to select the manifold that lies closest to the orbital plane at the time at which the manifold reaches the energy transition point. Plots of the manifold selection and energy transition point are shown in Appendix D. Based on these plots, it was determined that the spacecraft reached the energy transition point after traveling along the manifold for 133 days.

Once the spacecraft reaches the halo departure point, a small perturbation in its position and velocity cause it to merge onto the exterior unstable manifold and travel towards the interplanetary injection point. The spacecraft spends 133 days on this trajectory before connecting to the interplanetary transfer. Figure 6.13 shows the route the spacecraft takes for from the halo departure point to the interplanetary injection point.
Figure 6.13: Trajectory of the spacecraft along the chosen exterior unstable manifold (green) to connect to the interplanetary transfer to depart Earth.

At the interplanetary injection point, the reference frame shifts to the Heliocentric Frame and it is now assumed that the spacecraft has the speed of the Earth relative to its rotation about the Sun. Both the indirect and direct optimization methods were used to find minimum fuel trajectories to reach Saturn’s stable manifold. Before determining the low-thrust trajectory, the departure and arrival dates were selected to determine the locations of each planet within their orbital planes. Several dates were tested between 2025 and 2035 for potential departure dates for the mission, with August 1, 2030 chosen as the date at which the spacecraft will reach the interplanetary injection point. The departure date and time of flight of the transfer determine
the location of Earth at departure and Saturn at arrival as well as the locations of the manifold endpoints in the Heliocentric Frame. Figures 6.14 shows the optimal trajectory found using the indirect method and Figure 6.15 shows the optimal trajectory found using the direct method.

![Figure 6.14: Optimal Interplanetary trajectory for the indirect method.](image)

Figure 6.14: Optimal Interplanetary trajectory for the indirect method.
The interplanetary trajectory found using the indirect approach shows the spacecraft cycling twice close to Earth before traveling out to the exterior stable Saturn manifold at a very steep angle. Due to the steep angle at which the spacecraft arrives at the stable manifold, a small correction may be needed to properly align the spacecraft’s thrusting vector with the manifold trajectory. The trajectory found using the direct method shows that the spacecraft completes one revolution near Earth before taking a wide angle to align the thrust vector of the spacecraft with the arrival angle of the stable manifold at Saturn. Table 6.11 shows the fuel mass percentage and time of flight necessary for the trajectories found for both optimization methods.
The results above show that the direct method trajectory requires 386 more days to reach the Saturn manifold, but needs 2% less fuel than for the indirect trajectory. Similar to the orbit raising optimization, fewer time steps are required for the collocation algorithm to solve for the optimal trajectory which means the spacecraft will need to thrust at fewer points along the orbit and less fuel used.

Upon arrival at the exterior stable manifold for the Saturn halo orbit, the spacecraft reaches another energy transition point where the problem shifts from a two-body problem in the Heliocentric Frame to another three-body problem in the Sun-Saturn Synodic Frame. At this point the spacecraft is deemed to have reached a velocity that is sufficient to link to the manifold without any additional maneuvers. The halo size chosen for the Sun-Saturn L1 halo orbit was 700,000 km. The size of this halo orbit is arbitrary for the Patched Conics with Manifolds method as any size halo orbit will have manifolds that align with the synodic x-y plane. Similar to Jupiter, due to the mass of the planet, Saturn L1 halo orbits on the order of hundreds of thousands of kilometers in amplitude are almost parallel to the synodic x-y plane, meaning that their manifolds will be more parallel to the x-y plane as well. Plots of the manifold selection and energy transition point selection can be found in Appendix D. Based on these plots, it was determined that the spacecraft reaches the halo injection point after traveling along the manifold for 2672 days.

Once the spacecraft reaches the interplanetary arrival point, it connects with the exterior stable manifold and travel towards halo orbit without the need for any ad-
ditional maneuvers, arriving at the halo injection point after 2672 days. Figure 6.16 shows the route the spacecraft takes from the interplanetary arrival point to the halo injection point along the exterior stable manifold in the Sun-Saturn Synodic Frame.

![Diagram of spacecraft trajectory](image)

**Figure 6.16: Trajectory of the spacecraft along the chosen exterior stable manifold (green) to leave the interplanetary transfer and arrive at the Sun-Saturn L1 halo.**

At the halo injection point, the mission will be considered complete for the purposes of this work; however, as mentioned with the Jupiter test case, the spacecraft could complete several other types of maneuvers from this location. The fuel mass percentages used and time of flight data for each stage of the transfer are compiled
in Table 6.12 and Table 6.13 for missions using the indirect and direct optimization methods, respectively.

**Table 6.12: Fuel Mass Used and Time of Flight (ToF) for Each Mission Phase for Saturn Mission (Indirect).**

<table>
<thead>
<tr>
<th>Transfer Phase</th>
<th>Remaining Mass (kg)</th>
<th>ToF (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbit Raising</td>
<td>425</td>
<td>2</td>
</tr>
<tr>
<td>Manifold Merge</td>
<td>276</td>
<td>—</td>
</tr>
<tr>
<td>Earth to Halo</td>
<td>276</td>
<td>388</td>
</tr>
<tr>
<td>Exterior Manifold</td>
<td>276</td>
<td>133</td>
</tr>
<tr>
<td>Interplanetary Phase</td>
<td>80</td>
<td>2500</td>
</tr>
<tr>
<td>Jupiter Arrival Phase</td>
<td>80</td>
<td>2672</td>
</tr>
</tbody>
</table>

**Table 6.13: Fuel Mass Used and Time of Flight (ToF) for Each Mission Phase for Saturn Mission (Direct).**

<table>
<thead>
<tr>
<th>Transfer Phase</th>
<th>Remaining Mass (kg)</th>
<th>ToF (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbit Raising</td>
<td>430</td>
<td>2</td>
</tr>
<tr>
<td>Manifold Merge</td>
<td>280</td>
<td>—</td>
</tr>
<tr>
<td>Earth to Halo</td>
<td>280</td>
<td>388</td>
</tr>
<tr>
<td>Exterior Manifold</td>
<td>280</td>
<td>133</td>
</tr>
<tr>
<td>Interplanetary Phase</td>
<td>87</td>
<td>2886</td>
</tr>
<tr>
<td>Jupiter Arrival Phase</td>
<td>87</td>
<td>2672</td>
</tr>
</tbody>
</table>

Based on the results from Tables 6.12 and 6.13, the spacecraft finishes with a final mass of 87 kg for the direct approach mission, and 80 kg for the indirect approach mission. The time of flight for the direct mission is also 386 days longer due to the extra time required for the interplanetary transfer.

To compare the results found using the Patched Conics with Manifolds method for an interplanetary transfer to the Manifold to Hyperbolic Escape Trajectory Method used in Rund and to traditional Hohmann transfers, Table 6.14 shows comparisons of fuel mass percentage used for each phase of the transfers and Table 6.15 shows the total time of flight required for each method.
Table 6.14: Fuel Mass Percentage Comparison between Methods (Saturn).

<table>
<thead>
<tr>
<th></th>
<th>Patched Conics (Indirect)</th>
<th>Patched Conics (Direct)</th>
<th>Manifold to Hyperbolic Transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbit Raising</td>
<td>15%</td>
<td>14%</td>
<td>–</td>
</tr>
<tr>
<td>Manifold Merge</td>
<td>35%</td>
<td>35%</td>
<td>66%</td>
</tr>
<tr>
<td>IP Phase (Stage 1)</td>
<td>71%</td>
<td>69%</td>
<td>86%</td>
</tr>
<tr>
<td>IP Phase (Stage 2)</td>
<td>–</td>
<td>–</td>
<td>45%</td>
</tr>
</tbody>
</table>

When the combination of the orbit raising and manifold merge phases of the Patched Conics with Manifolds method are compared to the manifold merge phase for the Manifold to Hyperbolic Escape Trajectory method, the latter requires 16-17% less spacecraft mass dedicated to fuel over the former, depending on the optimization method used. When comparing the interplanetary phases of each method, the Patched Conics with manifolds method only requires one stage that also uses at least 15% less fuel than the other two methods. Both the Manifold to Hyperbolic Escape and Hohmann Transfer methods also require second stages, although the amount of spacecraft mass dedicated to fuel for this stage is much less than for the Earth to Jupiter mission.

Table 6.15: Time of Flight Comparison between Methods (Saturn).

<table>
<thead>
<tr>
<th></th>
<th>Patched Conics (Indirect)</th>
<th>Patched Conics (Direct)</th>
<th>Manifold to Hyperbolic Transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>ToF (days (yrs))</td>
<td>5695 (15.6)</td>
<td>6081 (16.7)</td>
<td>2578 (7.1)</td>
</tr>
</tbody>
</table>

Again, although there are major savings on fuel for the Patched Conics with Manifolds method, the time of flight is much higher for this method compared to the methods compared by Rund. This method requires a total flight time of over fifteen and a half years from Earth to Saturn, while the Manifold to Hyperbolic Escape and Hohmann transfer methods require about 7 and 6 years respectively.
Lastly, Table 6.16 shows the number of points used for each of the low-thrust optimized maneuvers used in the mission profile along with the run times of the optimization algorithms.

### Table 6.16: Number of Points and Run Times for Each Optimized Low-Thrust Maneuver.

<table>
<thead>
<tr>
<th>Maneuver</th>
<th>Number of Points</th>
<th>Run Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbit Raising (Indirect)</td>
<td>5625</td>
<td>00:01:10</td>
</tr>
<tr>
<td>Orbit Raising (Direct)</td>
<td>400</td>
<td>05:06:15</td>
</tr>
<tr>
<td>Interplanetary (Indirect)</td>
<td>981</td>
<td>00:00:14</td>
</tr>
<tr>
<td>Interplanetary (Direct - 1st Leg)</td>
<td>300</td>
<td>01:22:39</td>
</tr>
<tr>
<td>Interplanetary (Direct - 2nd Leg)</td>
<td>300</td>
<td>02:03:52</td>
</tr>
</tbody>
</table>

Similar to the Jupiter test case, the run times for the direct method are significantly longer for both the orbit raising and interplanetary maneuvers. This is again due to the indirect algorithm only needing to check the constraints at the boundaries of the problem, while the direct algorithm must check constraints at each point along the trajectory. Also, the interplanetary transfer requires fewer points to estimate the trajectory and shorter run times as was observed with the Jupiter case. This is again because the orbit raising maneuver requires the spacecraft to complete several cycles around Earth to get to the target orbit, while the spacecraft only has to cycle around Earth once for the interplanetary transfer before maneuvering out towards the Saturn exterior stable manifold.
7.1 Conclusions

This thesis further explored the possibility of using the dynamics of the Circular Restricted Three-Body Problem (CRTBP) to reduce the amount of fuel required for interplanetary missions. The method of using the invariant manifolds of halo orbits to create transfers from low Earth orbit to interplanetary trajectories using patched conics was used to test missions to Jupiter and Saturn. Two optimization methods, an indirect method employing Euler-Lagrange Theory and a direct method using collocation, were implemented for parts of each mission to assist in further reducing the amount of fuel required for these types of transfers.

The results from this work have shown that performing low-thrust orbit raising prior to merging onto a manifold to reach a Sun-Earth L1 halo orbit can significantly reduce the amount of fuel needed to reach the initial halo orbit. When compared to the Manifold to Hyperbolic Escape Trajectory method used by Rund, 13-14% less fuel was required to reach the halo for the Jupiter test case and 16-17% less fuel was required for the Saturn test case depending on the optimization method used.

For the interplanetary phase of the transfer, applying low-thrust maneuvers rather than impulse transfers also showed that a reduction can be made in the amount of fuel needed to complete transfers to Jupiter and Saturn by reducing the number of stages from two to one. In addition, for the Jupiter case the single stage needed 23-27% less fuel than the first stage for an impulse transfer, while for the Saturn case the single stage needed 15-17% less fuel depending on the optimization method used.
Despite the savings in fuel created by using the Patched Conics with manifolds method, this method requires much longer flight times. For the Jupiter and Saturn cases, 6.5 and 8.5 more years are required when compared to the Manifold to Hyperbolic Escape trajectory method, respectively.

When comparing the indirect and direct optimization methods, the direct method was able to find optimized minimum fuel trajectories that required about 1-4% less fuel than the trajectories produced by the indirect method. This was mainly due to the ability of the collocation scheme to find trajectories that required fewer time steps, and thus fewer thrusting periods, to create the optimal trajectories. Further research into both the dynamics of the CRTBP and optimization could reduce the amount of fuel needed for these types of transfers using manifolds while also improving on overall time of flight to expand the mission applications.

### 7.2 Future Work

This work expanded upon the field of creating transfer trajectories using the dynamics of the CRTBP by applying standard optimization techniques to different phases of interplanetary missions that use these dynamics. Although the goal of reducing fuel requirements for interplanetary transfers using manifolds was achieved, some improvements could be made and further investigation could take place to continue to explore interplanetary transfers using the CRTBP.

The main focus of this thesis was on applying optimization to phases of the interplanetary transfer that could be modeled as circular-to-circular maneuvers because the indirect and direct algorithms applied were more suited to this application. To adjust the Earth Orbit to Halo Orbit Transfer Phase of the Patched Conics with Manifolds method mission profile, the low-thrust optimized trajectory could be found strictly
along the manifold using the CRTBP dynamics rather than having to perform the
orbit raising maneuver. This would allow the spacecraft to travel from the initial
Earth orbit to the halo orbit along an optimized trajectory only using manifolds.
This was attempted in the early stages of the research for this work; however, the
indirect method proved to be ill-equipped for solving such an optimization problem
due to the volatility in the costates when the spacecraft approaches Earth. Different
optimization methods could be explored to determine if this is a feasible method for
performing the first phase of the transfer method. The point at which the spacecraft
enters the manifold along the target orbit could also be optimized by incorporating
a more robust search algorithm.

Furthermore, in terms of optimization, more modern optimization techniques such as
evolutionary algorithms could be used to attempt to find more efficient trajectories
than those found with indirect or direct optimization methods. Incorporating inter-
planetary transfers that use manifolds into an optimization suite such as the STOpS
suite developed at Cal Poly for optimizing interplanetary transfers by using evolu-
tionary algorithms. The optimization could also be modeled in three-dimensions to
gain more accurate results with the true positions of the Earth and destination planet
taken into account.

As this work only ran through each test case with one set of spacecraft mass, thrust,
and specific impulse values, different spacecraft parameters, for example a higher
spacecraft mass, could be investigated. This would allow for a more thorough analysis
of how spacecraft mass, thrust, and specific impulse values affect each other for these
types of missions. In addition, the interplanetary transfers explored in this paper that
combine using manifolds and low-thrust maneuvers could be compared to low-thrust
interplanetary transfers that do not use manifolds. This would show how much using
manifolds factors into reducing the overall fuel requirements of a mission that uses a low-thrust transfer rather than an impulse transfer for an interplanetary mission.

Lastly, as both this work and the work of Rund only tested missions to Jupiter and Saturn, more expansive missions with more complex mission profiles could be tested to determine their feasibility. For example, a mission that visits both Jupiter and Saturn in one trip using manifolds could be explored. This would require the spacecraft to enter into a halo orbit around Jupiter first and then connect to another exterior unstable manifold to link to a new interplanetary transfer that takes the spacecraft form Jupiter to Saturn. This could be compared to a mission that uses flybys of other planets to reach multiple destinations in one mission such as the Voyager spacecraft.
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    trajectory optimization with applications. Abstract and Applied Analysis -

A.1 Earth-Centered Inertial Frame

The Earth-Centered Inertial (ECI) Frame is a fixed two-body frame where the origin is the center of mass of the Earth and the spacecraft is the secondary body. This frame is used for the orbit raising portion of the Earth Orbit to Halo Orbit Transfer Phase. A diagram of the ECI frame is shown in Figure A.1.

![Figure A.1: The Earth-Centered Inertial Frame.](image)

The canonical units and their comparable conventional units used for analysis done in the ECI frame are shown in Table A.1:
Table A.1: Conversion Between Canonical and Conventional Units for ECI Frame.

<table>
<thead>
<tr>
<th>Description</th>
<th>Canonical Unit</th>
<th>Conventional Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance</td>
<td>1 DU</td>
<td>6378 km</td>
</tr>
<tr>
<td>Time</td>
<td>1 TU</td>
<td>806.8 sec</td>
</tr>
<tr>
<td>Velocity</td>
<td>1 DU/TU</td>
<td>7.9 km/s</td>
</tr>
<tr>
<td>Gravitational Parameter ($\mu$)</td>
<td>1 DU$^3$/TU$^2$</td>
<td>3.986 km$^3$/s$^2$</td>
</tr>
</tbody>
</table>

A distance unit (DU) for this system is defined as the radius of the Earth. The gravitational parameter ($\mu$) is set as 1 DU$^3$/TU$^2$ and is equivalent to the gravitational parameter of the Earth. From these two parameters, a time unit (TU) can be defined as 806.8 seconds, or approximately 13.5 minutes, and a velocity unit (VU) can be defined as 7.9 km/s, which is approximately the maximum speed that an object can travel while orbiting Earth in Low Earth Orbit (LEO).

A.2 Sun-Earth Synodic Frame

The Sun-Earth Synodic (SES) Frame is a three-body rotating frame with the origin at the center of mass of the two primary bodies, the Sun and the Earth, and the spacecraft as the secondary body. The equations of motion for the CRTBP discussed in Chapter 3 can be used to model motion of the spacecraft. This frame is used for the interior stable manifold traversal of the Earth Orbit to Halo Orbit Transfer Phase and the exterior unstable manifold traversal of the Earth Departure Phase. A diagram of the Sun-Earth Synodic Frame is shown in Figure A.2.
The canonical units and their comparable conventional units used for analysis done in the SES Frame are shown in Table A.2. The masses of the primary bodies and the mass ratio $\mu^*$ are also shown in Table A.3.

### Table A.2: Conversion Between Canonical and Conventional Units for SES Frame.

<table>
<thead>
<tr>
<th>Description</th>
<th>Canonical Unit</th>
<th>Conventional Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance</td>
<td>1 DU</td>
<td>1.496e8 km</td>
</tr>
<tr>
<td>Time</td>
<td>1 TU</td>
<td>5.0227e6 sec</td>
</tr>
<tr>
<td>Velocity</td>
<td>1 DU/TU</td>
<td>29.7845 km/s</td>
</tr>
<tr>
<td>Gravitational Parameter ($\mu$)</td>
<td>1 DU$^3$/TU$^2$</td>
<td>1.3271e11 km$^3$/s$^2$</td>
</tr>
</tbody>
</table>

### Table A.3: Mass Parameters for the SES Frame

<table>
<thead>
<tr>
<th>Description</th>
<th>Parameter Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of Sun</td>
<td>1.989e30 kg</td>
</tr>
<tr>
<td>Mass of Earth</td>
<td>5.974e24 kg</td>
</tr>
<tr>
<td>Mass Ratio</td>
<td>3.0035e-06</td>
</tr>
</tbody>
</table>

A distance unit (DU) is defined as the average distance between the Sun and the Earth. The gravitational parameter ($\mu$) is set as $1\ DU^3/TU^2$ and is equivalent to the
gravitational parameter of the Sun. From these two parameters, a time unit (TU) can be defined as 5.0227e6 seconds, or approximately 58.13 days, and a velocity unit (VU) can be defined as 29.7845 km/s, which is approximately the speed at which the Earth orbits the Sun. The mass ratio of the two primary bodies, 3.0035e-6, is also used in the equations of motion for the CRTBP system rather than the gravitational parameter itself.

A.3 Heliocentric Frame

The Heliocentric Frame is a two-body inertial frame where the origin is the center of mass of the Sun and the spacecraft is the secondary body. This frame is used to create the low-thrust interplanetary transfers for the Interplanetary Transfer Phase of the mission. A diagram of the Heliocentric Frame is shown in Figure A.3.

![Figure A.3: The Heliocentric Frame.](image)

The same conversions between canonical and conventional units used for the SES Frame are used for the Heliocentric Frame because the Sun remains the primary
body of the system. The canonical units and their comparable conventional units used for analysis done in the Heliocentric Frame are shown in Table A.4:

Table A.4: Conversion Between Canonical and Conventional Units for Heliocentric Frame.

<table>
<thead>
<tr>
<th>Description</th>
<th>Canonical Unit</th>
<th>Conventional Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance</td>
<td>1 DU</td>
<td>1.496e8 km</td>
</tr>
<tr>
<td>Time</td>
<td>1 TU</td>
<td>5.0227e6 sec</td>
</tr>
<tr>
<td>Velocity</td>
<td>1 DU/TU</td>
<td>29.7845 km/s</td>
</tr>
<tr>
<td>Gravitational Parameter ($\mu$)</td>
<td>1 DU$^3$/TU$^2$</td>
<td>1.3271e11 km$^3$/s$^2$</td>
</tr>
</tbody>
</table>

A.4 Sun-Destination Planet Synodic Frame

The Sun-Destination Planet Synodic Frame is a three-body rotating frame with the origin at the center of mass of the two primary bodies, the Sun and the destination planet (either Jupiter or Saturn for this thesis), and the spacecraft as the secondary body. The equations of motion for the CRTBP discussed in Chapter 3 can be used to model motion of the spacecraft. This frame is used for the final exterior stable manifold traversal to take the spacecraft to the arrival halo orbit as part of the Destination Planet Arrival Phase. A diagram of the Sun-Destination Planet Synodic Frame is shown in Figure A.4.
The canonical units and their comparable conventional units used for analysis done in the Sun-Jupiter Synodic (SJS) Frame are shown in Table A.5 and the mass parameters for the system are shown in Table A.6. Unit conversions and mass parameters for the Sun-Saturn Synodic (SSS) Frame are shown in Table A.7 and Table A.8. Definitions equivalent to those of the Sun-Earth Synodic Frame are observed for these two frames.

Table A.5: Conversion Between Canonical and Conventional Units for SJS Frame.

<table>
<thead>
<tr>
<th>Description</th>
<th>Canonical Unit</th>
<th>Conventional Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance</td>
<td>1 DU</td>
<td>7.78ee8 km</td>
</tr>
<tr>
<td>Time</td>
<td>1 TU</td>
<td>5.957e7 sec</td>
</tr>
<tr>
<td>Velocity</td>
<td>1 DU/TU</td>
<td>13.0606 km/s</td>
</tr>
<tr>
<td>Gravitational Parameter (μ)</td>
<td>1 DU³/TU²</td>
<td>1.3271e11 km³/s²</td>
</tr>
</tbody>
</table>

Table A.6: Mass Parameters for the SJS Frame

<table>
<thead>
<tr>
<th>Description</th>
<th>Parameter Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of Sun</td>
<td>1.989e30 kg</td>
</tr>
<tr>
<td>Mass of Jupiter</td>
<td>1.899e27 kg</td>
</tr>
<tr>
<td>Mass Ratio</td>
<td>9.5384e-04</td>
</tr>
</tbody>
</table>
Table A.7: Conversion Between Canonical and Conventional Units for SSS Frame.

<table>
<thead>
<tr>
<th>Description</th>
<th>Canonical Unit</th>
<th>Conventional Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance</td>
<td>1 DU</td>
<td>1.433e9 km</td>
</tr>
<tr>
<td>Time</td>
<td>1 TU</td>
<td>1.489e8 sec</td>
</tr>
<tr>
<td>Velocity</td>
<td>1 DU/TU</td>
<td>9.6234 km/s</td>
</tr>
<tr>
<td>Gravitational Parameter (\mu)</td>
<td>1 DU³/TU²</td>
<td>1.3271e11 km³/s²</td>
</tr>
</tbody>
</table>

Table A.8: Mass Parameters for the SSS Frame

<table>
<thead>
<tr>
<th>Description</th>
<th>Parameter Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of Sun</td>
<td>1.989e30 kg</td>
</tr>
<tr>
<td>Mass of Jupiter</td>
<td>5.685e26 kg</td>
</tr>
<tr>
<td>Mass Ratio</td>
<td>2.8582e-04</td>
</tr>
</tbody>
</table>
Appendix B

DIRECT METHOD VERIFICATION

To verify that the collocation schemes used in this work were implemented properly, an optimization scheme was set up to minimize the final time of a circular-to-circular orbit transfer. To accomplish this, several different final times were given as initial guesses for the optimization scheme and it was shown that for any guess, the optimization scheme would reach the same minimum. This was done rather than testing several initial guesses for the maximum final mass case used for the test cases because the optimize mass problems take much longer to solve, so it was not deemed feasible to test several initial guesses in the time needed to complete this work.

The problem set up to verify the collocation scheme was a transfer from 319 km altitude circular orbit about Earth (equal to 1.05 DU) to a 6378 km altitude circular orbit (equal to 2 DU). The initial guess for the optimization scheme was a Hohmann transfer from a true anomaly of 0° at the initial orbit to a 180° true anomaly at the final orbit that takes 5.92 TU, or approximately 79.48 minutes.

Both the trapezoidal and Hermite-Simpson methods were used as the integration methods and initial guesses between 1 TU and 20 TU were given. For each initial guess, the number of iterations it took to find the optimal solution was found. Figure B.1 shows the results for the Trapezoidal method while Figure B.2 shows the results for the Hermite-Simpson method.
Figure B.1: Trapezoidal method results.
From both Figure B.1 and Figure B.2, it can be seen that the algorithm finds the same minimum value despite the initial guess. For the trapezoidal method, it takes up to 47 iterations for the algorithm to find the solution, while for the Hermite-Simpson method it takes up to 24 iterations. This makes sense as the Hermite-Simpson method is twice as efficient as the Trapezoidal method because it employs the collocation point at each segment in the mesh. In addition, the optimal time found for each method was compared to the optimal time found by the indirect analytical method to ensure that the minimum time found was indeed a viable solution. Table 5.1 shows a comparison of the minimal times found for each of the three optimization methods compared to the guess for the Hohmann transfer.
Table B.1: Comparison of Minimal Transfer Times Found

<table>
<thead>
<tr>
<th>Description</th>
<th>Minimum Time (Canonical)</th>
<th>Minimum Time (Minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Guess</td>
<td>5.92</td>
<td>79.48</td>
</tr>
<tr>
<td>Indirect Method</td>
<td>5.75</td>
<td>77.26</td>
</tr>
<tr>
<td>Trapezoid</td>
<td>5.87</td>
<td>78.85</td>
</tr>
<tr>
<td>Hermite-Simpson</td>
<td>5.91</td>
<td>79.37</td>
</tr>
</tbody>
</table>

Based on the results shown in Table 5.1, it can be seen that the minimum times found by all three optimization methods are all slightly less than the initial guess time and are relatively close. For example, the minimum time found for the Hermite-Simpson method was within 3% of the minimum time found for the Indirect method. It should also be noted that the minimum solutions found here are local minimums based on the initial guess given for the problem. If a different initial guess was given, particularly for the collocation methods, different minimum values would be found for the transfer between the two circular orbits.
Figure C.1: View of the spacecraft’s trajectory (green) during the orbit raising and manifold merge portions of the Earth Orbit to Halo Orbit Transfer Phase (Direct).
Figure C.2: Exterior unstable manifold endpoints (blue) in relation to the synodic x-y plane, along with the closest point (red) for a 250,000 km Sun-Earth L$_1$ Halo.
Figure C.3: Two-body energy associated with the exterior unstable manifold selected for departure from the Sun-Earth L₁ Halo.
Figure C.4: Exterior stable manifold endpoints (blue) in relation to the synodic x-y plane, along with the closest point (red) for a 900,000 km Sun-Jupiter $L_1$ Halo.
Figure C.5: Two-body energy associated with the exterior stable manifold selected for arrival at the Sun-Jupiter L\textsubscript{1} Halo.
Appendix D

EARTH TO SATURN SUPPLEMENTAL PLOTS

Figure D.1: View of the spacecraft’s trajectory (green) during the orbit raising and manifold merge portions of the Earth Orbit to Halo Orbit Transfer Phase (Direct).
Figure D.2: Exterior unstable manifold endpoints (blue) in relation to the synodic x-y plane, along with the closest point (red) for a 300,000 km Sun-Earth $L_1$ Halo.
Figure D.3: Two-body energy associated with the exterior unstable manifold selected for departure from the Sun-Earth L₁ Halo.
Figure D.4: Exterior stable manifold endpoints (blue) in relation to the synodic x-y plane, along with the closest point (red) for a 700,000 km Sun-Saturn L$_1$ Halo.
Figure D.5: Two-body energy associated with the exterior stable manifold selected for arrival at the Sun-Saturn L₁ Halo.