TROPICALIZATION OF CANONICAL CURVES

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ABSTRACT

Tropicalization of Canonical Curves

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A classic result in the study of nonsingular projective curves is their complete classification by way of the canonical embedding. In this paper we analyze the canonical embedding for nonsingular projective curves of genus $g \geq 3$. Our main goal is to explore how the canonical map interacts with the process of tropicalization. For the purposes of our investigation, we will mainly use an explicit projective plane curve of genus six as a test case.
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Chapter 1

INTRODUCTION

“This is unavoidably a bit messy, but just to be able to brag, I think it is a good idea to be able to say, ‘I have seen every curve once.’” - David Mumford [9]

A classic result in the study of nonsingular projective curves is their complete classification by way of the canonical embedding. In this paper we analyze the canonical embedding for nonsingular projective curves of genus $g \geq 3$. Our main goal is to explore how the canonical map interacts with the process of tropicalization. For the purposes of our investigation, we will mainly use an explicit projective plane curve of genus six as a test case.

There is a fair bit of background material that needs to be covered before the canonical map can be properly defined. We begin in Chapter 2 with a brief overview of classical algebraic geometry, focusing first on affine varieties and then projective varieties. In Sections 2.4 and 2.5, we give an overview of divisors and differentials, respectively, aiming to understand them in the context of nonsingular projective curves. We then proceed to study the canonical map, especially focusing on explicit examples.

In Chapter 4, we give an overview of tropical geometry, including the origin story and analysis of the semiring structure of $\mathbf{T}$. We also delve into the competing notions for tropical variety: bend loci and congruence varieties. We also touch on how each can be used to give tropical analogues of classical varieties, a process we call tropicalization.
Finally, in Chapter 5, we take the first steps in answering our central question: how does tropicalization interact with the canonical map? In particular, we study whether the process of tropicalization commutes with the canonical map.

We have written the paper assuming the reader has an undergraduate background in abstract algebra. While graduate level algebra and algebraic geometry may be necessary to understand some of the more advanced topics, we have simplified some of the more complex and subtle issues to make this paper a good read for most levels of mathematics. If the reader would like to learn more about one of these topics, we have listed references throughout the paper that should serve as excellent starting points.
The field of algebraic geometry began as the study of geometric objects defined by algebraic equations, specifically polynomial equations. A simple example of this is the standard plane parabola defined by $y = x^2$. The parabola is the geometric object and its defining equation carries the algebraic information. As we will see, it turns out to be more convenient to write such equations as the vanishing of polynomials. In the case of our parabola we can write it as $y - x^2 = 0$. So, the geometric objects are the vanishing loci of polynomials. Conversely, given any subset $S$ of $n$-dimensional space, we can consider the collection of all polynomials in $n$ variables that vanish identically on $S$. This gives rise to the algebra-geometry correspondence. Further study of these details lead to the construction of affine and projective varieties, and their later generalizations to schemes, algebraic spaces, and stacks.

In this chapter we give a brief overview of affine and projective varieties, highlighting the facts and definitions we need for the task at hand.

2.1 Affine Algebraic Geometry

We begin with affine varieties over a field $k$. A good reference for this section is [3].

**Definition 1.** Given a collection of polynomials $I \subseteq k[x_1, ..., x_n]$, the **affine variety** determined by $I$ is the set

$$ V(I) = \{ a \in k^n \mid f(a) = 0 \text{ for all } f \in I \} \subseteq k^n. $$
This set is sometimes also called the **vanishing locus** or **zero locus** of $I$. If $I = \{f_1, \ldots, f_n\}$ is a finite collection of polynomials, we write simply $V(f_1, \ldots, f_n)$.

**Example 2.** The standard real plane parabola is the affine variety $V(y - x^2)$ in $\mathbb{R}^2$:

```
Example 2.
```

![Parabola](image1.png)

**Example 3.** The standard unit circle is the affine variety $V(x^2 + y^2 - 1)$:

```
Example 3.
```

![Unit Circle](image2.png)

**Example 4.** The affine variety $V(z^2 - x^2 - y^2)$ is a cone through the origin:

```
Example 4.
```

![Cone](image3.png)
Remark. To be careful, one should specify the space in which the zero locus is defined, or equivalently the ring in which the polynomial lives. For example, if we took a look at the variety $V(y - x^2)$ in $\mathbb{R}^3$ then it is in fact the parabolic cylinder along the $z$-axis rather than the standard parabola in $\mathbb{R}^2$. We could avoid confusion by either writing $V_{\mathbb{R}^3}(y - x^2)$, or by noting that the polynomial is $f(x, y, z) = y - x^2 \in \mathbb{R}[x, y, z]$.

The affine variety associated to a collection of polynomials is our method for moving from algebraic information (polynomials) to geometric information (varieties). We next describe the converse process.

**Definition 5.** Given a subset $S \subseteq k^n$, we define the set

$$I(S) = \{f \in k[x_1, \ldots, x_n] \mid f(a) = 0 \text{ for all } a \in S\} \subseteq k[x_1, \ldots, x_n].$$

In other words, $I(S)$ is the collection of polynomials that vanish everywhere on $S$.

**Example 6.** By definition, $I(\mathbb{C}^n)$ consists of all polynomials that vanish everywhere on $\mathbb{C}^n$. The only such polynomial is the zero polynomial, so $I(\mathbb{C}^n) = \{0\}$.

**Example 7.** If $S$ is the single point $0$ in $\mathbb{C}$, then $I(0)$ is the collection of all polynomials $f \in \mathbb{C}[x]$ such that $f(0) = 0$. Using the Factor Theorem from algebra, we know that these are precisely the polynomials of the form $f(x) = xg(x)$, where $g \in \mathbb{C}[x]$. Thus, $I(0)$ is the ideal generated by $x$ in $\mathbb{C}[x]$.

**Example 8.** If $S = \mathbb{Z}$ inside of $\mathbb{C}$, then $I(S) = \{0\}$. This is because a nonzero polynomial can only have a finite number of distinct zeros.
We now have a connection between subsets of $k^n$ and subsets of $k[x_1, \ldots, x_n]$:

$$\{\text{subsets of } k^n\} \leftrightarrow \{\text{subsets of } k[x_1, \ldots, x_n]\}$$

$$S \mapsto I(S)$$

$$V(I) \mapsto I$$

This connection between algebra and geometry has many nice properties, a few of which are listed below.

**The Algebra-Geometry Correspondence.**

1. The set $I(S)$ is always an ideal in $k[x_1, \ldots, x_n]$.

2. If an ideal $I$ is generated by $f_1, \ldots, f_k$, then $V(I) = V(f_1, \ldots, f_k)$.

3. There is a correspondence between standard operations on ideals and standard geometric operations. For example, intersection of varieties corresponds to sum of ideals. A table listing the main such correspondences is given below.

4. The maps $V$ and $I$ are inclusion reversing. In other words, if $I_1 \subset I_2$, then $V(I_2) \supset V(I_1)$. Similarly, if $V_1 \subset V_2$, then $I(V_2) \supset I(V_1)$.

5. The sets $V(I)$ form the closed sets of a topology on $\mathbb{C}^n$, called the **Zariski topology**. We call $\mathbb{C}^n$ with this topology **affine $n$-space**, and denote it $A^\mathbb{C}_n$ (or simply $A^n$).

### 2.1.1 The Big Questions of Classical Affine Algebraic Geometry

Our maps above give us a connection between ideals and varieties. There are several questions one is naturally led to consider at this point. For instance:
1. When is $V(I)$ empty?

2. For an ideal $I$, what is $I(V(I))$?

3. For a variety $V$, what is $V(I(V))$?

We first note that if we don’t work over an algebraically closed field, there are all sorts of issues.

**Example 9.** The ideal $(x^2 + 1) \subseteq \mathbb{R}[x]$ is a proper ideal, and yet $V_{\mathbb{R}}(x^2 + 1) = \emptyset$. Of course, the corresponding variety is not empty when working over $\mathbb{C}$; indeed, $V_{\mathbb{C}}(x^2 + 1) = \{-i, i\}$.

**Example 10.** In a similar vein to the previous example, the polynomials $\Phi_4(x) = x^2 + 1$ and $\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$ are irreducible over $\mathbb{Q}$ (they are the minimal polynomials over $\mathbb{Q}$ for the roots of unity $\zeta_4$ and $\zeta_5$, respectively). They therefore generate distinct proper ideals, but each of these polynomials has no rational roots, so the corresponding varieties over $\mathbb{Q}$ are empty. On other other hand, those varieties are not empty when working over $\mathbb{C}$.

These kind of issues don’t appear when working over an algebraically closed field. Indeed, our first question is answered by the following:

**Theorem 1** (The Weak Nullstellantz). *Suppose $k$ is an algebraically closed field. If $I \subseteq k[x_1, \ldots, x_n]$ is an ideal, then $V(I) = \emptyset$ if and only if $I = k[x_1, \ldots, x_n]$. Equivalently, $V(I) = \emptyset$ exactly when $1 \in I$.*

In other words, when working over an algebraically closed field (like $\mathbb{C}$), every proper ideal has a nonempty vanishing locus. There are still other issues, however. The example below shows that the map $V$ is definitely not injective.
Example 11. Working in $\mathbb{C}[x]$, we have a proper inclusion $(x^2) \subset (x)$, and yet $V(x^2) = V(x) = \{0\}$.

The above example illustrates the only real issue. We first recall the following definition from algebra:

Definition 12. An ideal $I$ is radical if whenever $f^m \in I$ for some integer $m \geq 1$, one has $f \in I$. For a general ideal $I$, we let $\sqrt{I}$ denote the smallest radical ideal that contains $I$. It is called the radical of $I$, and consists of all elements $f$ such that $f^m \in I$ for some integer $m \geq 1$.

Example 13. With our previous example in mind, one can show $\sqrt{(x^2)} = (x)$.

Remark. The above example is a bit misleading in the sense that it appears straightforward to compute radical ideals. However, computing radical ideals is a nontrivial problem. For further reference, one can look at [3] to see how radical ideals can be computed via Groebner Bases.

One of the fundamental results of classical affine algebraic geometry is the following, which answers our second question:

Theorem 2 (Hilbert’s Nullstellanz). Suppose $I \subseteq k[x_1, \ldots, x_n]$ is an ideal, where $k$ is an algebraically closed field. Then $f \in I(V(I))$ if and only if $f^m \in I$ for some integer $m \geq 0$. In other words,

$$I(V(I)) = \sqrt{I}.$$  

Example 14. Continuing with our previous example, if $I = (x^2) \subseteq \mathbb{R}[x]$, then $V(I) = \{0\}$ and $I(\{0\}) = (x) = \sqrt{(x^2)}$. 
It turns out that the answer to our third question is straightforward: if $V \subseteq k^n$ is a variety, then $\mathbf{V}(\mathcal{I}(V)) = V$. More generally, if $S \subseteq k^n$ is any subset, then $\mathbf{V}(\mathcal{I}(S))$ is the smallest variety that contains $S$, i.e., is the closure of $S$ in $\mathbb{A}^n$.

**Example 15.** Suppose $S \subseteq \mathbb{C}^2$ is the collection of points of the form $(x, x)$ with $x$ nonzero. Geometrically, this set $S$ is the line $y = x$ with the origin removed. By continuity, any polynomial that vanishes on $S$ must also vanish at $(0, 0)$, so the point $(0, 0) \in \mathbf{V}(\mathcal{I}(S))$. In fact, one can show that $\mathbf{V}(\mathcal{I}(S))$ is precisely the variety $\mathbf{V}(y - x)$.

In light of everything above, we now have a bijective correspondence:

\[
\begin{align*}
\{\text{affine varieties in } \mathbb{A}^n\} & \leftrightarrow \{\text{radical ideals in } k[x_1, \ldots, x_n]\} \\
V & \mapsto \mathcal{I}(V) \\
\mathbf{V}(I) & \leftrightarrow I
\end{align*}
\]

Below is a summary of some of the finer aspects of this correspondence.

**Correspondence between ideal operations and geometric operations.**

1. radical ideals $\leftrightarrow$ varieties
2. addition of ideals $\leftrightarrow$ intersection of varieties
3. product and intersection of ideals $\leftrightarrow$ union of varieties
4. quotient of ideals $\leftrightarrow$ difference of varieties
2.1.2 The Ring of Regular Functions on a Variety

Let $V \subseteq k^n$ be an affine variety. Given any polynomial $f \in k[x_1, \ldots, x_n]$, evaluation defines a function $f : k^n \to k$, which we can restrict to $V$ to obtain a function $f : V \to k$. We call such functions on $V$ regular functions.

When do two polynomials $f$ and $g$ define the same function on $V$? Exactly when $f - g$ is identically zero on $V$, i.e., when $f - g \in I(V)$. We are therefore led to make the following definition:

**Definition 16.** Let $V \subseteq k^n$ be an affine variety. The ring of regular functions (or coordinate ring) of $V$ is

$$k[V] = k[x_1, \ldots, x_n]/I(V).$$

The property of the ideal $I(V)$ being radical is equivalent to the ring $k[V]$ being nilpotent free. The fact that $k[V]$ is a quotient of $k[x_1, \ldots, x_n]$ is equivalent to $k[V]$ being a finitely generated $k$-algebra. If we replace our algebraic information of the ideal $I(V)$ instead with the ring $k[V]$, then we have a bijective correspondence

$$\{\text{affine varieties}\} \leftrightarrow \{\text{nilpotent-free, finitely-generated } k\text{-algebras}\}$$

The set on the right hand side is a special subset of commutative rings. The Algebra-Geometry Correspondence suggests the existence of geometric objects that corresponds to commutative rings. These objects are affine schemes, which we briefly define next.
2.1.3 Affine Schemes

One can show the maximal ideals of \( k[V] \) are in bijective correspondence with the points of \( V \). This brings rise to the question: what about the prime ideals?

**Definition 17.** For any commutative ring \( A \), the *spectrum* of \( A \) is the set of prime ideals in \( A \), denoted \( \text{Spec} \ A \).

One can equip \( \text{Spec} \ A \) with a **Zariski topology** analogous to the one defined above on affine space. Moreover, on this topological space one can define a *sheaf* of rings, called the *structure sheaf*. By definition, the structure sheaf assigns to every open set \( U \) on \( X \) the ring of regular functions on \( U \).

**Definition 18.** The pair \( (X, \mathcal{O}_X) \), where \( X = \text{Spec}(A) \) with the Zariski Topology and \( \mathcal{O}_X \) is the structure sheaf on \( X \), is called an **affine scheme**.

Here are a few important results that follow from this construction:

1. The points of \( \text{Spec} \ A \) corresponding to maximal ideals of \( A \) are called *closed points*.

2. There is a bijection between the points of an affine variety \( V \) and the closed points of \( \text{Spec} \ k[V] \).

3. There is a bijective correspondence between affine schemes and commutative rings. (In fact, there is an equivalence between the relevant categories.)

**Example 19.** The affine scheme \( X = \text{Spec} \ C \) consists of one point, corresponding to the zero ideal. The affine scheme \( Y = \text{Spec} \ C[x]/(x^2) \) also consists of one point, corresponding to the ideal \((x)\). (Note that the zero ideal is not prime in this ring!) However, this scheme is distinguished from the scheme \( X \) by its structure sheaf, since \( \mathcal{O}_Y(Y) = C[x]/(x^2) \) while \( \mathcal{O}_X(X) = C \).
Example 20. The affine scheme $\text{Spec} \mathbb{R}[x,y]/(y^2)$ contains a closed point for (the image in the quotient ring of) every maximal ideal of the form $(x-a, y-b)$ with $b = a^2$. There is also one non-closed point, corresponding to the zero ideal.

2.1.4 Affine Plane Curves

Affine plane curves are the vanishing loci of non-constant polynomials $f \in k[x,y]$, with $f$ uniquely determined (when $k$ is algebraically closed) up to multiplication by a nonzero constant. When thinking about curves one may asks: does the curve have any singularities?

Definition 21. Let $k$ be a field of characteristic zero. Suppose $C = \text{V}(f)$, where $f \in k[x,y]$ is nonconstant. We say $C$ has a singularity at point $p$ if $\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = f(p) = 0$.

Example 22. Consider the curve real plane curve $xy + x^5 + y^5 = 0$, shown below:

This curve seems to have a singularity at $(0,0)$. We have that $\frac{\partial f}{\partial x} = y + 5x^4$ and $\frac{\partial f}{\partial y} = x + 5y^4$, and so we do indeed have $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = f(0,0) = 0.(0,0)$. This is an example of a type of singularity called a node.

Example 23. Consider the real plane curve $C$ defined by $x^2 - y^3 = 0$, shown below:
Once again, there appears to be a singular point at \((0, 0)\). To verify this, first observe that \(\frac{\partial f}{\partial x} = 2x\) and \(\frac{\partial f}{\partial y} = -3y^2\). Therefore, \(\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = f(0, 0) = 0\), verifying there is a singularity at \((0, 0)\). This type of singularity is called a \textit{cusp}.

**Example 24.** Consider the cubic curve \(C\) defined by \(x^3 - xy = 0\). Notice that the defining equation for \(C\) factors, as \(x(x^2 - y) = 0\), so \(C\) is the union of the line given by \(x = 0\) and the parabola given by \(y = x^2\). These two curves intersect at \((0, 0)\), so we expect that to be a singular point. To verify this, first observe that \(\frac{\partial f}{\partial x} = 3x^2 - y\) and \(\frac{\partial f}{\partial y} = -x\). Therefore, \(\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = f(0, 0) = 0\), verifying there is a singularity at \((0, 0)\).

**Example 25.** Consider the quartic curve \(C\) defined by \(x^4 + xy + y = 0\). We claim this curve is nonsingular. Observe that \(\frac{\partial f}{\partial x} = 4x^3 + y\) and \(\frac{\partial f}{\partial y} = x + 1\). The only point at which both partials vanish is \((-1, 4)\), which is not on the curve. Thus, \(C\) is nonsingular.

An important characteristic of a curve is its \textit{genus}. There are two competing definitions of genus in algebraic geometry, namely the \textit{arithmetic genus} and the \textit{geometric genus}, but fortunately in the case of nonsingular curves the two are equivalent.
“It is a miracle that for a complex curve the geometric genus is the same as the topological genus and the arithmetic genus.” Ravi Vakil [12]

Conveniently, there is a formula to compute the genus for nonsingular plane curves, called the **Genus-Degree Formula**. It states that the genus \( g \) of a nonsingular plane curve of degree \( d \) is

\[
g = \frac{(d-1)(d-2)}{2}.
\]

(There is a modified version of this formula for singular curves, which takes into account the effect of the singularities.)

**Example 26.** The curve defined by \( x^3 + y^3 = 1 \) has genus 1.

**Example 27.** The curve defined by \( x^5 + y^5 + 1 = 0 \) has genus 6.
2.2 Projective Algebraic Geometry

One of the most useful properties a topological space can have is compactness. Unfortunately, affine varieties do not enjoy this feature. Even more troublesome, the usual notion of compactness is not well suited to schemes in general. Instead, a more suitable notion is that of a projective variety. We first define the projective analogue of affine space.

There are many equivalent ways to define projective space. We begin with what is probably the oldest definition, since it is very geometric and directly connected to affine space. Define an equivalence relation \( \sim \) on the nonzero points of \( k^{n+1} \), by saying

\[
(x'_0, \ldots, x'_n) \sim (x_0, \ldots, x_n)
\]

if there is some nonzero \( c \in k \) such that \( (x'_0, \ldots, x'_n) = (cx_0, \ldots, cx_n) \). We then define:

**Definition 28.** The \( n \)-dimensional projective space over \( k \) is the set of equivalence classes of \( \sim \) on \( k^{n+1} \setminus \{0\} \). We denote this as \( P^n_k \), or simply \( P^n \). In other words,

\[
P^n = k^{n+1} / \sim.
\]

We denote the class of a point \( (x_0, \ldots, x_n) \in k^{n+1} \) by \( [x_0 : \cdots : x_n] \), which we call the homogeneous coordinates of the point in projective space.

**Example 29.** The homogeneous coordinates \( [\sqrt{3} : 0 : i] \) and \( [3i : 0 : -\sqrt{3}] \) denote the same point in \( P^2 \), since \( (3i, 0, -\sqrt{3}) = (\sqrt{3}i \cdot \sqrt{3}, \sqrt{3}i \cdot 0, \sqrt{3}i \cdot i) \).
The points of $P^n_k$ are in one-to-one correspondence with lines through the origin in $k^{n+1}$. Indeed, each line through the origin in $k^{n+1}$ is determined by a point on that line away from the origin, uniquely up to scaling.

For each index $i$, consider the subset $U_i \subseteq P^n$ consisting of all points $[x_0 : \cdots : x_n]$ with $x_i \neq 0$. Each such point can be represented uniquely by homogeneous coordinates with $x_i = 1$. We call these subsets the \textbf{standard affine charts} for projective space. Each $U_i$ is in bijection with $k^n$, but in fact we have something even stronger. The subsets $U_i$ are subvarieties, each isomorphic to $A^n$. (In terms of schemes we have $U_i \cong \text{Spec } \mathbb{C}[y_0, \ldots, \hat{y}_i, \ldots y_n]$, where $y_j \mapsto x_j/x_i$.)

**Example 30.** Continuing the previous example, the given point is contained in both $U_0$ and $U_1$. In $U_0$, we can represent it by $[1 : 0 : i\sqrt{3}]$; in $U_2$, we can represent it by $[-\frac{3i}{\sqrt{3}} : 0 : 1]$.

We next define varieties in projective space. In order to do this, we first need to talk about homogeneous polynomials.

**Example 31.** Consider the polynomial $f(x_0, x_1) = x_0^2 - x_1 \in \mathbb{C}[x_0, x_1]$. While this induces a function $f : \mathbb{C}^2 \to \mathbb{C}$, this function is not constant on equivalence classes. For example, observe that $f(2, 2) = 2$ while $f(1, 1) = 0$. So while evaluation does define a map $f : \mathbb{C}^2 \to \mathbb{C}$, it does not induce a map $f : P^1 \to \mathbb{C}$. Similarly, we cannot talk about the zero locus of $f$ in $P^1$.

The issue in the example above is that $f$ is not \textbf{homogeneous}, i.e., it has monomial terms of different degrees. On the other hand, if $f$ is a homogeneous polynomial, then we can at least talk about the zero locus of $f$ in projective space.

**Example 32.** Consider now the polynomial $f(x_0, x_1) = x_0^2 - x_0x_1$. We still cannot use evaluation to define a function on projective space since, for instance, $f(4, 2) =$
8 \neq 2 = f(2,1)$. However, we can talk about the zero locus of $f$ in projective space. Indeed, observe that $f(cx_0,cx_1) = c^2 f(x_0,x_1)$, so $f(cx_0,cx_1) = 0$ if and only if $f(x_0,x_1) = 0$.

**Definition 33.** Given a finite collection of homogeneous polynomials $f_1,\ldots,f_j \in \mathbb{C}[x_0,\ldots,x_n]$, we define the **projective variety**

\[ V(f_1,\ldots,f_j) = \{a \in \mathbb{P}^n \mid f_i(a) = 0 \text{ for all } 1 \leq i \leq j\}. \]

The zero locus of a single homogeneous polynomial is called a **hypersurface**. If the polynomial is linear, we call the zero locus a **hyperplane**.

**Example 34.** In $\mathbb{P}^2$, the hyperplane $H = V(x_0)$ is one of the so-called **hyperplanes at infinity**. From the perspective of the affine chart $U_0 \cong \mathbb{A}^2$, the hyperplane $H$ looks like a projective line “out at infinity.”

**Example 35.** In $\mathbb{P}^3$, the projective variety $V(x_0^3 + x_1x_2^2)$ is a cubic hypersurface.

### 2.2.1 Homogeneous Ideals and Graded Rings

Earlier we saw that affine varieties in $\mathbb{k}^n$ correspond to (radical) ideals in $\mathbb{k}[x_1,\ldots,x_n]$. However, there is a distinction that we need to note on the algebra side of the Projective Algebra-Geometry Dictionary. We defined projective varieties as the common zeros to collections of homogeneous polynomials. However, observe that if we add two homogeneous polynomials, the sum might not be homogeneous. This gives rise to the notion of homogeneous ideals.

**Definition 36.** An ideal $I$ in $\mathbb{k}[x_1,\ldots,x_n]$ is **homogeneous** if for each $f \in I$, the homogeneous components of $f$ are in $I$ as well.
Example 37. Suppose \( f(x_0, x_1) = x_1 - x_0^3 \) and \( I = (f) \). The homogeneous components of \( f \) are \( f_1 = x_1 \) and \( f_3 = -x_0^3 \). Neither of these polynomials is in \( I \), since neither is divisible by \( x_1 - x_0^3 \). Therefore, \( I \) is not a homogeneous ideal.

Theorem 3. Let \( I \subset k[x_0, ..., x_n] \) be an ideal. Then the following are equivalent:

1. \( I \) is a homogeneous ideal; and

2. \( I \) can be generated by homogeneous polynomials.

Definition 38. For any homogeneous ideal \( I \subset k[x_0, ..., x_n] \), we define the projective variety

\[
V(I) = \{ a \in \mathbb{P}^n : f(a) = 0 \text{ for all } f \in I \}.
\]

To each projective variety \( V \), we can also associate a homogeneous ideal.

Proposition 1. Let \( V \subset \mathbb{P}^n \) be a projective variety and let

\[
I(V) = \{ f \in k[x_0, ..., x_n] : f(a) = 0 \text{ for all } a \in V \}.
\]

Then \( I(V) \) is a homogeneous ideal in \( k[x_0, ..., x_n] \).

Similar to the affine case, we now have a connection between subsets of \( \mathbb{P}^n \) and subsets of \( k[x_0, ..., x_n] \):

\[
\{ \text{subvarieties of } \mathbb{P}^n \} \leftrightarrow \{ \text{homogeneous ideals in } k[x_0, ..., x_n] \}
\]

\[
V \mapsto I(V)
\]

\[
V(I) \leftrightarrow I
\]

Below we summarize some of the standard results from this correspondence.
The Projective Algebra-Geometry Correspondence

1. If a homogeneous ideal $I$ is generated by $f_1, \ldots, f_k$, then $V(I) = V(f_1, \ldots, f_k)$.

2. The maps $V$ and $I$ are inclusion reversing. In other words, if $I_1 \subset I_2$, then $V(I_2) \supset V(I_1)$. Similarly, if $V_1 \subset V_2$, then $I(V_2) \supset I(V_1)$.

3. The radical homogeneous ideal is defined as $\sqrt{I} = \{ f \in k[x_0, \ldots, x_n] : f^n \in I \text{ for some } n \geq 1 \}$.

4. If $I$ is a homogeneous ideal, so is $\sqrt{I}$.

2.2.2 Projective Curves

A projective curve is a projective variety of dimension one. Every affine curve gives rise to a projective curve, as follows.

Proposition 2. [3] Let $f(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ be a polynomial of total degree $d$.

1. Let $f = \sum_{i=0}^{d} f_i$ be the expansion of $f$ as the sum of its homogeneous components, where $f_i$ is homogeneous of degree $i$. Define

$$f^h(x_0, \ldots, x_n) = \sum_{i=0}^{d} f_i(x_1, \ldots, x_n)x_0^{d-i}$$

Then $f^h$ is a homogeneous polynomial of total degree $d$ in $k[x_0, \ldots, x_n]$. We call $f^h$ the homogenization of $f$.

2. The homogenization of $f$ can be computed using the formula

$$f^h(x_0, \ldots, x_n) = x_0^d \cdot f \left( \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \right)$$
This proposition allows us to projectivize any affine curve. Throughout the paper we will use the notation \( x_0, \ldots, x_n \) to denote that the polynomial is a projective homogeneous polynomial.

**Example 39.** Consider the complex affine curve in Example 25, defined by \( x^4 + xy + y = 0 \). Applying the process of projectivization we find that the corresponding projective curve is defined by \( x_0^4 + x_0x_1x_2^2 + x_1x_2^3 = 0 \) in \( \mathbb{P}^2 \).

One might think that projectivizing a nonsingular affine curve produces a nonsingular projective, but unfortunately this is not necessarily the case.

**Example 40.** Consider the curve defined by \( y^2 - x^3 - 1 = 0 \). This is a nonsingular plane curve. The corresponding projective curve is given by \( x_1^2x_2 - x_0^3 - x_2^3 = 0 \). Let’s use the affine charts to detect any singularities. On \( U_0 \), we have \( x_0 \neq 0 \) so the curve is given by \( u_1^2u_2 - 1 - u_2^3 = 0 \), where the local coordinates \((u_1, u_2)\) satisfy \( u_1 = \frac{x_1}{x_0} \) and \( u_2 = \frac{x_2}{x_0} \). If we let \( g(u_1, u_2) = u_1^2u_2 - 1 - u_2^3 \), then \( \frac{\partial g}{\partial u_1} = 2u_1u_2 \) and \( \frac{\partial g}{\partial u_2} = u_1^2 - 3u_2^2 \). The only point where both of these partials vanish is \((0, 0)\), which does not satisfy \( g(0, 0) = 0 \) since \((0, 0)\) does not lie on the curve. Therefore, there are no singularities on \( U_0 \).

On \( U_1 \), we have \( x_1 \neq 0 \) and a similar computation yields \( v_2 - v_1^3 - v_2^3 = 0 \) where the local coordinates on \( U_1 \) are \( v_1 = \frac{x_0}{x_1} \) and \( v_2 = \frac{x_2}{x_1} \). If we let \( h(v_1, v_2) = v_2 - v_1^3 - v_2^3 \), then \( \frac{\partial h}{\partial v_1} = -3v_1^2 \) and \( \frac{\partial h}{\partial v_2} = 1 - 3v_2^2 \). The point in \( U_1 \) where \( v_1 \) and \( v_2 \) are both zero is the point \([0 : 1 : 0]\) where one can see \( v_1 = \frac{3}{2}v_2 \) and the constant line \( v_1 = 0 \) intersect.

On \( U_2 \), we use the local coordinates are \( w_1 = \frac{x_0}{x_2} \) and \( w_2 = \frac{x_1}{x_2} \) and obtain the polynomial \( w_2^2 - w_1^3 = 0 \). Similarly, we will see that the parabola \( w_2 = \pm\sqrt{w_1^3} \) and \( w_1 = 0 \) intersect, but this time at the point \([0 : 0 : 1]\). Therefore, this is a singular projective curve with the additional singularities on charts \( U_1 \) and \( U_2 \).
Example 41. Consider the affine curve defined by $x^5 + y^5 + 1 = 0$. The corresponding projective curve is given by $x_0^5 + x_1^5 + x_2^5 = 0$. If we let $f(x_0, x_1, x_2) = x_0^5 + x_1^5 + x_2^5$, then we quickly find that the only point at which partials of $f$ vanish is when $x_0 = x_1 = x_2 = 0$, which does not correspond to a point in $\mathbb{P}^2$. This curve is also known as Fermat’s curve.

Remark. All of our examples have been projective plane curves, so we have been able to use the genus formula mentioned earlier in the paper. We should note, however, that plane curves do not fully exhaust curves of every genus.

The following table contains some nonsingular and singular projective curves with their genus. We may use the nonsingular projective curves later when looking at the canonical embedding.

**Some Projective Curves**

1. The curve $x_0^3 - x_0x_1x_2 = 0$ is singular.

2. The curve $x_0^4 + x_0x_1x_2^2 + x_1x_2^3 = 0$ is singular.

3. The curve $x_0^5 + x_1^5 + x_2^5 = 0$ is nonsingular and has genus 6.
4. The curve $x_0^6 + x_1^6 + x_2^6 = 0$ is nonsingular and has genus 10.

2.3 A Word on Schemes and Sheaves

Modern algebraic geometry began with the generalization of varieties to schemes. In a rough sense, schemes can be thought of as geometric objects one obtains from gluing together affine schemes. One of the necessary ingredients required in this construction is the notion of a sheaf, something we mentioned earlier in connection with affine schemes but never actually defined.

Sheaves were developed to keep track of local information, where by “local” we mean in terms of the open sets in a topology on the space. For example, on a complex manifold the sheaf of holomorphic functions is the information, for every open set $U$, of the collection of all functions that are holomorphic on $U$. There are two main properties this collection of information satisfies, in terms of open covers $\{U_i\}_i$ of an open set $U$:

1. (Identification) If $f$ and $g$ are holomorphic functions on $U$ whose restrictions agree on every $U_i$, then $f = g$; and

2. (Gluing) If for every $i$ one has a holomorphic function $f_i$ on $U_i$ such that $f_i$ and $f_j$ agree when restricted to $U_i \cap U_j$, then there is a holomorphic function $f$ on $U$ that restricts $f_i$ on each $U_i$.

Sheaves are a generalization of this example. For further detail, we suggest the reader check out [7] or [12]. Fortunately for this paper, we do not need the full power of scheme theory, nor all of the details involving sheaves.
2.4 Divisors

In this section we give a quick overview of divisors, generally following the treatment presented in [7]. Divisors are an important tool for studying maps from a variety into projective space. There are two competing definitions for the concept of “divisor,” namely Weil divisors and Cartier divisors. The second is more general than the first, but fortunately in the case of nonsingular varieties the two concepts are equivalent. As such, we stick to Weil divisors, which are defined more geometrically than Cartier divisors.

Definition 42. Suppose $X$ is a nonsingular projective variety. A prime divisor on $X$ is a closed subvariety of codimension one. A Weil divisor on $X$ is a formal integer linear combination of prime divisors, i.e. an expression of the form

$$D = \sum_{P} n_{P}P,$$

where the sum runs over all prime divisors $P$ on $X$ and the $n_{P}$ are integers, only finitely many of which are nonzero. If every coefficient is nonnegative, we say $D$ is an effective divisor. The degree of a divisor is the sum of its coefficients.

Example 43. An example of a divisor on $\mathbb{P}^1$ is $D = 1 \cdot [0 : 1] - 2 \cdot [1 : 1] + 3 \cdot [1 : 2]$.

Example 44. Let $C$ be a nonsingular projective curve in $\mathbb{P}^2_C$. For each line $L \subset \mathbb{P}^2_C$, consider $L \cap C$, which is a finite set of points on $C$. If $C$ is a curve of degree $d$, and if we count the points with proper multiplicity, then $L \cap C$ will consist of exactly $d$ points. We then write $L \cap C = \sum n_{i}P_{i}$ where $P_{i} \in C$ are the intersection points and $n_{i}$ are the multiplicities. This formal sum is a divisor on $C$.

Below is an illustration of this process, in the case $C$ is a parabola. We can see the green line produces a divisor on $C$ consisting of two distinct points, while the blue line...
(which is tangent to $C$) produces a divisor consisting of a single point of multiplicity two.

![Figure 2.2: $C = V(y - x^2)$](image)

Throughout the rest of this paper we will assume that we are working with nonsingular projective varieties. If $X$ is a nonsingular projective variety, then the set of divisors on $X$ forms a free abelian group, which we denote $\text{Div}_X$. There are two subsets of divisors of particular importance, as well. The first is the subset of effective divisors, which we will denote $\text{EDiv}_X$. The second is the set of so-called principal divisors. To each rational function $f$ defined on $X$, we associate the divisor

$$(f) = \sum_{P \subset X} \nu_P(f)P,$$

where the sum runs over all prime divisors $P$ on $X$, and $\nu_P$ is the valuation of $f$ along $P$. One can easily check that the subset of principal divisors forms a subgroup of $\text{Div}_X$.

**Example 45.** On $\mathbb{A}^1_C$, consider the rational function $f(x) = \frac{(x-1)(x+2)}{(x-3)^2}$. The associated principal divisor is $(f) = 1 \cdot [1] + 1 \cdot [-2] - 2[3]$. Here we are using square brackets to denote the associated point in $\mathbb{A}^1_C$; e.g., $[1] = V(x - 1)$ is the point $x = 1$. 

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Example 46. On $\mathbb{A}^2_C$, consider the rational function $f(x, y) = \frac{x^4 - x^2}{y^2}$. The associated principal divisor is $(f) = 1 \cdot C_1 + 1 \cdot C_2 - 2 \cdot C_3$, where $C_1 = V(x)$, $C_2 = V(y - x)$, and $C_3 = V(y)$.

Example 47. On $\mathbb{P}^2$, consider $f(x_0, x_1, x_2) = \frac{x_0^2 x_1 x_2 + x_0 x_1^2 x_2}{x_0^4 - x_2^4}$. Then the associated principal divisor $(f) = 1 \cdot C_1 + 1 \cdot C_2 + 1 \cdot C_3 + 1 \cdot C_4 - 1 \cdot C_5 - 1 \cdot C_6$ where $C_1 = V(x_0), C_2 = V(x_1), C_3 = V(x_2), C_4 = V(x_0 + x_1), C_5 = V(x_0 - x_2)$, and $C_6 = V(x_0 + x_2)$.

If two divisors $D_1$ and $D_2$ differ by a principal divisor, we say they are linearly equivalent. As the name suggests, this is an equivalence relation on $\text{Div}_X$. There is a bijection between invertible sheaves on $X$ (up to sheaf isomorphism) and divisors on $X$ (up to linear equivalence).

2.5 Differentials

As previously stated, our goal is to understand the equations that define canonically embedded curves. To understand the canonical embedding, we must first understand differentials. As such, we briefly review the theory of differentials, following the development in [7]. Since we are only considering complex projective curves, we can avoid dealing with some of the more subtle issues. In particular, the sheaf of differential forms is essentially the same as the dual of the tangent bundle defined in differential geometry. However, following Hartshorne we will first define the differentials by a purely algebraic method.

2.5.1 Kähler Differentials

Let $A$ be a commutative ring with unity, $B$ an $A$-algebra, and $M$ a $B$-module.
**Definition 48.** An *A-derivation* of $B$ into $M$ is a map $d : B \to M$ such that

1. $d$ is additive;
2. $d(bb') = b \cdot db' + b' \cdot db$; and
3. $da = 0$ for all $a \in A$.

Usage of the term derivation is not a coincidence, as it is meant to be an algebraic extension of the concept of the derivative. Property (1) is the usual linearity property of differentiation; property (2) is a direct analogue of the product rule; and property (3) is the equivalent of constants having derivative equal to zero.

**Example 49.** Suppose $A$ is the field of complex numbers, and both $B$ and $M$ are the polynomial ring $C[x, y]$. One derivation $d_x : C[x, y] \to C[x, y]$ is partial differentiation with respect to $x$, i.e., the map $d_x(f) = \frac{\partial f}{\partial x}$. Another derivation is the map $d_y$ which is partial differentiation with respect to $y$.

We will now take a look at the *universal object* which captures every possible $A$-derivation from $B$.

**Definition 50.** The *module of relative differential forms* of $B$ over $A$ is a $B$-module $\Omega_{B/A}$, together with an $A$-derivation $d : B \to \Omega_{B/A}$, which satisfies the following universal property:

for any $B$-module $M$ and $A$-derivation $d' : B \to M$, there exists a unique $B$-module homomorphism $f : \Omega_{B/A} \to M$ such that $d' = f \circ d$:

\[
\begin{array}{ccc}
B & \xrightarrow{d} & \Omega_{B/A} \\
\downarrow{d'} & & \downarrow{f} \\
M & \xrightarrow{f} & M
\end{array}
\]
Remark. The commutative diagram above shows us that $\Omega_{B/A}$ is an initial object in the category $\text{Der}_M(A)$, whose objects are pairs $(M, d')$, where $M$ is a $B$-module and $d'$ is an $A$-derivation.

Of course, this definition hinges upon the fact that such a universal object exists. Fortunately, it is not a difficult construction. First, one considers the free $B$-module $F$ generated by the symbols $\{db \mid b \in B\}$, and then one quotients out the submodule generated by all the expressions of the form:

1. $d(b + b') - db - db'$ for $b, b' \in B$ (additivity);
2. $d(bb') - bdb' - b'db$ for $b, b' \in B$ (Leibniz rule); and
3. $da$ for $a \in A$ (triviality of scalars).

Then define $d : B \to \Omega_{B/A}$ by $b \mapsto db$. In particular, note that $\Omega_{B/A}$ is always generated as a $B$-module by $\{db \mid b \in B\}$.

Remark. The notion of module of relative differential forms can be extended to schemes as a sheaf of relative differentials, which can be done on each affine chart. It is also possible to define the sheaf without reference to affine charts.

Example 51. Suppose $A = \mathbb{C}$ and $B = \mathbb{C}[x_1, \ldots, x_n]$. We want to show that $\Omega_{B/A} \cong B^n$ where $B^n$ is a free $B$-module of rank $n$. First consider the natural map $\partial : B \to B^n$ defined by $\partial f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n})$. The consider the map $\phi : \Omega_{B/A} \to B^n$ defined by $\phi(df) = (\frac{\partial f}{\partial x_i})$, where $df = \sum (\frac{\partial f}{\partial x_i})dx_i$.

Claim: $\phi$ is an isomorphism.

Note that $\phi$ is surjective, since $\phi(dx_i) = (0, \ldots, 1, \ldots, 0)$. (Let’s denote this last element by $1_i$.) It remains to show that $\phi$ is injective. $B^n$ is a free module and has
basis \{1_i\} which means that \(\phi\) will take \(dx_1, \ldots, dx_n\) to a basis of \(B^n\) where \(x_i = 1\) if \(i = j\) and \(x_i = 0\) if \(i \neq j\). The inverse of \(\phi\) is the map \(\psi : B^n \to \Omega_{B/A}\) where
\[
\psi((b_i)) = \sum (b_i)dx_i.
\]
Check it out: \(\phi \circ \psi((b_i)) = \phi(\sum (b_i)dx_i) = \sum (b_i)\phi(dx_i) = (b_i)\) and \(\psi \circ \phi(df) = \psi((\frac{\partial f}{\partial x_i})) = \sum (\frac{\partial f}{\partial x_i})dx_i = df\). Therefore, \(\phi\) is an isomorphism and \(\Omega_{B/A} \simeq B^n\).

One can show the following fact: If \(B\) is generated by \(f_1, \ldots, f_n\), that satisfy relations \(r_1, \ldots, r_m\) then \(\Omega_{B/A}\) is generated by \(df_1, \ldots, df_n\) which satisfy relations \(dr_1, \ldots, dr_m\). The idea is that we start from a free \(A\) algebra that is a surjection onto \(B\), where the standard basis maps to each of the \(f_i\)'s. The kernel of this surjection will be generated by \(r_1, \ldots, r_m\).

**Proposition 3** ([12]). *If \(B\) is generated over \(A\) as an algebra by \(f_i \in B\) (where \(i\) lies in some index set, possibly infinite), subject to some relations \(r_j\) (where \(j\) lies in some index set, and each is a polynomial in the \(f_i\)’s), then the \(B\)-module \(\Omega_{B/A}\) is generated by the \(df_i\), subject to the relations \(dr_j = 0\).*

In short, we needn’t take every single relation of \(B\); we can take a generating set. And we needn’t take every single relation among these generating elements; we can take the generators of the relations.

**Example 52.** Let \(A = \mathbb{C}\), \(B = \mathbb{C}[x, y]\), \(I = (y - x^2)\), and \(C = B/I\). Then \(C\) is generated by \(\bar{x}\) and \(\bar{y}\), subject to the relation \(\bar{y} = \bar{x}^2\), so \(\Omega_{C/A}\) is generated as a \(C\)-module by \(d\bar{x}\) and \(d\bar{y}\), subject to the relation \(d\bar{y} = 2\bar{x} d\bar{x}\).

For schemes (affine or projective), one can define a sheaf of differentials by using the algebraic description above on each affine chart. There is also a more global definition in terms of sheaves, but we will not need that. We direct the interested reader to [7] or [12].
2.6 The Canonical Divisor

Of central importance to us is a very special divisor, called the **canonical divisor**. A canonical divisor on a curve $C$ is any divisor in the linear equivalence class of the divisor associated to the sheaf of differentials (which is an invertible sheaf). We denote a canonical divisor on $C$ by $K_C$. It is unique up to linear equivalence, i.e., up to the difference of a principal divisor. However, rather than get into the technical details of the definition, we instead illustrate how one computes a canonical divisor in several examples.

**Example 53.** Let $z_1, z_2$ be global coordinates on $A^2$, and let $\omega = dz_1 \wedge dz_2$. This differential has no zeroes or poles, so $K_{A^2} \equiv 0$.

**Example 54.** Suppose $X = P^1$, with (global) projective coordinates $[x_0 : x_1]$. Then $X$ is covered by two open affine charts, which we denote $U_0$ and $U_1$. As usual, $U_i$ is the subset consisting of all points $[x_0 : x_1]$ with $x_i \neq 0$. In particular, on $U_0$ we can use the local coordinate $u_0 = \frac{x_1}{x_0}$. We now consider the 1-form $\omega$ that on $U_0$ is defined by $\omega = du_0$. This defines $\omega$ everywhere except on $P^1 \setminus U_0$. Fortunately, we can use the second open chart $U_1$ to cover that part of the space. On $U_1$ we can use the local coordinate $u_1 = \frac{x_0}{x_1}$. On $U_0 \cap U_1$ we have that $u_0 = \frac{1}{u_1}$, so on that overlap we have that $\omega = du_0 = d \left( \frac{1}{u_1} \right) = -\frac{1}{u_1^2} du_1$. This forces the definition of $\omega$ on $U_1$. We now have that $\omega$ is defined on all of $P^1$.

What is the associated divisor? We must find the zeros and poles of $\omega$. Looking first on the chart $U_0$, we have $\omega = du_0$, which has no zeros or poles. On the chart $U_1$, however, we have $\omega = -\frac{1}{u_1^2} du_1$, which has a pole of order two where $u_1 = 0$. The point in $U_1$ where $u_1 = 0$ is the point $[0 : 1]$. Let’s denote this point $P_\infty$, since it corresponds to the point at infinity from the point of view of the chart $U_0$. We
therefore have

\[ K_X \equiv -2P_\infty. \]

**Example 55.** Suppose \( X = \mathbb{P}^3 \), with (global) projective coordinates \([x_0 : x_1 : x_2 : x_3]\). Then \( X \) is covered by four open affine charts which we denote \( U_0, U_1, U_2, \) and \( U_3 \). As usual, \( U_i \) is the subset consisting of all points with \( x_i \neq 0 \). In particular, on \( U_0 \) we can use the local coordinates \( u_0 = \frac{x_1}{x_0}, \) \( u_1 = \frac{x_2}{x_0} \) and \( u_2 = \frac{x_3}{x_0} \). We now consider the 3-form \( \omega \) on \( U_0 \) defined by \( \omega = du_0 \wedge du_1 \wedge du_2 \). This defines \( \omega \) everywhere except on \( \mathbb{P}^3 \setminus U_0 \). However, proceeding similarly to the last example. On \( U_1 \) we can use the local coordinates \( v_1 = \frac{x_0}{x_1}, v_2 = \frac{x_2}{x_1}, \) and \( v_3 = \frac{x_3}{x_1} \). On \( U_0 \cap U_1 \) we have that \( u_0 = \frac{1}{v_1}, u_1 = \frac{x_0}{v_1}, u_2 = \frac{x_2}{v_1}, \) so on that overlap we have that \( \omega = du_0 \wedge du_1 \wedge du_2 = d\left(\frac{1}{v_1}\right) \wedge d\left(\frac{x_0}{v_1}\right) \wedge d\left(\frac{x_2}{v_1}\right) = -\frac{1}{v_1^4} dv_1 \wedge dv_2 \wedge dv_3 \). This forces the definition of \( \omega \) on \( U_1 \). We now have that \( \omega \) is defined everywhere except \( \mathbb{P}^3 \setminus (U_0 \cup U_1) \). We could proceed to chart \( U_2 \) and \( U_3 \), or we could note that divisors are only defined to codimension 1, so we are free to stop here.

What is the associated divisor? We are again looking for the zeros and poles of \( \omega \). Scanning the first chart \( U_0 \), we have \( \omega = du_0 \wedge du_1 \wedge du_2 \), which has no zeros or poles. On the chart \( U_1 \), however, we have \( \omega = \frac{1}{v_1^4} dv_1 \wedge dv_2 \wedge dv_3 \), which has a pole of order four where \( v_1 = 0 \). This corresponds to the plane consisting of all points \([0 : 1 : x_2 : x_3]\). If we call this plane \( H \), then we have

\[ K_X \equiv -4H. \]

We conclude the section with an important proposition.

**Proposition 4 ([11]).** Let \( C \) be a nonsingular projective curve of genus \( g \). The degree of the canonical divisor \( K_C \) is \( 2g - 2 \).
2.6.1 Canonical Divisors on Projective Curves

The canonical map will require a basis of *globally defined* differentials $\omega_1, \ldots, \omega_g$, where $g$ is the genus of the curve. This means we need a method of writing down differentials that have *effective* associated divisors, i.e., do not have any poles. We illustrate how this is done in an example.

**Example 56.** Consider the curve $C$ defined by $x_0^3 - x_1^2x_2 + x_0x_2^3 = 0$. This curve has genus 1, so the canonical divisor has degree zero. As usual, we consider our standard affine charts. Note that the point $[1:0:0]$ is not on the curve, so $C$ is completely covered by affine charts $U_1$ and $U_2$. On $U_1$, the curve is defined by $v_1^3 - v_2 + v_1v_2^2 = 0$, while on $U_2$ it is defined by $w_1^3 - w_2^2 + w_1 = 0$. For reference, note $v_1 = \frac{x_1}{x_0}$, $v_2 = \frac{x_2}{x_0}$, $w_1 = \frac{x_0}{x_2}$, and $w_2 = \frac{x_1}{x_2}$.

On each affine chart, we get the differential relationships $(3v_2^2 + v_2^3)\,dv_1 + (2v_1v_2 - 1)\,dv_2 = 0$ and $(3w_2^2 + 1)\,dw_1 - (2w_2)\,dw_2 = 0$, respectively. We first define a global differential $\omega$ on $U_1$. Define the subset $U_{1,1} = \{(v_1, v_2) \mid 2v_1v_2 - 1 \neq 0\}$. On this subset, define $\omega = dv_1$. However, $U_{1,1}$ misses some points on $U_1$, in particular, the points where $2v_1v_2 - 1 = 0$. Through a simple substitution one can find that the points on $C \cap U_1$ missed by $U_{1,1}$ are $p_1 = \left[\sqrt{\frac{1}{2}} : 1 : \sqrt{\frac{1}{2}}\right]$, $p_2 = \left[-\sqrt{\frac{1}{2}} : 1 : -\sqrt{\frac{1}{2}}\right]$, $p_3 = \left[i\sqrt{\frac{1}{2}} : 1 : -i\sqrt{\frac{1}{2}}\right]$, and $p_4 = \left[-i\sqrt{\frac{1}{2}} : 1 : i\sqrt{\frac{1}{2}}\right]$. At these points, $\omega$ is defined by the conversion $\omega = dv_1 = \frac{-(2v_1v_2 - 1)}{3v_1^2 + v_2^3} \,dv_2$. (This follows directly from the differential relationship, above.) We now see that $\omega$ has zeroes at the points $p_1, \ldots, p_4$.

How is $\omega$ defined on $C \setminus U_1$? Since the curve is completely covered by $U_1$ and $U_2$, we have only missed the points on the curve where $x_2 = 1$ and $x_1 = 0$; i.e., when $w_2 = 0$. On $U_2$, these points must satisfy $w_1^3 + w_1 = 0$. Solving for $w_1$, we see that the points on $C$ missed by $U_1$ are $p_5 = [0:0:1]$, $p_6 = [i:0:1]$, and $p_7 = [-i:0:1]$.

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Since $v_1 = \frac{w_1}{w_2}$, we have $d v_1 = d\left(\frac{w_1}{w_2}\right) = \frac{d w_1}{w_2} - \frac{w_1}{w_2^2} d w_2$. We can write this final expression in terms of $d w_2$ alone. Here’s why: on the subset $U_{2,1} = \{(w_1, w_2) \mid w_2 \neq 0\}$, $d w_1$ generates $\Omega_C$. However, for the points where $w_2 = 0$ we must use the differential $d w_2$. Using the relation above we have that $d w_1 = \frac{2 w_2 d w_2}{3 w_2^4 + 1}$ and substituting this in for $d w_1$ we get that the differential on $U_2$ is $d v_1 = \left(\frac{2 w_2^2 - 3 w_2 (3 w_2^4 + 1)}{w_2 (3 w_2^4 + 1)}\right) d w_2$.

Upon first glance, it looks like the point $[0 : 0 : 1]$ is a double pole of $\omega$. Unfortunately, there is no trivial way to observe that it is not a double pole. However, use the Implicit Function Theorem on $C \cap U_2$ to write $w_1$ in terms of $w_2$. We will do this to show $[0 : 0 : 1]$ is not a double pole.

First note that $w_1 = w_2 = 0$. By implicit differentiation of the equation $w_1^3 - w_2^2 + w_1 = 0$, we get can compute the Taylor series expansion for $w_1$ in terms of $w_2$. We ultimately find

$$w_1 = 0 + 0 \cdot w_2 + w_2^2 + \ldots$$

Substitution into the differential on $U_2$ above yields:

$$d v_1 = \frac{2 w_2^2 - [(w_2^2 + \ldots)(3(w_2^2 + \ldots)^2 + 1)]}{w_2^3[3(w_2^2 + \ldots)^2 + 1]} d w_2 = \frac{w_2^2 + \ldots + 3 w_6 + \ldots}{w_2^3(3(w_2^2 + \ldots)^2 + 1)} d w_2 = \frac{1 + \ldots}{1 + \ldots} d w_2.$$

Therefore, $[0 : 0 : 1]$ is not a double pole. In fact, it is neither a zero nor a pole of the differential.

Finally, we analyze the points $p_6$ and $p_7$. Thankfully, for these two points we do not need to go through an exhaustive calculation. If we substitute $w_1 = i$ and $w_2 = 0$ into $\frac{2 w_2^2 - w_1 (3 w_2^4 + 1)}{w_2^3 (3 w_2^4 + 1)} d w_2$ we can see that the zero in the denominator comes from the $\frac{1}{w_2^3}$ term suggesting that $p_6$ is a double pole. Through a similar calculation, we find that $p_7$ is a double pole, as well.
In the first chart we found four zeroes and in the second chart we found two double poles. But this begs the question: are the zeroes we found simple zeroes or do they have order bigger than one? Again, this is a nontrivial calculation and involves a calculation similar to the one used above. We claim that \( p_1, p_2, p_3, \) and \( p_4 \) are simple zeroes.

We will do the case for the point \( p_1 = \left[ \sqrt{\frac{1}{2}} : 1 : \sqrt{\frac{1}{2}} \right] \), since the other three points follow in a similar fashion. Recall that \( \omega \) on this part of \( U_1 \) was defined as \( \omega = d v_1 = \frac{-(2v_1v_2-1)}{3v_1^2+v_2^2} \, d v_2 \). We only need to consider the numerator and show that this point is indeed a root. A local parameter for \( C \) near this point is \( (v_2 - \sqrt{\frac{1}{2}}) \). Using implicit differentiation in the equation \( v_1^3 - v_2 + v_1v_2^2 = 0 \), we get the following expansion for \( v_1 \):

\[
v_1 = \sqrt{\frac{1}{2}} + \frac{\sqrt{\frac{1}{2}}(v_2 - \sqrt{\frac{1}{2}})^2}{2} + \cdots
\]

Upon substitution and simplification, we see that

\[
2v_1v_2 - 1 = 2v_1 \left( v_2 - \sqrt{\frac{1}{2}} \right) + 2v_1 \sqrt{\frac{1}{2}} - 1
= \left( v_2 - \sqrt{\frac{1}{2}} \right) \left[ 2\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{2}} \left( v_2 - \sqrt{\frac{1}{2}} \right)^3 + \cdots \right].
\]

The above calculation shows us that \( p_1 \) is a simple zero of this differential. One may proceed similarly to show the other three roots are indeed simple zeroes, as well.

In addition to finding the global differential, we can also now write the canonical divisor associated to this \( \omega \):

\[
K_C \equiv p_1 + p_2 + p_3 + p_4 - 2p_6 - 2p_7
\]

Note that \( \deg K_C = 0 \), as predicted.
**Example 57.** For our next example, we will look at the Fermat curve defined by $$x_0^5 + x_1^5 + x_2^5 = 0.$$ We will spare some of the details for this curve, as a lot of the techniques are similar to those used in the last example. Again, the point $[1 : 0 : 0]$ is not on the curve, so we will define the global differential using the charts $U_1$ and $U_2$. First, we have that $C \cap U_1$ is defined by $v_1^5 + 1 + v_2^5 = 0$ and $C \cap U_2$ is defined by $w_1^5 + w_2^5 + 1 = 0$. We have the differential relationships $5v_1^4 d v_1 + 5v_2^4 d v_2 = 0$ and $5w_1^4 d w_1 + 5w_2^4 d w_2 = 0$.

Following a similar treatment to the last example, we define $\omega$ on $U_{1,1}$ by $\omega = d v_1$. However, this chart misses the points on $C$ where $v_2 = 0$. These are the points that also satisfy $v_1^5 + 1 = 0$. Solving for $v_1$, we find that the following five points are missed by $d v_1$:

$$p_1 = [-1 : 1 : 0], \quad p_2 = [\zeta \sqrt{-1} : 1 : 0], \quad p_3 = [\zeta^2 \sqrt{-1} : 1 : 0],$$

$$p_4 = [\zeta^3 \sqrt{-1} : 1 : 0], \quad p_5 = [\zeta^4 \sqrt{-1} : 1 : 0],$$

where $\zeta$ is the fifth root of unity. At these points, we use the conversion $d v_1 = \frac{-v_2^4}{v_1^4} d v_2$.

So, these five points are zeroes of order four for this differential.

The only points on $C$ that are not $U_1$ are those where $x_2 = 1$ and $x_1 = 0$. Translating this to the curve on $U_2$, we find that these points satisfy $w_1^5 + 1 = 0$. This yields the five following points:

$$p_6 = [-1 : 0 : 1], \quad p_7 = [\zeta \sqrt{-1} : 0 : 1], \quad p_8 = [\zeta^2 \sqrt{-1} : 0 : 1],$$

$$p_9 = [\zeta^3 \sqrt{-1} : 0 : 1], \quad p_{10} = [\zeta^4 \sqrt{-1} : 0 : 1].$$

Notice that $w_2 = 0$ for all of these points, which means we must write our differential $d v_1$ in terms of the differential $d w_2$. Through a similar computation to last example,
we find that \( d v_1 = \frac{(-w_2^5 - w_4^5)}{w_2^3 w_1^4} \) d \( w_2 \). It follows that our final five points are double poles of our differential.

The associated canonical divisor is

\[
K_C \equiv 4p_1 + 4p_2 + 4p_3 + 4p_4 + 4p_5 - 2p_6 - 2p_7 - 2p_8 - 2p_9 - 2p_{10}.
\]

Again, note that our curve is genus 6 and \( \deg K_C = 10 = 2 \cdot 6 - 2 \).

In the examples above, the differentials we constructed had poles. The canonical embedding, however, is defined using effective differentials. The next example demonstrates a choice of \( \omega \) that is effective.

**Example 58.** Again consider the curve defined by \( x_0^5 + x_1^5 + x_2^5 = 0 \). Similarly to above, we will use affine charts \( U_1 \) and \( U_2 \) to cover the curve. When we defined the differential on affine chart \( U_1 \), we saw that the points on \( C \) it missed yielded five zeroes of order four in the part of \( U_1 \) where \( v_2 = 0 \). However, when we converted to \( U_2 \) to see what points we missed, in this chart we found five double poles.

To avoid the poles in the second chart, this time we define \( \omega = \frac{1}{v_2} d v_1 \). Then on \( U_{1,1} \) there are no zeroes and poles. Switching to \( d v_2 \) to cover the missing points where \( v_2 = 0 \), we get that \( \omega = -\frac{d v_2}{v_1^2} \), yielding no zeroes or poles in this part of \( U_1 \), as well.

The points missed on \( U_2 \) are exactly the points it missed in the above example. Using the conversion \( v_1 = \frac{w_1}{w_2} \) and \( v_2 = \frac{1}{w_2} \), we find that \( \omega = \frac{w_2^3 d w_3}{w_1^4 w_2} = \frac{w_2^3 d w_2}{w_1^4} \). The five points missed are now five double zeroes. We have successfully defined an effective global differential. The associated effective canonical divisor is

\[
K_C \equiv 2p_1 + 2p_2 + 2p_3 + 2p_4 + 2p_5.
\]
3.1 The Canonical Map

In this section we discuss the canonical map and use the curves defined above to explicitly show how the map works. We first motivate the definition.

Suppose $f_1, \ldots, f_g$ are regular functions on $C$. We can attempt to define a map to projective space using these functions, namely $C \to \mathbb{P}^{g-1}$ by $p \mapsto [f_1(p) : f_2(p) : \cdots : f_g(p)]$. However, this attempt will to be defined at any points $p$ at which all of our functions vanish, since $[0 : \cdots : 0]$ is not a point in projective space. So, we will need to make sure the $f_i$ do not vanish at any one point.

We would like to do this for a basis of effective differentials $\omega_1, \ldots, \omega_g \in \Omega(C)$, obtaining a map from $C \to \mathbb{P}^{g-1}$ defined by $p \mapsto [\omega_1(p) : \cdots : \omega_g(p)]$. However, these differentials are not regular functions so this map doesn’t make sense. Fortunately, the sheaf $\Omega$ is an invertible sheaf, which means that locally it is isomorphic the the structure sheaf of $C$. So, locally the differentials can be identified with functions.

To see how this is done, we start on the affine charts $U_i$ of our curve. If we look at $\Omega(U_i \cap C)$, we have an isomorphism with the ring of regular functions on $C$. For example, consider the curve $C = V(x_1x_2 - x_0^2)$. Then on the chart $U_2$ we have that $U_2 \cap C \simeq V(y - x^2) \subset \mathbb{A}^2$. We know precisely what the differentials are here. We saw that they are generated as an algebra by $d\bar{x}$ where $B = \text{Spec} \, \mathbb{C}[x, y]/(y - x^2)$. It follows that the differentials of $C$ are $U_2$ can be written in the form $f(\bar{x})d\bar{x}$, where $f$ is a regular function on $U_2$. 

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In the general case, if $\omega_1, \ldots, \omega_g \in \Omega(C)$, then locally their restrictions look like $f_1(w_i) \, d\, w_i, \ldots, f_g(w_i) \, d\, w_i$, where $w_i$ is a local parameter for $C$. We can then use $f_1, \ldots, f_g$ to locally define a map to $\mathbb{P}^g$, and it turns out that these local maps agree and collectively define a map $C \to \mathbb{P}^g$. This is how we will build our canonical map.

The first step to understanding the canonical map is to write down a basis for $H^0(\Omega(C))$, the vector space of effective differential forms. In the proof of Petri’s Theorem (below), one chooses a basis by first picking $g$ points in “general position” on $C$, say $p_1, \ldots, p_g$. Then one chooses the so-called dual basis of effective differential forms, $\omega_1, \ldots, \omega_g$. The basis is constructed so that $\{\omega_1, \omega_2\}$ vanish to order one on points $p_3, \ldots, p_g$. However, the canonical map is essentially “choice free,” as it turns out that choosing a different basis changes the image of the canonical map by an automorphism of $\mathbb{P}^g$. We will show an example of how exactly to find this dual basis that Petri uses. However, we will then use a much simpler basis of differentials for our analysis of the canonical map. Our choice will be based on the following result.

**Proposition 5 ([12]).** Let $\omega_0$ be a globally defined differential on a nonsingular projective plane curve $C$ of degree $d$. Then a basis for $\Omega(C)$ is given locally by $\{u_1^i u_2^j \omega_0 \mid i, j \geq 0, i + j \leq d - 3\}$, where $u_1, u_2$ are affine local coordinates.

**Example 59.** In this example we will show how Petri chooses his dual basis. Consider the curve given by $x_0^4 + x_1^4 + x_2^4 = 0$. On the chart $U_2$, this curve is defined by $w_1^4 + 1 + w_2^4 = 0$. It follows that the differentials on $U_2$ are generated by $d\, w_1$ and $d\, w_2$, subject to the relation

$$4w_1^3 \, d\, w_1 + 4w_2^3 \, d\, w_2 = 0.$$
Define $\omega_0 \in \Omega(C)$ such that $\omega_0|_{U_{1,1}} = \frac{dw_1}{w_2}$. Using the proposition above, a basis for $\Omega(C)$ is then $\left\{ \frac{dw_1}{w_2}, \frac{w_1 dw_1}{w_2}, \frac{w_2 dw_1}{w_2} \right\} = \{\omega_0, w_1 \omega_0, w_2 \omega_0\}$.

Petri chooses $g$ distinct points in general position on $C$ and finds his differentials that are dual to these points. For the sake of an example, let us suppose the first point is $p_1 = (0, \zeta)$, where $\zeta = e^{\frac{a}{4}}$. We will find all $\omega \in \Omega(C)$ such that $\omega(p_1) = 0$.

We first write $\omega = \left(c_1 \frac{1}{w_2} + c_2 \frac{w_1}{w_2} + c_3 \frac{w_2}{w_2}\right) w_1 = \frac{c_1 + c_2 w_1 + c_3 w_2}{w_2} w_1$. Then $\omega(p_1) = \frac{c_1 + c_3 \zeta}{\zeta^4}$, so $\omega(p_1) = 0$ exactly when $c_1 + c_3 \zeta = 0$. Therefore, $c_3 = -c_1 \zeta^{-1}$ and $\omega = \frac{c_1 + c_2 w_1 - c_1 \zeta^{-1} w_2}{w_2} dw_1$.

Next, say we choose the point $p_2 = (1, \alpha)$, where $\alpha$ satisfies $a^4 + 2 = 0$. We now describe all $\omega \in \Omega(C)$ such that $\omega(p_1) = \omega(p_2) = 0$. Using $\omega = \frac{c_1 + c_3 w_1 - c_1 \zeta^{-1} w_2}{w_2} w_1$, from the equality $\omega(p_2) = 0$, we have $c_1 + c_2 - c_1 \zeta^{-1} \alpha = 0$. Solving for $c_2$, we see that $c_2 = (\zeta^{-1} \alpha - 1)c_1$. Therefore, $\omega = c_1 \frac{1 + (\zeta^{-1} \alpha - 1) w_1 - \zeta^{-1} w_2}{w_2} w_1$. Let us denote by $\omega_1$ the differential we obtain when we set $c_1 = 1$.

For the last point, suppose we choose $p_3 = (2, \beta)$, where $\beta$ satisfies $\beta^4 + 17 = 0$. We now find all $\omega$ such that $\omega(p_1) = \omega(p_3) = 0$. Proceeding similarly to the $p_2$ case, above, we find that all such $\omega$ are of the form $\omega = c_1 \frac{1 + (\zeta^{-1} \alpha - 1) w_1 - \zeta^{-1} w_2}{w_2} w_1$. Let us denote by $\omega_2$ the differential we obtain when we set $c_1 = 1$.

Lastly, we need to find $\omega \in \Omega(C)$ such that $\omega(p_2) = \omega(p_3) = 0$. We know that $\omega(p_2) = 0$ when $c_1 + c_2 + c_3 \alpha = 0$ and $\omega(p_3) = 0$ when $c_1 + 2c_2 + c_3 \beta = 0$. The solution to this system of equations is $c_1 = (-2\alpha + \beta)c_3$ and $c_2 = (-\beta + \alpha)c_3$, where $c_3$ is free. Thus, all such $\omega$ are of the form $\omega = c_3 \frac{-2\alpha + \beta + (\beta + \alpha) w_1 + w_2}{w_2} w_1$. Let us denote by $\omega_3$ the differential we obtain when we set $c_3 = 1$.

We have now constructed a basis $\{\omega_1, \omega_2, \omega_3\}$ for $\Omega(C)$ that is dual to the set of points $\{p_1, p_2, p_3\}$. We can now define a canonical map $C \cap U_{1,1} \to \mathbb{P}^2$ by mapping $(w_1, w_2)$...
to the point

\[
\begin{bmatrix}
1+(\zeta^{-1}\alpha-1)w_1 - \zeta^{-1}w_2 & 1+(\frac{\zeta^{1/2}-1}{2})w_1 - \zeta^{-1}w_2 & (\frac{\zeta\beta-1}{2})w_1 - \zeta^{-1}w_2 & (\frac{-2\alpha+\beta}{2})+(-\beta+\alpha)w_1 + w_2
\end{bmatrix}.
\]

As usual, one can use the coordinate-change method to see how this map is defined for points outside the chart \(U_2\).

**Example 60.** Now consider the curve \(C\) defined by \(x_0^5 + x_1^5 + x_2^5 = 0\). On \(U_1\), this curve is defined by \(v_1^5 + 1 + v_2^5 = 0\). It follows that the differentials on \(U_1\) are generated by \(dv_1\) and \(dv_2\), subject to the relation \(5v_1^4 dv_1 + 5v_2^4 dv_2 = 0\). We have seen that an effective differential is \(\omega_0 = \frac{dv_1}{v_1}\), and so a basis for the differentials is given by \(\left\{\frac{dv_1}{v_2}, \frac{v_1 dv_1}{v_2^2}, \frac{v_2 dv_1}{v_2^2}, \frac{v_1^2 dv_1}{v_2^3}, \frac{v_2^2 dv_1}{v_2^3}\right\}\). Using this basis, the canonical map is \(\phi : C \cap U_1 \to \mathbb{P}^5\) where \([v_1 : v_2] \mapsto [1 : v_1 : v_2^2 : v_1 v_2 : v_2^2]\).

In the previous example we looked at the canonical map for our genus 6 curve from one of the affine charts. However, the canonical map for projective plane curves can be described globally quite simply.

**Example 61.** Again, consider the curve \(C\) defined by \(x_0^5 + x_1^5 + x_2^5 = 0\) and assume the same basis as above. One can show that the canonical map is globally given by \(\phi : C \to \mathbb{P}^5\), where \([x_0 : x_1 : x_2] \mapsto [x_1^2 : x_0 x_1 : x_1 x_2 : x_0^2 : x_0 x_2 : x_2^2]\). Observe, we have the injections \(C \hookrightarrow \mathbb{P}^2 \hookrightarrow \mathbb{P}^5\), which correspond to the ring morphisms

\[
\mathbb{C}[x_0, \ldots, x_5] \xrightarrow{f} \mathbb{C}[x_0, x_1, x_2] \xrightarrow{\pi} \mathbb{C}[x_0, x_1, x_2]/(x_0^5 + x_1^5 + x_2^5).
\]

The map of projective spaces corresponding to \(f\) is called the Veronese embedding.
Using computer software, one can show that the kernel of the above ring morphism is generated by the following nine polynomials:

\[ x_4^2 - x_3x_5, \quad x_2x_4 - x_1x_5, \quad x_2x_3 - x_1x_4, \quad x_2^2 - x_0x_5, \quad x_1x_2 - x_0x_4, x_1^2 - x_0x_3, \]
\[ x_6^2x_2 + x_3^2x_4 + x_5^3, \quad x_6^2x_1 + x_3^2 + x_4x_5^2, \quad x_6^3 + x_1x_3^2 + x_2x_5^2. \]

The first six polynomials are actually generators for the kernel of the Veronese map, while the final three are unique to our canonical map. In any case, the canonical model of \( C \) is the curve in \( \mathbb{P}^5 \) defined by the vanishing of these nine polynomials.

### 3.1.1 Petri’s Theorem

The fundamental theorem in the study of the canonical map is the theorem below, which not only tells us when the map is an embedding, but also describes the equations that define the image. The result is originally due to [10], but here we provide the modern formulation as presented in Saint-Donat.

**Theorem 4 ([11]).** Let \( C \) be a complete non-singular curve of genus \( \geq 3 \) over an algebraically closed field \( k \). Let \( S^*\Omega(C) \) denote the symmetric algebra of \( \Omega(C) \). Then the map

\[
\phi : S^*\Omega(C) \to \bigoplus_{n \geq 0} \Omega^n(C)
\]

is surjective. Moreover, if we let \( I = \ker(\phi) \), then:

1) \( I \) is generated by its elements of degree 2 and of degree 3; and

2) \( I \) is generated in the by its elements of degree 2, except in the following cases:

   a) \( C \) is a non-singular plane quintic;

   b) \( C \) is trigonal, i.e., \( C \) is a triple covering of \( \mathbb{P}^1 \).
The map $\phi$ in the above theorem is a “choice free” description of the canonical map. Finding a tropical analogue of this result would be an interesting avenue for future research. A fair amount of tropical algebra machinery would probably need to be built, however, before one could make much of an attempt.
Chapter 4

TROPICAL ALGEBRAIC GEOMETRY

Tropical geometry is a field that is currently still in development. Part of the drive is coming from its applications to classical algebraic geometry, as tropical geometry can often give combinatorial interpretations of classical arguments. A great example of this is the tropical analogue of the Hilbert’s Nullstellensatz [1], or the tropical proof of the Brill-Noether theorem [2].

In this chapter we give a brief overview of tropical geometry, highlighting the key facts of the tropical setting. These key ingredients should help give us some insight into understanding what happens when we tropicalize canonical curves.

4.1 Origin of the Tropical Graph

Consider the map

\[ \log: (\mathbb{C}^*)^2 \to \mathbb{R}^2 \]

\[ (z_1, z_2) \mapsto (\log |z_1|, \log |z_2|) \]

For a complex plane curve \( C \), the image of \( C \cap (\mathbb{C}^*)^2 \) under this map is called an amoeba. The reason for the name is immediately evident from the example below, which is the image of the complex line defined by \( z_1 + z_2 = 1 \) under this logarithmic map. To understand why the amoeba has the shape it does, first note that every point on \( C \) with nonzero coordinates is of the form \( (1 - z_2, z_2) \) for \( z_2 \neq 0 \). The point \( (1,0) \) can be thought of as the limit of these points as \( z_2 \to 0 \), while the point \( (0,1) \)
can be viewed as the limit as $z_2 \to 1$. In view of the map above, we can see that the image of the point $(1 - z_2, z_2)$ tends towards $(0, -\infty)$ as $z_2 \to 0$, while the image tends towards $(-\infty, 0)$ as $z_2 \to 1$. These correspond to the "tentacles" pointing downward and to the left, respectively. As for the tentacle heading up and to the right, notice that when $z_2$ is very large, we have that $|z_1| = |1 - z_2| \approx |z_2|$, so the image points are very close to the line $y = x$. It turns out that the images of every complex plane curve have this general shape, with "tentacles" off in the three directions. In fact, for a curve of degree $d$, there are generally $d$ "tentacles" in each direction. Hidden within these amoebas is a "skeleton" that contains the basic piecewise-linear structure of the shape. We can extract this skeleton as follows.

Define, for each positive real number $t$, a map

$$
\log_t : (\mathbb{C}^*)^2 \to \mathbb{R}^2
$$

$$(z_1, z_2) \mapsto \left(-\frac{\log |z_1|}{\log t}, -\frac{\log |z_2|}{\log t}\right)$$

As we let $t \to 0$, the amoeba shrinks down onto its skeleton. In the case of our ongoing example, we find the following:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{amoeba.png}
\caption{\textbf{log($C$) for the curve $C$ defined by $z_1 + z_2 = 1$}}
\end{figure}
Such a figure is called a \textit{tropical curve}. All tropical curves are piecewise-linear objects, like this example.

4.2 Tropical Algebra

We next consider how one could obtain curves as above through a construction similar to classic algebraic geometry. In other words, what type of algebraic structure (and corresponding algebra-geometry dictionary) gives rise to such curves?

We begin with defining the algebra that takes place in the tropical setting. One of the distinguishing factors of tropical geometry is the way addition and multiplication are defined. We begin with equipping $\mathbb{R} \cup \{-\infty\}$ with two operations, called \textit{tropical addition} and \textit{tropical multiplication}. For tropical addition, we define $a \oplus b$ to be the real maximum of $a$ and $b$. For tropical multiplication, we define $a \odot b$ to be real addition of $a$ and $b$.

\textit{Remark.} Keeping in mind the origin of tropical curves as images of a logarithm map, these new tropical operations are not entirely unexpected. After all, the logarithm converts multiplication into addition, i.e., $\log(ab) = \log(a) + \log(b)$. The maximum operation is less obvious, but follows roughly from the approximation $\log(a + b) \approx \max\{\log(a), \log(b)\}$.
Although most of the usual algebraic properties of a ring still hold, notice that there is no additive inverse in the tropical setting, as the max function is not injective. Instead, we have the structure of a **semiring**. We will denote this semiring $\mathbf{T}$. (It is actually a semifield.) The additive identity in $\mathbf{T}$ is $-\infty$, which we will sometimes denote $0_T$; the multiplicative identity is the real number zero, which we will sometimes denote $1_T$.

We can define tropical polynomials just as we would for any semiring.

**Example 62.** Consider the polynomial $f(x) = 2x^3 + x^2 + 1$. In the tropical landscape the corresponding polynomial is $\tilde{f}(x) = (2 \odot x^3) \oplus (x^2) \oplus 1_T$, which as a function corresponds to $\tilde{f}(x) = \max\{2 + 3x, 2x, 0\}$.

![Figure 4.2: Graph of the tropical polynomial $\tilde{f}(x) = (2 \odot x^3) \oplus (x^2) \oplus 1_T$](image)

One important distinction between the tropical setting and classical setting is that multiple polynomials can define the same function on tropical space.

**Example 63.** Consider the tropical polynomials $f(x) = x^2 \oplus x \oplus 1_T$ and $g(x) = x^2 \oplus 1_T$. The green curve in the diagram corresponds to the tropical graph of both of these functions.
4.3 Bend Loci and Congruence Varieties

There are two general methods of constructing tropical geometric objects: congruence varieties and bend loci. We start with congruence varieties. A good reference is [1].

**Definition 64.** Given a subset \( S \subseteq T^n \), we define the set

\[
E(S) = \{ (f, g) \mid f(a) = g(a) \text{ for all } a \in S \} \subseteq T[x] \times T[x].
\]

This relationship is inclusion-reversing. Given a set \( E \subseteq T[x] \times T[x] \), we also define

\[
V(E) = \{ a \in T^n \mid f(a) = g(a) \text{ for all } (f, g) \in E \} \subseteq T^n.
\]

We call \( V(E) \) the **congruence variety** associated to \( E \).

**Example 65.** The tropical equation \((x^{\odot 2} \odot y) \oplus (2 \odot x^{\odot 3}) \oplus (x^{\odot 3}) = 1_T\) corresponds to the congruence variety \( V \left((x^{\odot 2} \odot y) \oplus (2 \odot x^{\odot 3}) \oplus (x^{\odot 3}), 1_T\right) = \{ (a, b) \mid \max \{2a + b, 2 + 3a, 3a\} = 0 \} \subseteq T^2.\)
Alternatively, any tropical polynomial can also produce a subset of tropical space, as follows:

**Definition 66.** Given a polynomial $f \in T[x]$, the **bend locus** (or **double-max locus**) of $f$ is the collection of points where the maximum (of the tropical function defined by $f$) is achieved at least twice. We denote this set $\text{Bend}(f)$.

**Example 67.** Consider the tropical polynomial $f(x) = (x^2) \oplus (3 \circ x) \oplus 2 = \max\{2x, 3 + x, 2\}$. There is a two-way tie for max when: 1) $2x = 3 + x$ and $2x > 2$; 2) $2x = 2$ and $2x > 3 + x$; and 3) $3 + x = 2$ and $3 + x > 2x$. These three scenarios reduce to the two points $x = -1$ and $x = 3$. Therefore, the $\text{Bend}(f) = \{-1, 3\}$. One can also see this immediately from the graph of $f$:

![Graph of f(x) = (x^2) \oplus (3 \circ x) \oplus 2]

**Example 68.** Consider the tropical polynomial $f(x, y) = (x^2 \circ y) \oplus 1_T = \max\{2x + y, 0\}$. There is a two-way tie for max when $2x + y = 0$, i.e., when $y = -2x$ (in real operations). Therefore, the $\text{Bend}(f) = \{(a, b) \mid 2a + b = 0\} \subseteq \mathbb{R}^2 \subseteq T^2$.

Suppose $C$ is a complex curve. From this point on, when we talk about a **tropicalization** of $C$, we mean a tropical analogue of $C$, either using a bend locus or a congruence variety. In particular, we do not mean the earlier process by which we compute a limit of logarithmic maps applied to $C$.

We have two main options. We can either view $C$ as the vanishing locus of some polynomials and look at the bend locus of the tropical analogues of those polynomials;
or we can rewrite all of the equations that define $C$ without any minus signs, and look at the congruence variety defined by tropical analogues of those equations.

**Example 69.** Consider the complex plane curve defined by $y = x^3 + x + 1$. Written as a zero locus, this is the vanishing of the polynomial $f(x, y) = x^3 + x - y + 1$. If we ignore the minus sign, we can consider the bend locus of the corresponding tropical polynomial $\tilde{f}(x, y) = (x^\odot 3) \oplus x \oplus y \oplus 1_T = \max\{3x, x, y, 0\}$. There is a two-way tie for max when: 1) $3x = x$ and $3x > y$, $3x > 0$; 2) $3x = y$ and $3x > x$, $3x > 0$; 3) $3x = 0$ and $3x > x$, $3x > y$; 4) $x = y$ and $x > 3x$, $x > 0$; 5) $x = 0$ and $x > 3x$, $x > y$; and 6) $y = 0$ and $y > 3x$, $y > x$.

On the other hand, we could also consider the tropical version of the defining equation of $C$, namely $y = x^\odot 3 \oplus x \oplus 1_T$. This defines a congruence variety. The congruence variety $V(x^\odot 3 \oplus x \oplus 1_T, y) = \{(a, b) : \max\{3a, a, 0\} = b\}$.

In this example, the congruence variety seems like the more natural choice, since the bend locus required arbitrarily changing a minus sign into a positive sign.

**Example 70.** Consider the curve $C$ defined by $x^3 + xy = 0$. If we let $f(x, y) = x^\odot 3 \oplus (x \odot y)$, then $\text{Bend}(f) = \{(a, b) \in T^2 : 3a = a + b\}$. As for the corresponding congruence variety, the corresponding tropical equation is $x^\odot 3 \oplus (x \odot y) = 0_T$, whose only solutions is $x = y = 0_T$.

In this example, the congruence variety seems like the more natural choice, since the bend locus required arbitrarily changing a minus sign into a positive sign.
Chapter 5

TROPICALIZATION AND CANONICAL CURVES

In this section we will introduce options for tropicalizing the canonical embedding. The classical canonical embedding gives us a classification of projective curves for genus \( \geq 3 \). For a given nonsingular projective curve \( C \), we will denote its image under the canonical map as \( \text{Can}(C) \). We will also denote by \( C_T \) a tropicalization of \( C \), keeping in mind our two options for tropicalizing a curve.

The first option we will explore is tropicalizing the image of the canonical map, i.e., the composition

\[
C \xrightarrow{\phi} \text{Can}(C) \xrightarrow{T} (\text{Can}(C))_T.
\]

The next option that we will explore is tropicalizing the curve first and then applying a tropical version of the canonical map, i.e.,

\[
C \xrightarrow{T} C_T \xrightarrow{\phi_T} \text{Can}(C_T).
\]

Ideally, the two processes would commute, but there is a lot of reason to suspect they might not. The next section will investigate the following:

1. Should we use the bend locus or congruence variety for \( C_T \)? Does either provide a more natural relationship in the above setting?

2. What (if any) restrictions need to be in place in order to have a tropical canonical map with properties similar to the classical canonical map?
5.1 Tropicalizing the Curves

While the canonical map is an embedding for all curves of genus $\geq 3$, we will stick to using our degree five Fermat curve as an explicit test case. We begin by considering the tropicalization of $C$ and its canonical model.

Starting with the Fermat curve $C$, it is quickly apparent that the congruence variety does not seem to give a good tropical analogue. That is because the tropical equation $x_0^{\circ 5} \oplus x_1^{\circ 5} \oplus x_2^{\circ 5} = 0_T$ only has one solution, namely $x_0 = x_1 = x_2 = 0_T$, which doesn’t correspond to a point in $\mathbb{P}^2_T$. In other words, that congruence variety is empty! So for a tropicalization of $C$, we will use the bend locus of the tropical polynomial $f(x_0, x_1, x_2) = x_0^{\circ 5} \oplus x_1^{\circ 5} \oplus x_2^{\circ 5}$.

We now consider the canonical model of $C$. Recall from Example 61 that the image of the canonical map is the vanishing locus of the following nine polynomials:

$$
\begin{align*}
x_4^2 - x_3 x_5, & \quad x_2 x_4 - x_1 x_5, \quad x_2 x_3 - x_1 x_4, \quad x_2^2 - x_0 x_5, \quad x_1 x_2 - x_0 x_4, x_1^2 - x_0 x_3; \\
x_0^2 x_2 + x_3^2 x_4 + x_5^3, & \quad x_0^2 x_1 + x_3^2 + x_4 x_5^2, \quad x_0^3 + x_1 x_3^2 + x_2 x_5^2.
\end{align*}
$$

To tropicalize this image we have two options:

1. Compute the intersection of the bend loci of the corresponding tropical polynomials; or

2. Compute the congruence variety defined by the corresponding tropical equations.

There are possible issues with both approaches. For the bend loci, we would need to change all the minus signs in the first six generators to plus signs, which seems
arbitrary. For this reason, we will try using the congruence variety, which is defined by the following nine tropical equations:

\[
\begin{align*}
\quad x_4 \circ 2 &= x_3 \circ x_5, & x_2 \circ x_4 &= x_1 \circ x_5, & x_2 \circ x_3 &= x_1 \circ x_4, \\
\quad x_2 \circ 2 &= x_0 \circ x_5, & x_1 \circ x_2 &= x_0 \circ x_4, & x_1 \circ 2 &= x_0 \circ x_3, \\
\quad (x_0 \circ 2 \circ x_2) \oplus (x_3 \circ 2 \circ x_4) \oplus x_5 \circ 3 &= 0_T, \\
\quad (x_0 \circ 2 \circ x_1) \oplus x_3 \circ 3 \oplus (x_4 \circ x_5 \circ 2) &= 0_T, \\
\quad x_0 \circ 3 \oplus (x_1 \circ x_3 \circ 2) \oplus (x_2 \circ x_5 \circ 2) &= 0_T.
\end{align*}
\]

We will see that the final three equations have their own issues.

### 5.2 Apply a Tropical Canonical Map to the Tropicalization

We now consider applying a tropical analogue of the canonical map to the tropicalization of \( C \). We first note that it seems reasonable to define the tropical canonical map as \( \phi_T : C_T \rightarrow \mathbb{P}_T^5 \) by \([x_0 : x_1 : x_2] \mapsto [2x_1 : x_0 + x_1 : x_1 + x_2 : 2x_0 : x_0 + x_2 : 2x_2] \).

Now, if we let \( f(x_0, x_1, x_2) = x_0 \circ 5 \oplus x_1 \circ 5 \oplus x_2 \circ 5 \), then one option for \( C_T \) is \( \text{Bend}(f) \). A sketch of \( C_T \cap U_0 \) is shown below. We note that although the bend locus of a general polynomial of degree five has five legs in the three main directions, in this case those legs can be thought of as having “come together.” In other words, each leg is a leg of “multiplicity five.”

We examine \( C_T \) on its different charts. The first chart we will analyze is \( U_0 \), when \( x_0 \circ T = 0 \). To analyze the downward leg, we let \( x_1 = 0 \) and let \( x_2 = t \), where \( t \) ranges from 0 to \(-\infty\). For this leg, our tropical canonical map \( \phi_T \) sends \((0, t) \mapsto [0 : 0 : t : 0 : t : 2t] \). Do the images of these points lie on a tropicalization of the
canonical model? As stated earlier, we have two options for what we could mean by a tropicalization of the canonical model.

To investigate, let’s first suppose we use congruence varieties for the tropicalization of the canonical model. In this case, our tropical canonical model is given by the nine equations

\[
\begin{align*}
    x_4 \circ_2 &= x_3 \odot x_5, \\
    x_2 \odot x_4 &= x_1 \odot x_5, \\
    x_2 \odot x_3 &= x_1 \odot x_4, \\
    x_2 \circ_2 &= x_0 \odot x_5, \\
    x_1 \odot x_2 &= x_0 \odot x_4, \\
    x_1 \circ_2 &= x_0 \odot x_3, \\
    (x_0 \circ_2 \odot x_2) \oplus (x_3 \circ_2 \odot x_4) \oplus x_5 \circ_3 &= 0_T, \\
    (x_0 \circ_2 \odot x_1) \oplus x_3 \circ_3 \oplus (x_4 \odot x_5 \circ_2) &= 0_T, \\
    x_0 \circ_3 \oplus (x_1 \odot x_3 \circ_2) \oplus (x_2 \odot x_5 \circ_2) &= 0_T.
\end{align*}
\]

It is straightforward to verify that the image points (of the downward leg) all satisfy the first six tropical equalities. For example, when \(x_0 = 0, x_1 = 0, x_2 = t, x_3 = 0, x_4 = t, \) and \(x_5 = 2t\), we have that \(x_4 \circ_2 = 2t \) and \(x_3 \odot x_5 = 0 + 2t = 2t\).
For the final three equations, however, there is a problem. None of the image points satisfy any of these equations. There is a possible fix, however. We will explain the fix first, before giving a possible argument for why such a fix might be needed.

**Idea:** Suppose we added one to both sides of the classical equation

\[ x_0^2 x_2 + x_2^2 x_4 + x_3^2 = 0. \]

This would not change the solutions in classical geometry. However, the corresponding tropical equation would become

\[ x_0^{\circ 2} \ominus x_2 \ominus x_3^{\circ 2} \ominus x_4 \ominus x_5^{\circ 3} \oplus 1_T = 1_T, \]

or in real operations

\[ \max \{2x_0 + x_2, 2x_3 + x_4, 3x_5, 0\} = 0. \]

For images of points on the downward leg, the left-hand side of this equality is

\[ \max \{t, t, 6t, 0\}, \]

which equals 0 (since \( t \) ranges from 0 to \(-\infty\) in our parameterization). Thus, the points in the image of the downward leg all satisfy the seventh equation after making this modification.

We now repeat the above analysis for another leg of \( C_T \), say the leg pointing to the right, where \( x_1 = x_2 = t \) and \( x_2 > 0 \). For this leg, the canonical map \( \phi_T \) sends \((t, t) \mapsto [2t : t : 2t : 0 : 2t] \). Once again, it is easy to verify that the image of this leg satisfies the first six tropical equations. However, once again the last three equalities do not hold, and again we can alleviate this issue if we add a term to both sides of the original equation before tropicalizing. For instance, if we add the polynomial \( x_1^6 \) to both sides of the equation before tropicalizing, then the image of the rightward leg satisfies the seventh tropical equation.

Finally, we analyze the last leg of \( C_T \), when \( x_2 = 0 \) and \( x_1 = t \), where \( t \) ranges from 0 to \(-\infty\). For this leg, the canonical map \( \phi_T \) sends \((t, 0) \mapsto [2t : t : t : 0 : 0 : 0] \). The story is the same as in the previous two cases: the image of this leg satisfies the first six tropical equations, but we need to add something to both sides of the last three equations before tropicalizing in order for the image to satisfy the last three tropical equations.
5.3 Final Thoughts

This example brings up points that are worth highlighting. First, we point out that there are issues with both tropical options, namely bend loci and congruence varieties. When using bend loci to give tropical analogues of classical curves, we sometimes need to deal with minus signs, perhaps arbitrarily changing them. When dealing with congruence varieties, we sometimes obtain tropical analogues that do not seem to reflect the original variety very well, possibly due to “missing” canceled terms.

We ran into both issues in our specific example. The tropical image for the first six generators in all the charts seem to work without any fail via the congruence variety method. However, the last three generators in all cases suggests that the congruence variety method is not the way to go, because the map seems to conditionally work by adding constants or polynomials to both sides of the equation. This could suggest that the canonical embedding in the classical setting maybe has some sort of cancellation that took place, however in the tropical setting there would be no cancellation of the elements because subtraction is still unclear in tropical geometry. Our proposed tropical map is carving some sort of tropical image, however it seems like the full canonical tropical curve is unable to be carved out by this map.

We could consider using bend loci for a tropical analogue of the canonical model. The issue with that option is justifying the arbitrary switch of every minus sign to a positive sign. Nevertheless, this would be a good avenue for future investigations.
BIBLIOGRAPHY


