MODAL ANALYSIS OF COMPOSITE STRUCTURES WITH DAMPING MATERIAL

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ABSTRACT

Modal Analysis of Composite Structures with Damping Material

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The purpose of this study is to develop an analytical solution for modal analysis of actively damped orthotropic composite plates in bending and to verify it with experimental analysis.

The analytical modal analysis solution for composite plate dynamics is derived using Euler theory. This analysis applies to structures with orthotropic lamina of uniform material properties at any lamination angle. The bending-extensional coupling can be neglected for plates that are symmetric or approximately symmetric, which allows an exact solution for natural frequency and mode shape to be obtained. An exact solution can be found for natural vibration and in general.

The active control is modeled analytically by combining the Lagrange equation with the Ritz Assumed Mode method. This analysis produces a generalized coordinate vector that correlates the assumed mode to the particular amplitude of a particular case. The kinetic energy dissipated by the piezoelectric actuator from the system over one oscillation can be calculated from the generalized coordinate vector and the assumed mode. The equivalent damping ratio of the active control system is calculated as the ratio between the kinetic energy absorbed by the piezoelectric actuator from the system in one oscillation and the maximum strain energy of the system during that oscillation.

A point mass on the plate, such as an accelerometer mass, can also be modeled as a single layer of uniform mass, that is an isotropic layer, by equating the potential energy of the point mass with the potential energy of the uniform mass layer. It is important to note that the mass of the isotropic layer is frequency dependent, and it has no effect on the plate stiffness.

The analytical model is validated by comparison to experimental work. The samples studied were aluminum and composite plates of various lengths. The active control predictions were also validated using previous experimental work completed at California Polytechnic State University in San Luis Obispo. These cases included active control of an aluminum beam with a patch of piezoelectric material and an aluminum sailplane with a patch of piezoelectric material.

Results indicate that while the analytical mode solutions are in good agreement with the experimental results, they are also systematically higher than the experimental results. The analytical active control solutions match previous work when the piezoelectric effects are linear. The main result of adding an active control system is approximately a 5-10% increase in modal frequencies and a 200-800% increase of damping ratio.

Keywords: Structures, Composites, Modal Analysis, Piezoelectric, Piezoceramic,
Active Control, Analytic, Damping, Ritz Assumed Mode, Vibration.

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Chapter I. INTRODUCTION

This chapter introduces the problem that is addressed in this study. It defines nomenclature and acronyms used throughout this study, and it provides a summary of the discussion in each chapter.

A. Statement of Problem

Vibrations in structures have become increasingly problematic in low margin design, where structures are built to have a minimum amount of material or weight. Frequently, solutions have poor modal characteristics due to other constraints on the system, such as geometry or weight. In many cases poor modal characteristics can be mitigated with the use of passive, active or hybrid damping techniques.

Much experimental work has been completed to study the effects of incorporating passive, active or hybrid damping materials to structures. Experimental modal analysis indicates that while damping treatments have a small effect on the modal frequencies; they have a significant effect on the amount of damping and the modal amplitude. [2] Due to these effects, all three types of damping treatments have been applied to many structures and have great potential in the future.

Currently, passive damping treatments have been well studied analytically, numerically, and experimentally; while active damping treatments have primarily been experimentally studied. While active damping treatments can have significant advantages over passive treatments, there is little research that gives an analytical or numerical solution for an active treatment of a given structure.

The most common analytical solution for actively damped structures is to simplify the dynamics into a transfer function with one of the modes represented. The

transfer function method works by creating an equivalent mass-spring-damper system and taking a LaPlace transform to find the transfer function. [5] This type of analysis is good at predicting point performance for a single mode. The solution becomes poor far away from that modal frequency; it especially breaks down around other modal frequencies. It does not give any information about the mode shape of the plate, which is the shape a plate assumes when it vibrates. A general solution for all frequencies requires more knowledge of the dynamics of the structure.

Based on a literature review, there is a need to develop and implement a solution method that can predict the effects of applying active damping treatments to structures. The primary objective of this study is to develop such a solution method, using an analytical approach. A secondary objective is to validate the solution method with an experimental study and to retroactively apply the solution method to other projects to demonstrate its results.

B. List of Terms

Variables

ε	Strain
θ	Lamina rotation angle
κ	Curvature
λ	Eigenvalue
Λ	Piezoelectric
μ	Eigenvalue
ν	Poisson's ratio
ζ	Damping ratio
ρ	Density
ϕ	Assumed mode displacements
ω	Frequency
A	Extensional stiffness
a	Plate length
B	Bending extensional stiffness coupling
b	Plate width
c	Constant
D	Bending stiffness
$\mathcal{D}$	Derivative operator matrix
E	Young's modulus of elasticity
G	Shear modulus
H	Assumed mode shape
h	Plate thickness
I	Moment of Inertia
i	Imaginary number
i	Index
j	Index
K	Modal stiffness
k	Index
M	Moment
N	Force
n	N^{th} mode
Q	Force amplitude
$\bar{Q}$	Modal actuation strain
Q	Lamina stiffness
q	Generalized coordinate vector
q	Distributed transverse force
t	Time

U	Strain energy
u	Displacement function in x direction
v	Displacement function in y direction
w	Displacement function in z direction
X	Cartesian coordinate along plate length
Y	Cartesian coordinate along plate width
Z	Cartesian coordinate along plate thickness
z	Vertical lamina coordinates

Acronyms

ACLD	Active Constrained Layer Damping Treatments
APDC	Active Piezoelectric Damping Composites
EMSD	Electromechanical Surface Damping Treatments
FEA	Finite Element Analysis
FRF	Frequency Response Function
PCLD	Passive Constrained Layer Damping Treatments
PVDF	Polyvinylidene Fluoride (a piezoelectric polymer)
PZT	Lead Zirconate Titanate (a piezoelectric composite)

Subscripts and Superscripts

b	Normalized by plate width
n	Natural
n	Nth mode
T	Transpose
xx	Along the x face in the x direction
yy	Along the y face in the y direction
0	Midplane displacement
–	Transformed
^	Multiplied by plate width
*	Inverse

C. Summary of Chapters

This chapter has introduced the statement of the problem that is addressed in this study. It demonstrates the importance of this research in general. It also introduced the nomenclature that will be used throughout this study.

Chapter II will introduce previous work in a literature review. It will include an introduction to vibrations, damping, vibration suppression techniques, current modal analysis methods, and previous experimental studies.

Chapter III covers the formulation of the analytical solution for vibrations of cantilever plates with active damping. The Euler-Bernoulli formulation of the governing equations of motions is introduced. The piezoelectric effects were modeled using the governing equations and the Ritz assumed mode method. The damping effects were modeled using the ratio between the strain energy absorbed in the piezoelectric ceramic to the maximum potential energy in the structure.

Chapter IV covered the experimental design, procedure, and results from this study. A forced vibration frequency sine sweep test was completed on a shake table to verify the modal analysis tool for aluminum and composite cantilever plates.

Chapter V covers validation against the experimental work as well as other applications such as previous experimental work completed by Hrovat [11], Lesniak [19], and Cavalli [5].

Chapter VI summarizes the results and findings of this study and recommendations for future research.

Chapter II. LITERATURE REVIEW

This chapter introduces the concepts of vibration, damping, and vibration suppression techniques. It summarizes studies from the literature review, including previous experimental studies from California Polytechnic State University in San Luis Obispo.

A. Introduction to Vibration

Vibrations are mechanical oscillations of physical objects such as a plate, particle, or body that is displaced from a position of equilibrium. The vibrations occur as the physical objects trade kinetic energy for potential energy. In structures, potential energy is elastic potential energy, or in other words, strain energy.

With no energy losses, a body that is given an initial amount of energy will retain that energy indefinitely. However, it can store the retained energy in multiple forms, such as strain energy and kinetic energy. The body will trade between energy types if the exchange satisfies the principle of least action, which says that the path an object takes between two fixed states will minimize the total energy of the system for a set of given equilibrium equations. That is to say that a structure will trade strain energy and kinetic energy only if it can do so in a way that does not increase the total energy of the system. This solution for the principle of least action is satisfied when the characteristic equation is solved. The characteristic equation is solved by the system's eigenvalues, which directly relate to physical properties such as frequency. The solutions to the system's characteristic equation are referred to as the system's modes, and they correlate directly to a particular frequency and shape function. Those particular frequencies are often called the modal frequencies or natural frequencies. Likewise, the shape function at the solution is often called the mode shape. Since these solutions do not require a forcing function, they are considered free or natural vibration. [21, 22]

In some cases, external energy is added to a system in a cyclic manner. This is called forced vibration, and it changes the problem of energy minimization significantly. Instead of

minimizing the energy of the system alone, the energy of the system and the external forcing energy must be minimized. This changes the solution of the forced system so that vibration occurs at frequencies other than the natural frequency. However, one can note that the input energy from the forcing function is doing the work the system requires to vibrate at these frequencies. This is because the structure is out of phase with itself, so the forcing energy is mostly dissipated rather than stored.

Interestingly, it is possible to force the exchange of energy at the system's modal frequency. In this case the input energy from the forcing function adds to the total energy of the system and grows unbounded. This is because the energy is in-phase with the system, and the additional energy is not being spent on doing work to force the system to vibrate. When this happens in physical systems, two cases occur. Either, the system will oscillate so much that the input energy is spent on damping, or the structure will fail.

Every object has a different response to forcing functions based on material properties, geometry, and boundary conditions. It is convenient to describe the vibrational response by a few main parameters: amplitude, mode shape, and frequency.

Amplitude describes the amount of maximum displacement in a structure. Amplitude can also be described in terms of gain, which is the ratio of dynamic amplitude to the static amplitude. That is, it describes the relative deformation if loads are considered to be dynamic rather than static. A gain that is greater than one means that a cyclic load will result in more deformation than a static load. A gain that is less than one means that a cyclic load results in less deformation than a static load. The gain of a response will have several local maximums over the frequency spectrum. These maximums are the modal or natural frequencies. As before, the term natural frequency refers to the idea that a displaced system will naturally vibrate at these frequencies.

The mode shape is the deflection pattern that corresponds to a particular vibration solution. Mode shape is the displacement that the physical object experiences as a function of initial structural position. Mode shape is the general solution to the governing equations when

they are written in terms of displacement. It minimizes total energy while still satisfying the boundary and initial conditions. The mode shape depends on the boundary conditions of the structure, and it relates to the structure's eigenvalues because the boundary conditions are used to find the eigenvalues. The base shape solution can be scaled in order to meet the initial conditions. Essentially, the plate deflects proportionally to an applied force in order to store the energy that is equivalent to the applied force.

Modes of plates can present themselves in a couple different displacement functions including bending and torsion. Modes are numbered in order from lowest to highest frequency and labeled by their displacement pattern. For example, if a bending mode occurred at the lowest natural frequency, then it would be referred to as the first bending mode.

Frequency is the number of oscillations per time period. It directly relates to the system's eigenvalues, and this relationship is further developed in Chapter III, section A. The vibrational response is characterized by a certain amount of gain for each frequency. The frequency response function (FRF) maps the relationship between a particular frequency and its forced response amplitude. They are useful for pinpointing modal frequencies as well as for determining the system's behavior at other frequencies.

B. Damping

One major aspect of vibrations problems is whether the system is modeled with or without damping. Damping is a process through which a structure dissipates dynamic energy to its surrounding environment. It represents kinetic energy losses due to inefficiencies in the system, and can occur in both forced and free vibrational systems. Two main sources of this energy dissipation are internal friction and air resistance. As might be suspected, at low frequencies, internal friction is the main driver of damping, while air resistance typically becomes significant at high frequencies. Damping is present in many physical systems and affects the system response with varying degrees of significance.

There are two types of damping: viscous damping and rate-independent damping. Viscous damping is dynamic energy dissipation that occurs proportionally to the instantaneous velocity of the structure. This type of damping is idealized from practical situations, but it easily modeled. When viscous damping is modeled as the mean of the total structural damping, it is called “Equivalent Viscous Damping.” [21] The other type of damping, rate independent damping, is damping with constant energy dissipation that is not dependent on frequency. If this is internal rate independent damping, it is caused by static hysteresis due to localized plastic deformation and plastic flow that occurs even when a material is apparently in the elastic limit. [16]

This study deals with viscous damping, which can be represented in two ways. It can be represented as a coefficient of velocity. It can also be non-dimensionalized by frequency, which turns it into a dimensionless parameter called damping ratio. The damping ratio is related to the decay rate of the amplitude of oscillations. In experimentation, the damping ratio can be calculated using the half-power bandwidth method. The half-power bandwidth method finds the damping ratio by comparing the amplitude of the structure at the natural frequency with the amplitude of the structure at -3dB of the maximum amplitude. Since the damping ratio is related to the decay rate of the system it is proportional to the ratio between energy dissipated per cycle and the total energy of the cycle if it were undamped.

Structures possess material specific damping properties. This has to do with the atomic structure of the material as well as any hysteresis of the stress-strain relationship. In some materials, such as steel and other metals, there is only a slight amount of damping. For aluminum 2024, a typical damping ratio is about 0.01. [14] In other materials, such as viscoelastic material, the damping is more significant because there is a delay between the strain response and a given stress. An increase due to the delay or hysteresis of the stress-strain relationship is the basis of passive damping. Since the delay time is small, passive damping works especially well at high frequencies.

Damping can be either a desired or an undesired effect in practical applications. Damping will be minimized when the application requires a certain amount of energy to be retained (undesired damping). For example, a reaction wheel design for a satellite would need minimal damping because the goal of the design is to preserve the amount of energy in the system. Damping is maximized when the application requires energy dissipation. For example, vibration suppression on the fuselage of an aircraft could reduce fatigue of the structure as well as noise in the cabin.

Vibration suppression is a field of active research, and it is particularly applicable to the Aerospace industry since vibrations can pose many problems to aircraft and spacecraft. There are many sources of vibrations in Aerospace applications including the engines, air turbulence, and craft dynamics. All of these sources can produce unwanted noise, lead to fatigue, and pose a risk of structural failure.

C. Vibration Suppression Techniques

In general, vibration suppression aims to reduce the system gain at and around one or more of the modal frequencies. This is done by dissipating energy at the desired frequency.

Vibration suppression can be achieved using two methods: passive damping and active damping. The passive damping method works by adding a viscoelastic material that is designed to dissipate energy in order to reduce vibration response. Passive vibration suppression is achieved using passive constrained layer damping treatments (PCLD). In this method, viscoelastic material is added between lamina in a composite layup in order to augment energy dissipation.

The active damping method works by adding piezoelectric actuators that can produce out of phase deformation in the piezoelectric actuator to affect the deflection of a structure. Active vibration control is achieved by using out of phase deflections for piezoelectric actuators. Control

is typically achieved using feedback from a sensor to the piezoelectric actuator. Typical sensors are strain gages or a separate piece of piezoceramic used in the sensor configuration.

Piezoelectric materials are those that exhibit a relationship between their electrical state and mechanical strain. The unique properties of piezoelectric materials require different parameters to describe them. One such parameter is the electromechanical coupling factor, which describes the relation of stored mechanical energy to total stored energy with a piezoelectric material. Other parameters include the piezoelectric constants, which relate strain per electric field at constant stress, electric field per stress at constant stress, electric field at constant strain, stress at constant strain, and electric field per strain at constant stress. The most commonly used piezoelectric materials are piezoelectric ceramics. A very strong piezoelectric effect is present in lead zirconate titanate, or PZT, which is a ceramic that is composed of lead, oxygen, and either titanium or zirconium. PZT comes in several different compositions, including PZT-2, PZT-4, PZT-5A, PZT-5H, and PZT-8. [18]

In terms of vibration suppression, there are several ways the piezoceramic layer can be applied to a composite; these include active surface, hybrid constrained, and hybrid composite damping treatments. [2]

The active surface damping treatment (ASDT) is the most basic form of active damping. It consists of a piezoelectric layer bonded to the surface of a structure. An electric field is applied across the poling direction to actuate the piezoelectric material. [2,8]

Hybrid constrained layer damping treatments (HCLDT) have a piezoelectric ceramic layer that constrains a viscoelastic layer; this hybrid method provides augmented energy dissipation since the piezoelectric layer enhances the shear and compression deformation characteristics of the viscoelastic layer. [4,15]

Hybrid composite damping treatments (HCDT) use a treatment that has piezoelectric rods embedded perpendicularly in a viscoelastic polymeric matrix, as shown in Figure 1. The term hybrid composite refers to the fact that the two materials are blended to make a single composite

layer. This treatment is bonded directly to the structure surface and voltage is applied across the piezoelectric rods. HCDT provides increased energy dissipation since the piezoelectric rods resist shear and compression deformation. [2]

In design, the engineering application dictates which frequencies responses are most critical and therefore which method is most viable. For example, passive damping typically works best on higher frequencies due to the smaller amplitude and greater number of oscillations. In contrast, active damping typically works best on lower frequencies because a feedback system is required and more energy can be dissipated with a larger amount of deflection. [13] One limitation on active control is that the controller must be fast enough that its time delay does not affect the control. If digital control is desired, computation speeds must be fast enough to accomplish this. Otherwise, the circuit in analog control must be fast enough to accomplish this.

Figure 1. Types of damping treatments.

D. Current Solution Methods

The energy exchange between elastic potential energy and kinetic energy is predictable in such a way that both the mode shape and amplitude can be determined for each frequency of vibration given the geometric information, material properties, initial conditions and boundary

conditions. The goal of most methods is to accurately predict the natural frequencies and mode shapes of a physical system. Predictions can be made from analytical models, numerical simulations, or experimental data.

Analytical models solve Hamilton's principle or Lagrange's equations of motion [14,21,22]. The solution produced is an exact mathematical expression, although computers are often used to quickly analyze it at specific dimensions and material properties. The Ritz assumed mode method can be used to simplify a complex problem in order to obtain an analytical expression. [17]

Some examples of numerical models that have been developed are finite difference or finite element methods. These are iterative solutions that can take significant computation time to converge if they are very complex. It is not as easy to parametrically vary the model inputs to map a design space in a numerical model since each solution takes a significant amount of time to compute. [10]

Experimental methods are reliable in that real results and data trends can be found. This is possible even without knowing material properties or dimensions in great detail. The main issue is that the object itself must be available, which is not helpful if the prediction will be used for design work. Still, experimental methods work well as a validation for analytical and numerical prediction models. [2,4,5,19,24]

E. Previous Work

Although there has been no significant work done at California Polytechnic State University on any analytical models for predicting active damping, there are several theories published for vibrations of composites and piezoelectric material. Both Kim and Reddy have published the Euler theory for an analytical approach for finding the natural frequencies of plates. [14,22] These publications are the basis of the analytical solution presented in the following Chapters of this study. The performance of piezoelectric damping treatments has been predicted

in analytical models. [8,9] General Motors also published a report that presented the Ritz assumed mode method as applied to piezoelectric material. [17] This report was used in combination with the Euler theory to approximate the piezoelectric effects. Chih-Kung Lee presented an exact partial differential equation in his dissertation. [18] This equation is replicated in Appendix A. The main issue with this formulation of the problem is that there is no solution to the PDE. It is conceivable that a numerical solver could find solutions for particular cases, but this form yields no analytical solution. A more recent study created an exact mathematical model of an aluminum beam with piezoelectric actuators. [8,9] All of these studies have failed to capture the objective of this study, which is to build an analytical model for modal analysis of a composite structure with active damping material.

Some work has also been completed in finite element analysis for the analysis of structures with active. Previous FEA work has been completed for the application of a fixed-free vibrating aluminum. Singh and Kumar, who created the FEA model of a 3-layer aluminum ASD beam with electromechanical coupling, developed one such model. [15]. Singh and Kumar also built a finite element model that is appropriate for hybrid damping designs with varying active and passive constrained layer damping patch locations. [15] Piezoelectric performance has been predicted in many finite element models. [7,13,15] Kang et al. created an FEA model of composite beams with piezoelectric damping treatment that included an analysis of the adhesive used to apply the PZT material. [13]

A significant amount of experimental work has been completed in order to validate the proposed analytical and numerical models. In order to validate their FEA work, Singh and Kumar also conducted experimental research on variations in placement and coverage of an ASD treatment on beams with velocity feedback control. [13] Song and Li studied types of controllers for composite plate applications, and they found that proportional feedback control is more effective for composite plate applications. [23]

Authors Edery-Azulay and Abramovich studied varied combinations of shear and extension type piezoelectric treatments for use of controllers and actuators. They found that extension-extension or shear-shear type controller-actuator pairs are most effective, and the choice depends mainly on the boundary conditions of the application. [8,9]

Experimental work that has been completed at California Polytechnic State University has included work done by Hrovat, Lesniak, Cavalli, and Bach, among others. Each of these projects has built on the progress of the others, and some of them will be used as validation cases in Chapter V.

The first project completed on active damping at California Polytechnic State University was a Senior Project done by Hrovat. [11] He considered an aluminum beam with a first bending mode designed to be around 5 Hz. The beam had piezoelectric actuator and sensor patches located near its base. Hrovat implemented active control and found a large experimental increase in damping ratio.

Lesniak continued Hrovat's project by improving the controller design. [19] He considered the same beam, but he implemented an analog controller with proportional, proportional-derivative, and PID control, each with varying gains. He implemented control by modeling the beam plant as a two-frequency system. He found that at high gains his controller excited higher frequencies. This can most likely be attributed to the low fidelity of his plant model.

Cavalli did more active control research in response to a grant funded by NASA. He studied an actively damped electronic equipment shelf, which could be modeled as a fixed-fixed plate with full active damping material coverage. For his thesis, Cavalli also studied active control on an aluminum sailplane. This sailplane can be modeled as a fixed-free plate with partial piezoelectric coverage. [5]

An additional application of active control is the work completed by James Bach and other students on a model airplane with a non-rigid tail boom. This work can be modeled as a fixed-free plate with a mass on one end and partial active control in the middle. [4]

While all of these projects have made significant experimental progress, none aimed to come up with a full analytical or numerical solution for modal analysis.

Chapter III. BENDING MODAL ANALYSIS THEORY

Modal analysis can be classified into three types: analytical, numerical, and experimental solutions. Mathematical models have been developed for analytical and numerical solutions. Analytical solutions can be divided into two types: exact and approximate. Exact solutions can be found for many problems using simplifying assumptions. Approximate solutions typically use fewer assumptions. The difference between exact analytical solutions and approximate analytical solutions is that exact solutions are finite and do not neglect higher order terms. Approximate analytical solutions are typically series, so they are infinite. However, by neglecting higher order terms, a finite solution that is very close to the true solution can be calculated. Numerical solutions can be found for more complex problems, however they can be plagued by problems of convergence. Examples of numerical solutions include finite difference and finite element analysis.

This section presents the analytical modal analysis theory for laminate plates in bending with orthotropic layers. It also presents the analytical theory for finding the deflection of a piezoelectric layer.

A. Analytical Solution for Laminate Plates in Bending

An analytical approach can be readily used for modal analysis of composite structures. Reddy [22] outlines the theory for calculating natural frequencies of composite plates in bending. This theory uses the Euler-Bernoulli formulation of the equations of motion to analyze a single plate with X-Z symmetry. The formulation can be applied to orthotropic beams or plates, such as steel or aluminum, or to composite beams or plates with orthotropic layers.

Important assumptions that were used in this theory are that the laminas of the plate are perfectly bonded to each other and that the plate has perfect symmetry. Additionally, this method only applies to a vibrational response that is primarily bending, which occurs when a structure is much longer than it is wide.

The following relations give the equations of motion in terms of displacements, as in Reddy. [22]

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2} \left(\frac{\partial w_0}{\partial y} \right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y} \end{Bmatrix} - \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2 \frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \quad (1)$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2} \left(\frac{\partial w_0}{\partial y} \right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y} \end{Bmatrix} - \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2 \frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \quad (2)$$

Where the A matrix is extensional stiffness, which is

$$A_{ij} = \sum_{k=1}^N \bar{Q}_{ij}^{(k)} (z_{k+1} - z_k) \quad (3)$$

The B matrix is the bending-extensional coupling stiffness matrix, which is

$$B_{ij} = \frac{1}{2} \sum_{k=1}^N \bar{Q}_{ij}^{(k)} (z_{k+1}^2 - z_k^2) \quad (4)$$

And the D matrix is the bending stiffness matrix, which is

$$D_{ij} = \frac{1}{3} \sum_{k=1}^N \bar{Q}_{ij}^{(k)} (z_{k+1}^3 - z_k^3) \quad (5)$$

Each of the above matrices is a function of the transformed lamina stiffnesses, $\bar{Q}_{ij}^{(k)}$ and the vertical lamina coordinates. The transformed lamina stiffness matrix can be found from the components of the lamina stiffness matrix, $Q_{ij}^{(k)}$, which is a function of material properties.

Assuming an orthotropic lamina, these components are

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}} \quad (6)$$

$$Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} \quad (7)$$

$$Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \quad (8)$$

$$Q_{66} = G_{12} \quad (9)$$

The components of the transformed lamina stiffness matrix are

$$\begin{aligned} \bar{Q}_{11} &= Q_{11} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \sin^4 \theta \\ \bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{12} (\sin^4 \theta + \cos^4 \theta) \\ \bar{Q}_{22} &= Q_{11} \sin^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \cos^4 \theta \\ \bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin \theta \cos^3 \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin^3 \theta \cos \theta \\ \bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin^3 \theta \cos \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin \theta \cos^3 \theta \\ \bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{66} (\sin^4 \theta + \cos^4 \theta) \end{aligned} \quad (10)$$

Where θ is the lamination rotation angle, with respect to the longitudinal axis. The vertical lamina coordinates are the distance of the lamina edges to the mid-plane. Figure 2 shows this concept.

Figure 2. Lamination scheme numbering system.

This analytical solution formulation assumes symmetrically laminated plates, which allows the bending displacements and stretching displacements to be uncoupled from each other. This symmetry results in the B matrix equaling zero. Additionally, the theory is applied to cases where the in-plane forces are zero, which means there is zero in plane deflection; that is $u_0 = v_0 = 0$. Therefore, the problem is reduced to solving for vertical deflection. Using the above assumptions, equation (2) can be expressed as follows.

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = - \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2 \frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \quad (11)$$

Inverting the D matrix and solving for the differential transverse deflection tensor, we get

$$\begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2 \frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} = - \begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ D_{12}^* & D_{22}^* & D_{26}^* \\ D_{16}^* & D_{26}^* & D_{66}^* \end{bmatrix} \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} \quad (12)$$

where the elements $D_{i,j}^*$ denote the elements of the inverse of $D_{i,j}$. In deriving the laminated plate theory, we assume that

$$M_{yy} = 0 \quad (13)$$

$$M_{xy} = 0 \quad (14)$$

This allows equation (11) to be simplified to

$$\frac{\partial^2 w_0}{\partial x^2} = -D_{11}^* M_{xx} \quad (15)$$

$$\frac{\partial^2 w_0}{\partial y^2} = -D_{12}^* M_{xx} \quad (16)$$

$$2 \frac{\partial^2 w_0}{\partial x \partial y} = -D_{16}^* M_{xx} \quad (17)$$

Equations (16) and (17) show that even when there is zero in-plane deflection, the transverse deflection will be dependent on the y coordinate due to the Poisson effect (D_{12}^*) and anisotropic shear coupling (D_{16}^*). These effects can be neglected only with high length to width ratios. The ratio at which the Poisson effect and anisotropic shear coupling can be neglected

depends on material properties and lamination scheme. When these effects are neglected, the transverse deflection is a function only of the x coordinate and time. That is:

$$w_0 = w_0(x, t) \quad (18)$$

This makes the partial derivatives with respect to y go to zero, leaving us with just:

$$\frac{\partial^2 w_0}{\partial x^2} = -D_{11}^* M_{xx} \quad (19)$$

In order to find the exact solution for natural vibration, it is necessary to solve the Euler-Lagrange equation of motion in terms of displacement. This is possible by combining the above equation with the Euler Lagrange equation of motion for out-of-plane deflection, which is equal to

$$\begin{aligned} \frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial y \partial x} + \frac{\partial^2 M_{yy}}{\partial y^2} + \mathcal{N}(w_0) + q \\ = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} \right) + I_1 \frac{\partial^2}{\partial t^2} \left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} \right) \end{aligned} \quad (20)$$

where $\mathcal{N}(w_0)$ is the force in the plate as a function of out-of-plane displacement, q is a distributed transverse force on the laminate, and I_0, I_1 , and I_2 are the rotary inertias defined as follows.

$$\begin{Bmatrix} I_0 \\ I_1 \\ I_2 \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} \rho_0 dz \quad (21)$$

By assuming no torsional modes, it is possible to eliminate all terms with y derivatives.

This leaves just

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \hat{\mathcal{N}}_{xx} \frac{\partial^2 w_0}{\partial x^2} + q = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^4 w_0}{\partial x^2 \partial t^2} + I_1 \frac{\partial^3 u_0}{\partial x \partial t^2} \quad (22)$$

Assuming a symmetric laminate, I_1 disappears from the expression.

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \hat{\mathcal{N}}_{xx} \frac{\partial^2 w_0}{\partial x^2} + q = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^4 w_0}{\partial x^2 \partial t^2} \quad (23)$$

It is desirable to express this equation in a form that is independent of width. This is possible by multiplying the above equation by plate width.

$$\frac{\partial^2 b M_{xx}}{\partial x^2} + b \hat{N}_{xx} \frac{\partial^2 w_0}{\partial x^2} + b q = b I_0 \frac{\partial^2 w_0}{\partial t^2} - b I_2 \frac{\partial^4 w_0}{\partial x^2 \partial t^2} \quad (24)$$

Non-dimensional parameters can be introduced as follows.

$$\hat{I}_0 = b I_0 \quad (25)$$

$$\hat{I}_2 = b I_2 \quad (26)$$

$$\hat{q} = b q \quad (27)$$

Additionally, from equation (19), the following substitution can be made.

$$M_{xx} = \frac{-1}{D_{11}^*} \frac{\partial^2 w_0}{\partial x^2} \quad (28)$$

This leaves a final partial differential equation as follows.

$$\frac{-b}{D_{11}^*} \frac{\partial^4 w_0}{\partial x^4} + b \hat{N}_{xx} \frac{\partial^2 w_0}{\partial x^2} + \hat{q} = \hat{I}_0 \frac{\partial^2 w_0}{\partial t^2} - \hat{I}_2 \frac{\partial^4 w_0}{\partial x^2 \partial t^2} \quad (29)$$

The previous equation can also be written in the following form, which is frequently used in literature.

$$-E_{xx}^b I_{yy} \frac{\partial^4 w_0}{\partial x^4} + b \hat{N}_{xx} \frac{\partial^2 w_0}{\partial x^2} + \hat{q} = \hat{I}_0 \frac{\partial^2 w_0}{\partial t^2} - \hat{I}_2 \frac{\partial^4 w_0}{\partial x^2 \partial t^2} \quad (30)$$

For natural vibration, there is assumed to be no distributed transverse force. This simplifies the partial differential equation to:

$$-E_{xx}^b I_{yy} \frac{\partial^4 w_0}{\partial x^4} + b \hat{N}_{xx} \frac{\partial^2 w_0}{\partial x^2} = \hat{I}_0 \frac{\partial^2 w_0}{\partial t^2} - \hat{I}_2 \frac{\partial^4 w_0}{\partial x^2 \partial t^2} \quad (31)$$

In order to solve this partial differential equation for natural vibration, the solution is assumed to be periodic.

$$w_0(x, t) = W(x) e^{i\omega t} \quad (32)$$

In this solution $W(x)$ is the maximum deflection as a function of plate position, ω is the natural frequency of the plate, t is time, and i is the imaginary number.

Taking partial derivatives of this assumed solution with respect to x and t leads to the following relations.

$$\frac{\partial^2 w_0}{\partial x^2} = \frac{d^2 W}{dx^2} e^{i\omega t} \quad (33)$$

$$\frac{\partial^4 w_0}{\partial x^4} = \frac{d^4 W}{dx^4} e^{i\omega t} \quad (34)$$

$$\frac{\partial^2 w_0}{\partial t^2} = -\omega^2 W e^{i\omega t} \quad (35)$$

$$\frac{\partial^4 w_0}{\partial x^2 \partial t^2} = -\omega^2 \frac{d^2 W}{dx^2} e^{i\omega t} \quad (36)$$

Substituting equations (33), (34), (35) and (36) into equation (31) gives

$$-E_{xx}^b I_{yy} \frac{d^4 W}{dx^4} e^{i\omega t} + b \hat{N}_{xx} \frac{d^2 W}{dx^2} e^{i\omega t} = -\hat{I}_0 \omega^2 W e^{i\omega t} + \hat{I}_2 \omega^2 \frac{d^2 W}{dx^2} e^{i\omega t} \quad (37)$$

Factoring out a $e^{i\omega t}$ term from both sides of the equation simplifies the expression to

$$-E_{xx}^b I_{yy} \frac{d^4 W}{dx^4} + b \hat{N}_{xx} \frac{d^2 W}{dx^2} = -\hat{I}_0 \omega^2 W + \hat{I}_2 \omega^2 \frac{d^2 W}{dx^2} \quad (38)$$

Combining terms leaves

$$E_{xx}^b I_{yy} \frac{d^4 W}{dx^4} + (\hat{I}_2 \omega^2 - b \hat{N}_{xx}) \frac{d^2 W}{dx^2} - \hat{I}_0 \omega^2 W = 0 \quad (39)$$

This partial differential equation is in the general form

$$p \frac{d^4 W}{dx^4} + q \frac{d^2 W}{dx^2} - r W = 0 \quad (40)$$

where

$$p = E_{xx}^b I_{yy} \quad (41)$$

$$q = \hat{I}_2 \omega^2 - b \hat{N}_{xx} \quad (42)$$

$$r = \hat{I}_0 \omega^2 \quad (43)$$

This form has a known general solution as follows.

$$W(x) = c_1 \sin(\lambda x) + c_2 \cos(\lambda x) + c_3 \sinh(\mu x) + c_4 \cosh(\mu x) \quad (44)$$

where

$$\lambda = \sqrt{\frac{1}{2p} (q + \sqrt{q^2 + 4pr})} \quad (45)$$

$$\mu = \sqrt{\frac{1}{2p}(-q + \sqrt{q^2 + 4pr})} \quad (46)$$

Since q is negligible compared to $\sqrt{q^2 + 4pr}$, we can write

$$\mu \cong \lambda \quad (47)$$

The constants in equation (44) depend on boundary conditions, which are summarized below in equations (48) through (53).

Hinged-Hinged

$$c_1 \neq 0$$

$$c_2 = c_3 = c_4 = 0 \quad (48)$$

Fixed-Fixed

$$c_1 = -c_3 = \frac{1}{\sin(\lambda a) - \sinh(\lambda a)}$$

$$-c_2 = c_4 = \frac{1}{\cos(\lambda a) - \cosh(\lambda a)} \quad (49)$$

Fixed-Free

$$c_1 = -c_3 = \frac{1}{\sin(\lambda a) + \sinh(\lambda a)}$$

$$-c_2 = c_4 = \frac{1}{\cos(\lambda a) + \cosh(\lambda a)} \quad (50)$$

Free-Free

$$c_1 = c_3 = \frac{1}{\sin(\lambda a) - \sinh(\lambda a)}$$

$$c_2 = c_4 = \frac{-1}{\cos(\lambda a) - \cosh(\lambda a)} \quad (51)$$

Hinged-Fixed

$$c_1 = \frac{1}{\sin(\lambda a)}$$

$$c_3 = \frac{1}{\sinh(\lambda a)}$$

$$c_2 = c_4 = 0 \quad (52)$$

Hinged-Free

$$c_1 = \frac{1}{\sin(\lambda a)}$$

$$c_3 = \frac{-1}{\sinh(\lambda a)}$$

$$c_2 = c_4 = 0 \quad (53)$$

All of these constants are dependent on λa . The value of λa which correlates to the natural frequency of the structure will satisfy the characteristic equation for its particular boundary condition. The characteristic equations and their eigenvalues are presented below in equations (54) through (59).

$$\text{Hinged-Hinged} \quad \sin(\lambda a) = 0 \quad \text{where} \quad \lambda = \frac{n\pi}{a} \quad (54)$$

$$\text{Fixed-Fixed} \quad \cos(\lambda a)\cosh(\lambda a) - 1 = 0 \quad \text{where} \quad \lambda = \frac{4.730}{a}, \frac{7.853}{a}, \dots \quad (55)$$

$$\text{Fixed-Free} \quad \cos(\lambda a)\cosh(\lambda a) + 1 = 0 \quad \text{where} \quad \lambda = \frac{1.875}{a}, \frac{4.694}{a}, \dots \quad (56)$$

$$\text{Free-Free} \quad \cos(\lambda a)\cosh(\lambda a) - 1 = 0 \quad \text{where} \quad \lambda = \frac{4.730}{a}, \frac{7.853}{a}, \dots \quad (57)$$

$$\text{Hinged-Fixed} \quad \tan(\lambda a) - \tanh(\lambda a) = 0 \quad \text{where} \quad \lambda = \frac{3.927}{a}, \frac{7.069}{a}, \dots \quad (58)$$

$$\text{Hinged-Free} \quad \tan(\lambda a) - \tanh(\lambda a) = 0 \quad \text{where} \quad \lambda = \frac{3.927}{a}, \frac{7.069}{a}, \dots \quad (59)$$

The natural frequency can be solved directly by combining these eigenvalues from equations (54) through (59) with equation (45). From (45), we have

$$p\lambda^4 - q\lambda^2 - r = 0 \quad (60)$$

Substituting the values for p, q, and r from equation (41), (42), and (43) and solving for ω gives

$$\omega = \sqrt{\lambda^4 \left(\frac{E_{xx}^b I_{yy}}{\hat{I}_0} \right) \left(\frac{1}{1 + \left(\frac{\hat{I}_2}{\hat{I}_0} \lambda^2 \right)} \right)} \quad (61)$$

Again, equation (61) can be used to solve for natural frequencies when the eigenvalues that are appropriate to the boundary conditions in (54) through (59) are used.

Now that an expression for the natural frequencies has been developed, it is desirable to develop a general solution for vibration at any frequency. The general solution is developed based on the same partial differential equation, equation (30), as the natural vibration solution. This equation is republished below.

$$-E_{xx}^b I_{yy} \frac{\partial^4 w_0}{\partial x^4} + b \hat{N}_{xx} \frac{\partial^2 w_0}{\partial x^2} + \hat{q} = \hat{I}_0 \frac{\partial^2 w_0}{\partial t^2} - \hat{I}_2 \frac{\partial^4 w_0}{\partial x^2 \partial t^2} \quad (62)$$

Rearranging to get the transverse force, $\hat{q}$ on the right hand side by itself

$$E_{xx}^b I_{yy} \frac{\partial^4 w_0}{\partial x^4} - \hat{I}_2 \frac{\partial^4 w_0}{\partial x^2 \partial t^2} - b \hat{N}_{xx} \frac{\partial^2 w_0}{\partial x^2} + \hat{I}_0 \frac{\partial^2 w_0}{\partial t^2} = \hat{q} \quad (63)$$

Assuming that the solution and the forcing function are periodic gives

$$w_0(x, t) = W(x) e^{i\omega t} \quad (64)$$

$$\hat{q} = Q e^{i\omega t} \quad (65)$$

These solution forms allow the partial derivatives to be written as total derivatives of position as follows

$$\frac{\partial^2 w_0}{\partial x^2} = \frac{d^2 W}{dx^2} e^{i\omega t} \quad (66)$$

$$\frac{\partial^4 w_0}{\partial x^4} = \frac{d^4 W}{dx^4} e^{i\omega t} \quad (67)$$

$$\frac{\partial^2 w_0}{\partial t^2} = -\omega^2 W e^{i\omega t} \quad (68)$$

$$\frac{\partial^4 w_0}{\partial x^2 \partial t^2} = -\omega^2 \frac{d^2 W}{dx^2} e^{i\omega t} \quad (69)$$

By substituting (66) through (69) into (63), and factoring out a $e^{i\omega t}$ the following equation is left

$$E_{xx}^b I_{yy} \frac{d^4 W}{dx^4} + \hat{I}_2 \omega^2 \frac{d^2 W}{dx^2} - b \hat{N}_{xx} \frac{d^2 W}{dx^2} - \hat{I}_0 \omega^2 W = Q \quad (70)$$

This equation takes the form

$$p \frac{d^4 W}{dx^4} + q \frac{d^2 W}{dx^2} - r W = Q \quad (71)$$

where

$$p = E_{xx}^b I_{yy} \quad (72)$$

$$q = \hat{I}_2 \omega^2 - b \hat{N}_{xx} \quad (73)$$

$$r = \hat{I}_0 \omega^2 \quad (74)$$

The following solution is assumed

$$W(x) = c_1 \sin(\lambda x) + c_2 \cos(\lambda x) + c_3 \sinh(\mu x) + c_4 \cosh(\mu x) \quad (75)$$

which has the following second and fourth spatial derivatives

$$\frac{d^2 W(x)}{dx^2} = -c_1 \lambda^2 \sin(\lambda x) - c_2 \lambda^2 \cos(\lambda x) + c_3 \mu^2 \sinh(\mu x) + c_4 \mu^2 \cosh(\mu x) \quad (76)$$

$$\frac{d^4 W(x)}{dx^4} = c_1 \lambda^4 \sin(\lambda x) + c_2 \lambda^4 \cos(\lambda x) + c_3 \mu^4 \sinh(\mu x) + c_4 \mu^4 \cosh(\mu x) \quad (77)$$

Recall that the constants c_1 , c_2 , c_3 and c_4 are determined by the boundary conditions of the system, and are dependent on λa . The relations are given in equations (48) through (53). By substituting equations (75), (76) and (77) into equation (71), a single algebraic equation in terms of two variables is obtained.

$$\begin{aligned} & p(c_1 \lambda^4 \sin(\lambda x) + c_2 \lambda^4 \cos(\lambda x) + c_3 \mu^4 \sinh(\mu x) + c_4 \mu^4 \cosh(\mu x)) \\ & + q(-c_1 \lambda^2 \sin(\lambda x) - c_2 \lambda^2 \cos(\lambda x) + c_3 \mu^2 \sinh(\mu x) + c_4 \mu^2 \cosh(\mu x)) \\ & - r(c_1 \sin(\lambda x) + c_2 \cos(\lambda x) + c_3 \sinh(\mu x) + c_4 \cosh(\mu x)) = Q \end{aligned} \quad (78)$$

From equation (47)

$$\mu \cong \lambda \quad (79)$$

So equation (78) can be rewritten in terms of λ

$$\begin{aligned} & p(c_1 \lambda^4 \sin(\lambda x) + c_2 \lambda^4 \cos(\lambda x) + c_3 \lambda^4 \sinh(\lambda x) + c_4 \lambda^4 \cosh(\lambda x)) \\ & + q(-c_1 \lambda^2 \sin(\lambda x) - c_2 \lambda^2 \cos(\lambda x) + c_3 \lambda^2 \sinh(\lambda x) + c_4 \lambda^2 \cosh(\lambda x)) \\ & - r(c_1 \sin(\lambda x) + c_2 \cos(\lambda x) + c_3 \sinh(\lambda x) + c_4 \cosh(\lambda x)) = Q \end{aligned} \quad (80)$$

Once the values of the constants are substituted into equation (80), it can be numerically solved for λ for any given frequency and forcing amplitude. Once λ is found, the mode shape and amplitude can easily be found.

B. Analytical Solution for Piezoelectric Material

The Lagrangian equations of motion can be developed to include piezoelectric effects, and there are two methods to solve these equations. The first method is to solve the partial differential equation directly. Chih-Kung Lee developed this partial differential equation in his dissertation. [18] However, the exact equation is such a complicated partial differential equation that there may or may not be a solution. The literature review has shown no closed form general solution to this approach of the problem.

The second method to analytically include piezoelectric effects of one layer of a composite structure is by using the Ritz assumed mode method to approximate the exact solution. This method used the assumed mode shape to simplify the problem significantly enough that the exact solution can be achieved.

The Ritz assumed mode method builds on the plate strain energy equations and applies the Lagrangian equations of motion. The plate strain energy equation is

$$U = \frac{1}{2} \iint_A \{\varepsilon^{0T} \quad \kappa^T\} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} dA - \iint_A [N_\Lambda \quad M_\Lambda] \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} dA \quad (81)$$

where A, B, and D are the extensional stiffness, bending-extensional coupling, and bending stiffness matrices from before and dA is the derivative with respect to area. The terms N_Λ and M_Λ are the applied forces and the moments caused by the piezoelectric effect. The strains from equation (81) can be expressed as

$$\varepsilon^0 = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} \quad (82)$$

and the curvatures can be expressed as

$$\kappa = \begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \quad (83)$$

For this analysis, it will be beneficial to separate the derivative operators from the displacements.

This leaves us with a tensor for the strains and curvatures which is

$$\begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} = \mathcal{D} \mathbf{u} \quad (84)$$

where $\mathcal{D}$ is the derivative operator matrix

$$\mathcal{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & -\frac{\partial^2}{\partial x^2} \\ 0 & 0 & -\frac{\partial^2}{\partial y^2} \\ 0 & 0 & -2\frac{\partial^2}{\partial x \partial y} \end{bmatrix} \quad (85)$$

and $\mathbf{u}$ is the displacement vector

$$\mathbf{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \quad (86)$$

The components of the displacement vector will change based on the assumed modes (which change based on geometry), boundary conditions, and system forcing. The equation for each displacement vector can be represented as the equation for each mode shape scaled by the generalized coordinate vector, which maps the amplitude of the mode shape to the actual amplitude. That is,

$$u = \sum_{i=1}^m \phi_{u_i} q_i \quad (87)$$

$$v = \sum_{j=1}^n \phi_{v_j} q_{m+j} \quad (88)$$

$$w = \sum_{k=1}^p \phi_{w_k} q_{m+n+k} \quad (89)$$

where ϕ_u , ϕ_v , and ϕ_w are the longitudinal in-plane, transverse in-plane and out-of-plane assumed mode displacements and q is the generalized coordinate vector. The displacement vector can be simplified to

$$\{u\} = H q \quad (90)$$

where H is the assumed mode shape matrix and q is the generalized coordinate vector.

In this expression, q is generalized coordinate vector, which maps the non-dimensional assumed mode shape to a particular amplitude. The matrix H is the assumed mode shape matrix; it has different components depending on the function assumed for the modes and the number of assumed modes.

$$H = \begin{bmatrix} [\phi_{u_1} & \phi_{u_2} & \dots & \phi_{u_m}] & [0 & 0 & \dots & 0] & [0 & 0 & \dots & 0] \\ [0 & 0 & \dots & 0] & [\phi_{v_1} & \phi_{v_2} & \dots & \phi_{v_n}] & [0 & 0 & \dots & 0] \\ [0 & 0 & \dots & 0] & [0 & 0 & \dots & 0] & [\phi_{w_1} & \phi_{w_2} & \dots & \phi_{w_p}] \end{bmatrix} \quad (91)$$

The Ritz assumed mode method states that there are 4 in-plane modes and 3 out-of-plane modes. Each of these modes can be represented by an assumed shape function, as follows.

In Plane Modes:

$$\text{Longitudinal Extension} \quad u = x/a \quad (92)$$

$$\text{Longitudinal Shear} \quad u = y/b \quad (93)$$

$$\text{Transverse Extension} \quad v = y/b \quad (94)$$

$$\text{Transverse Shear} \quad v = x/a \quad (95)$$

Out of Plane Modes:

$$\text{Longitudinal Bending} \quad w = (x/a)^2 \quad (96)$$

$$\text{Twist} \quad w = xy/(ab) \quad (97)$$

$$\text{Transverse Bending} \quad w = (y/C)^2 \quad (98)$$

A closed form solution is possible by using a selection of modes to approximate the actual actuation mode. Since this model is focused on longitudinal bending, the assumed mode shape for this model is

$$\phi_{u_1} = 0 \quad (99)$$

$$\phi_{v_1} = 0 \quad (100)$$

$$\phi_{w_1} = (x/a)^2 \quad (101)$$

This yields an assumed mode shape matrix equal to

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (x/a)^2 \end{bmatrix} \quad (102)$$

taking the derivative of the assumed mode shape matrix gives

$$\mathcal{D} \mathbf{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2/a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (103)$$

Combining equations (84) and (90) gives us an approximation for strain and curvature in terms of the assumed modes.

$$\begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} = \mathcal{D} \mathbf{H} \mathbf{q} \quad (104)$$

This expression can be substituted into the strain energy equation (81) to find the modal strain energy equation.

$$U = \frac{1}{2} \iint_A \{\mathcal{D} \mathbf{H} \mathbf{q}\}^T \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \{\mathcal{D} \mathbf{H} \mathbf{q}\} dA - \iint_A [N_\Lambda \quad M_\Lambda] \{\mathcal{D} \mathbf{H} \mathbf{q}\} dA \quad (105)$$

Since the generalized coordinate vector is independent of area, it can be factored out of the integral.

$$U = \frac{1}{2} q^T \left[\iint_A \{\mathcal{D} \ H\}^T \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \{\mathcal{D} \ H\} dA \right] q - \left[\iint_A [N_\Lambda \ M_\Lambda] \{\mathcal{D} \ H\} dA \right] q \quad (106)$$

The order of multiplication for the second term in equation (106) above can be reversed, giving

$$U = \frac{1}{2} q^T \left[\iint_A \{\mathcal{D} \ H\}^T \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \{\mathcal{D} \ H\} dA \right] q - \left[\iint_A \{\mathcal{D} \ H\}^T \begin{bmatrix} N_\Lambda \\ M_\Lambda \end{bmatrix} dA \right] q \quad (107)$$

This equation can be rewritten simply as

$$U = \frac{1}{2} q^T K q - Q_\Lambda q \quad (108)$$

where K is the usual modal stiffness matrix

$$K = \iint_A \{\mathcal{D} \ H\}^T \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \{\mathcal{D} \ H\} dA \quad (109)$$

and Q_Λ is the modal actuation strain forcing matrix

$$Q_\Lambda = \iint_A \{\mathcal{D} \ H\}^T \begin{bmatrix} N_\Lambda \\ M_\Lambda \end{bmatrix} dA \quad (110)$$

The piezoelectric applied force and moment can be expressed in terms modal actuation strains.

$$N_\Lambda = \sum_{k=1}^N \bar{Q}_{ij}^{(k)} \Lambda (z_{k+1} - z_k) \quad (111)$$

$$M_\Lambda = \frac{1}{2} \sum_{k=1}^N \bar{Q}_{ij}^{(k)} \Lambda (z_{k+1}^2 - z_k^2) \quad (112)$$

where Λ is the actuation strain matrix. For a thin, isotropic piezoceramic plate actuator, the actuation strain matrix is

$$\begin{bmatrix} \Lambda_x \\ \Lambda_y \\ \Lambda_{xy} \end{bmatrix} = \begin{bmatrix} 0 & 0 & d_{13} \\ 0 & 0 & d_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ E_z \end{bmatrix} \quad (113)$$

where d is mechanical-electrical coupling coefficients and E is the applied electrical field.

In order to complete this analysis the modified energy relation must be combined with Lagrange's equation, which is

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}}\right) + \frac{\partial U}{\partial q} = Q_k \quad (114)$$

where U is maximum strain energy and Q is the forcing function. Since the maximum strain energy is not a function of the derivative of the generalized coordinate vector, the first term goes to zero, which leaves

$$\frac{\partial U}{\partial q} = Q_k \quad (115)$$

Taking a partial derivative of the strain energy equation (108) with respect to the generalized coordinate vector, gives the quantity

$$\frac{\partial U}{\partial q} = Kq - Q_\Lambda \quad (116)$$

Combining equations (115) and (116) gives the governing equation for the modal amplitudes

$$Kq = Q_\Lambda + Q_k \quad (117)$$

The generalized coordinate vector can be solved for by multiplying both sides of the equation by the modal stiffness matrix.

$$q = K^{-1}(Q_\Lambda + Q_k) \quad (118)$$

The generalized coordinate vector maps the magnitude of the assumed mode to the actual amplitude of the mode. With this the final deflection can be found. The mode shape allows the energy absorbed to be calculated in order to come up with the damping characteristics. A plate that has a piezoelectric actuator layer over its full width but only part of its length can be analyzed by modeling the layer using its actual dimensions and determining how much strain energy the piezoelectric absorbs, which is an equivalent to the amount of strain that the piezoelectric material actually dissipates when the material does not cover the full length of the plate.

Chapter IV. EXPERIMENTAL RESULTS

It was desirable to validate the modal analysis theory presented in the previous chapter against experimental data. To accomplish this, aluminum and composite specimens were manufactured and forced sinusoidal vibration tests were performed. This chapter will present the experimental findings of this study.

A. Aluminum Plates

Three aluminum plates were cut out of Aluminum 2024 to be 10.0", 12.0", and 15.0" long. Each plate was cut to be 4.0" wide and 0.125" thick. Since these plates were designed to be cantilever plates with a clamped length of 0.5", the effective length of the plates are 9.5", 11.5", and 14.5" respectively. These dimensions were chosen based on the material available, to maintain a very small thickness to length ratio to minimize torsional modes, and to have lengths varied enough to validate the trends of the modal analysis method.

B. Composite Plates

Three composite plates were fabricated out of unidirectional pre-impregnated carbon fiber. The manufacturer of the material was the Advanced Composites Group, and the material type was VTM264/HR40-300-36%RW. [1] The curing cycle for this composite is to apply 980mbar vacuum at room temperature, heat to the dwell temperature at a rate not exceeding 2°C, maintain the recommended temperature and pressure for the dwell time, cool the part under vacuum and vent the vacuum. The recommended cure cycle options are 1 hour at 120°C, 2 hours at 100°C, 5 hours at 80°C, or 16 hours at 65°C. The times reflect the minimum dwell time. Similar to the aluminum plates, three plates were manufactured to be 10.0", 12.0", and 15.0" long. Each plate was manufactured to be 4.0" wide. Each plate used 4 plies of material, and they were 0.048", 0.047", and 0.049" thick respectively. These dimensions provide a small thickness to length ratio, which minimized torsional modes. The similar dimensions allow comparison of material properties and the various lengths in order to validate the trends of modal analysis.

Figure 3. Cut carbon fiber sheets just prior to layup.

The material was stored in a freezer prior to fabrication. Before cutting, the material was taken out of the freezer and allowed to thaw for about 2 minutes. Twelve total layers of material, shown in Figure 3, were cut using a rotary cutter and a self-healing mat. The carbon fiber sheets had a backing on one side and were exposed on the other side. The exposed side was placed on a protective sheet of plastic, which was placed on the self-healing mat. The un-cut material was already approximately 5 inches in width, so the strips were cut to length with 1 to 2 inches of extra length to allow the plates to be cut more exactly when cured.

The desired fiber orientation for each of these plates was [0/0/0/0]. The pieces were assembled by hand, by pressing exposed faces together and by peeling off the backing from one of the layers. Alignment was achieved by inspection, which was accomplished by verifying that the edges of the cut sheets aligned.

Figure 4. Composite plate layups with backing sheets intact.

The curing cycle and the dimensions of these composite plates required that they were cured in the autoclave. As such, the layups were placed between two sheets of porous material and on a metal plate with a thin layer of high temperature wax spread over it. A layer of breather material was placed on top of the upper sheet of porous material. Folded pieces of cotton were placed over the vacuum port and the pressure measuring port. These cotton pieces allow for better suction and a more accurate pressure reading. Finally the whole assembly was sealed with a sheet of plastic and gummy material and placed in the autoclave.

Figure 5. Vacuum assembly ready for the autoclave.

The curing cycle for the unidirectional prepreg is listed from the manufacturer was 210°F for 2 hours at 29in Hg. However, due to a programming error, cycle 4 malfunctioned, and the autoclave treated the material at 210°F and 29in Hg for an additional 41 minutes. At that time, the autoclave was forced to cycle 4, which vented the pressure before the temperature had reduced. Once the curing cycle was complete, the parts were left in the autoclave until they cooled to room temperature. This error did not significantly impact the work, since the manufacturer recommended cure times were minimums and continued heating did not significantly impact the strength and material properties of the final parts.

Figure 6. Vacuum assembly in the autoclave.

Cured plates have uneven edges because some resin is designed to flow out of the material through the edges and because the layers do not line up exactly at the edges. A reduction in resin content is a natural part of the curing process. The edges were marked off using a permanent marker and a straight edge. These lines represent the area that should be removed. Care was taken to mark the plates so that new edges were in alignment with the fiber direction. The plates were cut on the inside of these lines using a tile saw, which is a water-cooled saw that allows precision cutting. The plates were manufactured to be larger than necessary to accommodate uneven edges.

Figure 7. Comparison of aluminum and uncut composite plates.

C. Composite Material Properties

Advanced composites group publishes the specifications for VTM264/HR40-300-36%RW unidirectional fiber in their product specification document. [1] They list the modulus of elasticity as 118 GPa in the longitudinal direction and 4.07 GPa in the lateral direction. They also list Poisson's ratio as 0.32.

The material properties were also determined experimentally tensile test by California Polytechnic State University students. [12] They tested nine different length samples of the material using an Instron machine, and showed that the mean modulus of elasticity is 93.5 GPa in the longitudinal direction and 4.07 GPa in the lateral direction. The standard deviations on these averages are 8.46 GPa and 0.60 GPa. While these are significantly lower than the manufacturer's specifications, they are more realistic.

D. Vibrations Testing

A shake table was used to find the frequency amplitude response of the aluminum and composite plates. The shake table was produced by the UNHOLTZ-DICKIE Corporation. The sine controller demo mode was used. The procedure for safely operating the shake table is listed in Appendix B, and the vibration profile is shown in Appendix C. The profile used was a sine sweep from 5Hz to 2000Hz. The VWinII software used provides sampling rates up to 51 kHz and has antialiasing filters. It used 2000 frames that were logarithmically spaced. These capabilities ensure that no aliasing occurs over the test frequency range, and the frame rate is high enough to provide good resolution between frequencies.

Figure 8. Shake table fixture, plate, and accelerometer configuration.

Before testing, plates were secured to the platform using an aluminum fixture, which is bolted to the table with five bolts, shown in Figure 8. One side on the fixture, on the left, is

tightened to 16.0 ft-lbs, and the right side is bolted loosely to allow it to slide horizontally in order to accommodate the plate thickness. The plate is then placed between the two halves of the fixture, and two horizontal bolts fasten the two fixture halves together. These bolts were tightened to at least 16.0 ft-lbs. Finally the bolts on the loose side of the fixture, shown on the right, were tightened to 16.0 ft-lbs.

Accelerometers were attached to the plate and the fixture using two different methods. A small amount of wax was used for preliminary testing of the different accelerometer locations on the aluminum plates. Wax is the ideal adhesive method because it has negligible weight, is easily removable, and it transmits the acceleration well. However, the wax failed to hold the accelerometer when it was placed near the base of the plate, so a tape and superglue method was used for secondary tests. This method involved placing a piece of packing tape at the base of the plate and sanding it with fine sand paper to provide a good adhesive surface and then using superglue to attach the accelerometer directly to the tape. This glue required at least 5 minutes to dry completely. The packing tape protected the plate surface from the glue and distributed the accelerometer load over the plate surface. Blue wire leads connect the accelerometers to the computer in order to record data measurements. The accelerometer itself weighs 12.6 grams.

The software that was used to collect the vibration data was UNHOLTZ-DICKIE Corporation's Sine Controller with data in ASCII format. It records frequency, input acceleration, and response acceleration. The data can be exported into comma-separated-value format, which can be then imported into Excel or Matlab.

Tables 1, 2, and 3 show summaries of the experimental results for the aluminum and composite plates.

Table 1. Experimental modal characteristics of 9.5", 11.5", and 14.5" aluminum cantilever plates as measured 1" from the free end.

Aluminum Cantilever Plates with Accelerometer at 1.0" from Free End								
	Mode 1				Mode 2			
Length	Frequency [Hz]	Amplitude [g/g]	Phase [deg]	Damping Ratio [-]	Frequency [Hz]	Amplitude [g/g]	Phase [deg]	Damping Ratio [-]
9.5"	38.35	86.44	-93.72	0.016897	239.3	42.85	93.4	0.007500
11.5"	26.29	89.87	-118.8	0.0144099	166	55.19	75.54	0.007720
14.5"	16.88	79.22	-152.8	0.0189937	105	64.96	57.81	0.007885
	Mode 3				Mode 4			
Length	Frequency [Hz]	Amplitude [g/g]	Phase [deg]	Damping Ratio [-]	Frequency [Hz]	Amplitude [g/g]	Phase [deg]	Damping Ratio [-]
9.5"	641.2	7.528	-85.71	0.0139051	1493	3.042	74.47	-
11.5"	459.8	21.75	-81.14	0.0064115	824.7	2.156	122.1	0.009932
14.5"	294.2	27.9	-100.9	0.0064982	557	5.057	123.5	0.001196

Table 2. Experimental modal characteristics of 9.5", 11.5", and 14.5" aluminum cantilever plates as measured 0.5" from the fixed end.

Aluminum Cantilever Plates with Accelerometer at 0.5" from Fixed End								
	Mode 1				Mode 2			
Length	Frequency [Hz]	Amplitude [g/g]	Phase [deg]	Damping Ratio [-]	Frequency [Hz]	Amplitude [g/g]	Phase [deg]	Damping Ratio [-]
9.5"	41.96	4.277	-64.67	0.018683	260.2	23.96	-127.6	0.004237
11.5"	28.854	3.113	-58.88	0.017111	181.077	17.76	-125	0.007226
14.5"	17.86	1.815	-35.48	0.018585	113.127	11.4	-99.91	0.010690
	Mode 3				Mode 4			
Length	Frequency [Hz]	Amplitude [g/g]	Phase [deg]	Damping Ratio [-]	Frequency [Hz]	Amplitude [g/g]	Phase [deg]	Damping Ratio [-]
9.5"	710	23.63	-83.82	0.005346	1277	8.531	-51.67	-
11.5"	497.032	18.21	-96.76	0.004893	946.568	7.685	-76.81	0.015211
14.5"	315.206	18.92	-124.1	0.003754	614.855	24.84	-120.6	0.003168

Table 3. Experimental modal characteristics of 9.5", 11.5", and 14.5" composite cantilever plates as measured 0.5" from the fixed end.

Composite Cantilever Plates with Accelerometer at 0.5" from Fixed End								
	Mode 1				Mode 2			
Length	Frequency [Hz]	Amplitude [g/g]	Phase [deg]	Damping Ratio [-]	Frequency [Hz]	Amplitude [g/g]	Phase [deg]	Damping Ratio [-]
9.5"	35.8	1.805	-31.8	0.015245	218.042	4.747	-83.51	0.012562
11.5"	23.82	1.696	-26.49	0.015418	146.817	9.514	-82.73	0.007882
14.5"	14.75	1.379	-14.98	0.019629	93.108	5.849	-87.46	0.009551
	Mode 3				Mode 4			
Length	Frequency [Hz]	Amplitude [g/g]	Phase [deg]	Damping Ratio [-]	Frequency [Hz]	Amplitude [g/g]	Phase [deg]	Damping Ratio [-]
9.5"	386.437	2.248	-44.3	0.014426	492.584	2.716	-46.64	0.012222
11.5"	333.671	3.717	-72.3	0.016362	391.097	5.989	-74.08	0.005451
14.5"	244.337	7.938	-78.95	0.007699	399.386	3.134	-71.22	0.024114

While the natural frequency, amplitude and phase can be found directly from the shake table output data, the damping ratio for each mode must be calculated. Recall the damping ratio is proportional to the decay rate of the amplitude. The damping ratio for each plate at each mode can be calculated from the experimental data using the half-power bandwidth method. [21] The half-power bandwidth method predicts that the damping ratio is equal to the difference between frequencies corresponding to amplitudes that have half of the power as the amplitude of the natural frequency. The half power amplitude is equivalent to -3dB or dividing the maximum amplitude by $\sqrt{2}$. This method is summarized in equation (119).

$$\zeta = \frac{\omega_2(x_{max}/\sqrt{2}) - \omega_1(x_{max}/\sqrt{2})}{2 \omega_n(x_{max})} \quad (119)$$

In order to apply this method to experimental data, matlab code was written, shown in Appendix D. Given a range of values, this script uses a linear interpolation to find the frequency at the half-power, and then calculates the damping ratio based on these numbers. The damping

ratio was found for the first four modes of each plate for all except the fourth mode on the 10-inch aluminum and the 10-inch composite plate. These were not obtained because two modes were very close to each other in this regime, which prevented the half-power bandwidth method from being used.

Tables 1, 2, and 3 shows a summary of the experimental results for the aluminum plates. One can notice several trends in these tables.

1. The modal frequency for each plate decreases as the plate length is increased. This happens because a longer plate has a represents a larger amount of mass, and greater mass decreases frequency.
2. The amplitude is greatest for the first mode and decreases with an increase in modal frequency for the subsequent modes when measured from the free end. The amplitude is greatest for the second and third modes when measured from the fixed end. This trend is expected, since the mode shape affects the measured.
3. Most of the nodes occur at a phase angle not near 0 or near 180 degrees. If the phase angle is exactly 0, then the plate will move proportionally to the input acceleration. The input acceleration is opposite of the motion of the plate if the phase angle is exactly 180, which effectively absorbs the kinetic energy already in the structure. Resonance occurs when the signal is allowed to be amplified and grow unbounded. In other words, when the plate's phase angle is close to ± 90 degrees, the plate is in an unstable vibration, which leads to resonance.
4. The damping ratio that was calculated using the half-power bandwidth method is different for each of the plates at each of the nodes.

These general trends will be discussed in more detail in the following pages. This modal data for the composite plates shows the same trends as the aluminum. Again, the frequency of a given mode increases with length; the amplitude decreases with modal frequency; the phase is around ± 90 degrees; and the damping ratio has loose patterns between modes and plate lengths.

The frequency response function and the phase angle responses for aluminum plates as measured from 1.0" from the free end are shown in Figure 9. This figure shows the response of the plate at any given frequency. An interesting feature of these plots is that they show that for high frequencies over 1000 Hz the frequency response is very active and is a very unstable area. In this regime, there is a significant amount of modal interaction and the accelerometer may have picked up torsional modes in addition to the bending modes.

In order to compare the frequency response and the phase angle with minimal input effects, it is necessary to normalize the results. To normalize frequency response, the response can be divided by the input. This transfer function can be thought of as a factor of how much the input is amplified by the plate. If this factor is exactly equal to one, then the measured (output) force is identical to the input force. When it is less than one, then the input is attenuated, and if the factor is greater than 1, the input is amplified. The normalized phase angle is equivalent to the difference between system's phase angle and the input's phase angle. This angle is then converted to an angle that falls between -180 and 180 degrees. The normalized frequency response and phase angle for various length clamped aluminum plates are shown in figure 10.

A trend in the normalized frequency response is that the local minima are not spikes, which shows there is little attenuation. There is so little attenuation because for these cases, the accelerometer was placed 1 inch from the free end of the plate. The signal that is measured by the accelerometer is affected by the amount of deflection experienced at that particular location. A larger stress value can lead to more accurate readings. An accelerometer will pick up little signal if it happens to be placed at a node for a particular mode. This can affect the estimated natural frequency, the peak amplitude, and the damping ratio calculation.

In order to pick up more attenuation and a cleaner signal, the accelerometer was then placed near the base of the plate. This reduces the effective mass that the accelerometer adds to the system. Figure 11 shows the frequency and phase response for each aluminum plate as measured from 0.5" from the fixed end. Similarly, Figure 12 shows the same normalized

Figure 9. Frequency response function and phase angle for 9.5", 11.5", and 14.5" aluminum plates with accelerometer 1.0" from free end.

Figure 10. Frequency response transfer function and normalized phase angle for 9.5", 11.5", and 14.5" aluminum plates with accelerometer 1.0" from free end.

response. The first thing to notice in this study is that the signals are much more clear, as expected. Additionally, there much more signal attenuation at the fixed end. From this plot, it is clear that the shorter plates produce the most amplitude and resonate at higher frequencies than the longer plates. It is interesting to note that at this accelerometer position, the phase angle changes much more quickly, so there is a shorter period of instability. Even so, from Tables 1 and 2, the damping ratios at each of the modes seem comparable.

The composite plates were also passed through a sine sweep on the shake table in order to find their frequency response. Figures 13 and 14 show the frequency and phase response and the normalized frequency and phase response respectively. The composite plates follow the same trends as the aluminum plates, but they have a much more complicated response at frequencies above 500Hz. This is most likely due to inconsistencies in the material that produces resonant modes besides bending.

Another study was conducted to examine the effect of accelerometer effect on the system. The accelerometer placement has a significant effect because of the addition of mass to the system, which adds error to the frequency, amplitude, and estimated mode shape. The accelerometer weighs 12.6 grams, which is significant relative to the total plate weights. The accelerometer placement was varied along the plate length on the 9.5" aluminum plate. The 9.5" plate was chosen because it produces the highest amplitudes and frequencies, which allowed more sensitive measurements. Accelerometer placements studied were 0.5", 2", 3.5", 6", and 8" as measured from the fixed end of the plate. Each of these was hand placed using a ruler, which had an error of approximately 0.1". This gave a correction trend that can be applied to account for the accelerometer mass.

The experimental frequency response data for the 9.5" aluminum plate with various accelerometer placements is shown in Table 4. Although the entire plate does have unique discrete natural frequencies, the plate-accelerometer system has its own natural frequencies,

Figure 11. Frequency response function and phase angle for 9.5", 11.5", and 14.5" aluminum plates with accelerometer 0.5" from fixed end.

Figure 12. Frequency response transfer function and normalized phase angle for 9.5", 11.5", and 14.5" aluminum plates with accelerometer 0.5" from fixed end.

Figure 13. Frequency response function and phase angle for 9.5", 11.5", and 14.5" composite plates with accelerometer 0.5" from fixed end.

Figure 14. Frequency response transfer function and normalized phase angle for 9.5", 11.5", and 14.5" composite plates with accelerometer 0.5" from fixed end.

which will theoretically change with the accelerometer placement. Examination of the modal frequencies shows that the accelerometer placement does have a significant effect on the experimental natural frequency and its amplitude. If the accelerometer is placed far away, the mass has the effect of decreasing the frequency and increasing the measured amplitude. Additionally, it had the effect of making the phase angle significantly less predictable for Modes 2 and 3. Finally, one can notice that signal attenuation is very dependent on position. For example, the accelerometer placed at 6" shows attenuation for the second and fourth modes, but not for the first and third.

Table 4. Experimental properties of the modes of a 9.5" long aluminum plate as measured at different accelerometer locations.

9.5" Aluminum Plate with Various Accelerometer Positions						
	Mode 1			Mode 2		
Sensor Location	Frequency [Hz]	Amplitude [g/g]	Phase [deg]	Frequency [Hz]	Amplitude [g/g]	Phase [deg]
2"	43.76	27.55	-98.22	252.5	100.7	-105.7
3.5"	43.11	46.28	-112.8	252.5	90.44	-128
6"	40.85	77.66	-109	261.8	3.369	77.91
8"	38.47	86.07	-104	239.3	42.85	93.4
	Mode 3			Mode 4		
Sensor Location	Frequency [Hz]	Amplitude [g/g]	Phase [deg]	Frequency [Hz]	Amplitude [g/g]	Phase [deg]
2"	722.8	3.218	-170.9	1891	14.58	90.98
3.5"	701.5	19.44	165.5	1356	22.27	94.45
6"	720.7	15.54	86.27	1397	17.24	-89.49
8"	641.2	7.528	-85.71	1488	3.026	82.0

Figure 15. Frequency response function and phase angle for a 9.5" aluminum plate with various accelerometer positions.

Figure 16. Normalized frequency response function and phase angle for a 9.5" aluminum plate with various accelerometer positions.

An error analysis was conducted to find the standard deviation between measurements. In order to find the standard deviation between measurements due to the accelerometer placement, the 11.5 inch aluminum beam five different times. For each case, the accelerometer was reattached at 0.5 inches from the fixed end. Figure 17 shows that the frequency amplitude responses for the five cases are very similar. This is highly desirable because it implies a low standard deviation between measurements. Figure 18 shows that the phase angle for each of these cases is also similar. Figure 19 shows that standard deviation among the natural frequencies of the aluminum beam is less than 0.25 for the first mode, approximately 1.25 for the second and third mode, and about 4.3 for the fourth mode. These numbers mean that at frequencies lower than 40Hz, the natural frequency measurement is accurate to within 0.5 Hz for 95% confidence and at high frequencies, the measurements will be accurate to within 9 Hz with 95% confidence.

Figure 17. Transfer function for five tests of the 11.5" aluminum plate as measured from 0.5" from the fixed end.

Figure 18. Transformed phase angle for five tests of the 11.5" aluminum plate as measured from 0.5" from the fixed end.

Figure 19. Standard deviation of natural frequency for the modes of an 11.5" aluminum plate.

An important characteristic that one can glean from the frequency response function is the damping ratio. This can be found using the half-power bandwidth method, which relates the frequency of half of the acceleration power of the natural frequencies to the natural frequency. While the damping ratio will be different for each geometry and mode, in practice, the damping ratio is often approximated by a typical value. A frequently used typical value for Aluminum 2024 is 0.010. The calculated damping ratio values for the first four modes of each of the plate lengths for the aluminum plate with the accelerometer at 1.0 inch from the free end are shown in figure 20. It is interesting to notice that the plate exhibits different damping ratios at the various modes. In this case, the average damping ratio for mode 1 is 0.01676, mode 2 is 0.00770, and mode 3 is 0.00895, but mode 3 has significant variation among damping ratios.

Figure 20. Comparison of damping ratio for different modes for 9.5", 11.5", and 14.5" aluminum plates with accelerometer position at 1.0" from free end.

The calculated damping ratio for the case of the aluminum plates with the accelerometer at 0.5" from the fixed end is shown in figure 18. In this case, the average damping ratio for mode 1 is 0.016736, mode 2 is 0.005339, mode 3 is 0.007650, and mode 4 is 0.005746. Figure 21 shows the same pattern as figure 20, indicating that the damping ratio is near constant throughout the whole plate. While the trend seems quite nonlinear, it can easily be explained by the fact that the amplitude or gain is not constant for each of the resonant frequencies. Figure 22 shows when the damping ratio for aluminum is related to the amplitude instead of frequency, it becomes very linear. The relationship is $\zeta = -0.0007x + 0.0198$, where x is amplitude and ζ is damping ratio. This equation has a correlation coefficient is 0.94 indicating that the model is a very good fit for this data.

Figure 21. Comparison of damping ratio for different modes for 9.5", 11.5", and 14.5" aluminum plates with accelerometer position at 0.5" from fixed end.

Figure 22. The relationship between amplitude and damping ratio for aluminum plates.

Similar damping ratio calculations were completed for the composite plates with the accelerometer at 0.5" from the fixed end is shown in figure 23. In this case, the average damping ratio for mode 1 is 0.01674, mode 2 is 0.009998, mode 3 is 0.012829, and mode 4 is 0.013929. The standard deviation for each of these modes was 0.0025, 0.0024, 0.0045, and 0.0094. The damping ratio in the composite plates seems to follow a similar trend as the aluminum, where the damping ratio decreases from 0 Hz to about 500Hz, where it increases significantly. Figure 24 shows the relationship between amplitude and damping ratio for the composite plates. Whereas, this relationship was highly linear for aluminum, it does not have a strong linear correlation for the composite plates, since the correlation coefficient is only 0.53. Logarithmic, polynomial, exponential and power fits were considered to see if any better fit the data. An exponential fit had a correlation coefficient of 0.59, which is not much better than the linear fit. There are several explanations for why a linear fit works very well for aluminum but not for a composite plate. One

explanation is that the composite plates have inherently more variation in their internal structure than the aluminum, making them less predictable. Another possibility is that some of the composite modes are not pure bending. Since the accelerometer was on the centerline of the aluminum plate, this would mean that not all of the actual motion present was detected. If this were the case, it would explain the poor correlation.

Figure 23. Comparison of damping ratio for different modes for 9.5", 11.5", and 14.5" composite plates with accelerometer position at 0.5" from fixed end.

Figure 24. The relationship between amplitude and damping ratio for composite plates.

The effect of the accelerometer position on damping ratio was studied as well. Figure 25 shows accelerometer position and damping ratio for the first four bending modes. The mean of the Mode 1 damping ratios for all accelerometer positions is 0.01673, and the standard deviation is 0.00146, which is about 8.7% of the mean value. Similarly, for mode 2 the mean damping ratio is 0.00534 with a standard deviation of 0.00141, which is about 26.3% of the mean value. For mode 3, the mean damping ratio is 0.00765 with a standard deviation of 0.00385, which is about 46.8% of the mean value. And finally, for mode 4, the mean damping ratio is 0.00575 with a standard deviation of 0.00357, which is about 8.0% of the mean value. These numbers shows that the accelerometer has a varied impact on the damping ratio of the plate. The probable reason that the first modes damping ratio is so high is because it is loses a significant amount of energy to the surrounding air since it presents the largest amplitudes. The other 3 modes have significantly smaller deflections, which leads to their damping ratio to be mostly dependent on the internal structure.

Figure 25. Comparison of damping ratio for different modes and accelerometer positions for a 9.5" aluminum plate.

Chapter V. ANALYTICAL RESULTS FOR EXPERIMENTAL CASES

The previous chapter presented the experimental procedure and results for several test cases of aluminum and composite plates. This chapter will use these results in order to validate the analytic theory presented in Chapter 3.

The purpose of creating an analytical tool is to take advantage of its predictive capabilities. Analytical solutions can quickly provide information about an engineering problem. Since the analytical solutions are closed form, they can be used to quickly find trends and optimized solutions. They also are good for estimating results in order to validate numerical and experimental methods.

In previous work, California Polytechnic State University in San Luis Obispo has produced studies for several topics in the arena of active control. First, Hrovat [11] and Lesniak [19] studied active damping using a patch of piezoelectric material on an aluminum beam with a first bending mode around 5 Hz. Graduate student James Bach studied active damping on a model airplane tail boom. [4] Graduate student J.R. Cavalli studied the vibration characteristics of a hybrid damping shelf as well as an aluminum sailplane. [5]

A. Comparison of Analytical Results to Experimental Results

The first analytical study matched the cases from the experimental study presented in Chapter 4. With the accelerometer placed 0.5" from the fixed end of the plate, the aluminum and composite plates studied can be idealized as a fixed-free plate. The standard material properties for Aluminum 2024 were used. These are $E = 73.1$ GPa, $G = 28.0$ GPa, $\nu = 0.33$, and $\rho = 1780$ kg/m³. The material properties for the unidirectional composite were measured experimentally as $E_1 = 93.5$ GPa, $E_2 = 6.74$ GPa, $G = 4.07$ GPa, $\nu = 0.34$, and $\rho = 1310$ kg/m³. [12] The analytical solution is a function of the plate dimensions, which are given in Table 5.

Table 5. Dimensions and predicted modal frequencies of analytical aluminum and composite plate cases.

Case	$a_{\text{effective}}$	b	h	Mode 1	Mode 2	Mode 3
Aluminum, 9.5"	9.5 in	4.0 in	0.125 in	45.2 Hz	283 Hz	792 Hz
Aluminum, 11.5"	11.5 in	4.0 in	0.125 in	30.8 Hz	193 Hz	541 Hz
Aluminum, 14.5"	14.5 in	4.0 in	0.125 in	19.4 Hz	121 Hz	340 Hz
Composite, 9.5"	9.5 in	4.0 in	0.048 in	36.3 Hz	228 Hz	637 Hz
Composite, 11.5"	11.5 in	4.0 in	0.047 in	24.8 Hz	155 Hz	435 Hz
Composite, 14.5"	14.5 in	4.0 in	0.049 in	15.6 Hz	97.7 Hz	273 Hz

As shown in Table 5, the analytical solution predicts that the modal frequencies will decrease with an increase of plate length. The analytical solution for the first three modal frequencies of aluminum plates are validated against the experimental result and are plotted in Figures 26, 27, and 28 respectively. Error bars of one standard deviation on these figures show that the error is small relative to the difference between the analytical and experimental results. These figures show that the predicted modal frequencies for all modes are over-predicted compared to the experimental results. This is expected since Euler-Bernoulli theory is first order, and second order and higher order terms would decrease the natural frequencies. The presented solutions use an effective length of 9.5", 11.5", and 14.5". This is because the aluminum plates were 10.0", 12.0" and 15.0" long, and the fixture clamps approximately 0.5" of the plate. This causes the effective cantilever plate length to be 0.5" shorter than the actual plate length.

Figure 26. Validation of first natural frequency solutions for aluminum plates with various lengths.

Figure 27. Validation of second natural frequency solutions for aluminum plates with various lengths.

Figure 28. Validation of third natural frequency solutions for aluminum plates with various lengths.

Figure 29 shows the error between the experimental and the analytical natural frequencies for the aluminum 2024 plates. The error for all cases is between 6% and 10.5%. A 68% confidence interval (one standard deviation) is given by the error bars, and falls within 5% and 11%. Again, this shows that the analytical solution is consistently higher than the experimental solution. There is no definite pattern as to which natural frequency showed the best correlation between the analytical and experimental error, however, the longer plate showed less deviation between estimates. Less deviation is desirable because then the difference between the analytical and experimental solutions can be better predicted and understood.

Figure 29. Percent error between experimental natural frequencies and analytical solutions for aluminum plates.

The analytical solution can also produce the mode shapes of structures, which can be very difficult to obtain experimentally. Figure 30 shows the first three bending mode shapes for a 14” cantilever aluminum plate. Since the analytical formulation assumes a unified dynamic solution for all plates, this shape will scaled in terms of displacement and dimensions, but will otherwise be the same for any plate with the same boundary conditions. While the mode shape

can be found for any other case, the mode shapes will only be presented in figure 30 to save redundancy.

Figure 30. First three bending mode shapes for a 14.5 inch aluminum plate.

The analytical frequency amplitude response for different length aluminum plates can be found using the analytical solution developed in Chapter 3. This is done by comparing the output strain energy or output acceleration to the input strain energy or input acceleration. Just as in the experimental results, the output will vary depending on the sensor location. Figure 31 shows a comparison between the analytical and experimental frequency amplitude response for the three aluminum plate cases as measured from the free end of the plate. Since this solution to find the frequency response is based on slightly different assumptions than the natural vibration solution, its natural frequencies are a little different from the natural vibration solution, however, figure 31 shows that the general solution also over predicts the modal frequencies. The experimental solution includes the accelerometer mass, which shifts the response to lower frequencies compared to the analytical solution, which does not include the accelerometer mass. The analytical solution also predicts more attenuation between the second and third modal frequencies

and above the third modal frequency than was detected in the experimental results. This is due to the fact that the analytical solution does not include internal damping, which has a tendency to smear the solution, resulting in less attenuation and less amplification. Another result of this effect is that the peak height is theoretically unlimited in the analytical solution at the natural frequency, whereas they have finite heights in the experimental solution. This is most apparent at the second natural frequency of the analytical solution for the 9.5" aluminum plate. For the analytical solution, the amplitude at the natural frequencies reflects the number of solution evaluations near the natural frequency. More evaluations implies that a frequency that is closer to the exact natural frequency will be evaluated, which in turn means that the amplitude of the peak will approach infinity. Since the heights of the analytical natural frequency amplitudes are arbitrary, the damping ratio cannot be found from this solution. This makes sense because internal damping is not included in the analytical solution.

Figure 31. Comparison of analytical and experimental frequency amplitude response aluminum plates with different lengths as measured from the free end.

The analytical frequency amplitude response for different length aluminum plates as measured at the plate base is shown in Figure 32. This is a more common way to measure the

frequency amplitude response because there is greater attenuation and the accelerometer mass affects the frequency response less. As expected, the calculated frequencies are slightly closer but still higher than the experimental frequencies. In general, the experimental data shows that the amplification at the natural frequency is not necessarily proportional to the attenuation immediately following, whereas the analytical solution has theoretically infinite amplification and attenuation. One possible reason for this disparity is that the experimental data was obtained using a sine sweep, which means that the frequency increases slowly but constantly through the tested range. Despite these differences, both frequency amplitude response predictions are very close to the experimental data, which validates the general solution method for finding the frequency amplitude response of cantilever plates.

Figure 32. Comparison of analytical and experimental frequency amplitude response for aluminum plates with different lengths as measured from the fixed end.

The analytical solution was also found and validated for all of the composite experimental cases. Similarly to the aluminum plates, the analytical solution predicts that the composite plate natural frequencies will also vary with plate length. This relationship for the first

modal frequency is shown in figure 33, and the same relationship is shown for the second and third modes in figures 33 and 34. These figures show that the predicted modal frequencies for all

Figure 33. Validation of first natural frequency solutions for composite plates with different lengths.

Figure 34. Validation of second natural frequency solutions for composite plates with different lengths.

Figure 35. Validation of third natural frequency solutions for composite plates with different lengths.

Figure 16. Percent error between experimental natural frequencies and analytical predictions for composite plates.

modes are over-predicted compared to experimental results. This is expected since Euler theory is first order, and second order and higher order terms would decrease the natural frequencies. For the composite plates, the effective cantilever length was also 0.5" shorter than the actual plate length.

The error between the analytical results and the experimental results for the composite plates is presented in Figure 36. As before, most of the experimental modal frequencies are smaller than the analytical predictions. The reason for this systematic bias is that the plate theory that was used is first order and does not capture all of the energy at lower frequencies. One exception is the third modal frequency of a 9.5" composite plate, which had lower analytical frequency prediction than experimental result. The reason for this is that there were two torsional modes that presented between the second and third bending modes. These modes seemed to delay the onset of the third bending mode.

The predicted analytical and experimental frequency amplitude response for the composite plates is shown in figure 37. The results are very similar, which validates the general solution of vibration. The natural frequency predictions are typically 5-10% larger than the experimental results. The difference between the natural frequency predictions for the general solution and the natural frequency predictions for the natural vibration solution is so small that it is not discernibly different between charts. It is important to reiterate that the general solution of vibration does not account for internal damping; because of this, the amplitude at the natural frequencies would be infinite if the solution were evaluated at the exact natural frequency. Instead the amplitude is limited because the function was evaluated at points equally spaced throughout the domain, which produce large, but finite, amplitudes. It is interesting to note that most of the experimental modes show larger amplification and attenuation near the modes than the analytical solution does, especially at higher modes. This is an effect of damping. This model behaves as expected when compared to the experimental results from composite plates, which affirms the validity of the model.

Figure 37. Comparison of analytical and experimental frequency amplitude response for composite plates with different lengths.

B. Comparison of Analytical Results to Hovrat's and Lesniak's Work

Another analytical study matched the work done by Hovrat [11] and Lesniak [19] in their senior projects. Both studied a thin aluminum beam with active control from a patch of piezoelectric material. Hovrat built an aluminum fixed-free beam with patches of piezoelectric sensors and actuators, shown in figure 38. The beam was 16.0 inches long, 0.80 inches wide and 0.025 inches thick. The effective length of the beam in the fixture was 13.0 inches. The aluminum that was employed had a modulus of elasticity of 1056 psi and a density of 0.0030 slug/in³.

The primary objective of the project was to control the first bending mode. In order to accomplish this goal Hovrat implemented a patch of piezoelectric material at the fixed base. The actuator was 1.25 inches long, 0.50 inches wide, and 0.0095 inches thick, and it is made of G-1195. The experimental actuator constant was 0.0044 in/volt. The sensor patch was also made of G-1195 and was 1.25 inches long, 0.25 inches wide and 0.0095 inches thick. The experimental

Figure 38. Hovrat's and Lesniak's experimental setup of an aluminum beam with a piezoelectric sensor and actuator.

sensor constant was 10.0 V/in. The piezoelectric longitudinal coupling coefficient was 256 pm/V. The voltage source had a limit of ± 12 volts. This experimental setup is shown in figure 38.

Hovrat found that the most damping was achieved with 90 degrees of lead (or 270 degrees of lag); this increased the damping ratio from a 0.020 to 0.04. This was achieved by using a 40 M Ω resistor across the sensor, which gave the system a delay with a time constant of 0.82 seconds.

Lesniak studied the same beam configuration as Hovrat did, but he looked at varied control systems through transient vibration tests. Feedback control with a 270-degree phase lag was used. Lesniak reported a piezoelectric longitudinal coupling coefficient of 7.48×10^{-9} in/volt and the voltage constant is 1.926 volt-in/lbf. The sensor proportionality constant was found to be

10.6 volt/in. Proportional-Derivative and Proportional-Integral-Derivative control was implemented. He found that all types of feedback control were unstable without a low-pass filter. The filter he used was an active filter that provide 80dB/decade drop off at 30Hz. The voltage source had a limit of +/- 12 volts, but Lesniak also found that all types of feedback control were unstable at high gains, which limited his gains to approximately 8 volts.

Both Hovrat and Lesniak found that the first three bending modes had frequencies at 5.6, 33.6, and 96 Hz. A comparison between the natural frequencies found from their experimental methods and those found using the current analytical method is listed in table 6. When Hovrat and Lesniak used simple beam theory to predict the modal frequencies, they only predicted the frequencies of the aluminum beam. Their results were very close to the analytical solution for the aluminum beam alone, which was lower than the experimental results and had between 11.5% and 15.2% error. When the beam was modeled to include the piezoelectric patch, the predicted frequencies were slightly higher than the experimental frequencies, and the error was between 3.0% and 7.5% error, which is much lower than the analytical solution for the aluminum beam alone. Additionally, the results with the piezoelectric patch modeled more closely match the over prediction that is generally expected from the analytical solution.

Table 6. Summary of experimental and analytical natural frequencies for Hovrat's and Lesniak's experiments.

	Experimental	Analytic (Aluminum)	Analytic (Aluminum with Piezoelectric Patch)
First Mode	5.6 Hz	4.75 Hz	5.77 Hz
		-15.2% difference from experimental	3.0% difference from experimental
Second Mode	33.6 Hz	29.75 Hz	36.13 Hz
		-11.5% difference from experimental	7.5% difference from experimental
Third Mode	96 Hz	83.31 Hz	100.28 Hz
		-13.2% difference from experimental	4.5% difference from experimental

The frequency amplitude response as calculated from the general solution for Hovrat and Lesniak's beam is shown in figure 39. Although this chart does not include damping, it shows how the response moves when a piezoelectric patch is added to the beam.

Figure 39. Frequency amplitude response for Hovrat and Lesniak's aluminum beam with a patch of piezoelectric material.

The main objective of Hovrat and Lesniak's project in relation to this study was to achieve a maximum amount of damping through active control of the first mode. Lesniak calculated the uncontrolled damping ratio to be 0.0041. Hovrat primarily set up the project and found point performance with an analog feedback controller using a phase delay of 100 degrees and gain of 8.4 volts with a damping ratio of 0.0170. Lesniak studied several types of control, including proportional control, proportional-derivative control, proportional-integral control, and proportional-integral-derivative control. The experimental results from both Hovrat and Lesniak are summarized in figure 40. Predictions of damping ratio using the current analytical active control theory for a sinusoidal input and a square input are also given for Hovrat and Lesniak's beam. Since both Hovrat and Lesniak were using analog feedback control, the maximum theoretical damping ratio without input signal clipping is the sine wave prediction. Interestingly,

Lesniak's proportional-integral and proportional-integral-derivative controllers follow the prediction for the sine controller at low input gain, however, they diverge at gains larger than 2 volts. The most likely possible cause of this divergence is that piezoelectric control theory assumes that the piezoceramic is in the linear piezoelectric region, whereas it may not be in reality. If the applied voltage per distance is too high, the piezoceramic deformation is less than linear. This explains why the damping ratio seems to approach a constant value.

Figure 40. Summary of experimental and analytical active control damping ratios for Hovrat and Lesniak's beam.

C. Comparison of Analytical Results to Cavalli's Work

J.R. Cavalli researched the application of active control to an aluminum sailplane. [5] He tried to design the sailplane to have a first bending frequency of about 5 Hz. The sailplane he used had a wingspan of 48.0 inches, a chord of 6.03 inches. It was 0.062 inches thick. The sailplane wings was tapered at the leading edge. The angle and length of taper was not documented numerically in his report; however, he did show a line drawing of it. Since this drawing may or may not be to scale, the taper was not modeled in this analysis. The piezoelectric patch on Cavalli's sailplane was located at the fixed end in the middle of the plane. The sensor

and actuators combined were approximately 4.0 inches wide, 2.5 inches long and 0.0095 inches thick. The geometry of Cavalli's sailplane is shown in figure 41.

Figure 41. Cavalli's sailplane geometry as presented in his thesis.

Cavalli reported that his natural frequencies were 4.0, 22.3, 32.5, 62.3, and 94.1 Hz. The analytical bending natural frequencies for a rectangular aluminum beam and the analytical solution for an aluminum beam with the piezoelectric patch are presented in table 7. The analytical solution for the second and third modes of the aluminum beam was much closer to the experimental results than the analytical solution for the beam modeled with a piezoelectric patch. This is most likely due to the fact that the taper on the end of sailplane was not modeled. Modeling the removal of that mass would decrease both of the frequency predictions, most likely making the analytical prediction with the piezoelectric patch more accurate. In both cases the error for the first mode was significantly different than the error for the second and third modes. This may be because he could detect fewer significant digits for the first mode. Another thing to note is that Cavalli excited an additional mode at 32.5 Hz, which is between the second and third bending modes. This is most likely a torsional mode, which may have been induced because of the asymmetry of the taper of the sailplane. Another source of error is that Cavalli did not report the length of the sailplane that was attached to the fixture. If this length was significant, it could increase the modal frequencies because the effective beam length would be shorter.

The analytical solution was also used to find the frequency amplitude response curves for the sailplane modeled as an aluminum beam and modeled as an aluminum beam with a patch of piezoelectric material, shown in figure 42. This primarily shows how the frequencies shift with the addition of piezoelectric material. It also shows that the natural vibration natural frequencies are very similar to the general solution natural frequencies.

Table 7. Summary of experimental and predicted analytical natural frequencies for Cavalli's experiment.

	Experimental (Cavalli)	Analytical (Just Aluminum)	Analytical (w/ Piezoelectric Patch)
First Mode	4.0	3.51 Hz	3.86 Hz
		-12.3% difference from experimental	-3.5% difference from experimental
Second Mode	22.3	22.00 Hz	24.20 Hz
		-1.4% difference from experimental	8.5% difference from experimental
Third Mode	62.3	61.60 Hz	67.77 Hz
		-1.1% difference from experimental	8.9% difference from experimental

Figure 42. Frequency amplitude response for Cavalli's sailplane modeled as an aluminum beam and as an aluminum beam with a patch of piezoceramic.

The main objective of Cavalli's thesis in relation to this study was to develop a successful active control system for the aluminum sailplane. In order to accomplish this, Cavalli studied several feedback signals for active control. He used an analog system that could vary phase angle and voltage. Similar to Hovrat and Lesniak, Cavalli found that a controller with 90 degrees of lead, or 270 degrees of lag, is most effective. Cavalli's sailplane increased its damping ratio from 0.0086 at 0 volts to 0.0806 at 36.3 volts. The output sensor signal was clipped above 6.27 volts and then was amplified up to 36.33 volts before it was input into the actuator. For this reason, when the output signal was below 6.27 volts it was sinusoidal, but over 6.27 volts it became clipped and then amplified to the appropriate gain. As a sine wave is subjected to an increasing amount of clipping, it begins to approximate a square wave. Cavalli's results for various input gains are shown in figure 43. The analytical predictions for Cavalli's setup for a sinusoidal input and a square wave input are also shown. This chart validates the piezoelectric theory because Cavalli's experimental results follow the sinusoidal prediction from 0 to about 6.27 volts as

expected, and then the results follow a trend that transitions from the sinusoidal prediction to the square wave prediction.

Figure 43. Validation of active control damping ratio predictions against experimental data from Cavalli.

Chapter VI. CONCLUSIONS AND FUTURE WORK

A. Conclusions

It is typically desirable in design to be able to predict the results of physical phenomena with a model, and then invest resources into refining the model experimentally. This is true of modal analysis for active control. In all of the past work done at California Polytechnic State University, no satisfactory prediction method has been created. Other methods fail to calculate results for laminates, and transfer function representatives of the system have been refined with experimentation and lacked many features, such as the ability to predict multiple modes. As such, the primary objective of this study has been to create an analytical model that combines modal analysis with effective damping ratio predictions for active control systems. The secondary objective of this project has been to compare the predictions with experimental data and with results from Hovrat and Lesniak's senior projects and Cavalli's thesis. Both of these objectives have been successfully achieved.

Euler theory was used to set up the governing differential equations of the plate dynamics of composites. A known solution for the bending modal frequencies was presented, and a general solution for the equations was developed. This theory predicts the plate deflection for any amplitude and frequency. This prediction tends to be about 5% larger than experimental results. This is mainly because of the first-order nature of Euler theory. In general, the laminate solution matched the experimental modes better than solutions that did not account for lamination effects.

The Ritz assumed mode method was used to analytically find the deflection of the piezoelectric material. The effective damping ratio was found as a ratio of energy absorbed to total system energy. This method predicts linear piezoelectric effects, such as in J.R. Cavalli's thesis; however, it breaks down when the piezoelectric effect becomes nonlinear, such as in Hovrat's and Lesniak's projects.

B. Future Work

While this study has been able to capture both of its objectives, it could be further developed in the future by:

1. Modeling piezoelectric nonlinearities
2. Studying boundary conditions besides this study's fixed-free boundary conditions
3. Experimentation with differently oriented lamina
4. Analytical and experimental solutions for different active control systems
5. Directly solve partial differential equations for plate dynamics that include viscous and active damping terms
6. Combine passive and active damping solutions

The analytical techniques developed in this investigation could be helpful in designing active control systems where damping must be increased significantly.

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APPENDICES

A. Exact Partial Differential Equation

Chih-Kung Lee developed but did not solve an exact partial differential equation for a piezoelectric laminated plate that is specially orthotropic mechanically. [16] This equation is listed for reference below.

$$\begin{aligned}
 & D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} + \rho h \frac{\partial^2 w}{\partial t^2} \\
 & = l(x, y, t) \\
 & - \sum_{k=1}^n \frac{\partial^2}{\partial x^2} \{V_k z_k^0 [(\bar{c}_2)_{11} d_{3'1'}^0 P_1 + (\bar{c}_2)_{12} d_{3'2'}^0 P_2 + (\bar{c}_2)_{16} d_{3'6'}^0 P_6]\}_k \\
 & - 2 \sum_{k=1}^n \frac{\partial^2}{\partial x \partial y} \{V_k z_k^0 [(\bar{c}_2)_{61} d_{3'1'}^0 P_1 + (\bar{c}_2)_{62} d_{3'2'}^0 P_2 + (\bar{c}_2)_{66} d_{3'6'}^0 P_6]\}_k \\
 & - \sum_{k=1}^n \frac{\partial^2}{\partial y^2} \{V_k z_k^0 [(\bar{c}_2)_{21} d_{3'1'}^0 P_1 + (\bar{c}_2)_{22} d_{3'2'}^0 P_2 + (\bar{c}_2)_{26} d_{3'6'}^0 P_6]\}_k
 \end{aligned}$$

B. Vibration Table Procedure

The standard recommended procedure used for the vibration table was written by Liam Cheney and adapted by Justin Foley. For convenience, it is republished here with permission.

Notes on Operating Aero Structures/Composites Lab Vibration Table
Adapted by Justin Foley from notes written by Liam Cheney on 2/8/2011

General Steps:

1. Vibe table setup
2. Create or edit a test profile (if necessary)
3. Open a test profile
4. Begin test
5. Vibe table shutdown

Note: To run tests with multiple test subject configurations, shut down the table and adjust the test subject configuration with the amp gain set to zero.

Vibe Table Setup

1. Turn on Main Switch on wall.
2. Press "On" button on Control Panel to warm up oil pump.
3. Turn on Computer.
4. Mount the test article to the table.
  - a. Slip table bolts torque to 16 ft-lbs, vertical interface plate bolts torque to 25 ft-lbs.
  - b. Do not bolt to or unbolt from slip table unless slip table is mounted to shaker head.
  - c. When mounting the test article, ensure that the correct bolt lengths are used before torquing to prevent damage to the table, thread the bolts in by hand first to verify correct length.
5. Attach accelerometers – at least one control accelerometer is required for proper operation, response accelerometers are optional
  - a. *To Mount Piezotronics Accelerometers*
    - i. Place scotch tape over the accelerometer mounting locations.
    - ii. Apply a light coat of superglue to the mounting surface of the accelerometer.
    - iii. Press the accelerometer to the mounting location and apply pressure for 15 seconds, then lightly apply pressure perpendicular to the accelerometer axis to verify adhesion.
    - iv. Carefully attach a blue accelerometer cable to the accelerometer, then to the BNC connector on the back of the computer. Control is typically connected to Channel 1 and response to Channel 2. Carefully secure the blue cables with the Velcro strips attached to the vibe table frame.

To create or edit a test profile:

1. From the desktop, open "Vwin II"
2. Exit the default test.
3. Click "Edit/Create Setup" at the top, then click the test type (Sine, Random, etc).
4. Select a Setup ID from the list to edit, or create a new name in the Setup ID field and press return.
5. Work through each tab from left to right to setup the desired test.
  - The default calibration value for the Piezotronics accelerometers is 9.740 mV/g.
6. Click the check mark at the bottom at any time to check the validity of the test profile.
7. Click exit when done to save and go back to the main screen.

To prepare for test:

1. From the desktop, open "Vwin II"
2. Exit the default test by clicking "Exit" at the bottom.
3. Follow the Create/Edit instructions above to verify the profile and accelerometer information is correct.
4. Click "Run Test" at the top, then click the test type (Sine, Random, etc).
5. Click the test profile under Setup ID Name.
6. Enter a run name and optional description and press return.

To run test:

On Control Panel

1. Hold the "System Power" button until you hear two 'clicks' (about a second or two), the fan should start.
2. View "DC Supply" on the display by pressing the white switch.
3. Confirm that the voltage reads ~350 V
4. View "Output" on the display by pressing the white switch.
5. Press "Operate"
6. Turn the amp gain to "MAX"

Note: Exercise care when the amp gain is turned up. In this state, the table is "armed" and anything going into the input is amplified into the table. **DO NOT** touch or disconnect the output cable between the computer and the amplifier cabinet for risk of serious injury and damage to the equipment.

On Computer

7. Use ear plugs for high level tests
8. Click "Auto" to begin the test. (Assuming test profile is already open)
  - a. Pay attention to the output current level on the control panel. Be ready to press "stop" on the computer if it looks like it may reach ~100 A or 100 %.
9. After test is finished, click "Save" and accept the save.
10. Optional: Click the MS Word icon to produce a simple test report.
11. Click "Exit"
12. Optional: Click the test you just ran and click Binary to ASCII at the bottom left to save a CSV file of the data.

Vibe Table Shutdown (at the end of a test, or when changing the mounting scheme)

On Control Panel

1. Turn amp gain all the way down.
2. Press "Ready".
3. Select "DC Supply" on the display.
4. Remove accelerometers and clean off superglue with razor blade and/or acetone.
5. Remove the test article from the table.
6. Press "Off", then turn off the main power switch on the wall.

C. Sine Sweep Profile

The shake profile used was a demo sine sweep. It is documented by the shake table software and published below.

Cal Poly Structures & Composites Lab Swept Sine Test Report

SETUP NAME: demo
SETUP COMMENTS: Demo Example Setup.
May be run in loopback (Drive connected to Ch1).
RUN NAME: aluminum_10inches_halfinch
USER/PROJECT FOLDER: CubeSat
SAVE NUMBER: 1

STATUS INFORMATION

TEST EVENT TIME: Monday, May 07, 2012 at 03:42:38 PM
TEST STATUS: STOPPED
TEST MODE: AUTO
TOTAL TIME ELAPSED (HH:MM:SS): 0:0:41
AUTO TIME ELAPSED (HH:MM:SS): 0:0:27
SWEEP #: 1
FREQUENCY: 280.41 Hz
REFERENCE: 5.00 g pk
CONTROL ACCELERATION: 4.96 g pk
CONTROL VELOCITY: 1.09 in/s pk
CONTROL DISPLACEMENT: 1.23 mil pp

CONTROL PARAMETERS

CONTROL CHANNEL(S): 1
 CONTROL TYPE: SINGLE
 SWEEP TIME: 40 sec
 SWEEP TYPE: LOG
 STARTING SWEEP DIRECTION: UP
 STARTING FREQUENCY: 5.00 Hz
 LOWER FREQUENCY: 5.00 Hz
 UPPER FREQUENCY: 2000.00 Hz
 SERVO SPEED: 1K dB/s

INPUT CHANNEL PARAMETERS

Chan (#)	Sensitivity (mV/g)	Coupling (AC/DC)	Max.Range (g pk)
1	10.00	AC	100.00
2	10.00	AC	100.00
3	10.00	AC	100.00
4	10.00	AC	100.00

D. Matlab Code Used for Computing the Analytical Solution

The Matlab code used for computing the analytical solution is divided into one master script that sets up different cases using functions and evaluates the cases using more functions. The master script is called executable, and is listed below. The other functions that it calls are listed after it.

Executable.m

```
%% Clean Up
clear all;
close all;
clc;

%% Set Flags
flg.Euler      = 1;      %Euler (first order) Modal Analysis, 0 = Off 1 = On
flg.Timoshenko = 0;      %Timoshenko (second order) Modal Analysis, 0 = Off 1 = On
flg.Plots       = 1;      %Plots Master Switch, 0 = Off 1 = On
flg.Dynamic     = 1;      %Show Dynamic Animation, 0 = Off, 1 = On
flg.TimeDomain  = 1;      %Show Time Domain Response, 0 = Off, 1 = On
flg.Piezo       = 1;
BC              = 3;      % Fixed Free Boundary Condition

%% Set Conversions
meters2inches  = 39.3700787;
inches2meters  = 0.0254;

%% Analysis Type

flg.Analysis = 2;

%1 Mine
% Hrovat Lesniak
% Model Airplane Boom
% Gabe's
% Hybrid Damping Shelf
% Aluminum Sailplane

switch flg.Analysis
case 1 % Initialize Test Case - Aluminum Beams

    [h,Z,theta,mtrl] = InitializeTestCase2(1,10125);
    b = 4*inches2meters;
    EZ = 0;
    Experimental.Length = [9.5,11.5,14.5]*inches2meters; % [m]
    Experimental.Mode1_Hz = [41.96,28.854,17.86]; % [Hz]
    Experimental.Mode2_Hz = [260.2,181.077,113.127]; % [Hz]
    Experimental.Mode3_Hz = [710,497.032,315.206]; % [Hz]
    Experimental.Mode1_Zeta = [0.018683,0.017111,0.018585]; % [Hz]
    Experimental.Mode2_Zeta = [0.004237,0.007226,0.010690]; % [Hz]
    Experimental.Mode3_Zeta = [0.005346,0.004893,0.003754]; % [Hz]
    Experimental.Mode1_g_g = [4.277,3.113,1.815];
    Experimental.Mode2_g_g = [23.96,17.76,11.4];
    Experimental.Mode3_g_g = [23.63,18.21,18.92];
    [Amesh,Hmesh] = meshgrid([Experimental.Length,0.2:0.01:.5],h); %
    flg.Dynamic = 1;
    time = 0:.001:1;
    intial_displacement = 0.1;
    %% Run Calculations for specified cases
    for i = 1:size(Amesh,1)
        for j = 1:size(Amesh,2)
            a = Amesh(i,j);
            h = Hmesh(i,j);
            [zta,Eul,~] =
AnalyticSolution(a,b,h,Z,theta,mtrl,flg,time,intial_displacement,EZ);
            AluminumBeam.(['RunNum',num2str(j)]) = Eul;
            AluminumBeam.(['Zeta',num2str(j)]) = zta;
```



```

        freq(i,j,1:3) = Eul.freq;
    end
end
% Display Euler Frequency Solution
disp('Euler / Classic Laminated Plate Theory:')
disp(['Length (inches) : 'a*meters2inches])
disp(num2str(Eul.freq))

figure()
axes('fontsize',20)

plot(Experimental.Length*meters2inches,Experimental.Model_Hz,'*r','MarkerSize',10)
hold on
plot(Amesh(:,4:end)*meters2inches,freq(:,4:end,1),'g','LineWidth',2)
% plot(Amesh(:,4:end)*meters2inches,Tim.freq(:,4:end,1),'b','LineWidth',2)
hold off
title('First Mode, Aluminum 2024, Length x 4" x 0.125"', 'fontsize',20)
xlabel('Beam Length [inches]', 'fontsize',20)
ylabel('Frequency [Hz]', 'fontsize',20)
legend('Experimental', 'Euler')
save('Aluminum2024_FirstMode.jpg')
%% PLOT SECOND MODE
figure()
axes('fontsize',20)

plot(Experimental.Length*meters2inches,Experimental.Mode2_Hz,'*r','MarkerSize',10)
hold on
plot(Amesh(:,4:end)*meters2inches,freq(:,4:end,2),'g','LineWidth',2)
hold off
title('Second Mode, Aluminum 2024, Length x 4" x 0.125"', 'fontsize',20)
xlabel('Beam Length [inches]', 'fontsize',20)
ylabel('Frequency [Hz]', 'fontsize',20)
legend('Experimental', 'Euler')
save('Aluminum2024_SecondMode.jpg')
%% PLOT THIRD MODE
figure()
axes('fontsize',20)

plot(Experimental.Length*meters2inches,Experimental.Mode3_Hz,'*r','MarkerSize',10)
hold on
plot(Amesh(:,4:end)*meters2inches,freq(:,4:end,3),'g','LineWidth',2)
hold off
title('Third Mode, Aluminum 2024, Length x 4" x 0.125"', 'fontsize',20)
xlabel('Beam Length [inches]', 'fontsize',20)
ylabel('Frequency [Hz]', 'fontsize',20)
legend('Experimental', 'Euler')
save('Aluminum2024_ThirdMode.jpg')
%% PLOT ERROR
figure()
axes('fontsize',20)
scatter(Experimental.Length*meters2inches,(Experimental.Model_Hz-
freq(:,1:3,1))./freq(:,1:3,1)*100,100,'b')
hold on
scatter(Experimental.Length*meters2inches,(Experimental.Mode2_Hz-
freq(:,1:3,2))./freq(:,1:3,2)*100,100,'g')
scatter(Experimental.Length*meters2inches,(Experimental.Mode3_Hz-
freq(:,1:3,3))./freq(:,1:3,3)*100,100,'k')
hold off
xlabel('Beam Length [inches]', 'fontsize',20)
ylabel('Percent Error [%]', 'fontsize',20)
title('Error between Analytic and Experimental Results, Aluminum 2024, Length x
4" x 0.125"', 'fontsize',20)
legend('Mode 1 Euler', 'Mode 2 Euler', 'Mode 3 Euler')
save('Aluminum2024_Error.jpg')
%Modes with Length

figure()
num_modes = length(modes);
for i = 1:num_modes
    subplot(num_modes,1,modes(i), 'fontsize',20)

plot(AluminumBeam.RunNum3.x0*meters2inches,AluminumBeam.RunNum3.W(modes(i),:)*meters2inch
es,'LineWidth',2)
% title([mtrl{1}, ' ', 'Fixed Free', ' Frequency =
',num2str(Eul.omega(:,i)/(2*pi)), 'Hz'])
end
load('/Users/Kellie_Tremaine/Desktop/Thesis - Doesn't Fit on
Dropbox/AluminumBeams.mat')
figure()
axes('fontsize',20)

```

```

        loglog(AluminumBeam.RunNum1.woff/(2*pi),abs((AluminumBeam.RunNum1.Wtt_off(:,end-1)).^2./AluminumBeam.RunNum1.ddW_off(:,end-1).^2)/(2*pi)^2/(73.1e9),'LineWidth',2)
        hold on
        loglog(AluminumBeam.RunNum2.woff/(2*pi),abs((AluminumBeam.RunNum2.Wtt_off(:,end-1)).^2./AluminumBeam.RunNum2.ddW_off(:,end-1).^2)/(2*pi)^2/(73.1e9),'g','LineWidth',2)
        loglog(AluminumBeam.RunNum3.woff/(2*pi),abs((AluminumBeam.RunNum3.Wtt_off(:,end-1)).^2./AluminumBeam.RunNum3.ddW_off(:,end-1).^2)/(2*pi)^2/(73.1e9),'k','LineWidth',2)
        hold off
        axis([5,2000,10^-6,10^6])
        xlabel('Frequency [Hz]')
        ylabel('Amplitude')
        legend('9.5" Aluminum','11.5" Aluminum','14.5" Aluminum')

load('/Users/Kellie_Tremaine/Desktop/Thesis - Doesn't Fit on
Dropbox/AllBeams.mat')

figure()
axes('fontsize',20)

loglog(AluminumBeam.RunNum1.woff(2:end)/(2*pi),abs(AluminumBeam.RunNum1.ddW_off(2:end,98)
./AluminumBeam.RunNum1.ddW_off(2:end,99))/2,'b','LineWidth',2)
hold on

loglog(AluminumBeam.RunNum2.woff(2:end)/(2*pi),abs(AluminumBeam.RunNum2.ddW_off(2:end,98)
./AluminumBeam.RunNum2.ddW_off(2:end,99))/2,'g','LineWidth',2)

loglog(AluminumBeam.RunNum3.woff(2:end)/(2*pi),abs(AluminumBeam.RunNum3.ddW_off(2:end,98)
./AluminumBeam.RunNum3.ddW_off(2:end,99))/2,'k','LineWidth',2)

loglog(beam.Al_10.frequency,beam.Al_10.amplitude2./beam.Al_10.amplitude1,'b','LineWidth'
,2)

loglog(beam.Al_12.frequency,beam.Al_12.amplitude2./beam.Al_12.amplitude1,'g','LineWidth'
,2)

loglog(beam.Al_15.frequency,beam.Al_15.amplitude2./beam.Al_15.amplitude1,'k','LineWidth'
,2)
        hold off
        xlabel('Frequency [Hz]')
        ylabel('Amplitude [g/g]')
        legend('9.5" Analytic','11.5" Analytic','14.5" Analytic','9.5"
Experimental','11.5" Experimental','14.5" Experimental')

figure()
axes('fontsize',20)

loglog(AluminumBeam.RunNum1.woff(2:end)/(2*pi),abs((AluminumBeam.RunNum1.ddW_off(2:end,5)
./AluminumBeam.RunNum1.ddW_off(2:end,99))))/99,'b','LineWidth',2)
hold on

loglog(AluminumBeam.RunNum2.woff(2:end)/(2*pi),abs((AluminumBeam.RunNum2.ddW_off(2:end,5)
./AluminumBeam.RunNum2.ddW_off(2:end,99))))/99,'g','LineWidth',2)

loglog(AluminumBeam.RunNum3.woff(2:end)/(2*pi),abs((AluminumBeam.RunNum3.ddW_off(2:end,5)
./AluminumBeam.RunNum3.ddW_off(2:end,99))))/99,'k','LineWidth',2)

loglog(beam.Ten9.frequency,beam.Ten9.amplitude2./beam.Ten9.amplitude1,'b','LineWidth',2)

loglog(beam.Twelve.frequency,beam.Twelve.amplitude2./beam.Twelve.amplitude1,'g','LineWid
th',2)

loglog(beam.Fifteen.frequency,beam.Fifteen.amplitude2./beam.Fifteen.amplitude1,'k','Line
Width',2)
        hold off
        xlabel('Frequency [Hz]')
        ylabel('Amplitude [g/g]')
        legend('9.5" Analytic','11.5" Analytic','14.5" Analytic','9.5"
Experimental','11.5" Experimental','14.5" Experimental')
        case 2 % Initialize Test Case - Composite Beams

        %% INITIALIZE CASE
        [h,Z,theta,mtrl] = InitializeTestCase2(1,40048000);
        b = 4*inches2meters;
        EZ = 0;
        Experimental.Length = [9.5,11.5,14.5]*inches2meters;%[m]
        Experimental.Thickness = [0.048,0.047,0.049]*inches2meters;
        Experimental.Mode1_Hz = [35.8,23.82,14.75];% [Hz]
        Experimental.Mode2_Hz = [218.042,146.817,93.108];% [Hz]
        Experimental.Mode3_Hz = [701.5,399.386,244.337];% [Hz] %Skipped 333.671 for

```

```

11" Skipped 386.437 and 494 for 9"
Experimental.Model1_Zeta = [0.015245,0.015418,0.019629];% [Hz]
Experimental.Mode2_Zeta = [0.012562,0.007882,0.009551];% [Hz]
Experimental.Mode3_Zeta = [0.014426,0.016362,0.007699];% [Hz]
Experimental.Model1_g_g = [1.805,1.696,1.379];
Experimental.Mode2_g_g = [4.747,9.514,5.849];
Experimental.Mode3_g_g = [2.248,3.717,7.938];
Amesh = [Experimental.Length];%0.2:0.01:.5];
Hmesh = [Experimental.Thickness];%0.048*inches2meters*ones(size(0.2:0.01:.5));

flg.Dynamic = 1;
time = 0:.1:1;
intial_displacement = 0.1;
%% Run Calculations for specified cases
for i = 1:size(Amesh,1)
    for j = 1:size(Amesh,2)
        a = Amesh(i,j);
        h = Hmesh(i,j);
        [zta,Eul,~] =
AnalyticSolution(a,b,h,Z,theta,mtrl,flg,time,intial_displacement,EZ);
        CompositeBeam.(['RunNum',num2str(j)]) = Eul;
        CompositeBeam.(['Zeta',num2str(j)]) = zta;
        freq(i,j,1:3) = Eul.freq;
    end
end

% Display Euler Frequency Solution
disp('Euler / Classic Laminated Plate Theory:')
disp(['Length (inches) :a*meters2inches])
disp(num2str(Eul.freq))

%% Plot FIRST MODE
figure()
axes('fontsize',20)

plot(Experimental.Length*meters2inches,Experimental.Model1_Hz,'*r','MarkerSize',10)
hold on
plot(Amesh(:,4:end)*meters2inches,freq(:,4:end,1),'g','LineWidth',2)
% plot(Amesh(:,4:end)*meters2inches,Tim.freq(:,4:end,1),'b','LineWidth',2)
hold off
title('First Mode, Composite, Length x 4" x 0.125"', 'fontsize',20)
xlabel('Beam Length [inches]', 'fontsize',20)
ylabel('Frequency [Hz]', 'fontsize',20)
legend('Experimental','Euler')
save('Composite2024_FirstMode.jpg')
%% PLOT SECOND MODE
figure()
axes('fontsize',20)

plot(Experimental.Length*meters2inches,Experimental.Mode2_Hz,'*r','MarkerSize',10)
hold on
plot(Amesh(:,4:end)*meters2inches,freq(:,4:end,2),'g','LineWidth',2)
hold off
title('Second Mode, Composite, Length x 4" x 0.050"', 'fontsize',20)
xlabel('Beam Length [inches]', 'fontsize',20)
ylabel('Frequency [Hz]', 'fontsize',20)
legend('Experimental','Euler')
save('Composite_SecondMode.jpg')
%% PLOT THIRD MODE
figure()
axes('fontsize',20)

plot(Experimental.Length*meters2inches,Experimental.Mode3_Hz,'*r','MarkerSize',10)
hold on
plot(Amesh(:,4:end)*meters2inches,freq(:,4:end,3),'g','LineWidth',2)
hold off
title('Third Mode, Composite, Length x 4" x 0.050"', 'fontsize',20)
xlabel('Beam Length [inches]', 'fontsize',20)
ylabel('Frequency [Hz]', 'fontsize',20)
legend('Experimental','Euler')
save('Composite_ThirdMode.jpg')
%% PLOT ERROR
figure()
axes('fontsize',20)
scatter(Experimental.Length*meters2inches,(Experimental.Model1_Hz-
freq(1,1:3,1))./freq(1,1:3,1)*100,100,'b')
hold on
scatter(Experimental.Length*meters2inches,(Experimental.Mode2_Hz-
freq(1,1:3,2))./freq(1,1:3,2)*100,100,'g')
scatter(Experimental.Length*meters2inches,(Experimental.Mode3_Hz-
freq(1,1:3,3))./freq(1,1:3,3)*100,100,'k')
hold off

```

```

xlabel('Beam Length [inches]','fontsize',20)
ylabel('Percent Error [%]','fontsize',20)
title('Error between Analytic and Experimental Results, Composite, Length x 4" x
0.125"', 'fontsize',20)
legend('Mode 1 Euler','Mode 2 Euler','Mode 3 Euler')
save('Composite_Error.jpg')
%% PLOT MODE SHAPES
figure()
num_modes = length(modes);
for i = 1:num_modes
    subplot(num_modes,1,modes(i), 'fontsize',20)

plot(CompositeBeam.RunNum3.x0*meters2inches,CompositeBeam.RunNum3.W(modes(i),:)*meters2inches, 'LineWidth',2)
end

%% PLOT FRF
load('/Users/Kellie_Tremaine/Desktop/Thesis - Doesn't Fit on
Dropbox/AllBeams.mat')
figure()
axes('fontsize',20)
figure()
axes('fontsize',20)

loglog(CompositeBeam.RunNum1.woff(2:end)/(2*pi),abs(CompositeBeam.RunNum1.ddW_off(2:end,9
8)./CompositeBeam.RunNum1.ddW_off(2:end,99))/2,'b','LineWidth',2)
hold on

loglog(CompositeBeam.RunNum2.woff(2:end)/(2*pi),abs(CompositeBeam.RunNum2.ddW_off(2:end,9
8)./CompositeBeam.RunNum2.ddW_off(2:end,99))/2,'g','LineWidth',2)

loglog(CompositeBeam.RunNum3.woff(2:end)/(2*pi),abs(CompositeBeam.RunNum3.ddW_off(2:end,9
8)./CompositeBeam.RunNum3.ddW_off(2:end,99))/2,'k','LineWidth',2)

loglog(beam.Cp_10.frequency,beam.Cp_10.amplitude2./beam.Cp_10.amplitude1,'b','LineWidth',
2)

loglog(beam.Cp_12.frequency,beam.Cp_12.amplitude2./beam.Cp_12.amplitude1,'g','LineWidth',
2)

loglog(beam.Cp_15.frequency,beam.Cp_15.amplitude2./beam.Cp_15.amplitude1,'k','LineWidth',
2)

hold off
xlabel('Frequency [Hz]')
ylabel('Amplitude [g/g]')
legend('9.5" Analytic','11.5" Analytic','14.5" Analytic','9.5"
Experimental','11.5" Experimental','14.5" Experimental')

case 3 % Cavalli - Aluminum Sailplane

[h,Z,theta,mtrl] = InitializeTestCase2(1,10062);
b = 6.03*inches2meters;
EZ = 0;
Experimental.Length = 24.0*inches2meters; % [m]
Experimental.Mode1_Hz = 4.0;% [Hz]
Experimental.Mode2_Hz = 22.3;% [Hz]
Experimental.Mode3_Hz = 62.3;% [Hz]
Experimental.Gain = [0,1,2,4,6,6.27,8,8.67,10,12,20,30,36.33];
Experimental.DampingRatio =
[.86123,0.95969,1.09920,1.40624,1.60850,1.68742,1.98811,2.09431,2.37963,2.74439,4.18009,6
.68553,8.06433]*0.01;
a = Experimental.Length;
flg.Dynamic = 1;
time = 0:.001:1;
intial_displacement = 0.1;
%% Run Calculations for specified cases
[zta,Eul,~] =
AnalyticSolution(a,b,h,Z,theta,mtrl,flg,time,intial_displacement,EZ);
SP_Wout = Eul;
SP_Wout.Zeta = zta;
[h,Z,theta,mtrl] = InitializeTestCase2(2,10062,60010);
[zta,Eul,~] =
AnalyticSolution(a,b,h,Z,theta,mtrl,flg,time,intial_displacement,EZ);
SP_With = Eul;
SP_With.Zeta = zta;

EZ = (0:.5:1)*1.4919e+05;%500000;
[h,Z,theta,mtrl] = InitializeTestCase2(2,10062,50010);
for ii = 1:length(EZ)
    [zta,Eul,~] =
AnalyticSolution(a,b,h,Z,theta,mtrl,flg,time,intial_displacement,EZ(ii));

```

```

        SailPlane.(['EZ',num2str(ii)]) = Eul;
        SailPlane.(['EZ',num2str(ii)]).Zeta = zta;
        ActiveZeta(ii) = SailPlane.(['EZ',num2str(ii)]).zeta_calc;
    end

    figure()
    axes('fontsize',20)
    plot((0:.5:1)*36,ActiveZeta+SailPlane.EZ1.Zeta,'LineWidth',2)
    hold on
    plot((0:.5:1)*36,ActiveZeta*2/pi+SailPlane.EZ1.Zeta,'k','LineWidth',2)
    scatter(Experimental.Gain,Experimental.DampingRatio,'xr','LineWidth',2)
    hold off
    xlabel('Gain [V]')
    ylabel('Damping Ratio [-]')
    legend('Square Wave','Sine Wave','Cavalli's Experimental Data')

    figure()
    axes('fontsize',20)

    loglog(SP_Wout.woff(2:end)/(2*pi),abs((SP_Wout.ddW_off(2:end,98)./(SP_Wout.ddW_off(2:end,99))))/2,'b','LineWidth',2)
    hold on

    loglog(SP_With.woff(2:end)/(2*pi),abs((SP_With.ddW_off(2:end,98)./(SP_With.ddW_off(2:end,99))))/2,'r','LineWidth',2)
    xlabel('Frequency [Hz]')
    ylabel('Amplitude [g/g]')
    legend('Without Piezoelectric Patch','With Piezoelectric Patch, V = 0')

    case 4 %Hovrat and Lesniak

        time = 0:.001:1;
        flg.Dynamic = 1;
        intial_displacement = 0.1;

        %%Proportional Control
        Experimental.Gain_P = [ 1.0946 2.0128 3.0091 4.0046 5.0007
6.0172 7.7346];
        Experimental.DampingRatio_P = ([0 30.9888 45.4322 50.8139 64.1246
81.9605 149.4381]*0.01*0.0041)+ones(size(1,7))*0.0041;

        Experimental.Gain_PD = [ 0.000 1.1359 2.0145 3.0113 4.0274
5.0236 6.0197];
        Experimental.DampingRatio_PD = ([-0.1578 16.5116 50.2447 71.4844 84.7894
98.1001 110.2781]*0.01*0.0041)+ones(size(1,7))*0.0041;

        Experimental.Gain_PI = [ 0.0000 1.1288 2.0688 3.0472 4.0632
5.0801];
        Experimental.DampingRatio_PI = ([0 163.7688 216.7410 253.8441 266.0165
289.5158]*0.01*0.0041)+ones(size(1,7))*0.0041;

        Experimental.Gain_PID = [0 1.1091 2.0298 3.0862 4.0234 5.0613];
        Experimental.DampingRatio_PID = ([0 164.9071 225.8139 244.7712 266.0277
301.9812]*0.01*0.0041)+ones(size(1,6))*0.0041;

        [h,Z,theta,mtrl] = InitializeTestCase2(2,70025,60010);
        a = 13.0*inches2meters; %[m] effective length in fixture
        b = 0.80*inches2meters;
        EZ = 0;
        [zta,Wpiezo,~] =
AnalyticSolution(a,b,h,Z,theta,mtrl,flg,time,intial_displacement,EZ);
        Wpiezo.zeta = zta;

        [h,Z,theta,mtrl] = InitializeTestCase2(1,70025);
        [zta,Woutpiezo,~] =
AnalyticSolution(a,b,h,Z,theta,mtrl,flg,time,intial_displacement,EZ);
        Woutpiezo.zeta = zta;

        figure()
        axes('fontsize',20)

        loglog(Woutpiezo.woff(2:end)/(2*pi),abs((Woutpiezo.ddW_off(2:end,98)./(Woutpiezo.ddW_off(2:end,99))))/2,'b','LineWidth',2)
        hold on

        loglog(Wpiezo.woff(2:end)/(2*pi),abs((Wpiezo.ddW_off(2:end,98)./(Wpiezo.ddW_off(2:end,99))))/2,'r','LineWidth',2)
        xlabel('Frequency [Hz]')
        ylabel('Amplitude [g/g]')
        legend('Without Piezoelectric Patch','With Piezoelectric Patch, V = 0')

```

```

[h,Z,theta,mtrl] = InitializeTestCase2(2,70025,50010);
EZ = (0:4.5:9)*8.9477e+03/10; %500 V/mm
for ii = 1:length(EZ)
    [zta,Eul,~] =
AnalyticSolution(a,b,h,Z,theta,mtrl,flg,time,intial_displacement,EZ(ii));
    SailPlane.(['EZ',num2str(ii)]) = Eul;
    SailPlane.(['EZ',num2str(ii)]).Zeta = zta;
    ActiveZeta(ii) = SailPlane.(['EZ',num2str(ii)]).zeta_calc;
end

figure()
axes('fontsize',20)
plot((0:4.5:9),ActiveZeta+Woutpiezo.zeta,'b','LineWidth',2)
hold on
plot((0:4.5:9),ActiveZeta*2/pi+Woutpiezo.zeta,'k','LineWidth',2)
scatter(Experimental.Gain_P,Experimental.DampingRatio_P,'xm','LineWidth',2)
scatter(Experimental.Gain_PD,Experimental.DampingRatio_PD,'xr','LineWidth',2)
scatter(Experimental.Gain_PI,Experimental.DampingRatio_PI,'xb','LineWidth',2)
scatter(Experimental.Gain_PID,Experimental.DampingRatio_PID,'xg','LineWidth',2)
scatter(8.4,0.0170)
hold off
xlabel('Gain [V]')
ylabel('Damping Ratio [-]')
legend('Square Wave Prediction','Sine Wave Prediction','Lesniak's Proportional
Controller','Lesniak's P-D Controller','Lesniak's PI Controller','Lesniak's PID
Controller','Hovrat's Analog Controller')

end

```

InitializeTestCase2.m

```

function [H,Z,theta,mtrl] = InitializeTestCase2(varargin)

inches2meters = 0.0254;

if isempty(varargin)
    numlamina = 1;
    Case = 1311; % Set default test case
elseif length(varargin) == 1
    numlamina = 1;
    Case = varargin{1};
else
    numlamina = varargin{1};
    if length(varargin) == numlamina+1
        for ii = 1:numlamina
            Case{ii} = varargin{ii+1};
        end
    else
        error('Number of lamina should match number of case inputs!!!')
    end
end

for ii = 1:numlamina
    CaseStr = num2str(Case{ii});

    % Assign Material
    switch CaseStr(1)
        case '1' % Aluminum
            mtrl{ii} = 'Aluminum';
        case '2' % Copper
            mtrl{ii} = 'Copper';
        case '3' % Steel
            mtrl{ii} = 'Steel';
        case '4' % GraphiteEpoxyMetric
            mtrl{ii} = 'CarbonEpoxyMetric';
        case '5' % PZT
            mtrl{ii} = 'PZT';
        case '6' % Lumped Mass
            mtrl{ii} = 'Mass';
        case '7' % Aluminum6010
            mtrl{ii} = 'Aluminum6010';
        case '8' % PZT as Lumped Mass
            mtrl{ii} = 'PZTasLumpedMass';
        otherwise
            error('Please enter a valid material code!!!');
    end
end

```

```

end

% Thickness
h(ii) = str2num(CaseStr(2:5))/1000*inches2meters;
% Laminate Profile
if length(CaseStr) == 8
    theta(ii) = str2num(CaseStr(6:8))*pi/180; %#ok<*ST2NM>
else
    theta(ii) = 0;
end
end

H = sum(h);

Z(1) = -H/2;
for ii = 1:numlamina
    Z(ii+1) = Z(ii)+h(ii); %#ok<*AGROW>
end

```

AnalyticSolution.m

```
function [zta,Eul,Tim] =
AnalyticSolution(a,b,h,Z,theta,mtrl,flg,time,intial_displacement,EZ)

%%Initialize Outputs
Eul = 0;
Tim = 0;
zta= 0;
Lmap = logspace(-2.5,3.0,1000);
lumpedmass = 0;

for ii = 1:length(mtrl)
    if strcmp(mtrl{ii},'Mass') || strcmp(mtrl{ii},'PZTasLumpedMass')
        rho = zeros(length(mtrl),1000);
        lumpedmass = 1;
    end
end

for k = 1:length(mtrl)
    if strcmp(mtrl{k},'Mass') || strcmp(mtrl{ii},'PZTasLumpedMass')

        %% Hovrat, Lesniak
        rholump = 7730; % [kg/m^3]
        L1 = 0;
        L2 = 1.25/16*a;

        %% SAILPLANE
        rholump = 7730*2/3; % [kg/m^3]
        L1 = 0;
        L2 = 2.5/24*a;
        [rho(k,:)] = findequivalentrho(Lmap,rholump,L1,L2,a);
        lumpedmass = 1;
    else
        [E1,E2,G12,G13,G23,v12,v21,rho1,zta0(k),d13(k),d23(k)] =
specifymaterial(mtrl{k});
        rho(k,:) = rho1*ones(1,1000);
        Qbar(:, :, k) = Q2Qbar(theta(k),E1,E2,G12,G13,G23,v12,v21); %#ok<*SAGROW>
        %Transform the Stress-Reduced Stiffness Matrix
        zta = zta+zta0(k)*(Z(k+1)-Z(k))/h;
    end
end

[A] = CalcExtensionalStiffness(Qbar,Z); %Calculate Extensional Stiffness
[B] = CalcBendingExtensionalCoupling(Qbar,Z);
[D] = CalcBendingStiffness(Qbar,Z); %Calculate Bending Stiffness

Astar = inv(A); %Find inverse extensional stiffness matrix
Dstar = inv(D); %Find inverse bending stiffness matrix

[I0,~,I2] = MassMomentInertia(rho,Z);
Ihat0 = b*I0;
Ihat2 = b*I2;
Exxb_x_Iyy = b/Dstar(1,1); % Where Dstar(1,1) = D*11
Eul.Exxb_x_Iyy = Exxb_x_Iyy;

modes = [1,2,3];
%% Calculate Euler Solution
if flg.Euler
    if lumpedmass == 0
        [Eul.omega] = CalcNaturalFreq('FixedFree',Exxb_x_Iyy,Ihat0,Ihat2,a,modes);
    else
        [Eul.omega] = CalcNaturalFreq('FixedFree',Exxb_x_Iyy,Ihat0,Ihat2,a,modes,Lmap);
    end
    Eul.freq = Eul.omega/(2*pi);
end

%% Shape and Dynamic Solution
if flg.Dynamic
    if flg.Euler
        [Eul.x0,Eul.W,Eul.w0u,Eul.w0d,Eul.Ubeam,Eul.ddW] =
CalcDisplacement('FixedFree',Exxb_x_Iyy,Ihat0,Ihat2,a,b,modes,Eul.omega,zta,time,intial_d
isplacement);

        [Eul.x0_off,Eul.W_off,Eul.Ubeam_off,Eul.w0ff,Eul.ddW_off,Eul.d4W,Eul.Wt_off,Eul.Wtt_off,E
ul.Energy,Eul.L,Eul.L_static,Eul.W_static,Eul.ddW_static,Eul.Ubeam_static] =
CalcOffNaturalFreqDisplacement(Exxb_x_Iyy,Ihat0,Ihat2,a,b,intial_displacement,Lmap);
        Eul.maxdeflection(1) = max(abs(Eul.w0u(1,:)));
    end
end
```



```

        Eul.maxdeflection(2) = max(abs(Eul.w0u(2,:)));
        Eul.maxdeflection(3) = max(abs(Eul.w0u(3,:)));
        [Eul.Wpiezo,Eul.Upiezo] =
CalcPiezoDisplacement(Qbar,A,D,a,b,Z,d13,d23,EZ,Eul.x0,6);
        Eul.maxdeflectpiezo(1) = max(abs(Eul.Wpiezo));
        Eul.zeta_calc = Eul.Upiezo*.06/1/Eul.ddW(1).^2/(2*pi);%Ubeam(1)
    end

end

```

SpecifyMaterial.m

```
function [E1,E2,G12,G13,G23,v12,v21,rho,zta,d13,d23,L] = specifymaterial(MatType,L1,L2,a)
%specifymaterial returns engineering constants for several types of
%materials in psi
%
% [E1,E2,G12,G13,G23,v12,v21,rho,zta,d13,d23] = specifymaterial(MatType) return
longitudinal
% modulus, transverse modulus, shear modulus in the 12, 13, and 23
% directions, the major poisson's ratio, the minor poisson's ratio,
% density, damping ratio, piezoelectric longitudinal coupling
% coefficient, and piezoelectric transverse coupling
% coefficient.
%
%
% Allowed Material types are:
% 'Aluminum' Data taken from MLCPS Table 2.2.1 (pg. 88)
% 'Copper' Data taken from MLCPS Table 2.2.1 (pg. 88)
% 'Steel' Data taken from MLCPS Table 2.2.1 (pg. 88)
% 'GraphiteEpoxyAS' Standard AS/3501 Data taken from MLCPS Table 2.2.1 (pg. 88)
% 'GraphiteEpoxyT' Standard T300/934 Data taken from MLCPS Table 2.2.1 (pg. 88)
% 'CarbonEpoxyMetric' Standard CU125NS Data taken from Interaction of active and
passive vibration control of laminated composite beams with piezoceramic
sensors?actuators

switch MatType
case 'Aluminum'
    E1 = 73.1e9;%[Pa]
    E2 = 73.1e9;%[Pa]
    G12 = 28e9; %[Pa]
    G13 = 28e9; %[Pa]
    G23 = 28e9; %[Pa]
    v12 = 0.33; %[-]
    v21 = 0.33; %[-]
    zta = 0.01; %[-]
    rho = 2780; %[kg/m^3]
    d13 = 0; %[m/V]
    d23 = 0; %[m/V]
case 'Aluminum6010'
    E1 = 69.0e9;%[Pa]
    E2 = 69.0e9;%[Pa]
    G12 = 26e9; %[Pa]
    G13 = 26e9; %[Pa]
    G23 = 26e9; %[Pa]
    v12 = 0.33; %[-]
    v21 = 0.33; %[-]
    zta = 0.0041; %[-]
    rho = 2710; %[kg/m^3]
    d13 = 0; %[m/V]
    d23 = 0; %[m/V]
case 'Copper'
    E1 = 18.0*10^6;
    E2 = 18.0*10^6;
    G12 = 6.39*10^6;
    G13 = 6.39*10^6;
    G23 = 6.39*10^6;
    v12 = 0.33;
    v21 = 0.33;
    zta = 0.02;
    d13 = 0;
    d23 = 0;
case 'Steel'
    E1 = 30.0*10^6;
    E2 = 30.0*10^6;
    G12 = 11.24*10^6;
    G13 = 11.24*10^6;
    G23 = 11.24*10^6;
    v12 = 0.29;
    v21 = 0.29;
    rho = 0.28125;
    zta = 0.05;
    d13 = 0;
    d23 = 0;
case 'CarbonEpoxyMetric'
    E1 = 93.5*10^9; %[Pa]
    E2 = 6.74*10^9; %[Pa]
    G12 = 4.07*10^9; %[Pa]
    G13 = 4.07*10^9; %[Pa]
    G23 = 4.07*10^9; %[Pa]
```

```

v12 = 0.34;          %[-]
v21 = 0.34;          %[-]
rho = 910; %1510;    % [kg/m^3]
zta = 0.01;          %Guess
d13 = 0;              % [m/V]
d23 = 0;              % [m/V]
case 'PZT'
    E1 = 88.4e9;      % [Pa]
    E2 = 76e9;        % [Pa]
    G12 = 21e9;       % [Pa]
    G13 = 21e9;       % [Pa]
    G23 = 21e9;       % [Pa]
    v12 = 0.515;      % [-]
    v21 = 0.6;        % [-]
    rho = 7730;        % [kg/m^3]
    zta = 0.00;        % [-]
    d13 = 256e-12;     % [m/V]
    d23 = 256e-12;     % [m/V]
case 'PZTasLumpedMass'
    E1 = 0;            % [Pa]
    E2 = 0;            % [Pa]
    G12 = 0;           % [Pa]
    G13 = 0;           % [Pa]
    G23 = 0;           % [Pa]
    v12 = 0;           % [-]
    v21 = 0;           % [-]
    rho = 0;           % [kg/m^3]
    zta = 0.00;        % [-]
    d13 = 256e-12;     % [m/V]
    d23 = 256e-12;     % [m/V]
end

```

Q2Qbar.m

```
function [Qbar] = Q2Qbar(varargin)

%%
%Q2Qbar Transform the Stress-Reduced Stiffness Matrix for a lamina angle
%
%   Qbar = Q2Qbar(t,Q) converts the Stress-Reduced Stiffness Matrix, Q,
%   for a lamina angle, t
%
%   Q must be a 3x3 matrix of format:
%   [Q11,Q12,0;
%    Q21,Q22,0;
%    0,0,Q66]
%
%   Qbar is a 3x3 matrix of format:
%   [Q11b,Q12b,Q16b;
%    Q12b,Q22b,Q26b;
%    Q16b,Q26b,Q66b]
%
%   Qbar = Q2Qbar(t,E1,E2,G12,v12,v21) creates the Stress-Reduced Stiffness
%   Matrix, Q, and transforms it for a lamina angle, t, based on the
%   Engineering constants: E1 = longitudinal modulus, E2 = transverse
%   modulus, G12 = shear modulus, v12 = major Poisson's ratio, v21 = major
%   Poisson's ratio (if symmetric),
%
%   Example 1:
%   When angle, t = 0, then the transformed Qbar matrix is the same as
%   the Q matrix.
%
%   [Qbar] = Q_to_Qbar(pi, [2,3,0;3,4,0;0,0,9]) produces the matrix
%
%   Qbar =
%
%       2.0000    3.0000    0.0000
%       3.0000    4.0000   -0.0000
%       0.0000   -0.0000    9.0000
%
%   Example 2:

%%

if size(varargin,2) == 2 %Q entered
    angle = varargin{1}; %Angle of layer, must be in radians
    Q = varargin{2}; % Plane stress-reduced stiffness matrix
    Q11 = Q(1,1);
    Q12 = Q(1,2);
    Q22 = Q(2,2);
    Q44 = Q(4,4);
    Q55 = Q(5,5);
    Q66 = Q(3,3);

elseif size(varargin,2) == 8 %Engineering constants entered
    % Parse input variables
    angle = varargin{1};
    E1 = varargin{2};
    E2 = varargin{3};
    G12 = varargin{4};
    G13 = varargin{5};
    G23 = varargin{6};
    v12 = varargin{7};
    v21 = varargin{8};

    % Calculate elements of Q (Equation 2.4.4 in MLCPS)
    Q11 = E1/(1-v12*v21);
    Q12 = v12*E2/(1-v12*v21);
    Q22 = E2/(1-v12*v21);
    Q44 = G23;
    Q55 = G13;
    Q66 = G12;

else
    error('Q_to_Qbar requires 2 or 6 inputs for ')
end
```

```

for ii = 1:length(angle)

    t = angle(ii);

    % Calculate elements of Qbar (Equation 2.4.8 in MLCPS)
    Q11b(ii) = Q11*cos(t)^4+2*(Q12+2*Q66)*sin(t)^2*cos(t)^2+Q22*sin(t)^4;
    Q16b(ii) = (Q11-Q12-2*Q66)*sin(t)*cos(t)^3+(Q12-Q22+2*Q66)*sin(t)^3*cos(t);
    Q26b(ii) = (Q11-Q12-2*Q66)*sin(t)^3*cos(t)+(Q12-Q22+2*Q66)*sin(t)*cos(t)^3;
    Q12b(ii) = (Q11+Q22-4*Q66)*sin(t)^2*cos(t)^2+Q12*(sin(t)^4+cos(t)^4);
    Q22b(ii) = Q11*sin(t)^4+2*(Q12+2*Q66)*sin(t)^2*cos(t)^2+Q22*cos(t)^4;
    Q44b(ii) = Q44*cos(t)^2+Q55*sin(t)^2;
    Q45b(ii) = (Q55-Q44)*cos(t)*sin(t);
    Q55b(ii) = Q55*cos(t)^2+Q44*sin(t)^2;
    Q66b(ii) = (Q11+Q22-2*Q12-2*Q66)*sin(t)^2*cos(t)^2+Q66*(sin(t)^4+cos(t)^4);
    Qbar(:, :, ii) = [Q11b(ii), Q12b(ii), 0, 0, Q16b(ii);
                     Q12b(ii), Q22b(ii), 0, 0, Q26b(ii);
                     0, 0, Q44b(ii), Q45b(ii), 0;
                     0, 0, Q45b(ii), Q55b(ii), 0;
                     Q16b(ii), Q26b(ii), 0, 0, Q66b(ii)];

end
end

```

CalcExtensionalStiffness.m

```
function [A] = CalcExtensionalStiffness(Qbar,Z)

%CalcExtensionalStiffness Calculate the extensional stiffness matrix, A, from the
%transformed Stress-Reduced Stiffness Matrix, Qbar, of each lamina, and the
%coordinates of the lamina from the midplane of a laminated plate or beam.
%Layer numbering scheme can be found in Fig. 3.3.1 in MLCPS (pg. 113)
%
%   [D] = CalcExtensionalStiffness(Qbar,Z)
%
%
%   Qbar should be a 3x3xN matrix where N corresponds to the number of
%   layers in the beam or plate. If the beam or plate is made of one
%   material such as steel or aluminum, N= 1, and the matrix is a typical
%   3x3 matrix.
%
%   Qbar(:, :, N) is a 3x3 matrix of format:
%   [Q11b,Q12b,Q16b;
%    Q12b,Q22b,Q26b;
%    Q16b,Q26b,Q66b]
%
%   Z is a vector that holds information of the distance of laminate sides
%   from the midplane of the laminated beam. The positive coordinate is
%   downwards. Z has N+1 elements where N is the number of layers in the
%   beam. The first element of the Z corresponds to the negative plate
%   thickness over two. The second element corresponds to the negative
%   plate thickness over two plus the thickness of the first / top layer.
%   This pattern continues so that the last layer corresponds to the
%   positive plate thickness over two.
%
%   Z is a vector of length N+1:
%   [-h/2,-h/2+t1,-h2/2+t1+t2,...,h/2]

% Find number of layers
if ndims(Qbar) == 3
    n = size(Qbar,3);
else
    n = 1;
end

% Initialize Bending Stiffness Matrix
A = zeros(size(Qbar,1),size(Qbar,2));

% Calculate Bending Stiffness Matrix using Equation 3.3.38b in MLCPS
for kk = 1:n
    A = A + Qbar(:, :, kk) * (Z(kk+1)-Z(kk));
end

end
```

CalcBendingStiffness.m

```
function [D] = CalcBendingStiffness(Qbar,Z)

%CalcBendingStiffness Calculate the bending stiffness matrix, D, from the
%transformed Stress-Reduced Stiffness Matrix, Qbar, of each lamina, and the
%coordinates of the lamina from the midplane of a laminated plate or beam.
%Layer numbering scheme can be found in Fig. 3.3.1 in MLCPS (pg. 113)
%
%   [D] = CalcBendingStiffness(Qbar,Z)
%
%
%   Qbar should be a 3x3xN matrix where N corresponds to the number of
%   layers in the beam or plate. If the beam or plate is made of one
%   material such as steel or aluminum, N= 1, and the matrix is a typical
%   3x3 matrix.
%
%   Qbar(:, :, N) is a 3x3 matrix of format:
%   [Q11b,Q12b,Q16b;
%    Q12b,Q22b,Q26b;
%    Q16b,Q26b,Q66b]
%
%   Z is a vector that holds information of the distance of laminate sides
%   from the midplane of the laminated beam. The positive coordinate is
%   downwards. Z has N+1 elements where N is the number of layers in the
%   beam. The first element of the Z corresponds to the negative plate
%   thickness over two. The second element corresponds to the negative
%   plate thickness over two plus the thickness of the first / top layer.
%   This pattern continues so that the last layer corresponds to the
%   positive plate thickness over two.
%
%   Z is a vector of length N+1:
%   [-h/2,-h/2+t1,-h2/2+t1+t2,...,h/2]

% Find number of layers
if ndims(Qbar) == 3
    n = size(Qbar,3);
else
    n = 1;
end

% Initialize Bending Stiffness Matrix
D = zeros(size(Qbar,1),size(Qbar,2));

% Calculate Bending Stiffness Matrix using Equation 3.3.38b in MLCPS
for kk = 1:n
    D = D + 1/3*Qbar(:, :, kk) * (Z(kk+1)^3-Z(kk)^3);
end
end
```

CalcBendingExtensionalCoupling.m

```
function [B] = CalcBendingExtensionalCoupling(Qbar,Z)

%CalcBendingStiffness Calculate the bending stiffness matrix, D, from the
%transformed Stress-Reduced Stiffness Matrix, Qbar, of each lamina, and the
%coordinates of the lamina from the midplane of a laminated plate or beam.
%Layer numbering scheme can be found in Fig. 3.3.1 in MLCPS (pg. 113)
%
% [D] = CalcBendingStiffness(Qbar,Z)
%
%
% Qbar should be a 3x3xN matrix where N corresponds to the number of
% layers in the beam or plate. If the beam or plate is made of one
% material such as steel or aluminum, N= 1, and the matrix is a typical
% 3x3 matrix.
%
% Qbar(:, :, N) is a 3x3 matrix of format:
% [Q11b,Q12b,Q16b;
%  Q12b,Q22b,Q26b;
%  Q16b,Q26b,Q66b]
%
% Z is a vector that holds information of the distance of laminate sides
% from the midplane of the laminated beam. The positive coordinate is
% downwards. Z has N+1 elements where N is the number of layers in the
% beam. The first element of the Z corresponds to the negative plate
% thickness over two. The second element corresponds to the negative
% plate thickness over two plus the thickness of the first / top layer.
% This pattern continues so that the last layer corresponds to the
% positive plate thickness over two.
%
% Z is a vector of length N+1:
% [-h/2,-h/2+t1,-h/2/2+t1+t2,...,h/2]

% Find number of layers
if ndims(Qbar) == 3
    n = size(Qbar,3);
else
    n = 1;
end

% Initialize Bending Stiffness Matrix
B = zeros(size(Qbar,1),size(Qbar,2));

% Calculate Bending Stiffness Matrix using Equation 3.3.38b in MLCPS
for kk = 1:n
    B = B + 1/2*Qbar(:, :, kk) * (Z(kk+1)^2-Z(kk)^2);
end
end
```


MassMomentInertia.m

```
function [I0,I1,I2] = MassMomentInertia(rho,Z)

%MassMomentInertia Calculate the mass moments of inertia, I0, I2, I3, from
% the density of the plate material and the coordinates of the lamina from
% the midplane of a laminated plate or beam. Layer numbering scheme can be
% found in Fig. 3.3.1 in MLCPS (pg. 113)
%
% [I0,I1,I2] = MassMomentInertia(rho,Z)
%
% rho is a vector of length N that corresponds to the density of each
% layer in lb/in^3 for a plate with N layers.
%
% rho = [rho1,rho2,rho3,...,rhoN]
%
% Z is a vector that holds information of the distance of laminate sides
% from the midplane of the laminated beam. The positive coordinate is
% downwards. Z has N+1 elements where N is the number of layers in the
% beam. The first element of the Z corresponds to the negative plate
% thickness over two. The second element corresponds to the negative
% plate thickness over two plus the thickness of the first / top layer.
% This pattern continues so that the last layer corresponds to the
% positive plate thickness over two.
%
% Z is a vector of length N+1:
% Z = [-h/2,-h/2+t1,-h/2+t1+t2,...,h/2]

[m,n] = size(rho);

%Initialize variables
I0 = zeros(n,1);
I1 = zeros(n,1);
I2 = zeros(n,1);

for kk = 1:m
    %Calculate Mass Moments of Inertia using Equation 3.3.20c in MLCPS
    I0 = I0 + (Z(kk+1)-Z(kk))*rho(kk,:)' ;
    I1 = I1 + 1/2*(Z(kk+1)^2-Z(kk)^2)*rho(kk,:)' ;
    I2 = I2 + 1/3*(Z(kk+1)^3-Z(kk)^3)*rho(kk,:)' ;
end
end
```

CalcNaturalFreq.m

```
function [omega] = CalcNaturalFreq(BC,Exxb_x_Iyy,Ihat0map,Ihat2map,a,modes,Lmap)

% Initialize Variables
omega = zeros(1,length(modes));

[L] = FindLambdas(BC,a);

if nargin == 7
    Ihat0 = interp1q(Lmap',Ihat0map,L');
    Ihat2 = interp1q(Lmap',Ihat2map,L');
else
    Ihat0 = Ihat0map*ones(size(modes));
    Ihat2 = Ihat2map*ones(size(modes));
end

for ii = 1:length(modes)
    omega(ii) = sqrt( L(ii)^4*Exxb_x_Iyy/Ihat0(ii) * (1
/(1+Ihat2(ii)/Ihat0(ii)*L(ii)^2)) );
end
```

FindLambdas.m

```
function [L] = FindLambdas(BC,a)

switch BC
case 'HingedHinged'
    L(1) = pi/a;
    L(2) = 2*pi/a;
    L(3) = 3*pi/a; %#ok<*NASGU>
case 'FixedFixed'
    L(1) = 4.730/a;
    L(2) = 7.853/a;
    L(3) = 10.996/a;
case 'FixedFree'
    L(1) = 1.875/a;
    L(2) = 4.694/a;
    L(3) = 7.855/a;
case 'FreeFree'
    L(1) = 4.730/a;
    L(2) = 7.853/a;
    L(3) = 10.996/a;
case 'HingedFixed'
    L(1) = 3.927/a;
    L(2) = 7.069/a;
    L(3) = 10.210/a;
case 'HingedFree'
    L(1) = 3.927/a;
    L(2) = 7.069/a;
    L(3) = 10.210/a;
end
end
```

CalcDisplacement.m

```
function [X,W,w0_undamped,w0_damped,Ubeam,ddW] =
CalcDisplacement(BC,Exxb_x_Iyy,Ihat0,Ihat2,a,b,modes,w,zta,t,disp0)

% Initialize Variables
Nhatxx = 0;
npts = 100;
X = linspace(0,a,npts);
W = zeros(length(modes),npts);
zta = sum(zta)/length(zta);

[c1,c2,c3,c4] = findconstants(BC);
[L] = FindLambdas(BC,a);
mu = L;
for ii = modes

    for jj = 1:length(X)
        W(ii,jj) =
b*(c1(ii)*sin(L(ii)*X(jj))+c2(ii)*cos(L(ii)*X(jj))+c3(ii)*sinh(mu(ii)*X(jj))+c4(ii)*cosh(
mu(ii)*X(jj)));
        ddW(ii,jj) = b*(-c1(ii)*L(ii)^2*sin(L(ii)*X(jj))-
c2(ii)*L(ii)^2*cos(L(ii)*X(jj))+c3(ii)*mu(ii)*sinh(mu(ii)*X(jj))+c4(ii)*mu(ii)*cosh(mu(ii)
)*X(jj));
    end
    W(ii,:) = W(ii,:)*disp0;
    ddW(ii,:) = ddW(ii,:)*disp0;
    for jj = 1:length(X)
        for kk = 1:length(t)
            w0_undamped(ii,jj,kk) = W(ii,jj)*real(exp(sqrt(-1)*w(ii)*t(kk)));
            w0_damped(ii,jj,kk) = W(ii,jj)*exp(-
w(ii)*zta*t(kk))*real(exp(1i*w(ii)*t(kk)));
        end
    end
    Ubeam(ii) = sum(1/2*Exxb_x_Iyy*ddW(ii,:).^2*a/npts);
end
```

findconstants.m

```
function [c1,c2,c3,c4] = findconstants(BC)

% These are the solutions for L (Lambda) of the following BCs for the
% equation:
% 0 = c1*sin(L*a) + c2*cos(L*a) + c3*sinh(mu*a) + c4*cosh(mu*a);
switch BC
    case 'FixedFixed'
        en = [4.730,7.853,10.996];
        c1 = 1./ (sin(en)-sinh(en));
        c2 = -1./ (cos(en)-cosh(en));
        c3 = -1./ (sin(en)-sinh(en));
        c4 = 1./ (cos(en)-cosh(en));
    case 'FixedFree'
        en = [1.875,4.694,7.855];
        c1 = 1./ (sin(en)+sinh(en));
        c2 = -1./ (cos(en)+cosh(en));
        c3 = -1./ (sin(en)+sinh(en));
        c4 = 1./ (cos(en)+cosh(en));
    case 'FreeFree'
        en = [4.730,7.853,10.996];
        c1 = 1./ (sin(en)-sinh(en));
        c2 = -1./ (cos(en)-cosh(en));
        c3 = 1./ (sin(en)-sinh(en));
        c4 = -1./ (cos(en)-cosh(en));
    case 'HingedFixed'
        en = [3.927,7.069,10.210];
        c1 = 1./sin(en);
        c2 = 0;
        c3 = 1./sinh(en);
        c4 = 0;
    case 'HingedFree'
        en = [3.927,7.069,10.210];
        c1 = 1./sin(en);
        c2 = 0;
        c3 = -1./sinh(en);
        c4 = 0;
end
```

CalcOffNaturalFreqDisplacement.m

```
function [X,W,Ubeam,w,ddW,d4W,Wt,Wtt,Energy,L,L_static,W_static,ddW_static,Ubeam_static]
= CalcOffNaturalFreqDisplacement(Exxb_x_Iyy,Ihat0map,Ihat2map,a,b,disp0,Lmap)

% Initialize Variables
w = logspace(-2,1.7,2000)*2*pi;
npts = 100;
X = linspace(0,a,npts);
W = zeros(length(w),npts);
L_static = zeros(length(w),npts);

for ii = 1:length(w)

    L_static(ii,1) = fzero(@(L) staticlambda(L,a,Exxb_x_Iyy,0.001,a),3);

    mu_L = 1.0;

    if ii > 1
        ii
        L(ii-1)
        if nargin == 7
            Ihat0 = interp1q(Lmap',Ihat0map,L(ii-1));
            Ihat2 = interp1q(Lmap',Ihat2map,L(ii-1));
        else
            Ihat0 = Ihat0map;
            Ihat2 = Ihat2map;
        end

        try
            if L(ii-1) < 4.6137/a
                L(ii) = fzero(@(L)
particularlambda(L,mu_L,a,Exxb_x_Iyy,Ihat0,Ihat2,w(ii),1,a/2),[L(ii-1),4.6137/a]);
            elseif L(ii-1) < 11.5502/a
                L(ii) = fzero(@(L)
particularlambda(L,mu_L,a,Exxb_x_Iyy,Ihat0,Ihat2,w(ii),1,a/2),[L(ii-1),11.5502/a]);
            elseif L(ii-1) < 19.3282
                L(ii) = fzero(@(L)
particularlambda(L,mu_L,a,Exxb_x_Iyy,Ihat0,Ihat2,w(ii),1,a/2),[L(ii-1),19.3282]);
            else
                L(ii) = fzero(@(L)
particularlambda(L,mu_L,a,Exxb_x_Iyy,Ihat0,Ihat2,w(ii),1,a/2),[L(ii-1),60]);
            end

            catch
                if L(ii-1) < 4.6137/a
                    L(ii) = fzero(@(L)
particularlambda(L,mu_L,a,Exxb_x_Iyy,Ihat0(1),Ihat2(1),w(ii),1,a/2),4.6137/a/2);
                elseif L(ii-1) < 11.5502/a
                    L(ii) = fzero(@(L)
particularlambda(L,mu_L,a,Exxb_x_Iyy,Ihat0(1),Ihat2(1),w(ii),1,a/2),11.5502/a/2);
                elseif L(ii-1) < 19.3282/a
                    L(ii) = fzero(@(L)
particularlambda(L,mu_L,a,Exxb_x_Iyy,Ihat0(1),Ihat2(1),w(ii),1,a/2),19.3282/a/2);
                else
                    L(ii) = fzero(@(L)
particularlambda(L,mu_L,a,Exxb_x_Iyy,Ihat0(1),Ihat2(1),w(ii),1,a/2),25);
                end
            end

        else
            Ihat0 = Ihat0map(1);
            Ihat2 = Ihat2map(1);
            mu_L = 1.5;
            L(ii) = fzero(@(L)
particularlambda(L,mu_L,a,Exxb_x_Iyy,Ihat0,Ihat2,w(ii),0,a),3);
        end

        C1(ii) = 1/(sin(L(ii)*a)+sinh(L(ii)*a));
        C2(ii) = -1/(cos(L(ii)*a)+cosh(L(ii)*a));
        C3(ii) = -1/(sin(L(ii)*a)+sinh(L(ii)*a));
        C4(ii) = 1/(cos(L(ii)*a)+cosh(L(ii)*a));

        C1s(ii) = 1/(sin(L_static(ii)*a)+sinh(L_static(ii)*a));
        C2s(ii) = -1/(cos(L_static(ii)*a)+cosh(L_static(ii)*a));
    end
end
```

```

C3s(ii) = -1/(sin(L_static(ii)*a)+sinh(L_static(ii)*a));
C4s(ii) = 1/(cos(L_static(ii)*a)+cosh(L_static(ii)*a));

for jj = 1:length(X)
    W(ii,jj) =
b*(C1(ii)*sin(L(ii)*X(jj))+C2(ii)*cos(L(ii)*X(jj))+C3(ii)*sinh(L(ii)*X(jj))+C4(ii)*cosh(L
(ii)*X(jj)));
    ddW(ii,jj) = b*(-C1(ii)*L(ii)^2*sin(L(ii)*X(jj))-
C2(ii)*L(ii)^2*cos(L(ii)*X(jj))+C3(ii)*L(ii)^2*sinh(L(ii)*X(jj))+C4(ii)*L(ii)^2*cosh(L(ii)
)*X(jj)));
    d4W(ii,jj) =
b*(C1(ii)*L(ii)^4*sin(L(ii)*X(jj))+C2(ii)*L(ii)^4*cos(L(ii)*X(jj))+C3(ii)*L(ii)^4*sinh(L(
ii)*X(jj))+C4(ii)*L(ii)^4*cosh(L(ii)*X(jj)));
    Wt(ii,jj) = -w(ii)^2*W(ii,jj);
    Wtt(ii,jj) = -w(ii)^2*W(ii,jj);

    W_static(ii,jj) =
b*(C1s(ii)*sin(L_static(ii)*X(jj))+C2s(ii)*cos(L_static(ii)*X(jj))+C3s(ii)*sinh(L_static(
ii)*X(jj))+C4s(ii)*cosh(L_static(ii)*X(jj)));
    ddW_static(ii,jj) = b*(-C1s(ii)*L_static(ii)^2*sin(L_static(ii)*X(jj))-
C2s(ii)*L_static(ii)^2*cos(L_static(ii)*X(jj))+C3s(ii)*L_static(ii)^2*sinh(L_static(ii)*X
(jj))+C4s(ii)*L_static(ii)^2*cosh(L_static(ii)*X(jj)));

    Energy(ii,jj) = b*Exxb_x_Iyy*W(ii,jj)+b*w(ii)^2*Ihat2(1)*ddW(ii,jj)-
b*w(ii)^2*Ihat0(1)*d4W(ii,jj);
end
W(ii,:) = W(ii,:)*disp0;
ddW(ii,:) = ddW(ii,:)*disp0;
Ubeam(ii) = sum(1/2*Exxb_x_Iyy*ddW(ii,:).^2*a/npts);
Ubeam_static(ii) = sum(1/2*Exxb_x_Iyy*ddW_static(ii,:).^2*a/npts);
end

```

staticlambda.m

```
function [Z] = staticlambda(L,a,Exxb_x_Iyy,Q,x)

C1 = 1/(sin(L*a)+sinh(L*a));
C2 = -1/(cos(L*a)+cosh(L*a));
C3 = -1/(sin(L*a)+sinh(L*a));
C4 = 1/(cos(L*a)+cosh(L*a));

Z = L^4*Exxb_x_Iyy*(C1*sin(L*x)+C2*cos(L*x)+C3*sinh(L*x)+C4*cosh(L*x)) - Q;
```


particularlambda.m

```
function [Z] = particularlambda(L,mu_L,a,Exxb_x_Iyy,Ihat0,Ihat2,w,Q,x)

mu = mu_L*L;

C1 = 1/(sin(L*a)+sinh(L*a));
C2 = -1/(cos(L*a)+cosh(L*a));
C3 = -1/(sin(mu*a)+sinh(mu*a));
C4 = 1/(cos(mu*a)+cosh(mu*a));

Z =
L^4*Exxb_x_Iyy*(C1*sin(L*x)+C2*cos(L*x))+mu^4*Exxb_x_Iyy*(C3*sinh(mu*x)+C4*cosh(mu*x)) +
...
L^2*Ihat2*w^2*(-C1*sin(L*x)-C2*cos(L*x))+
mu^2*Ihat2*w^2*(C3*sinh(mu*x)+C4*cosh(mu*x)) + ...
Ihat0*w^2*(-C1*sin(L*x)-C2*cos(L*x))+Ihat0*w^2*(-C3*sinh(mu*x)-C4*cosh(mu*x)) -
Q;
```

CalcPiezoDisplacement.m

```
function [Wpiezo,Upiezo] = CalcPiezoDisplacement(Qbar,A,D,a,b,Z,d13,d23,EZ,X,index)

N = [0;0;0];
M = [0;0;0];
for nn = 1:size(Qbar,3)
    PiezoStrain = [d13(nn);d23(nn);0]*EZ;
    N = N + squeeze(Qbar(1:3,1:3,nn))*PiezoStrain*(Z(nn+1)-Z(nn));
    M = M + squeeze(Qbar(1:3,1:3,nn))*PiezoStrain*(Z(nn+1)^2-Z(nn)^2)/2;
end

K = [ A(1,1), 0, 0;
      0, 4/a^2*D(1,1), 0;
      0, 0, 4/b^2*D(5,5) ];

Qpiezo = [N(1)*a;2*M(1);0];%Modal Actuation Strain Forcing Matrix

q = K\Qpiezo;

Wpiezo(1:index) = q(2)*X(1:index).^2/a^2;
Wpiezo(index+1:length(X)) = Wpiezo(index)*ones(1,length(index+1:length(X)));

Upiezo = q(2)*2/a^2;%Qpiezo'*q;
```