**Verification and Expansion of Single-Degree-of-Freedom
Transformation Factors for Beams Using a Multi-Degree-of-
Freedom Non-Linear Numerical Analysis Method**

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TITLE: Verification and Expansion of Single-Degree-of-Freedom
Transformation Factors for Beams Using a Multi-Degree-
of-Freedom Non-Linear Numerical Analysis Method

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ABSTRACT

Verification and Expansion of Single-Degree-of-Freedom Transformation Factors for Beams Using a Multi-Degree-of-Freedom Non-Linear Numerical Analysis Method Takayuki Yokoyama

The single-degree-of-freedom (SDOF) transformation factors as tabulated in John Biggs' textbook (Biggs 1964) are accepted as the equivalent factors for simplifying and analyzing a component's response to blast. The study validates the stiffness and mass transformation factors through multi-degree-of-freedom (MDOF) numerical methods. After validating the transformation factors, the MDOF numerical method is used to create new stiffness and mass transformation factors for loading cases not already included in Biggs' textbook.

The validated factors, as well as the newly developed factors are used in SDOF analyses. The deflections from the SDOF responses accurately predict the MDOF responses as long as the component behaves elastically; however, the deflections diverge when inelastic behavior is involved. The diverged deflections indicate that the SDOF inelastic response analysis method can be improved.

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LIST OF NOMENCLATURE

D	distance from blast (ft)
E	modulus of elasticity (ksi)
$F(t)$	force varying by time (kips)
F_e	effective force
$F_{eff}(t)$	effective force varying by time (kips)
F_r	point load
I	moment of inertia (in ⁴)
K_{eff}	effective stiffness
K_L	stiffness factor
K_M	mass factor
L	length (in)
M_e	effective mass (kips*msec ² /in)
M_p	plastic moment (k-in)
M_r	lumped mass
P_{SO}	side-on peak pressure (psi)
$P_S(t)$	side-on pressure at time = t (psi)
R	resistance (kips)
R_m	maximum resistance (kips)
T_N	n-th period
W	equivalent TNT weight (lbs)
Z_{XX}	plastic section modulus (in ³)
Z	scaled distance of TNT (ft/lbs ³)
a_0	factor for mass damping
a_1	factor for stiffness damping
c	damping (kips*msec/in)
f_y	yield stress of steel under quasi-static loading (ksi)
f_{dy}	yield stress of steel under dynamic loading (ksi)
f_u	ultimate stress of steel under quasi-static loading (ksi)
f_{du}	ultimate stress of steel under dynamic loading (ksi)
f_1	factor – used in Newmark’s Average Acceleration Method
f_2	factor – used in Newmark’s Average Acceleration Method
i_s	side-on impulse
k	stiffness (kips/in)
$\hat{k}$	Incremental stiffness (kips/in)
k_e	stiffness matrix of an element
k_i	stiffness for step i (kips/in)
k_{i+1}	stiffness for step i+1 (kips/in)
l	length (in)
m	distributed mass (kips*msec ² /in)
m_e	stiffness matrix of an element
$\hat{p}_i$	incremental load (k/in)
p	distributed load (k/in)
p_0	initial loading (kips)

p_i	loading for step i (kips)
p_{i+1}	loading for step i+1 (kips)
s	normalized shape array
t	time (msec)
t_A	arrival time of blast wave (msec)
t_0	duration of positive pressure (msec)
u	displacement (in)
u_i	displacement at step i (in)
u_{i+1}	displacement at step i+1 (in)
$\bar{u}$	displaced shape array
$\ddot{u}$	acceleration (in/msec ²)
$\ddot{u}_i$	acceleration for step i (in/msec ²)
$\ddot{u}_{i+1}$	acceleration for step i+1 (in/msec ²)
$\dot{u}$	velocity (in/msec)
$\dot{u}_0$	initial velocity (in/msec)
$\dot{u}_i$	velocity at step i (in/msec)
$\dot{u}_{i+1}$	velocity at step i+1 (in/msec)
$w(x,t)$	distributed load (kips/in) varying in location (in) and time (msec), (kips/in)
y_e	yield displacement (in)
α	angle of incidence (degrees)
β	factor – used in Newmark’s Average Acceleration Method
δ	deflection (in)
δ_{max}	max deflection (in)
ϕ	shape function
λ	factor – used in Newmark’s Average Acceleration Method
$\Delta \hat{p}_i$	change in incremental load at step i (k/in)
Δt	timestep (msec)
Δv_i	change in velocity at step i (in/msec)
Δx_i	change in displacement at step i (in/msec)
ω_m	frequency of m-th mode
ω_n	frequency of n-th mode
ξ_m	damping ratio for the m-th mode
ξ_n	damping ratio for the n-th mode

1.0 INTRODUCTION

This project studies the equivalent single-degree-of-freedom (SDOF) factors presented by John Biggs (1964). The study presents the derivation of these factors, a validation of the factors using a multi-degree-of-freedom (MDOF) numerical analysis, the accuracy of the SDOF analysis compared to the MDOF analysis, and suggests factors for other cases to expand on what is commonly known as the “Biggs Factors” table.

The material in this study is most applicable to the practice of blast analysis of steel beams. To put the study into context, a literature review and the state of practice of the blast analysis and design industry follows the introduction. Equivalent SDOF factors are discussed in **Section 2**. Numerical analysis methods are discussed in **Section 3**, while their application and procedures based on a SDOF idealization and MDOF idealization are discussed in **Sections 4 and 5**. **Section 6** covers the set-up for the analyses and the assumptions built in to the set-up. **Section 7** presents the verification process of the Matlab algorithms. **Section 8** discusses the results from the analyses, while **Section 9** discusses the conclusions of the study. All derivations or code not presented within the Sections are located in the appendices.

1.1 Literature Review and State of Practice

The equivalent SDOF method is regularly used in the blast analysis and design field. Blast loads are dynamic, and ductility is a key parameter in material response (Marchand and Alfawakhiri 2005); therefore, the use of dynamic analysis methods is

imperative in modeling and understanding the behavior of the component under blast loads.

John Biggs, in his textbook *Introduction to Structural Dynamics*, outlined the equivalent SDOF method for analyzing component behavior. To convert a component into an equivalent SDOF system, equivalency factors are used to factor the assumed properties of the system. These equivalency factors are tabulated in John Biggs' textbook and are referred to commonly as Biggs' factors. This analysis method is accepted in the blast analysis and design industry as the first pass at estimating component behavior under blast loading. The field refers to this as the Simple SDOF Analysis Method (Conrath et al. 1999). The same theory in equivalent SDOF dynamics is included in other texts (Smith and Hetherington 1994), (Baker et al. 1983), and (Clough and Penzien 1975).

The United States Army Corps of Engineers – Protective Design Center distributes a program called SBEDS (Single-degree-of-freedom Blast Effects Design Spreadsheet). This excel spreadsheet is based on the theory presented by Biggs' dynamics and the document provided by the government titled *Structures to Resist the Effects of Accidental Explosions* (also referenced as Army TM 5-1300, NAVFAC P-397, and AFR 88-22). This spreadsheet is a norm in the industry as a SDOF analysis tool for component behavior due to blast.

The SBEDS manual provides a concise summary of blast loading, which is used to create the loading demands outlined in this study. This manual includes time-history loading equations and charts for the peak pressure and total impulse that can be expected

due to given distance to charge ratios. Similar information can be found in textbooks by Smith and Hetherington 1994, Baker et al. 1983, and Mays and Smith 1995.

As a quick and rough method of analysis, the SDOF method has weaknesses that lead to inaccuracy in analysis results. The greatest weaknesses that Krauthammer (Krauthammer 1998) attributes to the SDOF system are the simple nonlinear structural behavior assumptions. One major assumption is the elastic-perfectly plastic assumption. Lawver (Lawver et al. 2003) exposed another weakness of the equivalent SDOF method through experiments. He subjected varying W14 columns to varying blast loads. He showed that the equivalent SDOF method is accurate for far or small charges, and that the SDOF method is unable to predict the component behavior for close or large charges. He concludes that the equivalent SDOF method could not capture local bending of the steel member web or flanges for the close or large charge conditions.

Dynamic nonlinear Finite Element analysis is the most accurate method available for studying component behavior under blast loading (Krauthammer 1998, Conrath et al. 1999). The FE method can capture local effects, and can allow the analyst to model a component response to match response observed through experiments. The FE method requires more time and expertise, and is often reserved for research, forensics, and for complex configurations that can't be modeled as SDOF systems. For the aforementioned reasons, and in order to cater the study for industry practice, the study focuses on the more economical equivalent SDOF method.

1.2 Objective

The first objective of this project is to validate Biggs' equivalent SDOF factors through the use of MDOF numerical analysis. The second objective is to validate the accuracy of the factors by comparing a SDOF response to the MDOF response. The third objective is to use the MDOF response to expand on the Equivalent SDOF Factors table – adding linear, triangular, and parabolic load functions. The different loading patterns may exist when there is a near field – low impulse blast, or where the tributary width varies along the beam. The set-up that is used for the two methods is a steel wide-flange beam subjected to an air-blast load.

In order to validate the factors and the accuracy of the factors, the SDOF and MDOF models were programmed in Matlab using the Newmark's Average Acceleration numerical method. Response for analysis will not include damping or initial deflection; however, these options are included in the program and available for use.

1.3 Limitations

Theoretical models produce theoretical solutions, and limitations are caused by the assumptions made in creating the theoretical model. Such assumptions and limitations for this project are as follows:

- The analyses assume air-blast on the component. Fragments and their effects on the components are not addressed.
- The US government routinely performs blast test on structural components; however, the tests are not available to the public so the computer analysis cannot be compared to testing.¹

¹ Personal correspondence between Shalva Marjanishvili, Ph.D., S.E., and Takayuki Yokoyama on March 25, 2008. Shalva Marjanishvili is the Technical Director of Hinman Consulting Engineers, Inc.

- Only the primary impulse load is considered, and subsequent impulse loads caused by diffractions are not considered.
- Shear action plays an important role in beam response (Nonaka 2000); however, shear failure and shear ductility are not considered.
- Failure of boundary conditions, or component supports, is not considered.
- Material mechanics are based on elastic, perfectly plastic.
- A pre-existing axial load would influence the response of the component, but no pre-existing forces in the member are considered.

2.0 SINGLE-DEGREE-OF-FREEDOM ANALYSIS

A Single-degree-of-freedom (SDOF) system is a mathematical model where the component can only move along one axis as shown below.

Figure A: Simple SDOF model

The system has a single force on the single mass with a stiffness value and a damping value. The motion of this model can be expressed using the equation shown below.

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = F_{eff}(t) \quad \text{Eq. 1}$$

While the mass, damping, and stiffness stay the same; the force, acceleration, velocity, and displacement vary over time. The SDOF system is a simple spring-mass system that has the same work, strain, and kinetic energies at any given time, as the real model. In order to create a similar maximum displacement at the midspan; effective mass, force, and resistance terms must be used.

2.1 Equivalent SDOF Factors

Equivalent SDOF factors are used to obtain the effective mass, force, and resistance terms, and to make the SDOF system response equivalent to that of the real system. These equivalent SDOF factors are tabulated below.

Table 1: Equivalent SDOF factors for simply supported beams

Source: Biggs 1964

Loading Diagram	Strain Range	Load Factor K_L	Mass Factor K_M	Maximum Resistance R_m	Spring Constant k
uniform	Elastic	0.64	0.50	$\frac{8M_p}{L}$	$\frac{384EI}{5L^3}$
	Plastic	0.50	0.33	$\frac{8M_p}{L}$	0
point	Elastic	1.00	0.49	$\frac{4M_p}{L}$	$\frac{48EI}{L^3}$
	Plastic	1.00	0.33	$\frac{4M_p}{L}$	0

Table 2: Equivalent SDOF factors for beams with fixed ends

Source: Biggs 1964

Loading Diagram	Strain Range	Load Factor K_L	Mass Factor K_M	Maximum Resistance R_m	Spring Constant k
uniform	Elastic	0.53	0.41	$\frac{12M_p}{L}$	$\frac{384EI}{L^3}$
	E-P	0.64	0.50	$\frac{8}{L}(M_{ps} + M_{pm})$	$\frac{384EI}{5L^3}$
	Plastic	0.50	0.33	$\frac{8}{L}(M_{ps} + M_{pm})$	0
point	Elastic	1.0	0.37	$\frac{4}{L}(M_{ps} + M_{pm})$	$\frac{192EI}{L^3}$
	Plastic	1.0	0.33	$\frac{4}{L}(M_{ps} + M_{pm})$	0

In order to solve for equivalent SDOF factors, deflected shape assumptions are made. Biggs uses the static deflected shape, normalized, so that the maximum deflection is one unit. This is called the shape function, and is used to solve for both mass and load transformation factors. The factors are then used as coefficients to the corresponding factors in **Eq. 1** on page 6. The m and k values are based on the static deflection shape with the maximum deflection normalized to one unit.

The following subsections describe the derivation process of the equivalent SDOF factors; an example derivation can be found in **Section 11.1**.

2.1.1 Mass Factor

The mass factor can be defined as “the ratio of equivalent mass to the actual total mass of the structure”; setting the mass factor, K_M , equal to M_e (equivalent mass) divided by M_t (actual total mass). The equation for M_e is shown below (Biggs 1964).

$$M_e = \sum_{r=1}^j M_r \phi_r^2 \quad \text{Eq. 2}$$

for lumped masses, or

$$M_e = \int m \phi^2(x) dx \quad \text{Eq. 3}$$

for a distributed mass.

The mass factor, K_M , then, is the effective mass, M_e , divided by the true mass.

$$K_M = \frac{\sum_r^r M_r \phi_r}{\sum_r M_r} \quad \text{Eq. 4}$$

for lumped masses, or

$$K_M = \frac{\int^L m \phi^2(x) dx}{mL} \quad \text{Eq. 5}$$

for a distributed mass.

This mass factor is used to convert the actual mass value to the equivalent system mass value in the SDOF equation of motion.

2.1.2 Load Factor

Like the mass factor, the load factor can also be defined as the ratio of the equivalent to the actual. The equation for the equivalent load, F_e , is shown below (Biggs 1964).

$$F_e = \sum_{r=1}^j F_r \phi_r \quad \text{Eq. 6}$$

for concentrated loads, or

$$F_e = \int^L p(x) \phi(x) dx \quad \text{Eq. 7}$$

For a distributed load.

The stiffness factor, K_L , then, is the effective stiffness, K_e , divided by the true stiffness.

$$F_e = \frac{\sum_r^j F_r \phi_r}{\sum_r F_r} \quad \text{Eq. 8}$$

For concentrated loads, or

$$K_L = \frac{\int_0^L p\phi(x)dx}{pL} \quad \text{Eq. 9}$$

For a distributed load.

This stiffness factor is used to factor the stiffness value in the SDOF equation of motion.

2.1.3 Resistance

Stiffness can be defined as the load needed to cause a unit displacement. The resistance of an element is defined as the load resisted by the component and the corresponding deflection. The maximum resistance for a failure mechanism is the load resisted by the component when a hinge forms. Due to this resistance and load relationship, the resistance (and stiffness) factor is equal to the load factor. Like the load and mass factor, the resistance factor is also the ratio of the equivalent resistance to the actual resistance.

2.2 Yield

With large impulse loads, the component will yield. This needs to be reflected in the SDOF analysis. The static force required to cause a yielding moment in the beam is set as the maximum resistance value. Once the force reaches this maximum resistance value, a new shape function is used, and along with it, new load, mass, and resistance factors. The result is a resistance versus displacement curve as shown in **Figure B** for a uniformly loaded beam with both ends fixed.

Figure B: SDOF system nonlinear behavior

The graph in **Figure B** shows the nonlinear behavior of the system. For the uniformly loaded beam with both ends fixed, there are three stages in behavior: elastic, elasto-plastic, and plastic. The system behaves elastically until the first resistance value, R_{m1} , is reached. At this resistance value, hinges form on either end of the beam, and the beam now behaves similar to a uniformly loaded, simply supported beam. The third hinge forms at R_{m2} , at which point the system loses its stiffness and behaves plastically.

With the established factors, the SDOF model can be run through the Newmark's Average Acceleration numerical procedure to estimate the maximum component deflection due to a blast load.

3.0 NUMERICAL ANALYSIS METHODS

Based on Section 2.0 a basic equation for solving a dynamics problem is shown below as **Eq. 10**.

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = F_{eff}(t) \quad \text{Eq. 10}$$

If the mass (m), damping (c), and stiffness (k) are constant over time (t); the acceleration ($\ddot{u}$), velocity ($\dot{u}$), and displacement (u) can be calculated for a given at time (t) by solving the equation. Because the loading varies over time, and the stiffness changes due to the material and geometric nonlinearity; the acceleration, velocity, and displacement have to be solved in time increments in order to capture the behavior of the system over time. Numerical analysis methods are used to perform the time-step evaluations of such systems.

Before selecting Newmark's Average Acceleration method, there were three other numerical analysis methods considered for this study: Central Difference, Newmark's Linear Acceleration, and Wilson- θ Method (Chopra 2007).

The results obtained from all numerical methods include computational errors caused by the different assumptions made by each method between every timestep.

Figure C below shows the comparison of the different numerical methods to the theoretical solution.

Figure C: Free vibration solution by four numerical methods vs. the theoretical solution

Source: Chopra 2007

Wilson- θ method shows the biggest diversion from the theoretical solution as the amplitude decreases every period. This amplitude decrease is caused by numerical damping which is built in to the method. While the central difference method, the average acceleration method, and the linear acceleration method all predict the displacement amplitude accurately, the period varies slightly from the theoretical solution.

Figure C from above shows that as the timestep increases relative to the natural period of the single-degree-of-freedom (SDOF) system, the central difference method's inaccuracy increases rapidly. For the SDOF system, Newmark's Average Acceleration method and the Linear Acceleration method appear to be the more accurate methods.

After ruling out Wilson- θ method and the central difference method from above, the two remaining method options are the average and linear acceleration methods. The

remaining methods are compared using a multi-degree-of-freedom (MDOF) system. The linear MDOF system produces similar results to the SDOF system, and therefore, the linear acceleration is the most accurate method; however, when the MDOF system becomes a nonlinear system, the linear acceleration method runs into computational stability issues. In a MDOF nonlinear system, the stability of the solution depends on the timestep to period ratio of the highest mode of the system. Even though the higher modes of the system have negligible contribution to the overall behavior of the system, they are important to the stability of the analysis. The highest mode of the system depends on the number of elements used in the MDOF model. This project discretizes the beams into at least 24 elements, so the periods of the higher modes become very small. The small period of the highest mode requires the timestep for the analysis to be even shorter or the solution “blows up” due to the computational instability. When the timestep becomes too small, the calculation time becomes too long and impractical.

Because of the numerical procedure stability issue, using an unconditionally stable method is practical. The Average Acceleration method is unconditionally stable and does not have numerical damping (Wilson- θ method); therefore, the average acceleration method is used in this project. The step-by-step procedure of Newmark’s Average Acceleration method is introduced and described in the SDOF and MDOF sections (Section **4.0** and Section **5.0**).

4.0 NEWMARK'S AVERAGE ACCELERATION SINGLE-DEGREE- OF-FREEDOM ANALYSIS PROCEDURE

The Newmark's Average Acceleration numerical procedure is shown in **Table 3**.

Table 3: Newmark's Average Acceleration Method

Source: Chopra 2007 - modified

For Average Acceleration, $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$	
1.0	Initial Calculations
1.1	$\ddot{u} = \frac{p_0 - c\dot{u}_0 - ku_0}{m}$
2.0	Iterative Calculations
2.1	Determine tangent stiffness k_i
2.2	$\hat{k}_i = k_i + \frac{\gamma}{\beta\Delta t}c + \frac{1}{\beta(\Delta t)^2}m$
2.3	$f_1 = \frac{1}{\beta\Delta t}m + \frac{\gamma}{\beta}c \text{ and } f_2 = \frac{1}{2\beta}m + \Delta t\left(\frac{\gamma}{2\beta} - 1\right)c$
2.4	$\Delta\hat{p}_i = \Delta p_i + f_1\dot{u}_i + f_2\ddot{u}_i$
2.5	$\Delta u_i = \frac{\hat{p}_i}{\hat{k}_i}$
2.6	$\Delta\dot{u}_i = \frac{\gamma}{\beta\Delta t}\Delta u_i - \frac{\gamma}{\beta}\dot{u}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}_i$
2.7	$u_{i+1} = u_i + \Delta u_i, \dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i$
2.8	$\ddot{u}_{i+1} = \frac{p_{i+1} - c\dot{u}_{i+1} - ku_{i+1}}{m}$
3.0	Repeat steps 2.1 through 2.8

The method outlined above is an iterative procedure where there is an initial acceleration, velocity, and displacement at every time step. Every time step involves several equations. Steps 2.1 and 2.2 are used to check the effective mass and stiffness needed for the time step. The effective mass and stiffness changes only when the expected yield deflection is reached. Steps 2.5 and 2.6 use the initial acceleration, velocity, and displacement of the time step to calculate the change in displacement and velocity in the time step. Note that for the Average Acceleration Method, the acceleration component of step 2.6 is equal to zero, and only the change in displacement and the velocity affect the change in velocity. Step 2.8 is a step that is modified compared to the procedure in (Chopra 2007). The procedure in (Chopra 2007) uses the equation $\ddot{u}_{i+1} = \ddot{u}_i + \Delta\ddot{u}_i$. For this study, the equation is replaced by step 2.8 which forces equilibrium to ensure there is minimal gain or loss of energy in the system. This step calculates the acceleration for the next time step, and completes the calculation for a time step.

5.0 MULTI-DEGREE-OF-FREEDOM ANALYSIS

In order to measure the accuracy of the Biggs' single-degree-of-freedom (SDOF) factors, the output from the SDOF system must be compared to the output of a more accurate method of analysis. The multi-degree-of-freedom (MDOF) system is used to create the output for comparison.

5.1 Multi-Degree-of-Freedom Model

This project uses a MDOF system to help create the advanced SDOF method. MDOF in this project means the component being studied is broken up into multiple elements, with each component having two degrees of freedom on either end as shown below in **Figure D**.

Figure D: MDOF system

The first degree of freedom for each component moves in the same axis as the SDOF system, and the second degree of freedom for each component is a rotational movement.

Eq. 1 still applies to the MDOF system; however, each variable is represented as a vector or a matrix. The stiffness matrix is comprised of uniform beam segment stiffness matrices. A stiffness matrix for a beam segment is shown below.

$$k_e = \frac{2EI}{L^3} \begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L \end{bmatrix} \quad \text{Eq. 11}$$

In the equation above, k_e represents the stiffness matrix of a single uniform beam segment while L represents the length of the same segment. The beam is expected to respond inelastically when it is loaded with a blast impulse. To capture the transition that occurs between elastic behavior and inelastic behavior, the internal forces and the stiffness matrices are evaluated at every time step. If the internal moment of an element exceeds the material and section capacity, a new stiffness matrix is formed for the particular element. The pin-fixed stiffness matrix, depending on which end becomes pinned, can be substituted by one of the two stiffness matrices shown below.

$$k_e = \frac{EI}{L^3} \begin{bmatrix} 3 & 0 & -3 & 3L \\ 0 & 0 & 0 & 0 \\ -3 & 0 & 3 & -3L \\ 3L & 0 & -3L & 3L^2 \end{bmatrix} \quad \text{Eq. 12}$$

or

$$k_e = \frac{EI}{L^3} \begin{bmatrix} 3 & 3L & -3 & 0 \\ 3L & 3L^2 & -3L & 0 \\ -3 & -3L & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Eq. 13}$$

When the first node, in this case the left node, becomes plastic or pinned, **matrix Eq. 12** is used to replace the original stiffness matrix. When the second node, or the right node, becomes pinned, **matrix Eq. 13** is used to replace the original stiffness matrix.

The mass matrix used in the study is shown below.

$$m_e = \frac{\bar{m}L}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \quad \text{Eq. 14}$$

(Clough and Penzien 1975) present two ways to form a mass matrix, the lumped-mass matrix and the consistent-mass matrix. For this study, the consistent-mass matrix is used. The consistent-mass matrix is more accurate than the lumped-mass matrix because it couples the adjacent degrees of freedom. The downside is that it also can take more computational effort compared to the lumped-mass matrix (Clough and Penzien 1975).

One situation where an explicit damping matrix is needed is for “nonlinear responses, for which the mode shapes are not fixed but are changing with changes of stiffness” (Clough and Penzien 1975). Because the above criterion is met in this project, the Rayleigh Damping method is used to form an explicit damping matrix. The damping matrix is formed by adding the factored mass and stiffness matrices together as shown below in **Eq.15**.

$$c = a_0 m + a_1 k \quad \text{Eq. 15}$$

Where a_0 and a_1 are factors that need to be solved for, and in order to do so, two natural frequencies (modes) and the damping ratios for those frequencies must be determined. The natural frequencies can be found by solving for the Eigen values, and the damping ratio can be assumed. Once the natural frequencies ω_m and ω_n for the m^{th} and n^{th} modes are selected along with the damping ratios ξ_m and ξ_n , the factors a_0 and a_1 can be solved using **Eq.16** below.

$$\begin{Bmatrix} \xi_m \\ \xi_n \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 1/\omega_m & \omega_m \\ 1/\omega_n & \omega_n \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} \quad \text{Eq. 16}$$

The factors a_0 and a_1 can then be inserted into **Eq. 15** for the damping matrix.

This method of deriving a damping matrix causes the damping effect to be high on modes that are less or significantly higher than the two selected significant modes. This may cause some inaccuracy to the behavior of the beam; however, damping out the higher modes allows the analysis to run with larger time steps.

The loading matrix is a two dimensional matrix with the rows corresponding to the degrees of freedom of the system and the columns corresponding to the loading pattern for each timestep. For this project, the magnitude of the load varies with time to follow the typical pressure time-history of a blast while the loading distribution over the length of the beam stays constant.

5.2 Newmark's Average Acceleration Method in MDOF

The Newmark's Average Acceleration numerical procedure for a multi-degree-of-freedom (MDOF) system is the same as the process for a single-degree-of-freedom (SDOF) system shown in **Table 3**, and shown again below in **Table 4**.

Table 4: Newmark's Average Acceleration Method

Source: Chopra 2007 - modified

For Average Acceleration, $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$

1.0 Initial Calculations

$$1.1 \quad \ddot{u} = m^{-1}(p_0 - c\dot{u}_0 - ku_0)$$

$$1.2 \quad f_1 = \frac{1}{\beta\Delta t}m + \frac{\gamma}{\beta}c \text{ and } f_2 = \frac{1}{2\beta}m + \Delta t\left(\frac{\gamma}{2\beta} - 1\right)c$$

2.0 Iterative Calculations

$$2.1 \quad \Delta\hat{p}_i = \Delta p_i + f_1\dot{u}_i + f_2\ddot{u}_i$$

2.2 Determine tangent stiffness k_i

$$2.3 \quad \hat{k}_i = k_i + \frac{\gamma}{\beta\Delta t}c + \frac{1}{\beta(\Delta t)^2}m$$

$$2.4 \quad \Delta u_i = \frac{\hat{p}_i}{\hat{k}_i}$$

$$2.5 \quad \Delta\dot{u}_i = \frac{\gamma}{\beta\Delta t}\Delta u_i - \frac{\gamma}{\beta}\dot{u}_i$$

$$2.6 \quad u_{i+1} = u_i + \Delta u_i, \quad \dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i$$

$$2.7 \quad \ddot{u}_{i+1} = m^{-1}(p_{i+1} - c\dot{u}_{i+1} - ku_{i+1})$$

3.0 Repeat steps 2.1 through 2.7

Most of the steps in the procedures between the MDOF and SDOF steps are the same with two exceptions. The main difference is the variables. MDOF uses matrices and vectors for every step instead of the single values used for the SDOF variables.

Another difference between the two systems is the re-evaluation process of the stiffness matrix for each iteration. In the SDOF system, the stiffness value was changed when the resistance in the system reached a certain force. This instant change in the stiffness value creates a bilinear system. In the MDOF system, the resistance is evaluated for each node of every element in the beam. The stiffness in each element is a bilinear system, but globally creates a curved transition from elastic to plastic.

Similar to the modification made to the SDOF procedure, the equation for solving the acceleration for the next time step is a modification to the procedure outlined in (Chopra 2007). This modification is made to ensure there is minimal gain or loss of energy in the system.

6.0 ANALYSIS SET UP

This Section outlines the analysis set up. The set up includes beam configuration, material properties, and loading. **Figure E** is a graphical representation of the basic component set up used for the analyses.

Figure E: Analysis set-up

The component is a single-span steel beam with the simulated load coming from above.

6.1 Beam Configuration

The beam configuration is shown in **Figure E**. The geometric properties of the beam are shown in **Table 5**.

Table 5: Initial configuration and section properties

Beam section	W24x131
Length	$L = 24$ ft
Moment of inertia	$I = 4020$ in ⁴
Plastic modulus	$Z_{XX} = 370$ in ³

Boundary conditions are either pinned or fixed at each end. The supports are infinitely rigid in translation. If the support is a pinned-support, there is no rotational stiffness, and if the support is a fixed-support, the rotational stiffness is infinite. The program allows the beam to carry initial dead and live loads, with an initial deflection caused by the static loads; however, this function is not included in the analysis for this project. The beam is assumed to be sufficiently braced laterally so that the beam behavior is based on the section properties.

There are two types of beam configurations that are used for analysis in this study:

Figure F: Pin-pin boundary configuration

Figure G: Fixed-fixed boundary configuration

The configurations above are used with different loading conditions.

6.2 Material Properties

Material properties of the steel beam are shown in **Table 6** and **Figure H**.

Table 6: Material properties

Steel	A572
Modulus of elasticity	$E_s = 29000$ ksi
Yield strength	$f_y = 50$ ksi

Figure H: Assumed Steel nonlinear behavior

The AISC Steel Construction Manual indicates that the expected yield strength of an A572 grade steel is 55 ksi. TM5-1300 indicates that the dynamic yield strength is greater than the quasi-static yield strength, and calls it a strain rate increase (U.S. Department of the Army 1990). Considering the two sources, the $f_y = 50$ ksi assumption made in **Table 6** is erroneous and yield strength should be around 65 ksi under impulse loads with short durations; however, this 50 ksi assumption will not influence the data as long as the yield strength is consistent between the multi-degree-of-freedom (MDOF) and single-degree-of-freedom (SDOF) analyses.

As shown in **Figure H**, the material is assumed to behave elastic, perfectly plastic. Elastic, perfectly plastic means that f_y is the absolute maximum yield stress the material can see, and there is no residual stiffness beyond yield.

6.3 Definitions of Blast Load types

For this study, an explosion is defined as “an extremely rapid release of energy in the form of light, heat, sound, and a shock wave” (FEMA 427). In structural blast analysis and design, the shock wave defines the loading criteria. This study uses detonation, a type of explosion, as the source for the blast load. In a detonation, the shock wave travels faster than the speed of sound and the wave front causes an instantaneous rise in pressure to the atmosphere.

For any explosion, the charge weight and the standoff distance are used to determine the shock wave load at a given point. The pressure time history is based on TNT explosive weight and standoff. If any other explosive is used, equivalency charts are used to determine the equivalent TNT weight.

In this project, the structural component being studied is oriented perpendicular to the direction of the shock wave. The loading in this case is referred to as a reflected blast load; however, free-air explosions and side-on explosions are presented first to help explain the dynamics of reflected blast loads.

6.3.1 Free-Air Explosions

Free-air explosions describe blast wave parameters in the open: There is no material obstructing the wave. The point of interest is closer to the blast source than any other surface. The pressure change over time at a point in space is shown in **Figure I**.

Figure I: Typical pressure-time history of an airblast in free air

Source: U.S. Army Corps of Engineers 2008

Figure I shows that the pressure suddenly increases as the blast wave-front hits the point of interest and decreases as the wave passes through the point of interest. Once the wave front passes through the point of interest, a negative phase is caused by a suction pressure right behind the wave. This suction pressure ends when the point of interest regains the atmospheric pressure.

6.3.2 Side-On Explosions

In most cases, there is a surface closer to the explosion than to the point in space under speculation, e.g. ground. This means that the point under speculation receives a reflected shock wave in addition to the direct shock wave. In this case, the two shock waves are assumed to travel simultaneously to form a hemispherical wave front. The

combined shock wave peak pressure is approximately equal to 1.8 times a free-air explosion at any given point (U.S. Army Corps of Engineers 2008).

6.3.3 Reflected Blast Loads

Side-on loads can be used for components parallel to the shock wave direction. When the component orientation is not completely parallel to the shock wave direction, the component experiences reflected pressure. The difference in the angle of the component orientation relative to the blast source is called the angle of incidence. The peak pressure loaded on a given point on a component is the side-on pressure multiplied by the reflected pressure coefficient depending on the angle of incidence.

6.4 Loading Assumptions

The loading demands used in this project are based on the blast loading values and equations outlined in the SBEDS manual. All blast loads in SBEDS assume a hemispherical surface burst, with the pressure-time history identical to **Figure I**.

The positive phase of the blast load acts in the same direction as the shock wave. This positive phase is modeled by the equation below (U.S. Army Corps of Engineers 2008).

$$P_s(t) = P_{so} \left[1 - \left(\frac{t - t_A}{t_0} \right) \right] e^{-\left(\frac{t - t_A}{\theta} \right)} \quad \text{Eq. 17}$$

Where: $P_s(t)$ = shock overpressure as a function of time
 P_{so} = peak side-on overpressure
 t = detonation time
 t_A = arrival time of initial shock front
 t_0 = positive phase duration
 θ = shape constant of pressure waveform

The shape constant of pressure waveform can be solved for by taking the integral of **Eq. 17** and setting it equal to the impulse; the parameters for this equation, along with the impulse value, can be found in **Figure J**.

Figure J: Positive phase shock wave parameters for a hemispherical surface burst of TNT

Source: U.S Army Corps of Engineers 2008

Where:

- Pr = peak reflected pressure (psi)
- Pso = peak side-on pressure (psi)
- Ir = reflected impulse (psi-ms/lb^{1/3})
- Is = side-on impulse (psi-ms/lb^{1/3})
- ta = arrival time of shock (ms/lb^{1/3})
- to = shock (positive phase) duration (ms/lb^{1/3})
- U = shock front velocity (ft/ms)
- Lw = shock wavelength (ft/lb^{1/3})

All the parameters can be read off **Figure J** given the scaled distance, Z . The scaled distance is a function of the blast standoff distance and the equivalent TNT mass of the blast. The equation to solve for the scaled distance is shown as:

$$Z = \frac{R}{W^{1/3}} \quad \text{Eq. 18}$$

Where: R = standoff distance from blast (ft)
 W = equivalent TNT mass (lbs)

The negative phase pressure time-history and equation in SBEDS is shown in **Figure K**.

Figure K: Negative phase load pressure history

Source: U.S Army Corps of Engineers 2008

$$t^- = t'^- Cp^- \quad \text{Eq. 20}$$

$$Cp^- = \frac{i^-}{i'^-} \quad \text{Eq. 21}$$

Where: P_o^- = peak negative pressure
 t_o^- = negative phase duration
 t^- = corrected time after first negative pressure determined from equation
 t'^- = time after first negative pressure
 Cp^- = correction factor for time after first negative pressure preserving actual impulse i^-
 i'^- = negative phase impulse calculated initially with integral of equation in **Eq. 19** with $t^- = t'^-$
 i^- = actual negative phase impulse from empirical prediction method from **Figure L**.

Some of the parameters for **Eq. 19** can be obtained from the figure below.

Figure L: Negative phase shock wave parameters for a hemispherical surface burst of TNT

Source: U.S Army Corps of Engineers 2008

Where:

- $-Pr$ = peak negative reflected pressure (psi)
- $-Pso$ = peak negative side-on pressure (psi)
- $-Ir$ = negative reflected impulse ($\text{psi-ms/lb}^{1/3}$)
- $-Is$ = negative side-on impulse ($\text{psi-ms/lb}^{1/3}$)
- $-to$ = shock (negative phase) duration ($\text{ms/lb}^{1/3}$)
- $-Lw$ = negative shock wavelength ($\text{ft/lb}^{1/3}$)

The pressure time history **Figure J** and **Figure L** are for side-on bursts. When the shock wave hits a component, blast loads are reflected, and thus the pressure-time history changes. Appropriate modifications can be made by examining the angle of incidence, α .

The angle of incidence is 0° when the blast wave front travels perpendicular to the component and 90 degrees when the blast wave front travels parallel to the component. The figure below shows values for the reflected pressure coefficient, C_{Ra} , which can be multiplied to the peak side-on pressure taken from **Figure J** and **Figure L**.

Figure M: Reflected pressure coefficient versus angle of incidence

Source: U.S. Army Corps of Engineers 2008)

If the angle of incidence is 0 degrees, the peak pressure can be found from **Figure J** and, as P_r . **Figure N** shows the scaled reflected impulse due to the angle of incidence.

Figure N: Scaled reflected impulse versus angle of incidence

Source: U.S. Army Corps of Engineers 2008

As indicated by the vertical scale on **Figure N**, the scaled reflected impulse can be multiplied to the cube root of the charge mass to obtain the reflected impulse.

Once the pressure history is derived, it is applied on the beam in several different patterns show below:

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Figure O: Uniformly distributed loading pattern

Figure P: Linearly distributed loading pattern

Figure Q: Triangular loading pattern

(With Peak Loading at the Midpoint of Beam)

Figure R: Parabolic loading pattern

7.0 MATLAB ALGORITHM VERIFICATION

Once the algorithm was written in Matlab, several checks were completed, and are shown in this Section. These checks were made to verify that the program works as intended and to minimize any code errors that may exist. The checks include: constant load, time-step variation, and the elastic limit, dynamic inelastic response vs. theoretical inelastic response.

All checks are done with the analysis set up described in Section 6.

7.1 Constant Static Load

With a constant static load applied during the analysis, the beam should deflect and oscillate with the average deflection around the theoretical static deflection. **Figure S** and **Figure T** below show the deflected shape of a 24 feet long, W24x131 beam, with both ends fixed, and with a uniform load of 12.1 klf.

Figure S: Deflected shape with static load

Figure T: Displacement time history with static load

The constant load is introduced after a few milliseconds as shown in the figures.

Once the load is applied, the system has an average period of 0.0115 seconds, and oscillates between 0.00408 inches and -0.31387 inches with an average deflection of 0.1549 inches. This is consistent with the theoretical first-mode period of

$$T_n = \frac{2\pi}{\sqrt{k/m}} = 0.0115 \text{ seconds (solved using equivalent SDOF model with Biggs' factors),}$$

the theoretical amplitude of $u(t) = (u_{st})_0 \left(1 - \cos\left(\sqrt{k/m} \times t\right) \right) = 0.31 \text{ inches}$, and the

$$\text{theoretical static deflection of } \delta(.5x) = \frac{wl^4}{384EI} = 0.1550 \text{ in.}$$

7.2 Time Step Variation

For Newmark's average acceleration method, (Weaver and Johnston 1987) suggest a timestep (Δt) equal to $T_n/5$, where T_n is the period of the highest mode, to ensure accuracy for all modes: however, also states that $T_n/10$ is commonly used. Chopra (2007) and Weaver and Johnston (1987) both show proof that the average acceleration method is unconditionally stable. This study includes the following exploration of timestep variation vs. accuracy using the average acceleration method for the beam shown below in **Figure U**.

Figure U: 24 Ft, 24-Element, uniformly loaded W24x131 beam with fixed ends

The beam in **Figure U** is a 24 foot, 24-element, uniformly loaded W24x131 with fixed ends. The analyses are tested on a TNT equivalent scaled distance (Z) of 12 ft/lbs³ (the pressure and impulse values are located on **Table 11**). For reference, the period of the 46th mode is 8.96×10^{-6} seconds (8.96×10^{-3} ms). Using the rule of thumb for Newmark's linear acceleration method, the timestep of $T_n/10$ (or 8.96×10^{-7} seconds) is used as the base timestep. Assuming $T_n/10$ will provide the exact answer for maximum displacement, the timestep variation study checks how the maximum displacement varies with increasing timesteps. Percent error is calculated relative to the response for timestep equal to $T_n/10$. **Figure V** below shows the percent error trend with varying timesteps.

Figure V: Δt vs. % error for 24 ft fixed end beam with uniform loading ($Z = 12$ ft/lbs³)

The percent error of the solution gradually increases until $\Delta t = 5.1 \times 10^{-6}$ sec (about $T_n/1.8$), at which point the analysis is no longer producing a reasonable solution.

This shows that the program is accurate at the suggested $\Delta t = T_n/5$ suggested by Weaver

and Johnston (1987), and this limit of $\Delta t = T_n/5$ is not an issue for the Matlab program. This accuracy graph should show a gradual increase in percent error², indicating a possible error in the program. In order to keep the solution accurate in all cases, the study uses a timestep of 5.0×10^{-7} sec. For the W24x131 spanning 24 feet with fixed ends, this timestep is equivalent to about $T_{46}/18$. For the W24x131 spanning 24 feet with pinned ends, this timestep is equivalent to about $T_{48}/18$. Note that the fixed degrees of freedom are taken out of the matrices in the analysis iterations. The study suggests that the 46th mode and the 48th mode for the fixed ends and pinned ends cases respectively, are the highest important modes for accuracy.

7.3 Elastic Limit

The elastic limit check shows that the system can switch back and forth between elastic behavior and inelastic behavior on a 24 feet long W24x131 beam, fixed on both ends. The loading pressure is scaled to a magnitude of 0.2 ksi, which forces the beam to barely reach the elastic limit state. This switch can be shown through the moment vs. rotation hysteresis, **Figure X**.

² Personal correspondence between Graham Archer, Ph.D., and Takayuki Yokoyama on March 31, 2011.

Figure W: Bending stress at fixed end vs. rotation

As expected, the moment vs. rotation figure shows that there is a linear relationship between the moment stress and rotation until the moment stress reaches 50 ksi. Once the moment stress reaches 50 ksi, the figure shows little inelastic behavior where only the rotation continues to increase, the element then snaps back to elastic behavior. While the ends go plastic, the midspan stays elastic as shown in **Figure X**.

Figure X: Bending stress at midspan vs. rotation

Two quick observations can be made from the figure above. The first is that the bending versus rotation relationship is not perfectly linear. This non-linear relationship is probably due to multiple modes being activated by the fast and high-impulse loading. The second observation is that the maximum bending stress is about 25 ksi as expected, which is half of the bending stress at the two fixed ends. The two observations can also be seen in **Figure Y**.

Figure Y: Bending stress time history with brief inelastic behavior

The figure above shows the bending stress time history at both ends and the midspan. Again, the stress limit is at 50 ksi and the ends of the beam shows a brief instance of inelastic behavior while the stress at the midspan is about half the stress compared to the two ends.

7.4 Inelastic Behavior

The inelastic behavior check shows the change in overall stiffness of the beam as it is loaded. **Figure Z** below shows the resistance (as defined in Section 2.1.3) versus displacement relationship for four time histories compared to the theoretical SDOF relationship defined by Biggs.

Figure Z: Resistance vs. displacement relationship

The figure above shows the resistance versus displacement time history for TNT scaled distances (Z) of 16 ft/lbs³, 14 ft/lbs³, 12 ft/lbs³, 10 ft/lbs³, and 8 ft/lbs³. Note that the resistance versus displacement time history goes as far as the maximum displacement, and the elastic oscillation after the peak deflection is not included in the plots. The fifth plot is the theoretical resistance vs. displacement relationship for a SDOF idealization as defined by Biggs. In all cases, the stiffness for the elastic range match the theoretical stiffness. The first kink in any line represents hinge formation at the beam ends, and the beam transitions into an elasto-plastic behavior. The second kink in the theoretical line indicates the hinge in the midspan of the beam, at which point the system theoretically becomes perfectly plastic. This perfectly plastic behavior assumption is not reflected in the MDOF response because hinges still resist rotation, and the other elements continue to absorb more force. The mismatch between the theoretical SDOF response and the

MDOF response after the midspan hinge is more of a reflection of the SDOF system's shortcoming in the nonlinear behavior assumption.

For the $Z = 8 \text{ ft/lbs}^3$ case, the nonlinear behavior starts at a low resistance value of about 320 k while the other cases show nonlinear behavior at around 700-800 k – this behavior is noticed in every case at low Z values (discussed in Section 8.3). This discrepancy is consistent with the observation of (Lawver et al. 2003) that the SDOF method is unable to predict the component behavior for close or large charges. **Figure AA** through **Figure MM** are shown below to further explore this $Z=8\text{ft/lbs}^3$ case.

Figure AA: Resistance vs. Midspan Deflection of 24 ft, uniformly loaded W24x131 with fixed ends, loaded by blast with a $Z = 8\text{ft/lbs}^3$

Figure BB: Bending stress diagram for $Z=8\text{ft/lbs}^3$ at $t = 0.000902$ seconds

Figure CC: Deflected shape diagram for $Z=8\text{ft/lbs}^3$ at $t = 0.000902$ seconds

Figure DD: Bending stress diagram for $Z=8\text{ft/lbs}^3$ at $t = 0.002046$ seconds

Figure EE: Deflected shape diagram for $Z=8\text{ft/lbs}^3$ at $t = 0.002046$ seconds

Figure FF: Bending stress diagram for $Z=8\text{ft/lbs}^3$ at $t = 0.002112$ seconds

Figure GG: Deflected shape diagram for $Z=8\text{ft/lbs}^3$ at $t = 0.002112$ seconds

Figure HH: Bending stress diagram for $Z=8\text{ft/lbs}^3$ at $t = 0.003133$ seconds

Figure II: Deflected shape diagram for $Z=8\text{ft/lbs}^3$ at $t = 0.003133$ seconds

Figure JJ: Bending stress diagram for $Z=8\text{ft/lbs}^3$ at $t = 0.003378$ seconds

Figure KK: Deflected shape diagram for $Z=8\text{ft/lbs}^3$ at $t = 0.003378$ seconds

This unexpected response is likely related to the load magnitude and application time. When the load is applied fast, there is no time to develop a quasi-static deflected shape with the same hinging sequence as the theoretical SDOF system. This is shown in **Figure BB**. The hinges on the ends occur as expected; however, the bending diagram and the deflected shape do not reflect what is expected from a quasi-static case. In other words, the fast loading affects beam stresses locally before the beam can distribute the load globally and respond like a statically loaded beam. As shown in **Figure Z**, the lesser magnitude and slower application of the load has enough time to develop the quasi-static deflected shape and follows the SDOF resistance curve.

The inelastic behavior check confirms that the MDOF Matlab program written for this study captures the inelastic response to blast demands better than the SDOF method.

8.0 ANALYSIS DATA

8.1 Calculating Equivalent SDOF Factors

The multi-degree-of-freedom (MDOF) numerical analysis allows for a new and different way, from Biggs, to derive the equivalent single-degree-of-freedom (SDOF) factors. The following subsections describe how the data taken from the MDOF is used to create the equivalent SDOF factors.

8.1.1 Stiffness Values

The Biggs method of creating the stiffness value was discussed in **Section 2** and an example derivation is given in **Section 11.1**. The stiffness values for a beam using Biggs and the MDOF method are the same because the values are based on static deflection.

The first study for producing stiffness values takes a simply supported beam with triangular loading, and varies the element mesh size. Using the MDOF method with static loads, the values using different mesh sizes are tabulated, in **Table 7**, against the theoretical stiffness values.

Table 7: Stiffness values for simply supported beam, triangular loading
- Theoretical vs. MDOF

E	29000	ksi		
I	4020	in ⁴		
peak load	2.4	klf	=	0.2 in/ft
length	24	ft	=	288 in
Total load (k)	28.8	k		

Mesh Size	Matlab Deflection (in)	X	Beam Chart X	% diff
6 ft (4 elements)	0.0884	66.78	60.00	11.30%
3 ft (8 elements)	0.0958	61.59	60.00	2.65%
2 ft (12 elements)	0.0972	60.70	60.00	1.17%
1.5 ft (16 elements)	0.0977	60.39	60.00	0.65%
1.2 ft (20 elements)	0.0979	60.25	60.00	0.42%
1 ft (24 elements)	0.0981	60.17	60.00	0.28%
0.5 ft (48 elements)	0.0983	60.04	60.00	0.07%
0.25 ft (96 elements)	0.0983	60.01	60.00	0.02%

Where $X = ([\text{total load}]/[\text{Matlab deflection}])(L^3/EI)$

Beam Chart $X = (EI/L^3)/[\text{max deflection per AISC}]$

The table above is represented in **Figure LL**.

Figure LL: Varying % error against theoretical stiffness value due to # of elements in beam for a simply supported, triangularly loaded, 24 ft long, W24x131

Table 7 and **Figure LL** show that as the mesh size decreases (meanwhile, the number of elements increase), the error of the stiffness value against the theoretical stiffness value also decreases. Given that the study is based on a 24ft beam, discretized to 24 elements, and that the stiffness value error is minimal, the rest of the stiffness values calculated by the MDOF program are also based on discretizing the beam into 24 elements.

To confirm that the stiffness value error is minimal for the other boundary condition and loading cases, the stiffness values for the cases below are compared to the theoretical stiffness values.

- Simply supported beam with uniformly distributed load (pin-pin uniform)
- Simply supported beam with linear loading starting with 0 klf on one end (pin-pin linear)

- Simply supported beam, distributed load with peak loading in the middle (pin-pin triangular)
- Beam with fixed end supports and uniformly distributed load (fix-fix uniform)

The cases listed above are tabulated in **Table 8**.

Table 8: Stiffness values - beam charts vs. MDOF

E	29000	ksi			
I	4020	in ⁴			
peak load	2.4	klf	= 0.2	in/ft	
length	24	ft	= 288	in	
Boundary Conditions and Loading Pattern	Total Load (k)	Matlab Deflection (in)	X	Beam Chart X	% diff
pin-pin uniform	57.6	0.1535	76.91	76.80	0.1%
fix-fix uniform	57.6	0.0307	384.00	384.00	0.0%
pin-pin linear	28.8	0.0767	76.91	76.69	0.3%
pin-pin triangle	28.8	0.0981	60.17	60.00	0.3%

Where $X = ([\text{total load}]/[\text{Matlab deflection}])(L^3/EI)$

Beam Chart $X = (EI/L^3)/[\text{max deflection per AISC}]$

Because the errors for the cases in **Table 8** are all minimal, we can derive some other stiffness values by using the program. The additional combinations of boundary conditions and loading patterns listed below are shown in **Table 9**:

- Simply supported beam with a parabolic distributed load (pin-pin parabola)
- Beam with fixed end supports, with linear loading starting with 0 klf on one end (fix-fix linear)
- Beam with fixed end supports, distributed load with peak loading in the middle (fix-fix triangular)
- Beam with fixed end supports, with a parabolic distributed load (fix-fix parabola)

Table 9: Stiffness values - beam charts vs. MDOF

E	29000	ksi	
I	4020	in ⁴	
peak load	2.4	klf	= 0.2 in/ft
length	24	ft	= 288 in

Boundary Conditions and Loading Pattern	Total Load (k)	Matlab Deflection (in)	X
fix-fix linear	28.8	0.0154	383.0
fix-fix triangle	28.8	0.0214	275.2
pin-pin parabola	19.2	0.0457	86.1
fix-fix parabola	19.2	0.0088	444.8

- Where $X = ([\text{total load}]/[\text{Matlab deflection}])(L^3/EI)$
- Beam Chart $X = (EI/L^3)/[\text{max deflection per AISC}]$

8.1.2 Stiffness Factor

The effective stiffness of the system is based on the generalized shape when the beam yields. The program records the time-step when the first node reaches yield. At the time step that the first node reaches yield stress, the generalized shape is taken and scaled so that the maximum deflection =1, and the effective stiffness of the system can be solved by using the following equation.

$$K_{eff} = s' K_s \quad \text{Eq. 22}$$

$$\text{Where } s = \frac{\bar{x}}{\delta_{\max}} \quad \text{Eq. 23}$$

K_{eff} is a scalar value and can be compared directly to the static stiffness values.

$$K_L = \frac{K_{eff}}{k} \quad \text{Eq. 24}$$

8.1.3 Mass Factor

The mass factor is derived with the same process as the stiffness factor. The equation for the effective mass is shown below.

$$M_{eff} = s' Ms \quad \text{Eq. 25}$$

K_{eff} is a scalar value and can be compared directly to the static stiffness values.

$$K_M = \frac{M_{eff}}{ml} \quad \text{Eq. 26}$$

8.1.4 Resistance Factor

The resistance factor can be calculated by using the scaled deflection vector used for the stiffness and mass factors, the stiffness matrix, and the deflection vector as shown in the equation below.

$$R_m = -s' Kx \quad \text{Eq. 27}$$

8.1.5 Factors Post Yield

The transformation factors used by Biggs assume that the beam has a new configuration after the first node(s) yield. For a uniformly loaded beam with fixed ends, the post-yield beam becomes a simply supported beam (as explained in **Section 2.2**). The same concept was applied in the MDOF numerical method; however, the factors derived from the Biggs' assumptions on nonlinear behavior are not consistent to the factors derived using the MDOF analysis. Derived values for a uniformly loaded beam with fixed ends are shown below.

Table 10: Secondary factor derivation attempt

Factor	Biggs	MDOF method	% difference
K_{L1}	0.53	0.5096	3.8%
K_{M1}	0.41	0.3900	4.5%
R_{m1}	12Mp/L	13.6Mp/L	13.3%
K_{L2}	0.64	2.9195	356%
K_{M2}	0.50	0.3513	29.7%
R_{m2}	16Mp/L	-	-

$Z = 10$ (See Table 11: Blast wave parameters used for analysis)

The secondary shape assumption, that the beam with fixed ends behaves like a simply supported beam after hinges form on either end, does not produce reasonable factors. This assumption does not work despite having a similar deflected shape. The figure below shows the inelastic deflected shape from the analysis (with the elastic shape subtracted) and the theoretical elastic deflected shape of a simply supported beam, both scaled to a max deflection of one unit.

Figure MM: Inelastic deflection from analysis (fixed ends) vs. theoretical elastic deflection (simply supported)

While having a similar deflected shape, **Eq. 22** through **Eq. 27** do not produce the expected factors close to Biggs' factors. Therefore, the K_{L2} and K_{M2} factors post-yield in this study are based on the elastic K_L and K_M factors of the assumed post-yield beam configuration. This means that in the case above, the K_L and K_M factors of a pin-pin configuration are used as the K_{L2} and K_{M2} factors for the post-yield calculations for a fix-fix beam with uniformly distributed loading. As seen in **figure Z** in **Section 7.4**, the elasto-plastic stiffness slope is similar to the theoretical SDOF stiffness slope, making this a decent assumption. Also as seen in **Figure Z**, there is no clear stiffness change indicating the next hinge forming, making the collapse mechanism difficult to model for a SDOF system. This study assumes the collapse mechanism that Biggs suggested;

however, the post yield behavior should be a topic to explore in order to better model the collapse progression.

8.2 Loading Parameters

In order to fill in the rest of the SDOF equivalent factors table, several loading cases were run through the MDOF numerical analysis algorithm. The blast wave parameters applied to each beam geometry and loading pattern are listed on **Table 11** below.

Table 11: Blast wave parameters used for analysis

Z	Pr	Ir	t ₀
30	0.0036	0.009	6.5
25	0.0046	0.013	7.34
20	0.0065	0.021	8.58
15	0.0105	0.038	10.36
14	0.0119	0.044	10.82
13	0.0137	0.052	11.34
12	0.0161	0.061	11.9
11	0.0193	0.074	12.5
10.5	0.0214	0.082	12.8
10	0.0239	0.091	13.12
9.5	0.0269	0.101	13.42
9	0.0307	0.113	13.72
8.5	0.0353	0.128	14
8	0.0412	0.146	14.24

Note: $Z = \frac{D}{W^{1/3}}$ where D=distance (ft) and W=equivalent TNT weight (lbs)

(See Section 6.4)

The Z -value stops at 8 ft/lbs^3 because the analyses showed that the factors become inconsistent between 13 ft/lbs^3 and 9 ft/lbs^3 . This data is, however, is consistent with the observation made in **Section 7.4**. When the impulse is too large and too fast, hinges form due to local forces, before the beam can develop the quasi-static deflected shape assumed by the Bigg's method. This reinforces the idea that the SDOF equivalent factors are only accurate for far or small charges, as stated in **Section 1** of this study.

8.3 Output and Factors for Beam with Both Ends Fixed, Uniformly Distributed Loading

The maximum deflection and the factors (calculated according to **Section 8.2**) are tabulated below in **Table 12**.

Table 12: Data for fix-fix, uniformly loaded beam

Z	K_L	K_M	R_m (M_p/L)	δ_{max} (in)
30	0.5105	0.3904	-	0.167
25	0.5114	0.3917	-	0.227
20	0.5129	0.3933	-	0.340
15	0.5454	0.4117	12.2	0.569
14	0.5369	0.4084	12.2	0.631
13	0.5274	0.4036	12.4	0.843
12	0.5132	0.3935	12.7	1.090
11	0.5077	0.3863	12.8	1.563
10.5	0.5082	0.3881	12.6	1.930
10	0.5125	0.3928	11.8	2.438
9.5	0.8992	0.4886	9.8	3.126
9	1.4337	0.5738	7.2	4.100
8.5	1.7339	0.5943	5.7	5.592
8	2.0666	0.5935	5.7	7.918
Average	0.52	0.40	12.4	-

Note: W24x131 beam, $L = 24$ ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

The factors that are similar in range are averaged and declared as the factors for the equivalent SDOF factors. In this case, K_L and K_M are only averaged between $Z=30$ ft/lbs³ and $Z=10$ ft/lbs³ because $Z=9.5$ ft/lbs³ seems to be the limit for accurate factors. Over the Z range of 10~30 ft/lbs³, the average K_L is 0.52, and K_M is 0.40. The average for R_m doesn't include $Z=20$ ft/lbs³ through $Z=30$ ft/lbs³ because the beam remains elastic at these demands. Otherwise, the average R_m is about 12.4Mp/L. Besides R_m , which

fluctuates with different cases, these values are further verified by running more cases altering beam size, beam length, and tributary area. The data for these other cases are tabulated below in **Table 13** and **Table 14**.

Table 13: Data for fix-fix, uniformly loaded beam

Z	K_L	K_M	R_m (M_p/L)	δ_{max} (in)
30	0.5084	0.3877	-	0.296
25	0.5102	0.3899	-	0.406
20	0.5465	0.4121	11.1	0.611
15	0.5101	0.3905	13.4	1.268
14	0.5067	0.3851	13.0	1.632
13	0.5077	0.3879	13.3	2.194
12	0.5199	0.3987	13.1	3.050
11	1.5435	0.5855	4.8	4.666
10.5	1.719	0.5936	4.8	5.908
10	1.9516	0.5960	4.7	7.534
9.5	2.2209	0.5904	4.4	9.741
9	3.1734	0.6175	3.0	12.841
8.5	3.6983	0.6337	2.8	17.199
8	5.2258	0.6473	2.1	23.393
Average	0.52	0.39	12.8	

Note: W21x93 beam, L = 24 ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

Table 14: Data for fix-fix, uniformly loaded beam

Z	K_L	K_M	R_m (M_p/L)	δ_{max} (in)
30	0.5054	0.3844	-	0.308
25	0.5063	0.3854	-	0.429
20	0.5080	0.3872	-	0.658
15	0.5228	0.3855	10.4	1.186
14	0.5102	0.3908	10.9	1.436
13	0.5049	0.3826	10.8	1.867
12	0.5050	0.3844	11.1	2.480
11	1.0711	0.5199	4.7	3.463
10.5	1.4567	0.5769	4.0	4.257
10	1.6582	0.5908	4.0	5.425
9.5	1.9123	0.5955	3.9	6.810
9	2.4088	0.5899	3.2	8.828
8.5	3.2461	0.6194	2.4	11.653
8	3.868	0.6384	2.3	15.790
Average	0.51	0.39	10.8	

Note: W24x131 beam, $L = 30$ ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

In both cases shown in **Table 13** and **Table 14**, the factors are only reasonable down to $Z = 12$ ft/lbs³. As mentioned in **Section 7.4** and **Section 8.2**, this is because the deflected shape at lower Z values are no longer consistent with the theoretical quasi-static deflected shapes as assumed by the Biggs' method.

All other cases discussed in **Section 8.1.1** are run and tabulated and located in **Table 21** through **Table 27** in the Appendix.

8.4 SDOF Results

In order to validate the Transformation Factors table, each case studied using MDOF is run through a SDOF analysis as well. The data for a Fix-fix beam under uniform load is shown in **Table 15** below.

Table 15: MDOF vs. SDOF for fix-fix, uniform load

Z	MDOF (in)	SDOF (in)	% Difference
30	0.167	0.164	-2.0
25	0.227	0.224	-1.7
20	0.340	0.335	-1.4
15	0.569	0.643	13.0
14	0.631	0.797	26.2
13	0.843	1.023	21.4
12	1.090	1.222	12.1
11	1.563	1.580	1.1
10.5	1.930	1.868	-3.2
10	2.438	2.265	-7.1
9.5	3.126	2.812	-10.1
9	4.100	3.618	-11.8
8.5	5.592	4.836	-13.5
8	7.918	6.651	-16.0

Note: W24x131 beam, L = 24 ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

The analysis was only run until $Z = 8 \text{ ft/lbs}^3$ to match the data range presented in Section 8.2. All other MDOF vs. SDOF comparisons are tabulated and located in **Table**

28 through **Table 34** in the Appendix. In all cases, there is less than 1% difference between the MDOF and SDOF methods in the elastic range. The displacements vary more in the inelastic response, up to about 35%.

9.0 CONCLUSION AND RECOMMENDATIONS

This project studies the equivalent single-degree-of-freedom (SDOF) factors presented by Biggs in 1964. The study presents the derivation of these factors, a validation of the factors using a multi-degree-of-freedom (MDOF) numerical analysis, the accuracy of the SDOF analysis compared to the MDOF analysis, and suggests factors for other cases to expand on what is known as the “Biggs Factors” table.

9.1 Verification of Biggs Values

There are four values from Biggs’ tables that the study tried to verify: load factor (K_L), mass factor (K_M), maximum resistance (R_m), and the spring constant (k). Of the four, the spring constant is based on a MDOF static deflection. The load factor, mass factor, and the maximum resistance value are solved for through using the MDOF numerical method and averaged over varying loading magnitude cases.

As seen on **Table 8**, the stiffness values taken from the MDOF for a 24-element beam model is no more than 0.3% different from the theoretical values based on the Euler-Bernoulli Beam Theory. The source of this small 0.3% error comes from the discretization assumption built into the MDOF method, specifically of the elements and the loading, as shown in **Table 7** and **Figure LL**. The deflection values from theory are based on a continuous beam, and while the MDOF attempts to mimic a continuous beam, it is still a discretized beam with material properties only applied to the nodes. By replacing the loading matrix with a consistent loading matrix, or formulating the loads so

that fixed end moments are applied to the end nodes, deflection values may be made more accurate with larger mesh sizes.³

In order to compare the study to the Biggs factors, the K_L and K_M factors from the study are rounded to two significant figures; therefore, the factors are rounded to two significant figures for consistency. After rounding, the average K_L and K_M values from the MDOF numerical analysis are the same as the K_L and K_M values from the Biggs tables.

The maximum resistance factors varied more between each case ran. The average resistance factor for the uniform load on a fixed-fixed beam hovered around 12 M_p/L between the (3) cases (**Table 12**: W24x131 over 24 ft, **Table 13**: W21x93 over 24 ft, **Table 14**: W24x131 over 30 ft), ranging between 10.8 M_p/L and 12.8 M_p/L . With enough cases, a statistical analysis may provide a reasonable average resistance value; however, running excessive cases is beyond the scope of the study. The three different beam cases, changing the length or the size of the beam, are enough to conclude that one case for each boundary condition and loading pattern combination is not enough data to validate or produce a resistance factor.

The load factor (K_L), mass factor (K_M), and the spring constant (k) from this study adequately validates Biggs' factors while the maximum resistance (R_m) factors derived from this study were too inconsistent to come to the same conclusion as the other factors.

³ Personal correspondence between Shalva Marjanishvili, Ph.D., S.E., and Takayuki Yokoyama 10/9/2011

9.2 Expanded Biggs Tables

The expanded Biggs' tables, based on the MDOF numerical analysis, is shown below.

Table 16: SDOF transformation factors for beams with pin-pin boundary conditions

Loading Diagram	Strain Range	Load Factor K_L	Mass Factor K_M	Maximum Resistance R_m	Spring Constant k
uniform	Elastic	0.64	0.50	$\frac{8M_p}{L}$	$\frac{384EI}{5L^3}$
	Plastic	0.50	0.33	$\frac{8M_p}{L}$	0
linear	Elastic	0.64	0.50	$\frac{7.6M_p}{L}$	$\frac{76.7EI}{L^3}$
	Plastic	0.50	0.33	$\frac{7.6M_p}{L}$	0
triangular	Elastic	0.81	0.50	$\frac{6M_p}{L}$	$\frac{60EI}{L^3}$
	Plastic	0.50	0.33	$\frac{6M_p}{L}$	0
Parabolic	Elastic	0.59	0.49	$\frac{8.47M_p}{L}$	$\frac{85.9EI}{L^3}$
	Plastic	0.50	0.33	$\frac{8.47M_p}{L}$	0

For a graphic of the pin-pin boundary condition, refer to **Figure F**.

Table 17: SDOF transformation factors for beams with fix-fix boundary conditions

Loading Diagram	Strain Range	Load Factor K_L	Mass Factor K_M	Maximum Resistance R_m	Spring Constant k
uniform	Elastic	0.52	0.40	$\frac{12M_p}{L}$	$\frac{384EI}{L^3}$
	E-P	0.64	0.50	$\frac{8}{L}(M_{ps} + M_{pm})$	$\frac{384EI}{5L^3}$
	Plastic	0.50	0.33	$\frac{8}{L}(M_{ps} + M_{pm})$	0
linear	Elastic	0.52	0.39	$\frac{10M_p}{L}$	$\frac{383.2EI}{L^3}$
	E-P (1)	0.65	0.44	$\frac{30}{7L}(2M_{pL} + M_{pR})$	$\frac{164.4EI}{L^3}$
	E-P (2)	0.64	0.50	$\frac{5.872}{L}(1.327M_{pM} - M_{pR} + 2.329M_{pL})$	$\frac{76.7EI}{L^3}$
	Plastic	0.50	0.33	$\frac{5.872}{L}(1.327M_{pM} - M_{pR} + 2.329M_{pL})$	0
triangular	Elastic	0.72	0.40	$\frac{9.7M_p}{L}$	$\frac{275.8EI}{L^3}$
	E-P	0.81	0.50	$\frac{6}{L}(M_{ps} + M_{pm})$	$\frac{60EI}{L^3}$
	Plastic	0.50	0.33	$\frac{6}{L}(M_{ps} + M_{pm})$	0
Parabolic	Elastic	0.45	0.39	$\frac{12M_p}{L}$	$\frac{447.2EI}{L^3}$
	E-P (1)	0.61	0.43	$\frac{10}{L}(M_{pL} + M_{pR})$	$\frac{172.6EI}{L^3}$
	E-P (2)	0.59	0.49	$\frac{3.132}{L}(2.7M_{pM} + 5.4M_{pR} + M_{pL})$	$\frac{85.9EI}{L^3}$
	Plastic	0.50	0.33	$\frac{3.132}{L}(2.7M_{pM} + 5.4M_{pR} + M_{pL})$	0

For a graphic of the fix-fix boundary condition, refer to **Figure G**.

9.3 Validity of Expanded Factors

Based on the MDOF analysis and the existing transformation factors table, the transformations factors K_L and K_M can be assumed accurate for practical use. This conclusion is based off the data on Section 8.3. The average K_L and K_M factors match the Biggs' table values to two significant digits in the elastic response cases. While the K_L and K_M values are confirmed based on the parallel test cases of the MDOF and SDOF methods, the transformation factors can only be assumed accurate for dynamic loads that result in the beam reacting elastically. The maximum deflection values diverge when the system behaves inelastically. This divergence is mild when the beam has an elasto-plastic response range, and large when the loading is too large and too fast for the beam to respond in the assumed quasi-static behavior. While the hinge property assumptions are consistent between the two analysis methods, the differences in deflection data, and the inelastic response comparison done in **Figure Z** in **Section 7.4**, suggest that the assumed simplified inelastic beam configuration for the SDOF is erroneous when the loading is too large and too fast.

The inelastic configuration is based on the elastic configuration of a beam with altered boundary conditions. Subsequently, the factors used are also based on the assumed elastic deflection of an altered beam configuration. As seen in **Figure MM**, the scaled inelastic deflected shape does not match the theoretical elastic deflected shape of the post-yield beam configuration. This difference in deflected shape causes the factors to vary post-yield; consequently, when the Biggs' assumptions are used in the SDOF method, the deflections also vary from the MDOF method when the beam behaves

inelastically. This approach for deriving post-yield assumed deflected shape and factors is acceptable for the elasto-plastic range, with minimal error.

While further study with significant data will help in establishing better stiffness transitions, this study shows that the MDOF approach produces the same factors as the static deflection assumptions found in (Biggs 1964) for the elastic and elasto-plastic responses. This study has also shown that compared to a MDOF analysis, the SDOF analysis is, as advertised, only a simplified method good enough to produce rough response information.

9.4 Opportunities for Further Study

The scope of the study was to verify Biggs' transformation factors by creating and using a MDOF numerical analysis program, and to expand upon the table using the same program. The study was unable to derive a consistent resistance value, which should be the first focus when expanding on this study.

Another focus for further study should be on the post-yield behavior of the components. The goal is to find assumptions that lead to SDOF analysis data to match the MDOF analysis data.

Once the above issues can be improved upon, further study can focus on adding variables to the MDOF system. Variables to consider include boundary fixity, effect of axial loads, and material properties.

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11.0 APPENDIX

11.1 Example Derivation of Equivalent SDOF Factors (Biggs' Method)

As introduced in **Section 2**, the Biggs' factors can be derived using the general deflection equation. The following is an example for a uniform load over a fix fixed beam.

Stiffness of the beam is defined as

$$k = \frac{P}{\delta_{\max}} \quad \text{Eq. 28}$$

Substitute in $P = wl$ and $\delta_{\max} = \frac{wl^4}{384EI}$ for k .

$$k = \frac{384EI}{l^3} \quad \text{Eq. 29}$$

A shape function is needed to derive the mass and stiffness factors. This shape function is based on the deflection equation for a uniform loading on a fix-fixed beam:

$$\delta(x) = \frac{wx}{24EI} (l^3 - 2lx^2 + x^3) \quad \text{Eq. 30}$$

The shape function is the deflection equation divided by the max deflection

$$\phi(x) = \frac{\frac{wx}{24EI} (l^3 - 2lx^2 + x^3)}{\frac{wl^4}{384EI}} \quad \text{Eq. 31}$$

$$\phi(x) = \frac{16x}{l^4} (l^3 - 2lx^2 + x^3) \quad \text{Eq. 32}$$

Plug the shape function into the K_M **Eq. 5** and the K_L **Eq. 9**

$$K_M = \frac{\int^L m \phi^2(x) dx}{mL} = 0.410 \quad \text{Eq. 33}$$

$$K_L = \frac{\int^L p \phi(x) dx}{pL} = 0.530 \quad \text{Eq. 34}$$

For the resistance function, the maximum bending value, M_p , is used.

$$M_p(0, L) = \frac{wl^2}{12} \quad \text{Eq. 35}$$

Solve for the load that causes M_p

$$wl = R_m = \frac{12M_p}{l} \quad \text{Eq. 36}$$

With the values calculated above, the elastic factors and values can be filled in on the Biggs Factors Table.

Table 18: Equivalent SDOF factors example

Loading Diagram	Strain Range	Load Factor K_L	Mass Factor K_M	Maximum Resistance R_m	Spring Constant k
uniform	Elastic	0.530	0.410	$\frac{12M_p}{L}$	$\frac{384EI}{L^3}$
	E-P				
	Plastic				

11.2 SDOF Average Acceleration Matlab program

The study uses the algorithm below for the analysis of a SDOF system. This algorithm has passed the verification process outlined in **Section 7**; however, there may

still be errors contained in the program. It is the responsibility of the user to verify the algorithm.

Table 19: Matlab algorithm for nonlinear numerical analysis of a SDOF system

File	Matlab Algorithm	Commentary
Main Script	<pre> %% User Input %%%%%%%%%% boundary = 'fix-fix' %'fix-fix' 'fix-pin' 'pin-pin' Loadpat = 'uniform' %% Section Properties W24x131 %%% weight = 131; % plf Ixx = 4020; % in^4 Sxx = 329; % in^3 Zxx = 370; % in^3 fy = 50; % ksi E = 29000; % ksi Mp = fy*Zxx; % k-in %%%%%%%% Beam Damping Ratio %%% c = .00 %%%%%%%% Beam Layout %%% L = 24*12; % in </pre>	<p>“%” are used as comment lines.</p> <p>Select boundary conditions: ‘fix-fix’ for both ends fixed ‘fix-pin’ for (1) fixed end and (1) pinned end ‘pin-pin’ for both ends pinned</p> <p>Loading type: - uniform = 'uniform', - point load = 'pointld', - linear starting at 0(left) and Pr(right) = 'linearR', - triangular with peak in middle = 'triangl' - parabolic with 0(left) and Pr(right) = 'parabol'</p> <p>Assumes the beam is a W24x131 and lists the section properties as well as the material properties.</p>

<pre> tribwidth = 24; % ft steps = 20000 deltat=.000002 %% Blast Properties %% Pr = .0412; % ksi ir = .146; tA = 0; % ms t0 = 14.24; % ms %% Script %% [k KL KM Res] = Boundary_conditions(boundary, Loadpat, E, Ixx, L, Mp) m = weight/1000*L/12/(32*12); % ksec2/in M = m*KM for i=1:3 K(1,i)=k(1,i)*KL(1,i); end omega = sqrt(K(1,1)/(M(1,1))) T1 = 2*pi()/omega Pressure_history = Loading_History(Pr, ir, tA, t0, deltat,steps); Phistory = Pressure_history*tribwidth*12*L; </pre>	<p>Define time step increments and the number of timesteps</p> <p>Reflected peak blast pressure Reflected blast impulse Time of blast arrival End of positive blast pressure</p> <p>Calls out the boundary conditions function to retrieve the stiffness factors, mass factors, and resistance values for the defined boundary conditions and loading pattern</p> <p>Mass</p> <p>Effective mass</p> <p>Effective stiffness</p> <p>Frequency Period</p> <p>Calls out the loading history function to define the loading time history</p> <p>Multiplies the loading time history with the tributary area of the beam</p>
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	<pre>[u R] = Newmark_Average(k,K,KL,Res,Phistory,c,M,delta t,steps);</pre>	<p>Calls out the Newmark Average function that runs the SDOF system through Newmark's Average Acceleration numerical analysis method</p>
Boundary Conditions	<pre>function [K KL KM Res] = Boundary_conditions(Boundary, Loadpat, E, Ixx, L, Mp) if Loadpat == 'pointld' if Boundary == 'fix-fix' K = [192*E*Ixx/L^3 192*E*Ixx/L^3 0] KL = [1 1 1] KM = [.37 .37 .33] Res = [8/L*Mp 0.001/L*Mp 0] end if Boundary == 'fix-pin' K = [107*E*Ixx/L^3 48*E*Ixx/L^3 0] KL = [1 1 1] KM = [.43 .49 .33] Res = [16*Mp/(3*L) 2/3*Mp/L 0] end if Boundary == 'pin-pin' K = [48*E*Ixx/L^3 48*E*Ixx/L^3 0] KL = [1 1 1] KM = [.49 .49 .33] Res = [4/L*Mp 0.001/L*Mp 0] end end if Loadpat == 'uniform' totalload = 1; if Boundary == 'fix-fix' K = [384*E*Ixx/L^3 384*E*Ixx/(5*L^3) 0] KL = [.53 .64 .5] KM = [.41 .5 .33] Res = [12*Mp/L 4*Mp/L 0] end if Boundary == 'fix-pin' K = [185*E*Ixx/L^3 384*E*Ixx/(5*L^3) 0] KL = [.58 .64 .5]</pre>	<p>Assigns the stiffness, stiffness factors, mass factors and resistance values based on the boundary conditions and loading pattern</p>

<pre> KM = [.45 .5 .33] end if Boundary == 'pin-pin' K = [384*E*Ixx/(5*L^3) 384*E*Ixx/(5*L^3) 0] KL = [.64 .64 .5] KM = [.5 .5 .33] Res = [8*Mp/L 0.001*Mp/L 0] end end if Loadpat == 'triangl' totalload = .5; if Boundary == 'fix-fix' K = [275.8*E*Ixx/L^3 60*E*Ixx/L^3 0] KL = [.72 .81 .5] KM = [.4 .5 .33] Res = [9.6/L*Mp 2.4/L*Mp 0] end if Boundary == 'pin-pin' K = [60*E*Ixx/L^3 60*E*Ixx/L^3 0] KL = [.81 .81 .5] KM = [.5 .5 .33] Res = [6/L*Mp 0.001/L*Mp 0] end end if Loadpat == 'linearR' totalload = .5; if Boundary == 'fix-fix' K = [383.2*E*Ixx/L^3 164.4*E*Ixx/*L^3 76.6*E*Ixx/L^3] KL = [.52 .65 .64] KM = [.39 .44 .5] Res = [10*Mp/L 2.9*Mp/L 2.7*Mp/L] end if Boundary == 'pin-pin' K = [76.6*E*Ixx/*L^3 76.6*E*Ixx/*L^3 0] KL = [.64 .64 .5] KM = [.5 .5 .33] Res = [7.6*Mp/L 0.001*Mp/L 0] end end end </pre>	
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	<pre> if Loadpat == 'parabol' totalload = 1/3; if Boundary == 'fix-fix' K = [447.2*E*Ixx/L^3 172.6*E*Ixx/*L^3 85.9*E*Ixx/L^3] KL = [.45 .61 .59] KM = [.39 .43 .49] Res = [12*Mp/L 8*Mp/L 10.5*Mp/L] end if Boundary == 'pin-pin' K = [85.9*E*Ixx/*L^3 85.9*E*Ixx/*L^3 0] KL = [.59 .59 .5] KM = [.49 .49 .33] Res = [8.47*Mp/L 0.001*Mp/L 0] end end end </pre>	
Pressure History	<pre> function Pressure_history = Loading_History(Pr, ir, tA, t0, deltat,steps) syms t q w Pressure_history = zeros(1,steps+2); Psr = Pr*(1-(t/t0))*exp(-(t/w)) Psrdt = int(Psr,t,0,t0) y = Psrdt - ir ynew =subs(y,'w',0:0.05:10); x= [0:0.05:10]; wnew = interp1(ynew,x,0) Pos = ceil(t0/1000/deltat); for i=1:Pos t=i*deltat*1000-deltat*1000; Pos_History(1,i) = Pr*(1-(t/t0))*exp(-(t/wnew)); end Arrival = ceil(tA/1000/deltat)+1 Pos_end = Pos + Arrival - 1 Pressure_history(1,Arrival:1:Pos_end) = Pos_History; </pre>	Creates the pressure time history as outlined in Section 6.3
	<pre>function [u R] =</pre>	Runs the numerical

Newmark Average Acceleration	<pre> Newmark_Average(k,K,KL,Res,Phistory,C,M,delta t,steps) gamma = 1/2; beta = 1/4; xi = 0; vi = 0; ai = 0; deltapd = 0; deltax = 0; deltav = 0; deltaa = 0; ures = 0; Rres = 0; for i=1:3 if k(1,i)>0 ulim(1,i) = Res(1,i)/k(1,i); else ulim(1,i) = 500; end end %%% R2=K2*U2, R2=K1*U2+b ==> solve for b, (b=R2-K1*U2) %%% subs into 0=K1*uresidual+b, or uresidual = - b/K1 %%% ratio = uresidual/U2 uresratio = (-(Res(1,2)- K(1,1)*ulim(1,2))/K(1,1))/ulim(1,2); Rswitch = 1; for i=1:steps+1 deltap(1,i) = Phistory(i+1)-Phistory(i); end P = Phistory; % Initial Calculations a0 = 0; </pre>	<p>analysis using Newmark's average acceleration method</p> <p>Gamma and beta numbers for average acceleration method</p> <p>Initial values</p> <p>Residual displacement</p> <p>Establish displacements corresponding with the resistance values and the stiffness values</p> <p>Indicates switches in elasticity/plasticity</p>
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<pre> ures1 = 0; ures = 0; R=zeros(1,steps+1) u = zeros(1,steps+1); umeas = zeros(1,3,steps); up = 0; upp = 0; pass = zeros(1,steps); uresidual=zeros(1,steps+1); rezero = 0; for i=1:steps [dum1 Rswitch j dum2 dum3 dum4 dum5] = hysteresis (u,up,upp,uresratio,ulim,ures,ures1,rezero,Rswitch,i ,umeas); khat = K(1,j) + (gamma/(beta*deltat))*C + (1/(beta*deltat*deltat))*M(1,j); f1 = (1/(beta*deltat))*M(1,j) + (gamma/beta)*C; f2 = (1/(2*beta))*M(1,j) + deltat*(gamma/(2*beta)- 1)*C; deltapd = deltap(1,i)*KL(1,j) + f1*vi + f2*ai; deltax = deltapd/khat; deltav = (gamma/(beta*deltat))*deltax - (gamma/beta)*vi + deltat*(1-(gamma/(2*beta)))*ai; xi = xi + deltax; upp = up; up = u(i); u(i+1)=xi; [umeas dum6 dum7 ures ures1 rezero pass(i)] = hysteresisssss(u,up,upp,uresratio,ulim,ures,ures1,re zero,Rswitch,i+1,umeas); for z=1:3 R(i+1) = R(i+1)+K(1,z)*umeas(1,z,i); </pre>	<p>Start of time-steps Calls out the hysteresis function to calculate the state of plasticity of the system</p> <p>2.2</p> <p>2.3</p> <p>2.4 calculate incremental effective load</p> <p>2.5 calculate Δx 2.6 calculate Δv</p> <p>2.7 new x</p>
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	<pre> end vi = vi + deltav; ai = (P(i+1)*KL(1,j)-C*vi-R(i+1))/M(1,j); uresidual(i) = ures; end </pre>	<p>2.7 new v 2.8 new a</p>
Hysteresis	<pre> function [Rswitch j] = hysteresis(u,up,upp,uresratio,ulim,ures,ures1,rezero ,Rswitch,i,umeas) if Rswitch == 1; %%%%elastic umeas(1,1,i) = u(i); j=1; if abs(u(i))>ulim(1,1) Rswitch = 2; end end if Rswitch == 2; if u(i)>up && up>upp if u(i)<(ulim(1,1)+ulim(1,2)) umeas(1,1,i) = ulim(1,1); umeas(1,2,i) = u(i) - ulim(1,1); ures1 = umeas(1,2,i)*uresratio; j = 2; end else umeas(1,1,i) = ulim(1,1); umeas(1,2,i) = ulim(1,2); umeas(1,3,i) = u(i) - ulim(1,1) - ulim(1,2); ures1 = ulim(1,2)*uresratio + umeas(1,3,i); j = 3; end end if u(i)<up && up<upp </pre>	<p>This function dictates what stage of the hysteresis loop the system is in. Rswitch = 1: elastic Rswitch = 2: yield Rswitch = 3: direction change</p> <p>j = 1: elastic j = 2: elasto-plastic j = 3: plastic</p>

	<pre> if u(i)>(ulim(1,1)+ulim(1,2)) umeas(1,1,i) = -ulim(1,1); umeas(1,2,i) = u(i) + ulim(1,1); ures1 = umeas(1,2,i)*uresratio; j = 2; pass = 4; else umeas(1,1,i) = -ulim(1,1); umeas(1,2,i) = -ulim(1,2); umeas(1,3,i) = u(i) + ulim(1,1) + ulim(1,2); ures1 = -ulim(1,2)*uresratio + umeas(1,3,i); j = 3; pass = 5; end end if u(i)<up && up>upp Rswitch = 3; end if u(i)>up && up<upp Rswitch = 3; end end if Rswitch == 3; if u(i)<up && up<upp hinge1 = rezero-2*ulim(1,1); hinge2 = rezero-2*ulim(1,1)-ulim(1,2); if u(i)>=hinge1 umeas(1,1,i) = u(i) - ures; umeas(1,2,i) = 0; umeas(1,3,i) = 0; j = 1; pass = 6; end if u(i)>=hinge2 && u(i)<hinge1 umeas(1,1,i) = hinge1-ures; umeas(1,2,i) = u(i)-hinge1; umeas(1,3,i) = 0; ures1 = ures + umeas(1,2,i)*uresratio; j = 2; pass = 7; </pre>	
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	<pre> end if u(i)<hinge2 umeas(1,1,i) = hinge1-ures; umeas(1,2,i) = hinge2-ures-umeas(1,1,i); umeas(1,3,i) = u(i) - hinge2; ures1 = ures+umeas(1,2,i)*uresratio + umeas(1,3,i); j = 3; pass = 8; end end if u(i)>up && up>upp hinge1 = rezero+2*ulim(1,1); hinge2 = rezero+2*ulim(1,1)+ulim(1,2); if u(i)<=hinge1 umeas(1,1,i) = u(i) - ures; umeas(1,2,i) = 0; umeas(1,3,i) = 0; j = 1; pass = 9; end if u(i)<=hinge2 && u(i)>hinge1 umeas(1,1,i) = hinge1-ures; umeas(1,2,i) = u(i) - hinge1; umeas(1,3,i) = 0; ures1 = ures+umeas(1,2,i)*uresratio; j = 2; pass = 10; end if u(i)>hinge2 umeas(1,1,i) = hinge1-ures; umeas(1,2,i) = hinge2-ures-umeas(1,1,i); umeas(1,3,i) = u(i) - hinge2; ures1 = ures+umeas(1,2,i)*uresratio + umeas(1,3,i); j = 3; pass = 11; end end if u(i)<up && up>upp umeas(1,.,i) = umeas(1,.,i-1); j = 1; ures = ures1; </pre>	
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	<pre> rezero = up; pass = 12; end if u(i)>up && up<upp umeas(1,:,i) = umeas(1,:,i-1); j = 1; ures = ures1; rezero = up; pass = 13; end end </pre>	
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11.3 MDOF Average Acceleration Matlab program

The study uses the algorithm below for the analysis of a SDOF system. This algorithm has passed the verification process outlined in **Section 7**; however, there may still be errors contained in the program. It is the responsibility of the user to verify the algorithm.

Table 20: Matlab algorithm for nonlinear numerical analysis of a MDOF system

File	Matlab Algorithm	Commentary
Main Script	<pre> %%%%%%%% User Input %%%%%%%%% %% Section Properties W24x131%% weight = 131; % plf Ixx = 4020; % in⁴ Sxx = 329; % in³ Zxx = 370; % in³ fy = 50; % ksi E = 29000; % ksi %% Beam Damping Ratio %% zetai = 0; zetaj = 0; </pre>	<p>“%” are used as comment lines.</p> <p>Assumes the beam is a W24x131 and lists the section properties as well as the material properties.</p> <p>Damping is not used for analysis in this study, but the program is capable of Rayleigh Damping</p> <p>1st mode damping 2nd mode damping</p>

Verification and Expansion of Single-Degree-of-Freedom Transformation Factors for Beams Using a Multi-Degree-of-Freedom Non-Linear Numerical Analysis Method

<pre> %% Beam Layout %% L = 24*12; % in tribwidth = 24; % ft num_elements = 24; nodes = num_elements + 1; n = nodes*2; mlf = weight/1000/12/(32*12); (ksec2/in/in) DL = -0; % ksf uniform = y; % 'y' or 'n' LoadDist = [1 n;0 0]; % kips %% Boundary Conditions %% % fixed = [1 2 n-1 n]; % free = [3:n-2]; % pinned-fixed beam % fixed = [1 2 n-1]; % free = [3:n-2 n]; % simply supported beam fixed = [1 n-1]; free = [2:n-2 n]; %% Blast Properties %% Pr = .0589; %ksi ir = .195; tA = 0; % ms t0 = 14.41; % ms type = 'triangl'; w = 1; % factor pt = 1; % factor %% Numerical Integration Options %% </pre>	<p>Beam geometry Length of beam Tributary width Number of finite elements</p> <p>Mass per lineal foot Dead load Dead load distribution – ‘y’ = uniform load, ‘n’ = point loads Point load location and load</p> <p>Boundary conditions Comment out the conditions not in use 1 and (2n-1) are vertical translation at the ends 2 and (2n) are rotation at the ends</p> <p>Blast Loading factors Peak pressure Total impulse Time of blast arrival End of positive blast pressure Loading type: uniform = 'uniform', point load = 'pointld', linear starting at 0(left) and Pr(right) = 'linearR', triangular with peak in middle = 'triangl'</p>
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<pre> steps = 80000; deltat = .0000005; % sec %% Script %% [K k] = Stiffness_Assembly(E, Ixx, L, num_elements, n, fixed); M = Mass_Assembly(E, Ixx, L, mlf, num_elements, n); [phifull, lambda] = eig(K,M); omega = sqrt(diag(lambda)); [omega z] = sort(omega); for i=1:8 T(i,1) = 2*pi()/omega(i,1); end [C a0 a1] = Damping_Assembly(k,M,omega,zetai,zetaj); q = size(z); q = q(1,1); for i=1:q Rdamp(i) = a0/(2*omega(i))+a1*omega(i)/2; end u = zeros(n,steps); [Pconst u(:,1)] = Initial_Displacement(uniform,LoadDist,K,weight,D L,L,tribwidth,num_elements,n,steps); 'Preliminary Calculations finished' Pspace = loading(type, w, pt, n, num_elements, L); </pre>	<p>Number of steps in analysis Seconds per step</p> <p>Calls out the Stiffness_Assembly function to assemble the stiffness matrix</p> <p>Calls out the Mass_Assembly function to assemble the mass matrix</p> <p>Calls out the eig function to calculate modeshapes and frequencies</p> <p>Calculates period from the frequencies</p> <p>Calls out the Damping_Assembly function to assemble the damping matrix. Damping is based on Rayleigh Damping.</p> <p>Sets displacement matrix</p> <p>Calls out the Initial_Displacement function to calculate initial displacement shape from Dead Load</p> <p>Calls out the loading function to establish</p>
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	<pre> Pressure_history = Loading_History(Pr, ir, tA, t0, deltat,steps); Phistory = Pressure_history*tribwidth*12*12; P = Pspace * Phistory; 'Loading finished' Kfree = K(free,free); Mfree = M(free,free); Cfree = C(free,free); Pspacefree = Pspace(free,1); [u Ma Keff count] = NewmarkMDOF(n,Kfree,Phistory,Pspacefree,Pcon st,Cfree,Mfree,deltat,steps,E,Ixx,Sxx,Zxx,fy,L,num _elements,fixed,free,u); 'Analysis Finished' </pre>	<p>loading distribution along the length of the beam</p> <p>Calls out the Loading_History function to calculate the pressuretime history</p> <p>Adjusts loading per tributary width</p> <p>Builds a full loading matrix over space and time</p> <p>Takes out the degrees of freedom</p> <p>Calls out the NewmarkMDOF function to run the numerical analysis of the beam behavior under a blast-type load</p>
Stiffness Matrix	<pre> function [K k] = Stiffness_Assembly(E, Ixx, L, num_elements, n, fixed) K = zeros(n,n); l = L/num_elements; for i=1:num_elements j = i*2-1; ke(:,i) = 2*E*Ixx/(l^3)*[6, 3*l, -6, 3*l; 3*l, 2*l^2, -3*l, l^2; </pre>	<p>Stiffness matrix assembly function</p> <p>Loop structure to insert the stiffness matrix of each segment into the global matrix.</p> <p>Stiffness matrix of a single segment</p>

	<pre> -6, -3*1, 6, -3*1; 3*1, 1^2, -3*1, 2*1^2]; loc = [j:j+3]; K(loc,loc) = K(loc,loc)+ke(:,i); end k=K; adj = max(max(K))*10^6; for i=1:length(fixed); j = fixed(i); K(j,j) = K(j,j) + adj; end </pre>	Adjust the stiffness of the boundary condition to be larger by 10^6
Mass Matrix	<pre> function M = Mass_Assembly(E, Ixx, L, mlf, num_elements, n) M = zeros(n,n); l = L/num_elements; mlf; for i=1:num_elements j = i*2-1; me = mlf*l/420*[156, 22*1, 54, -13*1; 22*1, 4*1^2, 13*1, -3*1^2; 54, 13*1, 156, -22*1; -13*1, -3*1^2, -22*1, 4*1^2]; loc = [j:j+3]; M(loc,loc) = M(loc,loc)+me; end </pre>	<p>Mass matrix assembly function</p> <p>Loop structure to insert the mass matrix of each segment into the global matrix.</p> <p>Mass matrix of a single segment</p>
Damping Matrix	<pre> function [C a0 a1] = Damping_Assembly(K,M,omega,zetai,zetaj) omegai = omega(1,1); omegaj = omega(2,1); a = (1/2*[1/omegai, omegai; 1/omegaj, omegaj])\[zetai;zetaj]; a0 = a(1,1); </pre>	<p>Damping matrix assembly function, based on Rayleigh Damping</p> <p>1st mode frequency 2nd mode frequency</p> <p>Solve for the damping ratios</p>

	<pre>a1 = a(2,1); C = a0*M + a1*K;</pre>	Use the damping ratios to create the Rayleigh damping factor
Initial Displacement	<pre>function [Pconst u] = Initial_Displacement(uniform,LoadDist,K,weight,D L,L,tribwidth,num_elements,n,steps) F = zeros(n,1); if uniform == 'y' lineload = DL*tribwidth+weight/1000; else lineload = weight/1000; end for i=1:length(LoadDist) j = LoadDist(1,i); F(j,1) = F(j,1)+LoadDist(2,i); end F(1,1) = lineload*(L/12)/num_elements/2; F(n-1,1) = F(1,1); for i=2:num_elements F(2*i-1,1) = lineload*(L/12)/num_elements+F(2*i-1,1); end Pconst = zeros(n,steps+2); for i = 1:steps+2 Pconst(:,i)=F; end u = K\F;</pre>	<p>Initial Displacement calculation function</p> <p>Calculate the initial displacement of the beam with a uniform distribution of load</p> <p>Locate the point loads along the length of the beam</p> <p>Calculate the displacement of the beam based on the point loads</p> <p>Construct a constant load time history to add to the impulse loading</p> <p>Initial displacement</p>
	<pre>function Pspace = loading(type, w, pt, n, num_elements, L) Pspace = zeros(n,1); if type == 'uniform' total_load = w*L/12; %(p)</pre>	<p>Assembles the loading distribution over the beam length</p> <p>For loading type of 'uniform', point loads are</p>

Loading Pattern	<pre> loadpernode = total_load/num_elements; for i = 1:num_elements+1 Pspace(i*2-1,1) = loadpernode; end Pspace(1,1) = loadpernode/2; Pspace(n-1,1) = loadpernode/2; end if type == 'pointld' Pspace(n/2,1) = pt; end if type == 'triangl' half = num_elements/2; for i = 1:half Pspace(i*2-1,1) = 2*((i- 1)/num_elements)*w/12*(L/num_elements); Pspace(2*num_elements+2-(i*2-1),1) = Pspace(i*2-1,1); end Pspace(1,1)=Pspace(3,1)/4; Pspace(num_elements*2+1,1) = Pspace(1,1); Pspace(num_elements+1,1) = 2*(w/12*(L/(2*num_elements))-Pspace(1,1)); end if type == 'linearR' for i = 1:num_elements Pspace(i*2-1,1) = ((i- 1)/num_elements)*w/12*(L/num_elements); end Pspace(1,1)=Pspace(3,1)/4; Pspace(2*num_elements+1,1) = w/12*(L/(2*num_elements))-Pspace(1,1); end if type == 'parabol' for i = 1:num_elements+1 Pspace(i*2-1,1) = ((i- 1)/num_elements)^2*L/num_elements*w/12; end </pre>	<p>distributed evenly along the length of the beam</p> <p>For loading type of 'pointld', a point load is applied at the center of the beam span</p> <p>For loading type of 'triangl', point load is max at center of beam span and minimal at the ends.</p> <p>For loading type of 'linearR', load increases linearly from node i to node j</p> <p>For loading type of 'parabol', the loading shape follows a parabolic increase (x²) with peak load at node j</p>
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	<pre> Pspace(1,1) = Pspace(1,1)/2; Pspace(2*num_elements+1,1)=Pspace(2*num_elements+1,1)/2; end </pre>	
Loading History	<pre> function Pressure_history = Loading_History(Pr, ir, tA, t0, deltat,steps) syms t q w Pressure_history = zeros(1,steps+2); Psr = Pr*(1-(t/t0))*exp(-(t/w)); Psrdt = int(Psr,t,0,t0); y = Psrdt - ir; ynew =subs(y,'w',0:0.05:10); x= [0:0.05:10]; wnew = interp1(ynew,x,0); Pos = ceil(t0/1000/deltat); for i=1:Pos t=i*deltat*1000-deltat*1000; Pos_History(1,i) = Pr*(1-(t/t0))*exp(-(t/wnew)); end Arrival = ceil(tA/1000/deltat)+1; Pos_end = Pos + Arrival - 1; Pressure_history(1,Arrival:1:Pos_end) = Pos_History; </pre>	Develops the pressure history of the blast wave based on the loading equations in the SBEDS manual
	<pre> function [u Ma Keff count] = NewmarkMDOF(n,K,Phistory,Pspace,Pconst,C,M, deltat,steps,E,Ixx,Sxx,Zxx,fy,L,num_elements,fixe d,free,u) gamma = 1/2; beta = 1/4; l=L/num_elements; flagcount = 0; f=0; maxed(1) = num_elements+2; Mp = fy*Zxx; </pre>	<p>Runs the numerical analysis using Newmark's average acceleration method</p> <p>Gamma and beta numbers for average acceleration method</p>

	<pre> [Ki, R, Ma(:, :, i)] = K_redo(E, Ixx, Mp, l, num_elements, Map, u(:, i), up, fixed); if i == 1 Keff(:, :, 1) = Ki; end khat = Ki(free, free) + (gamma/(beta*deltat))*C + (1/(beta*deltat*deltat))*M; deltax = khat\deltapd; deltav = (gamma/(beta*deltat))*deltax - (gamma/beta)*vi + deltat*(1-(gamma/(2*beta)))*ai; deltaa = 4/(deltat^2)*deltax-4/deltat*vi-2*ai; xi = xi + deltax; xtem = zeros(n,1); xtem(free) = xi; [dum1, R, dum1] = K_redo(E, Ixx, Mp, l, num_elements, Ma(:, :, i), xtem, u((:, i), fixed); vi = vi + deltav; dif = fix(R(free) - Ki(free, free) * xi); ai = M\ (P(:, i+1) - C*vi - R(free)); Map = Ma(:, :, i); flag = 0; for s = 1:(num_elements) mom(s) = abs(Ma(2, s, i)); end mom(num_elements+1) = abs(Ma(4, num_elements, i)); mom(num_elements+2) = 0; z = length(mom); mom(maxed) = zeros; for b = 1:z if mom(b) == Mp f=f+1; </pre>	<p>2) Calls out the K_redo function</p> <p>3) tangent stiffness</p> <p>4) calculate delta x</p> <p>5) calculate delta v</p> <p>6) calculate delta a</p> <p>7) new x, v, and a</p> <p>Records the timestep when any node goes nonlinear, for the purpose of shape functions</p>
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	<pre> maxed(f) = b; flag = 1; end end if flag == 1 flagcount = flagcount+1; count(flagcount) = i; Keff(:, :, flagcount+1) = Ki; end up = u(:,i); u(free,i+1) = xi; end </pre>	
Stiffness and Mass Matrix Assembly	<pre> function [Ki, R, Ma] = K_redo(E,Ixx,Mp,l,num_elements,Map,u,up,fixed) n = length(u); Ki = zeros(n,n); R = zeros(n,1); Ma = zeros(4,num_elements); for i = 1:num_elements ke = 2*E*Ixx/(l^3)*[6, 3*l, -6, 3*l; 3*l, 2*l^2, -3*l, l^2; -6, -3*l, 6, -3*l; 3*l, l^2, -3*l, 2*l^2]; loc = [2*i-1 2*i 2*i+1 2*i+2]; eleu = u(loc); eleup = up(loc); Mtr = Map(:,i) + ke * (eleu-eleup); Ma(:,i) = Mtr; if abs(Mtr(2))>= Mp ke = (E*Ixx/l)*[3/l^2, 0, -3/l^2, 3/l; 0, 0, 0, 0; -3/l^2, 0, 3/l^2, -3/l; 3/l, 0, -3/l, 3]; Ma(2,i) = sign(Mtr(2)) * Mp; Ma(4,i) = Mtr(4); Ma(1,i) = (Ma(2,i) + Ma(4,i)) / l; end end </pre>	<p>Re-assembles the stiffness matrix and mass matrix at every timestep</p> <p>New stiffness matrix for plastic moment at node i of segment</p>

<pre> Ma(3,i) = -Ma(1,i); end if abs(Mtr(4)) >= Mp ke = (E*Ixx/l) * [3/l^2, 3/l, -3/l^2, 0; 3/l, 3, -3/l, 0; -3/l^2, -3/l, 3/l^2, 0; 0, 0, 0, 0]; Ma(2,i) = Mtr(2); Ma(4,i) = sign(Mtr(4)) * Mp; Ma(1,i) = (Ma(2,i) + Ma(4,i)) / l; Ma(3,i) = -Ma(1,i); end Ki(loc,loc) = Ki(loc,loc) + ke; R(loc) = R(loc) + Ma(:,i); end adj = max(max(Ki)) * 10^6; for i=1:length(fixed); j = fixed(i); if Ki(j,j) == 0; Ki(j,j) = Ki(j,j); else Ki(j,j) = Ki(j,j) + adj; end end end </pre>	<p>New stiffness matrix for plastic moment at node j of segment</p>
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11.4 MDOF Response Data

The following are tabulated response data as discussed in **Section 8.3**.

Table 21: Data for pin-pin, uniform load

Z	K_L	K_M	R_m (M_p/L)	δ_{max} (in)
30	-	-	-	0.456
25	0.6282	0.4936	-	0.648
20	0.6284	0.4942	-	1.018
15	0.6280	0.4937	7.2	1.953
14	0.6320	0.4960	7.5	2.386
13	0.6403	0.5041	7.9	3.112
12	0.6560	0.5113	8.2	4.065
11	0.6683	0.5167	8.6	5.673
10.5	0.6623	0.5141	8.5	6.795
10	0.6353	0.5006	7.6	8.349
9.5	0.6285	0.4927	7.3	10.375
9	0.6261	0.4873	6.5	13.185
8.5	0.6292	0.4922	6.6	17.161
8	0.6346	0.4987	7.0	22.737
Average	0.64	0.50	7.5	-

Note: W24x131 beam, $L = 24$ ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

Table 22: Data for pin-pin, linear load

Z	K_L	K_M	R_m (M_p/L)	δ_{max} (in)
30				0.228
25	0.6379	0.4939		0.324
20	0.6411	0.4948		0.509
15	0.6464	0.4972		0.884
14	0.6473	0.4977		1.013
13	0.6482	0.4982		1.183
12	0.6354	0.4992	7.6	1.401
11	0.6497	0.4986	7.5	1.762
10.5	0.6400	0.4950	7.2	2.018
10	0.6375	0.4975	7.5	2.355
9.5	0.6388	0.5016	7.9	2.804
9	0.6540	0.5084	7.9	3.384
8.5	0.9467	0.4482	5.0	4.258
8	1.0592	0.4377	5.1	5.606
Average	0.64	0.50	7.6	

Note: W24x131 beam, L = 24 ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

Table 23: Data for fix-fix, linear load

Z	K_L	K_M	R_m (M_p/L)	δ_{max} (in)
30	0.5181	0.3913		0.084
25	0.5171	0.3925		0.114
20	0.5174	0.3941		0.170
15	0.5180	0.3953	8.3	0.284
14	0.5182	0.3955	9.4	0.323
13	0.5184	0.3958	10.9	0.375
12	0.5335	0.4061	12.5	0.436
11	0.5542	0.4097	9.9	0.522
10.5	0.5497	0.3990	10.2	0.586
10	0.5549	0.3863	10.2	0.671
9.5	0.6415	0.3713	8.2	0.792
9	0.7441	0.3646	6.5	0.953
8.5	0.9369	0.3598	5.4	1.231
8	1.1918	0.3506	4.5	1.665
Average	0.52	0.40	10.7	

Note: W24x131 beam, L = 24 ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

Table 24: Data for pin-pin, triangular load

Z	K_L	K_M	R_m (M_p/L)	δ_{max} (in)
30	0.815	0.5019		0.287
25	-	-		0.408
20	0.8143	0.5015		0.643
15	0.8123	0.5003		1.119
14	0.8119	0.5001	5.8	1.284
13	0.8085	0.4978	6.0	1.560
12	0.8130	0.5008	6.1	1.933
11	0.8137	0.5012	6.1	2.589
10.5	0.8087	0.4979	5.9	3.078
10	0.8056	0.4954	5.8	3.697
9.5	0.8053	0.4950	5.8	4.466
9	0.8060	0.4958	5.9	5.569
8.5	0.8085	0.4979	5.9	7.138
8	0.812	0.5007	6.1	9.322
Average	0.81	0.50	6.0	

Note: W24x131 beam, L = 24 ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

Table 25: Data for fix-fix, triangular load

Z	K_L	K_M	R_m (M_p/L)	δ_{max} (in)
30	0.7188	0.3960		0.115
25	0.7189	0.3961		0.156
20	0.7191	0.3962		0.234
15	0.7193	0.3963		0.392
14	0.7194	0.3963		0.446
13	0.7193	0.3963	9.7	0.515
12	0.7218	0.3977	9.6	0.598
11	0.7212	0.3974	9.6	0.767
10.5	0.7211	0.3973	9.7	0.893
10	0.7194	0.3964	9.7	1.069
9.5	0.7174	0.3952	9.7	1.319
9	0.7170	0.3949	9.6	1.666
8.5	0.7182	0.3957	9.7	2.165
8	0.7206	0.3970	9.8	3.043
Average	0.72	0.40	9.7	

Note: W24x131 beam, $L = 24$ ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

Table 26: Data for pin-pin, parabolic load

Z	K_L	K_M	R_m (M_p/L)	δ_{max} (in)
30	0.5837	0.486		0.137
25	0.5836	0.4875		0.194
20	-	-		0.305
15	0.5908	0.4902		0.529
14	0.5919	0.4907		0.606
13	0.5935	0.4915		0.708
12	0.5945	0.4913		0.825
11	0.5954	0.4915		0.991
10.5	0.5958	0.4915		1.094
10	0.5981	0.4921	7.9	1.211
9.5	0.5687	0.4982	8.3	1.364
9	0.6024	0.5045	8.3	1.576
8.5	0.5807	0.4873	7.5	1.835
8	0.5767	0.4949	8.2	2.221
Average	0.59	0.49	8.0	

Note: W24x131 beam, L = 24 ft, Tributary width = 24 ft, f_y = 50 ksi, angle of incidence = 0.

Table 27: Data for fix-fix, parabolic load

Z	K_L	K_M	R_m (M_p/L)	δ_{max} (in)
30	0.4476	0.3829		0.048
25	0.4460	0.3866		0.065
20	0.4461	0.3904		0.097
15	0.4462	0.3942		0.162
14	0.4462	0.3944		0.184
13	0.4461	0.3947		0.214
12	0.4461	0.3945		0.249
11	0.4461	0.3945		0.298
10.5	0.4461	0.3944	11.2	0.329
10	0.4777	0.4060	11.5	0.365
9.5	0.4741	0.4143	13.3	0.407
9	0.4983	0.4062	10.3	0.459
8.5	0.7205	0.3472	7.8	0.533
8	0.9178	0.3184	6.3	0.653
Average	0.46	0.39	11.6	

Note: W24x131 beam, $L = 24$ ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

11.5 MDOF vs. SDOF Response Comparison Data

The following are tabulated MDOF vs. SDOF response data as discussed in Section 8.4.

Table 28: MDOF vs. SDOF for pin-pin, uniform load

Z	MDOF (in)	SDOF (in)	% Difference
30	0.456	0.454	0.31%
25	0.648	0.645	0.37%
20	1.018	1.015	0.29%
15	1.953	1.665	14.71%
14	2.386	1.907	20.15%
13	3.112	2.295	26.25%
12	4.065	2.841	30.11%
11	5.673	3.839	32.32%
10.5	6.795	4.591	32.43%
10	8.349	5.567	33.32%
9.5	10.375	6.821	34.26%
9	13.184	8.561	35.07%
8.5	17.161	11.078	35.44%
8	22.737	14.637	35.62%

Note: W24x131 beam, L = 24 ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

Table 29: MDOF vs. SDOF for pin-pin, linear load

Z	MDOF (in)	SDOF (in)	% Difference
30	0.228	0.228	0.18
25	0.324	0.323	0.22
20	0.509	0.509	0.14
15	0.884	1.884	0.05
14	1.013	1.013	0.01
13	1.183	1.183	0.04
12	1.401	1.412	3.38
11	1.762	1.564	11.25
10.5	2.018	1.719	14.83
10	2.355	1.919	18.51
9.5	2.804	2.178	22.32
9	3.384	2.545	24.79
8.5	4.258	3.088	27.47
8	5.606	3.869	30.99

Note: W24x131 beam, L = 24 ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

Table 30: MDOF vs. SDOF for fix-fix, linear load

Z	MDOF (in)	SDOF (in)	% Difference
30	0.084	0.083	3.05
25	0.114	0.113	2.34
20	0.170	0.169	1.69
15	0.284	0.282	1.12
14	0.323	0.322	1.02
13	0.375	0.381	1.36
12	0.436	0.467	6.69
11	0.522	0.602	14.94
10.5	0.586	0.705	19.88
10	0.671	0.834	23.72
9.5	0.792	0.993	24.82
9	0.953	1.202	25.64
8.5	1.231	1.476	19.31
8	1.665	1.831	9.46

Note: W24x131 beam, L = 24 ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

Table 31: MDOF vs. SDOF for pin-pin, triangular load

Z	MDOF (in)	SDOF (in)	% Difference
30	0.287	0.289	0.66
25	0.408	0.411	0.69
20	0.643	0.647	0.64
15	1.119	1.125	0.49
14	1.284	1.289	0.41
13	1.560	1.444	7.47
12	1.933	1.632	15.55
11	2.589	1.952	24.60
10.5	3.078	2.186	29.00
10	3.697	2.484	32.81
9.5	4.466	2.865	35.85
9	5.569	3.395	39.04
8.5	7.138	4.157	41.76
8	9.322	5.241	43.78

Note: W24x131 beam, L = 24 ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

Table 32: MDOF vs. SDOF for fix-fix, triangular load

Z	MDOF (in)	SDOF (in)	% Difference
30	0.115	0.114	0.52
25	0.156	0.155	0.51
20	0.234	0.233	0.47
15	0.392	0.390	0.43
14	0.446	0.444	0.43
13	0.515	0.537	4.23
12	0.598	0.672	12.30
11	0.767	0.891	16.18
10.5	0.893	1.020	14.20
10	1.069	1.131	5.76
9.5	1.319	1.275	3.37
9	1.666	1.480	11.14
8.5	2.165	1.782	17.67
8	3.043	2.234	26.59

Note: W24x131 beam, L = 24 ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

Table 33: MDOF vs. SDOF for pin-pin, parabolic load

Z	MDOF (in)	SDOF (in)	% Difference
30	0.137	0.139	1.47
25	0.194	0.197	1.39
20	0.305	0.309	1.25
15	0.529	0.536	1.21
14	0.606	0.614	1.17
13	0.708	0.717	1.21
12	0.825	0.834	1.18
11	0.991	1.002	1.18
10.5	1.094	1.107	1.18
10	1.211	1.225	1.15
9.5	1.364	1.340	1.78
9	1.576	1.461	7.31
8.5	1.835	1.637	10.78
8	2.221	1.893	14.75

Note: W24x131 beam, L = 24 ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.

Table 34: MDOF vs. SDOF for fix-fix, parabolic load

Z	MDOF (in)	SDOF (in)	% Difference
30	0.048	0.047	1.66
25	0.065	0.065	1.07
20	0.097	0.097	0.41
15	0.162	0.162	0.19
14	0.184	0.184	0.11
13	0.214	0.213	0.05
12	0.249	0.249	0.12
11	0.298	0.298	0.10
10.5	0.329	0.329	0.09
10	0.365	0.368	0.82
9.5	0.407	0.426	4.62
9	0.459	0.503	9.55
8.5	0.533	0.681	27.79
8	0.653	0.738	13.09

Note: W24x131 beam, L = 24 ft, Tributary width = 24 ft, $f_y = 50$ ksi, angle of incidence = 0.