Smooth Representation of Functions on Non-Periodic Domains by Means of the Fourier Continuation Method

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Introduction

The Fourier Continuation (FC) method[1] provides a smooth, periodic continuation to a non-smooth function. This allows for a Fourier series to be used in the solution of Partial Differential Equations where periodicity is not applicable. Furthermore this provides a smooth continuous interpolation of the function throughout the domain. In this investigation the accuracy of the Fourier series and it's derivative was compared to the analytical solution.

Background

Fourier series was initially developed to solve the heat equation in the 1820s. These methods approximate functions, $f(x)$, in terms of a series of sine and cosine. The Fourier series is defined as:

$$f(x) \approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi x) + b_n \sin(2n\pi x)$$

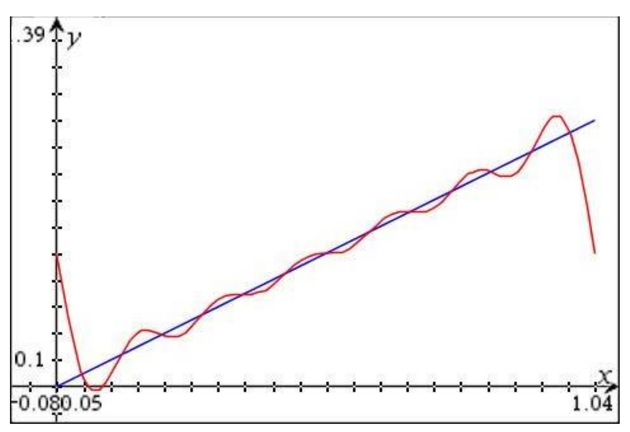
where the coefficients a_n and b_n are given by:

$$a_0 = \frac{2}{T}\int_c^{c+T} f(x)\,dx$$

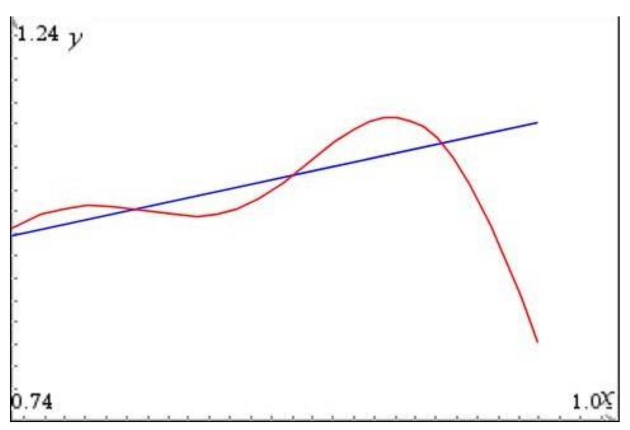
$$a_n = \frac{2}{T}\int_c^{c+T} f(x)\cos\left(\frac{2n\pi x}{T}\right)dx$$

$$b_n = \frac{2}{T}\int_c^{c+T} f(x)\sin\left(\frac{2n\pi x}{T}\right)dx$$

However, something known as the Gibbs phenomenon occurs at the endpoints or discontinuities of a function, especially non-periodic functions (e.g. linear) when it is expressed as this series of waves. This phenomenon is an oscillation around a "jump" which results in an underestimation or overestimation of the value of this function at these points of interest.



[2]Fig 1: Fourier Series Representation of the function $f(x) = x$ on the interval [0,1]



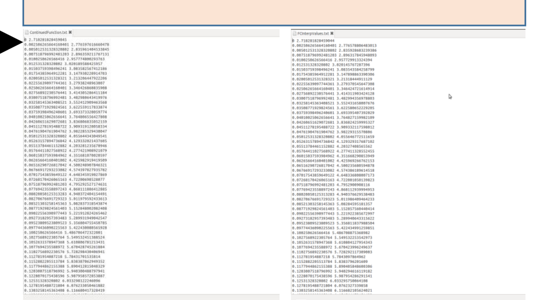
[2]Fig 2: Close-up of the Gibbs Phenomenon

Methods

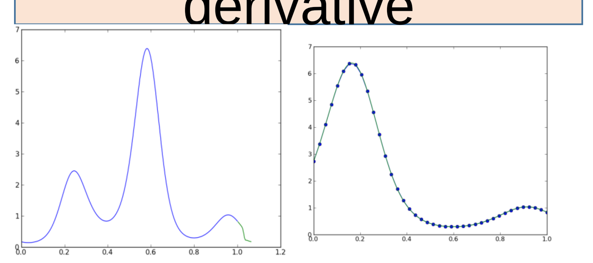
Using a C++ code, the Fourier Continuation algorithm was applied to various functions on different, non-periodic domains. Python scripts were then used to plot the graphs of three different aspects of the given functions: 1) the newly formed smooth and periodic extension of the original function, 2) and interpolation of the original function using the new continued function $f^c(x)$, and 3) the derivative of the original function using the Fourier Coefficients. Focusing on the interpolation and the derivative calculated by the program, in particular, the accuracy of the algorithm was determined by means of the absolute error.

Apply Fourier Continuation (FC) Algorithm to function

Collect data stored in text documents



Generate plots of continued function, interpolation and derivative



Determine accuracy by the L2 Norm

N	h	L2	Convergence
16	6.2500E-02	7.3227E+00	
32	3.1250E-02	1.3811E-01	5.728444688
64	1.5625E-02	7.3116E-03	4.239522067
128	7.8125E-03	6.4368E-04	3.50577694
256	3.9063E-03	4.2983E-05	3.904489972
512	1.9531E-03	2.7239E-06	3.980059647
1024	9.7656E-04	1.6256E-07	4.066574421
2048	4.8828E-04	5.2987E-08	1.61728951
4096	2.4414E-04	1.1222E-07	-1.082640266
8192	1.2207E-04	2.2266E-07	-0.988514984

Results and Discussion

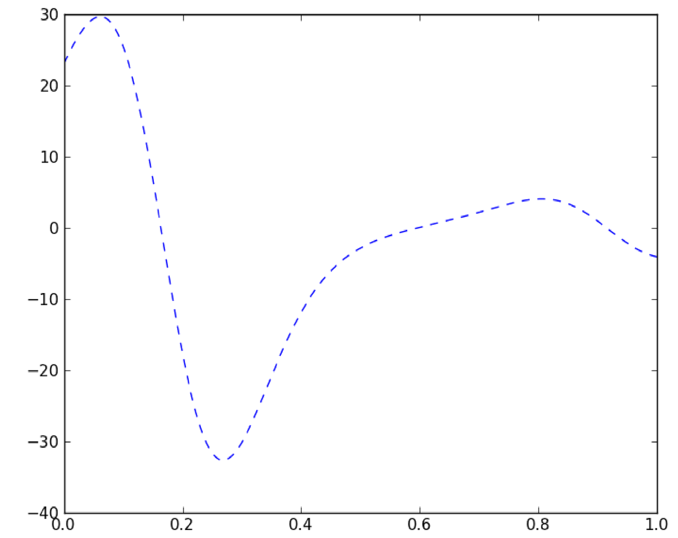


Fig 3: Derivative of $f(x) = e^{\sin(2.7\pi x)+\cos(\pi x)}$ as calculated by the FC algorithm.

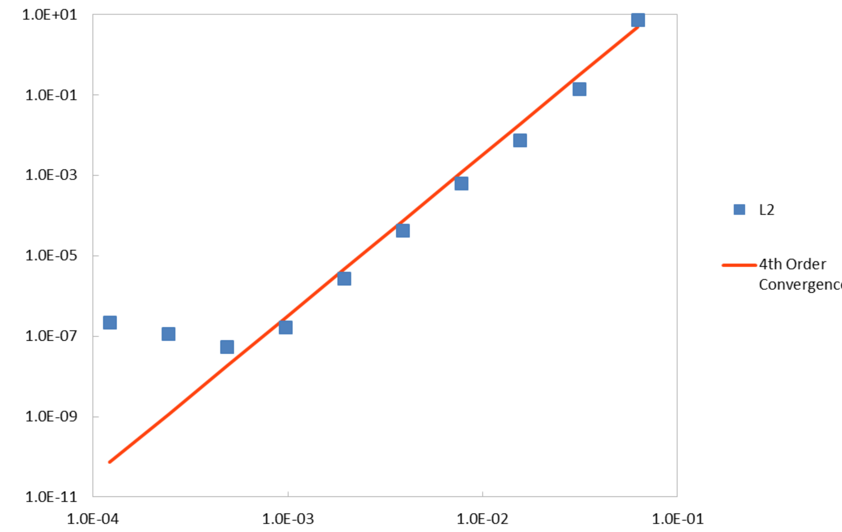


Fig 4: L2 norm convergence plot of the derivative of the Fourier series.

The program produced a smooth, periodic extension of the given function. It also provided data for the interpolation of the function and the derivative of the function. The majority of the functions converged to quickly for accurate convergence rates to be collected. One exception was the function $f(x) = e^{\sin(2.7\pi x)+\cos(\pi x)}$, where we saw an absolute error of the derivative converge to approximately 10^{-7} at around a 4th order convergence rate. This rate was anticipated because the order of the Gram Polynomials used were 4th order.

Conclusions

The results given by this algorithm are a stepping stone to prove the viability and accuracy of the FC method. Further study should investigate its numerical efficiency (especially in regards to interpolation) and its applications to solving PDEs, in particular, the Vlasov and Poisson's equations.

References

[1] High-order unconditionally stable FC-AD solvers for general smooth domains I. Basic Elements

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