Space-time adaptive processing with multi-stage Wiener filter and principal component signal dependent algorithms

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Abstract

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By

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Space-time Adaptive Processing (STAP) is a two-dimensional filtering technique for antenna array with multiple spatial channels. The name "space-time" describes the coupling of these spatial channels with pulse-Doppler waveforms. Applications for STAP includes ground moving target indicator (GMTI) for airborne radar systems.

Today, there are strong interests to develop STAP algorithms for operations in "sample starved" environments, where intense environmental interference can reduce STAP capacity to detect and track ground targets. Careful applications of STAP can effectively overcome these conditions by suppressing these interferences and maximize the signal to interference plus noise ratio (SINR). The Multi-stage Wiener filter (MWF) and principal component signal dependent (PC-SD) algorithm are two such methods that can suppress these interference through truncation of the signal subspace.

This thesis makes contribution in several ways. First it details the importance of rank compression and sample compression for effective STAP operations in "sample starved" environments. Second, it shows how MWF and PC-SD could operate in this type of environment. Third it details how a "soft stop" technique like diagonal loading (DL) could improve STAP performance in target detection for MWF and PC-SD. Fourth, this thesis contrasts the performance of several existing "hard stop" techniques in rank compression and introduces a new one using a-priori knowledge.

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ACRONONYM AND ABBREVIATIONS

A/D	Analog-to-digital converter
AIC	Akaike Information Criterion
CG	Conjugate Gradients
CGLS	Conjugate Gradient Least Squares
CMT	Covariance Matrix Taper
CPI	Coherent Processing Interval
CSM	Cross Spectral Metric
CV	Cross Validation
dB	Decibel
DL	Diagonal Loading
ESA	Electronically Scanned Array
EVD	Eigenvalue Decomposition
GDP	Generalized Discrepancy Principle
GSC	Generalized Sidelobe Canceller
HREE	Hanke-Raus Error Estimate
IID	Independent and Identically Distributed
IOPF	Input-Output Power Function
ISMSE	Independent Sample Mean Square Error
JNR	Jammer-to-Noise Ratio
LNR	Loading-to-Noise Ratio
MDL	Minimum Description Length
MHREE	Modified Hanke-Raus Error Estimate
MMSE	Minimum Mean Square Error
MSE	Mean Square Error
MVDR	Minimum Variance Distortionless Response
MWF	Multi-stage Wiener Filter
PC	Principal Components
PC-SD	Signal Dependent Principal Components
PC-SI	Signal Independent Principal Components
PRF	Pulse Repetition Frequency
SINR	Signal-to-Interference-plus-Noise Ratio
SMI	Sample Matrix Inversion
SMSE	Sample Mean Square Error
SNR	Signal-to-Noise Ratio
STAP	Space Time Adaptive Processing
SVD	Singular Value Decomposition
ULA	Uniform Linear Array
WNGC	White Noise Gain Constraint

Chapter 1

Introduction

First published in the 1970s by Lawrence E. Brennan and Irving S. Reed, Space-Time Adaptive Processing (STAP) is a signal processing technique mostly used in radar systems to aid in target detection. It uses adaptive processing algorithms to suppress interferences such as jamming, ground clutter and noise to increase system sensitivity to target returns.

STAP commonly uses a phased-array antenna with multiple elements (i.e. spatial channels). The term “*space-time*” derives from the coupling of these spatial channels with pulse Doppler waveforms thus creating a two dimensional filter to suppress stated interferences. Adding the statistics of interference environment (which can be known or unknown), an *adaptive* STAP weight vector can be estimated to improve target detection.

A common application is an airborne STAP platform used in detecting ground moving targets, also known as ground moving target indicator (GMTI). One purpose of this thesis is to demonstrate how adaptive filtering of target signals can be achieved via multi-stage Wiener filter (MWF) and Principal Component signal dependent (PC-SD) algorithms.

This thesis will also demonstrate how MWF performance can be improved with techniques such as diagonal loading (DL) and modified Hanke-Raus error estimation (MHREE) technique. Simulations will later show that MWF generally offer superior rank and sample support compression than the more commonly used PC-SD.

This thesis makes contributions in several areas. First it details how MWF and PC contribute to rank and sample support compression. Second it shows how diagonal loading can be added to improve MWF and PC performances. Lastly it shows how hard stop techniques like modified Hanke-Raus error estimation (MHREE) can reduce processing time required for the MWF.

1.1 Background

MWF is an adaptive filtering algorithm that seeks to minimize the mean square error (MSE). More precisely it seeks to derive a weight vector (a user controllable parameter) that can yield the highest possible signal to interference plus noise ratio (SINR). This will yield the highest probability of target detection. The stated signal power is the received target returns, and the interference plus noise are self explanatory. In most environments and applications, target signal returns are very weak. This can be due to many factors such as low transmission power, array gain, target size,

target range, interferences, etc. In realm of signal processing we cannot alter most of these factors; however, we can effectively reduce the impact of interferences to increase target detection. Interferences contaminate our desired signal returns and can be effectively suppressed. One technique to accomplish this is to use the Wiener filter (WF).

The original Wiener filter is a linear, discrete time, adaptive finite impulse response filter. It is the fundamental building block of linear adaptive filtering

Figure 1: The classic Wiener filter

Figure 1 shows a block diagram of the classic Wiener filter where $d_0(k)$ and $\hat{d}_0(k)$ are scalars. The output minimizes the mean square error by subtracting a weighted signal $\mathbf{x}_0$ from the desired signal $\mathbf{d}_0$ or

$$MSE = \mathbf{E}[|\mathbf{d}_0(k) - \mathbf{w}^H \mathbf{x}_0(k)|^2] = \sigma_{d_0}^2 - \mathbf{w}^H \mathbf{r}_{x_0 d_0} - \mathbf{r}_{x_0 d_0}^H \mathbf{w} + \mathbf{w}^H \mathbf{R}_{x_0} \mathbf{w}$$

Equation 1.1

where

$$\mathbf{R}_{x_0} = \mathbf{E}[\mathbf{x}_0 \mathbf{x}_0^H]$$

Equation 1.2

$$\mathbf{r}_{x_0 d_0} = \mathbf{E}[\mathbf{x}_0 \mathbf{d}_0^*]$$

Equation 1.3

and

$$\sigma_{d_0}^2 = \mathbf{E}[\mathbf{d}_0 \mathbf{d}_0^*]$$

Equation 1.4

If we take the gradient of both sides of Equation 1.1 with respect to the weight vector $\mathbf{w}$ and set it equal to 0, we get

$$\nabla_{\mathbf{w}}(MSE) = -\mathbf{r}_{x_0 d_0} + \mathbf{R}_{x_0} \mathbf{w} = 0$$

Equation 1.5

This leads us to the classic Wiener filter equation:

$$\mathbf{w}_{opt} = \mathbf{R}_{x_0}^{-1} \mathbf{r}_{x_0 d_0}$$

Equation 1.6

In Equation 1.6, the weight vector is dependent on the given environment; hence it is “adaptive”. Though this equation seems relatively simple in theory, it’s a different story in practice. Both $\mathbf{R}$ and $\mathbf{r}$ are actually unknown and are estimated from acquired samples of the environment. Due to this inconvenient truth, system performance is heavily dependent on sample support.

Today, Application for Wiener filter includes interference cancellation, equalization, and multisensory array processing, which is the focus of this thesis.

One of the most interesting forms of multisensory array processing is the adaptive array shown in figure 2.

Figure 2: Block diagram of adaptive array filter

Adaptive array filter collects data samples from each spatial channel (element receiver) and jointly process the information to acquire useful knowledge of the environment.

In practice, large arrays are generally impractical due to its computational burden, cost and sample support requirements in weight training, therefore spawning motivations for partial adaptive array processing.

Partial adaptive array utilizes a reduced rank transform (RRT) on the collected data to reduce its adaptive degree of freedom (DOF). This reduces the computational complexities and training sample requirements.

Figure 3: Reduce Rank Transformation in block diagram

Both principal component signal dependent (PC-SD) and multi-stage Wiener filter (MWF) are reduced rank transformations (shown in Figure 3), but they use different methods to accomplish this goal.

Many known techniques for reduce rank processing such as PC and cross spectral metric (CSM) uses statistical transformation that transforms element-based data (output of array elements) to eigenspace beam-former. MWF is another example of statistical reduce rank algorithm but can operate in a more compact subspace than PC and CSM. This results to superior rank compression. Though this rank compression is not apparent for spatial array processing (SAP), it is very noticeable in space-time adaptive processing (STAP).

The motivation for many STAP derived algorithms is to reduce system complexity (hence cost) while improve performance in target detection. Translating these objectives into design goals would result into the following:

1. Reduce computation requirement for STAP
2. Reduce training sample requirements
3. Increase performance in target detection

1.2 Problem Statement

This thesis is derived from Raytheon's University Research project with CalPoly on SAR image target tracking. The intent of this project is to expose CalPoly students to SAR and its applications. For example, my predecessor Jessica Kiefer compared target centroid tracking performances of the Kalman and H Infinity filter on Moving and Stationary Target Acquisition and Recognition (MSTAR) images.

Though SAR image tracking has much application appeal and performed admirably, it's not without shortcomings. Disadvantages to tracking with SAR image processing include delays from image synthesizing, overcoming typical image distortion effects and need for classifiers. For example, typical SAR image may need 3 to 5 seconds to be synthesized, therefore all target identification and tracking algorithms needs to wait at least that long. These factors create an environment where critical information is not produced in "real-time" which is crucial for fast decision making. This thesis will explore an alternative – space-time adaptive processing (STAP).

First a few clarifications are needed. STAP makes no attempt at forming images nor does it classify targets. It is capable of *detecting* target returns with information on target range and direction. It has many desirable traits: first, it is adaptive, meaning that it can function in any environment because it adapts to the environment. Second, STAP is faster than most alternatives such as SAR image tracking because it does not require heavy computations. Third, it is relatively simple meaning that it requires no specialized hardware. Fourth it is versatile to implement, meaning it can work using a variety of algorithms (such as PC and MWF) on many platforms ranging from space satellites to a small low flying unmanned aerial vehicle (UAV).

This thesis will present a general walkthrough of STAP followed by descriptions of how RRT like PC and MWF impact STAP in practice. Afterward I will detail why the MWF is better than PC in accomplishing our stated goals. Later I will detail how augmenting techniques like diagonal loading (DL) and modified Hanke-Raus error estimation (MHREE) technique improves MWF performance for STAP.

1.3 Overview of this thesis

This thesis is broken down into seven chapters. Chapter 2 will discuss adaptive array processing. It covers many historical facts relevant to this thesis and details how STAP came to be. Chapter 3 discusses the significance of reduce rank transformations (RRT). RRT such as PC, CSM and MWF will be translated from theory into practice.

Chapter 4 will chronicle the importance of rank and sample support compression for STAP. This chapter also details a few “natural” factors that contribute to rank compression. It also includes MATLAB simulations that will contrast PC and MWF performance.

Chapter 5 explains how a “soft stop” technique like diagonal loading (DL) improves MWF mean square error (MSE) performance. Additional simulations will demonstrate these improvements.

Chapter 6 details how “hard stop” techniques like white noise gain constraint (WNGC) and modified Hanke-Raus error estimation (MHREE) improves MSE performance. Four other “hard stop” techniques will also be referenced. A comparison of these techniques will be conducted.

Chapters 7 summarizes this thesis and proposes future work.

Chapter 2

History of array processing

In the early days, mechanically scanned arrays operated in frequency range from 30 to 200 MHz like the US Army's 100 MHz SCR-270 that detected the Japanese attack on Pearl Harbor. That array was a collection of Half-wave dipoles that generated a broadside main beam to scan nearby airspace for planes.

Once the radar frequency moved into the microwave region, antenna arrays were dropped in favor of parabolic reflector. In the 1970s, development of waveguide radiator technologies made it possible for antenna arrays to have ultra low side lobes in microwave frequencies. This created a resurgence of interest into antenna arrays. A modern day example of that would be the AWACS (Airborne Warning and Control System) used by the U.S. Air Force.

Though the antenna array had low side lobes it was still very limited. Modern military applications demands faster response time and more flexibility than what a mechanically scanned array allowed. Therefore interests rose for electronically scanned arrays (ESA).

In the 1960s, development of solid state devices like phase shifters made it possible for antenna arrays to have very fast and flexible beam forming architecture. Hence giving birth to electronically scanned arrays (or phased arrays). These usually appeared for some more demanding military applications such as an instance where a single array must perform multiple functions simultaneously such as tracking and surveillance. Beam steering is accomplished by altering the respective phase of electrical current fed to individual elements, instead of mechanical apparatuses like gimbals, motors and joints.

At introduction, electronically scanned array (ESA) were complex and prohibitive costly. Advancement of computer processing technology alleviated some of that cost and therefore ESA are now much more widespread.

2.1 Passive and active ESA

Figure 4 shows the architecture of a passive electronically scanned array (ESA). It contains phase shifters at each element to facilitate beam forming. The transmitter and receiver are centrally located. The feed block can feed to and from the array and it can perform a constrained feed like a waveguide or a space feed. In a passive ESA, its transmitters

and amplifiers are centrally located therefore its phase shifters must be able to handle high power.

Figure 4: Passive ESA architecture

Figure 5 shows generic blocks of an active ESA. Its architecture is dubbed the distributed transmitter architecture, which means each element contains a transmitter/receiver (T/R) module that provides amplification. In addition, the T/R module also has a phase shifter and a low noise amplifier. The active ESA retains a centralized receiver.

Figure 5: Active ESA architecture

Though ESA is much faster and more flexible than mechanically scanned array, it still lacked when dealing with one important real world problem: interferences. Interferences are mostly jammer sources, noise, and clutter (in STAP only). These distort the desired signal returns therefore making target detection difficult. Jammers are typically considered

stationary omni-directional transmitters, clutters are the unwanted echoes of our transmitted signals and noise is the thermal noise of elements.

Figure 6: ESA beam-former

Figure 6 shows an example beam former of ESA. It does not adapt to interferences thus performs poorly in environments where jammers exists. Hence we need a beam former that can *adapt* to the environment.

An *adaptive* array electronically steers not only the beams but also the nulls as shown in Figure 7.

Figure 7: Adaptive array beam-former

These nulls are steered toward the directions of the jammers and nullify its detrimental effects. Adaptive arrays typically first sample the environment to estimate the environmental interferences. A weight vector can then be calculated to modify the side lobes for effective null steering and suppress the interferences. Once these interferences are effectively suppressed, we can maximize the signal to interference plus noise ratio (SINR) and find our targets. Because of its ability to adapt to any environment, adaptive arrays are now a very attractive option for applications in interference heavy environments.

2.2 Adaptive array

There are generally two types of adaptive arrays: fully adaptive and partially adaptive.

Figure 8: Fully adaptive array

Figure 8 shows the receive-only version of a fully adaptive array. Each receive element of the array has an analog to digital (A/D) converter.

Their outputs are then individually multiplied by a pre-calculated complex weight. The results are then linearly combined to form the beam former. The number of adaptive degree of freedoms is equal to the number of weights.

A partially adaptive array is similar to a fully adaptive array except that a rank reducing transform is applied to the data beforehand. Figure 9 shows an example.

Figure 9: Partially adaptive array

This reduces the number of degree of freedoms in data processing and weight calculation. This reduction lessens the computational burden in data processing, which can be significant due to the typically enormous workload and fast response time in the realm of STAP. Recent and more compelling motivation for using partially adaptive arrays is that they can also provide superior performance than the fully adaptive array for real world applications. For example, in some battlefield conditions, heavy jamming interference makes it difficult to acquire adequate sample support for a fully adaptive array. By contrast, a partially adaptive array needs fewer samples to operate and hence is the more attractive option (18).

There are generally two ways to implement reduce rank transformations (RRT): sub-arraying, which is considered by most to be costly, or signal processing (48). RRT implemented through signal processing can be done through deterministic approaches like beam space processing or through statistical approaches like principal component and the multistage Wiener filter (70, 77). We will discuss these statistical approaches more fully in the next chapter.

2.3 Signal Models

Before we get into STAP we must first understand its predecessor, spatial array processing (SAP). In SAP there are generally three types of signals: desired target signal, jammer signal and noise signal. Thus the total signal received by our array is $\mathbf{x}$, which equals the superposition of the three signals.

$$\mathbf{X} = \mathbf{X}_s + \mathbf{X}_j + \mathbf{X}_n$$

Equation 2.1

The jammer and noise are classified as interference. This undesired interference signal is

$$\mathbf{X}_u = \mathbf{X}_j + \mathbf{X}_n$$

Equation 2.2

and modeled as a zero mean Gaussian random variable. Therefore the covariance matrix of this undesired signal would be $\mathbf{R}_u = \mathbf{E}[\mathbf{x}_u \mathbf{x}_u^H] = \mathbf{R}_n + \mathbf{R}_j$. And because these are Gaussian, their covariance matrix fully describes their statistical properties.

2.3.1 Desired target signal

Unlike the undesired signals which are random signals with unknown amplitude and phases, the desired signal is a narrowband (i.e. signal frequency bandwidth is small compare to carrier freq) sinusoidal waveform, or

$$\tilde{x}(t) = \alpha e^{j(2\pi f_c t - \phi)}$$

Equation 2.3

where α is the unknown amplitude, f_c is the carrier frequency and ϕ is the unknown phase shift.

Figure 10: Wave-front impacting array

Figure 10 shows a wave front impacting a uniform linear array (ULA). For positive values of θ , our wave front first arrive at the left most element. This wave front will then have to travel an additional path length of $d \sin \theta$ in order to reach the next element and so forth. Since this signal is assumed narrowband we can approximate the propagation delay between the elements as phase shifts.

As mentioned earlier, the adaptive array has a receiver behind each element. These receivers digitize the received signal. The combined output of these receivers is an $\mathbf{N}$ dimensional vector in the form of

$$\mathbf{x} = x[e^{j0}, e^{j2\pi\frac{d}{\lambda}\sin\theta}, \dots, e^{j2\pi(N-1)\frac{d}{\lambda}\sin\theta}]^T = x\mathbf{v}_\theta$$

Equation 2.4

where x is a complex baseband signal received at the left most element and $\mathbf{v}_\theta$ is known as the spatial array manifold vector (AMV) given by

$$\mathbf{v}_\theta = [e^{j0}, e^{j2\pi\frac{d}{\lambda}\sin\theta}, \dots, e^{j2\pi(N-1)\frac{d}{\lambda}\sin\theta}]^T$$

Equation 2.5

T is the transpose operator, d is the inter element distance and λ is the center frequency wavelength.

One important assumption from above is that the signals are narrowband, or that the signal bandwidth is small compare to the carrier

frequency. This assumption can be enforced as a design parameter on the some system's operating bandwidth. For other applications where this is not possible, one can still satisfy the narrowband assumption through channelization.

In this channelized processing architecture, a wideband received signal can first be transformed to the frequency domain and then broken down to multiple parallel narrowband beam formers. The output of these beam formers can then be recombined into a single composite beam former output as shown in Figure 11.

Figure 11: Channelization of wideband signal

2.3.2 Noise signal

The first interference we model is noise. In our receiver array, each element produces thermal noise. These thermal noises can be modeled

as zero mean Gaussian random processes thus uncorrelated with each other. The noise covariance matrix is

$$\mathbf{R}_n = \sigma_n^2 \mathbf{I}$$

Equation 2.6

where $\mathbf{I}$ is the identity matrix of size of $\mathbf{N} \times \mathbf{N}$ dimension, $\mathbf{N}$ is the total number of elements and σ_n^2 is each element's thermal noise power. This noise power is equal to $\sigma_n^2 = kT_n B$, where k is the Boltzmann constant, T_n is noise temperature (in Kelvin), and B is the receiver bandwidth in Hz. Typically in analysis, the element noise power is normalized to unity for convenience, meaning thermal noise $\sigma_n^2 = kT_n B$ is normalized to 1.

2.3.3 Jamming signal

The jammers are modeled as spatial point sources that constantly transmit a high power omni-directional interference signal. This signal's covariance matrix is equal to

$$\mathbf{R}_J = \sigma_J^2 \mathbf{v}_{\theta_j} \mathbf{v}_{\theta_j}^H$$

Equation 2.7

where σ_J^2 is the jammer's power and $\mathbf{v}_{\theta_j}$ is the array manifold vector

associated with jammer's direction of arrival. If dealing with N_j jammers,

their covariance matrix would add because jammers are mutually uncorrelated.

$$\mathbf{R}_{J,Total} = \sum_{i=1}^{N_j} \mathbf{R}_J(i)$$

Equation 2.8

2.3.4 Signal to noise ratio

The signal to noise ratio is a metric that most should be familiar with.

$$SNR = \frac{\sigma_s^2}{\sigma_n^2}$$

Equation 2.9

The σ_s^2 is the received signal power at element level and σ_n^2 is thermal power at element level.

2.3.5 Jammer to noise ratio

Likewise, the jammer to noise ratio is defined as

$$JNR = \frac{\sigma_j^2}{\sigma_n^2}$$

Equation 2.10

Similarly, the σ_j^2 is the received jammer power at element level and σ_n^2 is same as above. Both SNR and JNR are defined at the element level and used to characterize the environment at the input of the array.

When dealing with N_j jammers, the total JNR is equal to

$$JNR_{Total} = \frac{\sum_{i=1}^{N_j} \sigma_j^2(i)}{\sigma_n^2}$$

Equation 2.11

where N_j is again the total number of jammers. Note that SNR and JNR should always be expressed in decibels (dB).

2.4 Beam forming fundamentals

The beam former output y of our array at Figure 10 is

$$y = w_1^* x_1 + w_2^* x_2 \dots + w_N^* x_N = x \sum_{i=1}^N w_i^* e^{j2\pi(i-1)\frac{d}{\lambda} \sin \theta} = x \mathbf{A}(\theta)$$

Equation 2.12

where $\mathbf{A}(\theta) = \sum_{i=1}^N w_i^* e^{j2\pi(i-1)\frac{d}{\lambda} \sin \theta}$ is the array factor, this can also be considered

the beam pattern $\mathbf{B}(\theta)$ since we are treating all element as omnidirectional. But typically in practice, element pattern is not omnidirectional and therefore the beam pattern is the product of array factor and element radiation pattern $\mathbf{B}(\theta) = \mathbf{A}(\theta) \mathbf{B}_e(\theta)$.

2.4.1 Minimum Variance Distortion less Response

A minimum variance distortion-less response (MVDR) beam former is also known as the optimal beam former which can accomplish two very desirable objectives:

1. Minimize the array output interference power.
2. Get the target desired signal without distortion.

Since our only controllable parameters are the complex weights, we must derive a weight vector $\mathbf{w}$ that can achieve the two objectives. Note that our array output in vector notation is $\mathbf{y} = \mathbf{w}^H \mathbf{x}$. This output is a combination of outputted desired signals and interference signals, or

$$\mathbf{y} = \mathbf{y}_s + \mathbf{y}_{i,n}$$

Equation 2.13

The interference output $\mathbf{y}_{i,n}$ is the sum of noise and jammer outputs or

$\mathbf{y}_{i,n} = \mathbf{y}_n + \mathbf{y}_j$ and $\mathbf{y}_s$ is the signal output. If we want to minimize the

interference output power then we must minimize $\mathbf{E}[|\mathbf{y}_{i,n}|^2]$. This can be

accomplished by reforming the equation as

$$\min\{\mathbf{E}[|\mathbf{y}_{i,n}|^2]\} = \min\{\mathbf{E}[|\mathbf{w}^H \mathbf{x}_{i,n}|^2]\} = \min\{\mathbf{w}^H \mathbf{E}[\mathbf{x}_{i,n} \mathbf{x}_{i,n}^H] \mathbf{w}\} = \min\{\mathbf{w}^H \mathbf{R}_{i,n} \mathbf{w}\}$$

Equation 2.13.1

The criteria in Equation 2.13.1 are the general building blocks of all STAP derived algorithms. If we can apply a MVDR *constraint*

$$\mathbf{w}^H \mathbf{s} = 1$$

Equation 2.14

on Equation 2.13.1, then our *constrained* minimum variance distortion-less response can be achieved if we calculate the MVDR weight vector by

$$\mathbf{w}^{MVDR} = \frac{\arg \min}{\mathbf{w}^H \mathbf{s} = 1} \{ \mathbf{w}^H \mathbf{R}_{i,n} \mathbf{w} \}$$

Equation 2.15

where $\mathbf{s} = \frac{\mathbf{v}_\theta}{\|\mathbf{v}_\theta\|}$ is the desired steering vector and $\|x\|$ is the 2-norm

operator. This weight vector $\mathbf{w}^{MVDR}$ can accomplish the two stated objectives.

Calculating the weight vector requires a method known as the Lagrange multiplier. This method can find the maximum or minimum of a function given constraints.

Figure 12: Function with constraint

Figure 12 shows a 3D function and its linear constraint. The multiplier can reform the function with new limitations set by the constraint. In that regard, our function transforms to

$$L = \mathbf{w}^H \mathbf{R}_{i,n} \mathbf{w} + \lambda(\mathbf{w}^H \mathbf{s} - 1) + \lambda^* (\mathbf{s}^H \mathbf{w} - 1)$$

Equation 2.16

If we take gradient of L with respect to $\mathbf{w}$ and set it equal to zero we have

$$\nabla_{\mathbf{w}^H} L = \mathbf{R}_{i,n} \mathbf{w} + \lambda \mathbf{s} = 0$$

Equation 2.17

Our desired weight vector can be calculated with the following equation

$$\mathbf{w} = -\lambda \mathbf{R}_{i,n}^{-1} \mathbf{s}$$

Equation 2.18

Inserting the linear constraint of $\mathbf{w}^H \mathbf{s} = 1$ and solve for the Langrage multiplier we would receive

$$\lambda = -\left(\frac{1}{\mathbf{s}^H \mathbf{R}_{i,n}^{-1} \mathbf{s}}\right)$$

Equation 2.19

Substitute above into Equation 2.18 yields the weight vector for optimal beam forming

$$\mathbf{w}^{MVDR} = \mathbf{R}_{i,n}^{-1} \left(\frac{\mathbf{s}}{\mathbf{s}^H \mathbf{R}_{i,n}^{-1} \mathbf{s}} \right)$$

Equation 2.20

Please note that $\mathbf{s}^H \mathbf{R}_{i,n}^{-1} \mathbf{s}$ from above is a normalizing factor and generally

Equation 2.21 is expressed in its most general form

$$\mathbf{w}^{MVDR} = \mathbf{R}_{i,n}^{-1} \mathbf{s}$$

Equation 2.21

2.4.2 Adaptive beam-forming

Optimal beam forming sounds really good in theory, unfortunately it sounds only good in theory. One important fact we must face in practice is that we do not have the interference covariance matrix or $\mathbf{R}_{i,n}$ which would require infinite number of samples. However, we can estimate it.

A *sampled* interference covariance matrix is

$$\hat{\mathbf{R}}_{i,n} = \frac{1}{K} \sum_{k=1}^K [\mathbf{x}_{i,n}(k) \mathbf{x}_{i,n}^H(k)]$$

Equation 2.22

K is the total number of training samples available and $\mathbf{x}_{i,n}(k)$ is the k th training sample. The sampled covariance matrix $\hat{\mathbf{R}}_{i,n}$ is the maximum likelihood estimate of the true covariance matrix $\mathbf{R}_{i,n}$ (70). With a sampled covariance matrix, our weight vector equation updates to

$$\mathbf{w}^{MVDR,SMI} = \hat{\mathbf{R}}_{i,n}^{-1} \left(\frac{\mathbf{s}}{\mathbf{s}^H \hat{\mathbf{R}}_{i,n}^{-1} \mathbf{s}} \right)$$

Equation 2.23

Likewise its most general form is

$$\mathbf{w}^{MVDR,SMI} = \hat{\mathbf{R}}_{i,n}^{-1} \mathbf{s}$$

Equation 2.24

The tag SMI stands for sample matrix inverse. Though this Equation 2.24 is practical, there are unfortunate consequences. For example, before the STAP system can be used for target detection, it must first be *trained* with data samples of the environment. These training samples must be “target free” and plentiful, both luxuries in practice.

There are two basic types of training data: “target free” training data, and “target in” training data. The “target free” training data are such that $\mathbf{x}_{i,n} = \mathbf{x}_n + \mathbf{x}_j$ and “target in” data is $\mathbf{x}_{s,i,n} = \mathbf{x}_s + \mathbf{x}_n + \mathbf{x}_j$. The covariance matrix that we wish to estimate is the interference plus noise covariance $\mathbf{R}_{i,n}$. In some applications like radar, “target free” training data is always available and will be considered plentiful in our simulations. Since radar can take measurements at ranges shorter and longer than the target, $\mathbf{x}_{i,n}$ is more easily obtained. However for applications such as passive sonar, it can only collect “target in” training data because it cannot do ranging. Applications such as sonar will have to depend on “target in” training data $\mathbf{x}_{s,i,n}$ and deal with its performance degradation effect as best as possible. Fortunately in this thesis we only deal with airborne radar system and we can assume that “target free” training data is always available.

Our second concern is the number of training samples. Typically, the more samples acquired the better, however in practice that is not a luxury to count on. With battlefields becoming more complex and adversaries becoming more sophisticated, good “target free” training samples are difficult to acquire thus spawning interest to develop systems that can perform well in “sample starved” environments. A common rule of thumb is $K \approx 2N$ samples are needed for good performance, or to be within 3dB of the optimum. However, it has been shown that MWF can achieve competitive performance with only $K \approx 1N$ samples.

Quality of training data is also a concern. The training data samples $\mathbf{x}_{i,n}$ are needed to estimate the interference covariance matrix $\mathbf{R}_{i,n}$. These samples are also realized from this covariance matrix. Hence $\mathbf{x}_{i,n}$ are formed from independent and identically distributed (IID) snapshots of $\mathbf{R}_{i,n}$. Generally, training samples are taken either from “target free” spatial locations or collected across time. In the first option, samples taken from different physical locations rely on an assumption of spatial homogeneity. For the second option, samples taken across time rely on the assumption of stationary interference (i.e. no moving jammers). In practice, our ability to acquire good IID samples that can satisfy either of these assumptions is severely limited. Hence algorithms like MWF that

can operate in sample starved environments draw strong practical interest.

2.4.3 Performance metrics

Here we detail two important performance metrics: the signal to interference plus noise ratio (SINR) and mean square error (MSE). In contrast to the previous described metrics SNR and JNR that describes interferences at element level; SINR and MSE describes performance at array level, more precisely the output of the array beam former.

In this section we take a closer look at these performance metrics. We will derive their expressions, show their relationship to each other and evaluate them. Simulations of these metrics will be given in later chapters to clarify their behavior in complex environments. These evaluations will demonstrates the advantage of the MWF, particularly in environments with jamming, clutter and low sample support.

For the remainder of this thesis, this author assumes that “target free” training data $\mathbf{x}_{i,n}$ is available, that there is no steering vector mismatch (i.e., the signal arrives from a known direction and the element positions are known), and that the distortion-less constraint is relative to the unit-norm steering vector (i.e. $\mathbf{w}^H \mathbf{s} = 1$ where $\mathbf{s}^H \mathbf{s} = 1$).

The SINR is similar to the SNR but also count the effect of jamming. Maximizing SINR is a criterion in most detection and estimation problems

(48). Radar detection performance is directly related to SINR. For example, with assumptions of Gaussian interference and known statistics, it can be shown that maximizing the SINR leads to maximizing the probability of detection (9). SINR can be computed at both the input (i.e., element level) and output of the array (i.e., beam former). This thesis is mostly concerned with SINR at array output.

The mean square error (MSE) is a metric of an adaptive filter's ability to cancel out interference. MSE is the most fundamental performance metrics. SINR is inversely proportional to MSE.

MSE for "target free" training is

$$MSE = \mathbf{E}[|\mathbf{w}^H \mathbf{x}_{i,n}|^2] = \mathbf{w}^H \mathbf{R}_{i,n} \mathbf{w}$$

Equation 2.25

where $\mathbf{R}_{i,n}$ is the known interference-plus-noise covariance matrix and $\mathbf{w}$ is the calculated weight vector from acquired samples.

The output SINR is the ratio of the signal power to the interference-plus-noise power at the output of the array

$$SINR = \frac{|\mathbf{w}^H \mathbf{x}_s|^2}{E[|\mathbf{w}^H \mathbf{x}_{i,n}|^2]} = \frac{\sigma_s^2 |\mathbf{w}^H \mathbf{v}_\theta|^2}{\mathbf{w}^H \mathbf{R}_{i,n} \mathbf{w}}$$

Equation 2.26

$\mathbf{v}_\theta$ is the spatial array manifold vector, recall that $\mathbf{s} = \frac{\mathbf{v}_\theta}{\|\mathbf{v}_\theta\|}$ which implies

that $\mathbf{v}_\theta = \sqrt{N} \mathbf{s}$. Substituting this into Equation 2.26 yields

$$SINR = \frac{N\sigma_s^2 |\mathbf{w}^H \mathbf{s}|^2}{\mathbf{w}^H \mathbf{R}_{i,n} \mathbf{w}}$$

Equation 2.27

Assuming distortion-less response $\mathbf{w}^H \mathbf{s} = 1$, Equation 2.27 is reduced to

$$SINR = \frac{N\sigma_s^2}{\mathbf{w}^H \mathbf{R}_{i,n} \mathbf{w}} = \frac{N\sigma_s^2}{MSE}$$

Equation 2.28

Therefore we conclude that our output SINR is inversely proportional to MSE.

2.5 Space-time adaptive processing

We are now ready to thoroughly discuss STAP. Before we get down into the equations there are few things we should keep in mind. First, recall in spatial array processing (SAP) our total degree of freedom (DoF) equals the number of elements because we can apply a complex weight to each receiver as shown in Figure 13. These weights give us a certain degree of control over the array output.

Figure 13: Receiver array with weight

For STAP, our transmitter emits a train of M pulses. Our receiver receives the echoes and can apply a different complex weight to each pulse return, therefore in STAP, our degree of freedom equals the total number of receiving elements (N) multiply the total number of pulses in the pulse-train (M) which equals to $N \times M$. Another important point we should consider is the progressive phase shift from these successive pulses. Like the previously discussed *spatial* array manifold vector which account for shift due to spatial displacement, another progressive shift due to *temporal* factors must be accounted, this requires us to add the *temporal* array manifold vector. We will discuss this temporal vector later in this section. The third and final note is the received data format. In STAP, the received data have the dimension of $N \times M \times R$ or cubic as shown in Figure 14.

Figure 14: Data cube

The dimension of R in this data cube corresponds to the number of range gates. Range gates are Doppler filters that can filter signal returns based on their Doppler shift due to range. Typically the data from the range gate under test is excluded from the training samples to enforce condition of “target free” training. Usually nearby range gates are also excluded as guard cells to excluded target side lobe energy from creeping in.

With these points given, we can now get into STAP. Recall that in SAP we used a spatial array manifold vector $\mathbf{v}_\theta$ to take account of spatial phase shift from inter-element displacement. In STAP we must also account the inter-pulse progressive phase shift as well. Hence we show the *temporal* array manifold vector,

$$\mathbf{V}_w = [e^{j0} e^{j2\pi w_n} \dots e^{j2\pi(M-1)w_n}]^T$$

Equation 2.29

In this equation, $w_n = \frac{f}{f_p}$ is the normalized Doppler frequency, f is the

Doppler of the signal and f_p is the pulse repetition frequency (PRF).

Dimension of temporal array manifold vector is $\mathbf{M} \times 1$ where $\mathbf{M}$ is number of pulses.

Since we are dealing with *space-time* adaptive processing, our steering vector must now account for both spatial and temporal phase shifts.

Therefore our space-time steering vector is

$$\mathbf{s} = \mathbf{v}_{st} = \mathbf{v}_w \otimes \mathbf{v}_\theta$$

Equation 2.30

where $\otimes$ is the Kronecker product. The Kronecker product of an $A \times B$ matrix and a $C \times D$ matrix result in a matrix with dimension of $AC \times BD$.

With the space-time steering vector $\mathbf{s}$, our weight vector

$\mathbf{w}^{MVDR,SMI} = \mathbf{R}_{i,n}^{-1} \mathbf{s}$, becomes a 2D space-time filter that filters received signals based on both the spatial steering vector and the temporal steering vector.

An added dimension of filtering comes with added dimensions of complexity. One of these additional complexities is the interference: clutter. The term clutter generally covers all unwanted echoes from the natural environment. We restrict our attention to land clutter only, which has zero inherent velocity. High reflectivity factors such as buildings and ground mineral deposits (a.k.a. discrete) and low reflectivity factors from plain ground falls into this category. This clutter interference, like jamming and noise, must also be suppressed for effective target detection.

Figure 15: Clutter ring

2.5.1 Clutter

For airborne mounted radar that moves with velocity v_p , received ground clutter is viewed as a “clutter ring” with the same radius as the desired target signal. Figure 15 shows an example. For simplicity sake we only model half the ring by assuming our array back lobes are heavily tapered. The width of this ring is also the range resolution of the radar

$$\Delta R = c / 2B$$

Equation 2.31

where c is speed of light and B is the array operating bandwidth. For our simulations we assume 180 clutter patches in this ring. Echo from each patch contains a Doppler shift due to radar platform velocity which can be calculated by

$$f_c = \frac{2v_p}{\lambda} \sin \theta_c$$

Equation 2.32

θ_c is the angle of clutter patch.

Figure 16: Clutter “ridge”

Figure 16 illustrates the resulting “clutter ridge” seen by radars in the normalized angle-Doppler domain. This “ridge” is the result of Doppler shift of received clutter due to platform velocity and hence is a linear function of the spatial frequency. The slope of this clutter ridge for an array with half-wavelength spacing is

$$\beta = \frac{4v_p}{\lambda f_r}$$

Equation 2.33

where f_r is the radar PRF. The term β dictates the number of times that the clutter Doppler spectrum aliases into the unambiguous Doppler space (63). Figure 16 corresponds to $\beta = 1$ (a.k.a. Displaced Phase Center Array

mode) where the Doppler space is fully populated but there is no aliasing.

Figure 17 illustrates this ridge behavior for other values of β .

Figure 17: Top view of clutter ridges of various β values

For simplicity sake, we assume $\beta = 1$ throughout this thesis. The *total* clutter power received by our array is the sum of clutter power from each clutter patch, with each patch modeled as a random Gaussian process.

Hence our space-time clutter covariance matrix is

$$\mathbf{R}_c = \sum_{i=1}^{N_r} \sum_{j=1}^{N_c} \sigma_c^2(i, j) [\mathbf{v}_{w_c}(i, j) \mathbf{v}_{w_c}(i, j)^H] \otimes [\mathbf{v}_{\theta_c}(i, j) \mathbf{v}_{\theta_c}(i, j)^H]$$

Equation 2.34

where N_c is the number of evenly distributed clutter patches in azimuth, N_r is the number of range ambiguities, $\sigma_c^2(i, j)$ corresponds to received clutter power from the ij^{th} patch, $\mathbf{v}_{w_c}$ and $\mathbf{v}_{\theta_c}$ are the temporal and spatial array manifold vectors associated with the ij^{th} clutter patch, and $\otimes$ is again the Kronecker product.

The total clutter to noise ratio would be

$$CNR = \frac{\sum_{i=1}^{N_r} \sum_{j=1}^{N_c} \sigma_c^2(i, j)}{\sigma_n^2}$$

Equation 2.35

where σ_n^2 is the element level received noise power.

The desired signal, noise, and jamming models discussed earlier for SAP must be extended for STAP. We use our new space-time array manifold vector

$$\mathbf{S} = \mathbf{V}_{st} = \mathbf{V}_w \otimes \mathbf{V}_\theta$$

Equation 2.36

to accomplish this. For example, the desired signal model for STAP would be

$$\mathbf{X}_s = x\mathbf{V}_{st} = x\mathbf{S}$$

Equation 2.37

where x is again the complex baseband signal received at leftmost element for the first pulse and $\mathbf{v}_{st}$ is the space-time array manifold vector.

The noise model in STAP changes little from its SAP origin and is

$$\mathbf{R}_n = \sigma_n^2 (\mathbf{I}_N \otimes \mathbf{I}_M)$$

Equation 2.38

where σ_n^2 is the thermal noise power at the n^{th} receiver, $\mathbf{I}_N$ and $\mathbf{I}_M$ are identity matrices of dimension $\mathbf{N} \times \mathbf{N}$ and $\mathbf{M} \times \mathbf{M}$ respectively. The element

level thermal noise is still modeled as a Gaussian random process because these noises are spatially and temporally uncorrelated.

Unfortunately jamming is spatially correlated but uncorrelated pulse to pulse. Therefore the jamming covariance matrix for a single jammer can be expressed as (63)

$$\mathbf{R}_j = \sigma_j^2 [\mathbf{I}_M \otimes [\mathbf{v}_{\theta_j} \mathbf{v}_{\theta_j}^H]]$$

Equation 2.39

where σ_j^2 is the jammer power, $\mathbf{v}_{\theta_j}$ is this jammer's spatial array manifold vector.

These undesired signals (noise, clutter, and jamming) for STAP are all assumed to be mutually uncorrelated. Therefore, the total interference plus noise covariance matrix $\mathbf{R}_{i,n}$ for STAP can be expressed as the sum of the individual covariance matrices

$$\mathbf{R}_{i,n} = \mathbf{R}_n + \mathbf{R}_c + \sum_{i=1}^{N_j} \mathbf{R}_j(i)$$

Equation 2.40

Chapter 3

Reduce rank processing

In this chapter I will describe several algorithms for reduced rank processing. My focus is on statistical approaches such as multi-stage Wiener filter (MWF), principal component (PC) and to a less extent cross spectral metric (CSM). All of these algorithms apply to the partially adaptive arrays discussed in Chapter 2. I will detail all three but the MWF is the focus in this thesis. Another rank reducing technique called diagonal loading (DL) will be discussed as well. I will also describe the relationships among these reduce rank techniques for clarity purposes.

The general role of a rank reducing transformation (RRT) is to convert the input data $\mathbf{x}$ from a full rank, $\mathbf{N}$ dimensional vector, to the reduced rank, r dimensional vector $\mathbf{z}$, where r is less than $\mathbf{N}$.

Figure 18: Reduce rank transformation

The adaptive filter then operates on the reduced dimensional space for desired results. Generally there are two categories for rank reducing transformation: deterministic transforms and statistically optimized transforms (35). Example of the first type is beam space processing and examples of latter include principal components (7, 69, 37), the cross-spectral metric (1, 73), and the multistage Wiener filter (16). This thesis primarily focuses on the MWF but we also discuss principal components to use it as a benchmark due to its popularity.

The deterministic and statistical approaches I mentioned thus far are also known as “hard stop” techniques because they physically truncate the data to a reduced dimensional space. There is an alternative, also known as “soft stop” technique. One example is diagonal loading (4, 11). Diagonal loading doesn’t reduce the physical dimensions of computational space but does reduce the required

number of adaptive degrees of freedoms needed to process. I will discuss more on DL in later sections. Finally, “soft stop” techniques such as DL can work in conjunction with “hard stop” techniques such as MWF to improve performance.

3.1 Principal Components

The concept of Eigen-space beam former has been studied by many authors for decades (69, 30, 51, 5, 8, 37, 15). A popular variant is the principal components (PC) (69, 37). PC uses the Eigen-value decomposition (EVD) to produce a low rank estimate of the sampled covariance matrix $\hat{\mathbf{R}}_x$. The selling point of this technique is that this lower rank estimate would still be a good approximation to the original but it would dramatically reduce the required computer processing power. This “speed for accuracy” trade off is widely accepted in the industry due to benefit of reduced cost. Consider the MVDR-SMI beam former we discussed earlier

$$\mathbf{w}^{MVDR,SMI} = \hat{\mathbf{R}}_x^{-1} \left(\frac{\mathbf{s}}{\mathbf{s}^H \hat{\mathbf{R}}_x^{-1} \mathbf{s}} \right)$$

Equation 3.1

where

$$\hat{\mathbf{R}}_{\mathbf{x}} = \frac{1}{K} \sum_{k=1}^K [\mathbf{x}_{i,n}(k) \mathbf{x}_{i,n}^H(k)]$$

Equation 3.2

and K is the number of training snapshots. An EVD of $\hat{\mathbf{R}}_{\mathbf{x}}$ would be

$$\hat{\mathbf{R}}_{\mathbf{x}} = \sum_{i=1}^N \lambda_i \mathbf{v}_i \mathbf{v}_i^H$$

Equation 3.3

where λ_i and $\mathbf{v}_i$ represent the i th eigenvalue and eigenvector of $\hat{\mathbf{R}}_{\mathbf{x}}$ and N is the total number of degree of freedoms.

The best reduced r rank approximation of $\hat{\mathbf{R}}_{\mathbf{x}}$ is formed by retaining the r largest eigenvalues and their corresponding eigenvectors and eliminate the rest. Therefore

$$\hat{\mathbf{R}}_{\mathbf{x}}^{(PC)} = \sum_{i=1}^{r_{PC}} \lambda_i \mathbf{v}_i \mathbf{v}_i^H$$

Equation 3.4

where r_{PC} is less than N but contains the *principal components* of $\hat{\mathbf{R}}_{\mathbf{x}}$ or the components with most signal power. Selecting the value for r_{PC} is completely arbitrary. A typical strategy to select r_{PC} is to find the number of eigenvalues that are above the noise floor. One assumption from Equation 3.4 is that the eigenvalues are ordered from highest to lowest. Meaning that the highest eigenvalue is at $i = 1$, the next highest eigenvalue is at $i = 2$, and so forth.

Figure 19 below is a block diagram example of PC-SI system.

Figure 19: Principal component signal independent

Principal component signal independent (PC-SI) algorithm is a data dependent signal *independent* rank reducing algorithm. It is considered to be data dependent because the data $\mathbf{X}$ is considered in weight vector calculation. It is considered signal independent because the steering vector $\mathbf{s}$ is not. One advantage of this algorithm is that it is simple. A disadvantage is that we lose performance by not taking $\mathbf{s}$ into account. Whether this is applicable really depends on the application, error tolerance and cost. The PC-SI weight vector would be

$$\mathbf{w}^{MVD, SMI, PC-SI} = (\hat{\mathbf{R}}_x^{PC})^{-1} \left(\frac{\mathbf{s}}{\mathbf{s}^H (\hat{\mathbf{R}}_x^{PC})^{-1} \mathbf{s}} \right)$$

Equation 3.5

Another PC variation is PC-SD, SD as in signal dependent. Figure 20 shows a block diagram example.

Figure 20: Principal component signal dependent

As the name suggest, PC-SD does take the steering vector $\mathbf{s}$ (or main lobe response) into account for rank reduction. The block $\mathbf{B}$ is a set of vectors that are orthonormal to $\mathbf{s}$ (i.e. side lobe responses). The steering vector for PC-SD would be

$$\mathbf{s}_{PC-SD} = \mathbf{s} - \mathbf{B}\mathbf{w}_a$$

Equation 3.6

where $\mathbf{w}_a$ is

$$\mathbf{w}_a = (\hat{\mathbf{R}}_x^{PC})^{-1} \mathbf{r}_{\mathbf{z}_0 \mathbf{d}_0}$$

Equation 3.7

and $\mathbf{r}_{\mathbf{z}_0 \mathbf{d}_0}$ is the cross correlation vector between $\mathbf{z}_0$ and $\mathbf{d}_0$. The space-time weight vector would be

$$\mathbf{w}^{MVDR, SMI, PC-SD} = (\hat{\mathbf{R}}_x^{PC})^{-1} \left(\frac{\mathbf{s}_{PC-SD}}{\mathbf{s}_{PC-SD}^H (\hat{\mathbf{R}}_x^{PC})^{-1} \mathbf{s}_{PC-SD}} \right)$$

Equation 3.8

3.2 Cross spectral metric

In this section we briefly describe the cross-spectral metric (CSM) algorithm for rank

reduction (1, 73). We will not discuss CSM too deeply because we need to focus on PC and MWF, however skimming this can provide clarity for other techniques.

Like the PC, CSM also uses an eigenvector basis for the rank reducing transformation. However, CSM selects those eigenvectors base on maximizing the signal to interference plus noise ratio (SINR) instead of high eigenvalues. This is explained as follows (35), recall that

$$SINR = \frac{N\sigma_s^2}{\mathbf{w}^H \mathbf{R}_{i,n} \mathbf{w}}$$

Equation 3.9

where N is the number of adaptive degrees-of-freedom, σ_s^2 is the element-level signal power, $\mathbf{w}$ is the weight vector of interest, and $\mathbf{R}_{i,n}$ is the interference plus noise covariance matrix.

Next, consider the optimal MVDR weight vector previously discussed

$$\mathbf{w}^{MVDR} = \mathbf{R}_{i,n}^{-1} \left(\frac{\mathbf{s}}{\mathbf{s}^H \mathbf{R}_{i,n}^{-1} \mathbf{s}} \right)$$

Equation 3.10

and substitute this into Equation 3.9 results to

$$SINR = N\sigma_s^2 (\mathbf{s}^H \mathbf{R}_{i,n}^{-1} \mathbf{s})^{-1}$$

Equation 3.11

Now if we perform an eigenvalue decomposition (EVD) of the covariance matrix, $\mathbf{R}_{i,n} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$, where $\mathbf{V}$ is the eigenvector matrix and $\mathbf{\Lambda}$ is the diagonal matrix of eigenvalues. The SINR can be transformed to

$$SINR = N\sigma_s^2 (\mathbf{s}^H \mathbf{V} \mathbf{\Lambda}^{-1} \mathbf{V}^H \mathbf{s}) = N\sigma_s^2 \left(\sum_{i=1}^N \frac{|\mathbf{v}_i^H \mathbf{s}|^2}{\lambda_i} \right)$$

Equation 3.12

Therefore CSM algorithm retains eigenvectors that maximize the

metric $\frac{|\mathbf{v}_i^H \mathbf{s}|^2}{\lambda_i}$. These eigenvectors will be different from the eigenvectors

chose by PC therefore the resulting weight vector is different as well.

Performance wise, CSM is generally considered superior to PC but lower than MWF. Aside from these we will not discuss CSM in any more detail.

3.3 Multi-stage Wiener filter

Rank reduction for Wiener filter is heavily dependent on the cross correlation vector as shown previously

$$\mathbf{w}_{opt} = \mathbf{R}_{x_0}^{-1} \mathbf{r}_{x_0 d_0}$$

Equation 3.13

where the weight vector is a function of both covariance matrix $\mathbf{R}_{x_0}$ and cross correlation vector $\mathbf{r}_{x_0 d_0}$. Derived from the original Wiener filter, the *multi-stage* Wiener filter was introduced in (16). Its constrained form structure is shown in Figure 21.

Figure 21: A three stage adaptive filter

In forward recursion, the filter decomposes the sampled data snapshot $\mathbf{x}$ with a sequence of orthogonal projection blocks like $\mathbf{B}_0$ (16). Rank reduction can be accomplished by truncating these decomposition stages to a desired number $\mathbf{r}^{mwf}$. The result is a reduced rank transformation basis that spans the Krylov subspace instead of the eigenvector basis like PC and CSM (54, 13).

$$K(\mathbf{s}, \mathbf{R}_x, \mathbf{r}^{mwf}) = \text{span}\{\mathbf{s}, \mathbf{R}_x \mathbf{s}, \mathbf{R}_x^2 \mathbf{s}, \dots, \mathbf{R}_x^{(\mathbf{r}^{mwf}-1)} \mathbf{s}\}$$

Equation 3.13

Since it tailors its basis selection to the desired steering vector $\mathbf{s}$, the MWF is able to operate in a more compact subspace than PC and CSM (35).

After the forward recursion is completed, the MWF computes a series of scalar weights (w_1, w_2, etc) at each stage and subsequently combine them to form the overall MWF weight vector

$$\mathbf{w}^{mwf} = \mathbf{s} - w_1 \mathbf{B}_0^H \mathbf{h}_1 + w_1 w_2 \mathbf{B}_0^H \mathbf{B}_1^H \mathbf{h}_2 - w_1 w_2 w_3 \mathbf{B}_0^H \mathbf{B}_1^H \mathbf{B}_2^H \mathbf{h}_3 + \dots$$

Equation 3.14

This technique has many desirable properties. First, its main computation operation is the simple vector cross correlation. Second, it does not form a covariance matrix which requires substantial computation workload (59). Last, it doesn't need matrix inversion or eigenvector decomposition, both of which are expensive operations (13, 70).

3.4 Diagonal loading

Diagonal loading is very common beamforming technique that provides a variety of benefits. For example, it can acquire beamforming solution in situations where samples are less than the degrees of freedoms. It can also add robustness to beamformers when dealing with mismatch problems from direction of arrival, element position error, gain/phase perturbations and statistical mismatch from finite sample support (4, 70).

Diagonal loading can also reduce rank. It is categorized as a "soft stop". Unlike the previously discussed "hard" stop algorithms like PC, CSM and MWF which forcefully reduces rank, diagonal loading gently "turn off" the eigenvectors that have small eigenvalues and therefore reduces the number of adaptive degrees of freedom.

A demonstration is as follows. Consider first the beamformer weight vector with diagonal loading

$$\mathbf{w}^{dl} = \tilde{\mathbf{R}}_x^{-1} \left(\frac{\mathbf{s}}{\mathbf{s}^H \tilde{\mathbf{R}}_x^{-1} \mathbf{s}} \right)$$

Equation 3.15

where

$$\tilde{\mathbf{R}}_x = \hat{\mathbf{R}}_x + \sigma_L^2 \mathbf{I}$$

Equation 3.16

$\hat{\mathbf{R}}_x$ is the sample covariance matrix and σ_L^2 is the loading value. One would immediately recognize from Equation 3.16 that a mere constant value of σ_L^2 is added to the diagonal elements of $\hat{\mathbf{R}}_x$. Why is this important? Consider that

$$\hat{\mathbf{R}}_x = \frac{1}{K} \sum_{k=1}^K [\mathbf{x}(k) \mathbf{x}^H(k)] = \frac{1}{K} \mathbf{X} \mathbf{X}^H = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^H$$

Equation 3.17

where $\mathbf{\Lambda}$ is a diagonal matrix of eigenvalues and $\mathbf{V}$ is the corresponding eigenvector matrix. So then

$$\tilde{\mathbf{R}}_x = \frac{1}{K} \mathbf{X} \mathbf{X}^H + \sigma_L^2 \mathbf{I} = \frac{1}{K} (\mathbf{X} \mathbf{X}^H + \beta^2 \mathbf{I})$$

Equation 3.18

where $\beta^2 = \sigma_L^2 K$. The eigenvalue decomposition (EVD) of $\tilde{\mathbf{R}}_x$ results to

$$[\mathbf{X} \mathbf{X}^H + \beta^2 \mathbf{I}] = \mathbf{V} \tilde{\mathbf{\Lambda}} \mathbf{V}^H$$

Equation 3.19

where $\tilde{\Lambda} = \Lambda + \beta^2 \mathbf{I}$. From Equation 3.19, a constant value of β^2 is added to all eigenvalues of Λ , hence “boosting” them all by the same amount.

The inverse of $\tilde{\mathbf{R}}_x$ can be manipulated in a similar fashion because

$$(\mathbf{V} \tilde{\Lambda} \mathbf{V}^H)^{-1} = \mathbf{V} \tilde{\Lambda}^{-1} \mathbf{V}^H$$

Equation 3.20

given that $\mathbf{V}$ is unitary, therefore $\mathbf{V}^H = \mathbf{V}^{-1}$.

The key in implementing diagonal loading (DL) is to load $\hat{\mathbf{R}}_x$ with the “right” level. Thus one can consider DL as weight factor that depends on the magnitude of the each eigenvalue. If the eigenvalues are too large relative to the diagonal loading level then the system performs as though no diagonal loading was applied and each eigenvector is retained for weight vector calculation. Conversely, if the eigenvalues are too small then all eigenvectors are essentially discarded from weight vector calculation (12). These two circumstances are generally labeled as under-loading and over-loading respectively. If the “right” loading level can be applied then we can effectively “turn on” eigenvectors that have important dynamics while “turn off” eigenvectors that does not. In practice, DL is typically chosen between 5 to 10dB above the environment noise level.

3.5 Relationships between DL, PC and MWF

Before we move on to the next chapter we should understand a concept known as filter factors and how it connects DL, PC and MWF. Recall from the full rank Wiener filter

$$\mathbf{w}_{a,opt}^{fr} = \hat{\mathbf{R}}_{x_0}^{-1} \mathbf{r}_{x_0 d_0}$$

Equation 3.21

where $\hat{\mathbf{R}}_{x_0}$ can be decomposed using the EVD into

$$\hat{\mathbf{R}}_{x_0} = \frac{1}{K} [\mathbf{X}_0 \mathbf{X}_0^H] = \frac{1}{K} [\mathbf{V} \mathbf{\Lambda} \mathbf{V}^H] = \frac{1}{K} \left[\sum_{i=1}^N \lambda_i \mathbf{v}_i \mathbf{v}_i^H \right]$$

Equation 3.22

and

$$\mathbf{r}_{x_0 d_0} = \frac{1}{K} [\mathbf{X}_0 \mathbf{d}_0^H]$$

Equation 3.23

There is another decomposition we can use, singular value decomposition (SVD), where a data snapshot can be transformed to

$$\mathbf{X}_0^H = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^H = \sum_{i=1}^N \mathbf{u}_i \sigma_i \mathbf{v}_i^H$$

Equation 3.24

where σ_i are the singular values, $\mathbf{u}_i$ and $\mathbf{v}_i$ are the left and right singular vectors. Combining Equation 3.22, 3.23 and 3.24 into 3.21 yields

$$\mathbf{w}_{a,opt}^{fr} = [\mathbf{X}_0 \mathbf{X}_0^H]^{-1} [\mathbf{X}_0 \mathbf{d}_0^H] = \sum_{i=1}^N \frac{\mathbf{u}_i^H \mathbf{d}_0^H}{\sigma_i} \mathbf{v}_i$$

Equation 3.25

Equation 3.25 shows that the weight vector $\mathbf{w}_a^{fr}$ is a weighted sum of the eigenvectors $\mathbf{v}_i$. This method can be applied to other algorithms as well, which results to:

1. For principal component:

$$w_a^{PC} = \sum_{i=1}^N f_i^{PC} \frac{\mathbf{u}_i^H \mathbf{d}_0^H}{\sigma_i} \mathbf{v}_i$$

Equation 3.26

2. For diagonal loading

$$w_a^{dl} = \sum_{i=1}^N f_i^{dl} \frac{\mathbf{u}_i^H \mathbf{d}_0^H}{\sigma_i} \mathbf{v}_i$$

Equation 3.27

3. For MWF

$$w_a^{MWF} = \sum_{i=1}^N f_i^{MWF} \frac{\mathbf{u}_i^H \mathbf{d}_0^H}{\sigma_i} \mathbf{v}_i$$

Equation 3.28

where $f_i^{PC} = 0$ if $i \leq r^{PC}$ and $f_i^{PC} = 1$ otherwise. $f_i^{dl} = \frac{\lambda_i}{\lambda_i + \beta^2}$ and

$f_i^{MWF,r} = 1 - \prod_{j=1}^r \frac{\theta_j^r - \lambda_i}{\theta_j^r}$. These are the filter factors. The θ_j^r for the last filter

factor are known as “Ritz values” which are the eigenvalues of $\hat{\mathbf{R}}_{x_0}$ in tri-

diagonal form, or $\hat{\mathbf{R}}_d$. Equations 3.26, 3.27 and 3.28 demonstrate that all

weight vectors are calculated from weighted sum of eigenvectors and

their respective filter factor. This is the relationship I wanted to show. This will not be discussed further in this thesis due to little relevance here. However if any curious reader is interested to know more, Rank-Deficient and Discrete ill-Posed Problems by P.C. Hansen is your best source.

Chapter 4

Rank and sample compressions

For STAP, the term “rank compression” refers to lowering the dimensions of operating subspace so that fewer adaptive degree of freedoms are required to process the data. This is highly desirable because computational complexities are proportional to filter rank thus reducing filter rank would save time and cost. The MWF has been shown that it can operate at lower rank than principal components and cross spectral metric in STAP (16, 35, 78, 55). This translates to strong interest in how to implement MWF for STAP applications.

Sample compression is another desirable trait. Sample compression reduces training sample requirements. In military applications, some system must operate in high tech environment with strong interferences. Typically in these “sample starved” environments, good training samples are difficult to acquire. Algorithms that can operate in these scenarios continue to draw strong interest from the industry. Unfortunately MWF excellent rank compression does not translate into noticeable sample compression.

In this chapter we will explore factors that contribute to rank and sample compressions. We will analyze MWF and understand why it offers superior rank compression than PC and CSM. We will also explain why this rank compression does not translate to sample compression.

4.1 Sample Compression

Typically $2\mathbf{N}$ ($\mathbf{N}$ is number of adaptive DoF) samples are needed for full rank minimum variance distortion-less response (66). But good number of quality samples is generally difficult to acquire, especially for large arrays. However research has shown that with reduce rank techniques, sample support does not follow the “ $2\mathbf{N}$ ” rule (2, 31, 37, 75). For example with diagonal loading, when there are N_s interferers, good performance can be achieved with $2N_s$ samples as opposed to $2\mathbf{N}$ samples (71, 75), where N_s is less than $\mathbf{N}$ and corresponds to the number of large eigenvalues (i.e.

interferers). This can be significant reduction in sample support requirement and it holds true for PC as well.

Unlike PC, the MWF rank compression is lower than the number of significant interferers. It was the hope of researchers that MWF excellent rank compression would translate to excellent sample compression as well, unfortunately this did not materialize. So the question became: was MWF sample requirement proportional to number of MWF stages or the number of significant interferers in the environment? In this thesis I will explain that MWF sample requirement is more related to the number of interferers.

Recall that “sample requirement” is the number of samples needed for SINR performance to be within 3dB of the optimum MVDR-SMI response and recall that MSE is

$$MSE = \mathbf{w}^H \mathbf{R}_x \mathbf{w}$$

Equation 4.1

where $\mathbf{R}_x$ is the true covariance matrix and $\mathbf{w}$ is the calculated weight vector. The SINR is related to the MSE by

$$SINR = \frac{N\sigma_s^2}{MSE}$$

Equation 4.2

where N is the number of adaptive degrees-of-freedom and σ_s^2 is the element-level desired signal power. For the sake of comparison we must normalize SINR with the optimum $SINR_{opt}$ or

$$\rho = \frac{SINR}{SINR_{opt}}$$

Equation 4.3

Combining Equations 4.1 and 4.2 with 4.3 gives us

$$\rho = \frac{MMSE}{MSE}$$

Equation 4.4

where MMSE is the minimum mean square error, which is associated with the optimum MVDR weight vector. With these metrics defined, I will simulate with the parameters in Table 1.

Table 1: Simulation parameters 1

Parameter	Value
N elements	8
M pulses/CPI	10
d (inter element distance)	$\lambda/2$
Clutter power (CNR)	10dB
# of effective Jammers	2
Jammer power (JNR)	30dB
Noise power	0dB
β (aka DPCA mode)	1
Monte Carlos trials	100

CMT type	ICM
b	5.7
wind speed	10 Mph
PRF	1 KHZ
fc (Carrier Freq)	1 GHZ

CPI is the coherent processing interval. CNR and JNR are the clutter and jammer to noise ratio respectively. β (Beta) is as described in chapter 2. CMT stands for covariance matrix tapers. We will not discuss CMT in great details here. All we need to know for now is that they account for mismatch errors such as channel mismatch, antenna dispersion and etc. ICM (internal clutter motion) is the only CMT we account for in these simulations. It is developed by Bell Labs which model errors due to clutter reflected from foliage in the air. The parameters of b , wind speed, PRF and carrier freq are all needed to derive the impact of ICM. From these four parameters one worth noting is the pulse repetition frequency (PRF). PRF controls the frequency of transmitting pulses. In many practical applications, signal trains are generated in multiple PRFs (a.k.a. a dwell) to resolve range and Doppler ambiguities. Though we will not be using dwells for our simulation, it is worth noting their importance.

Figure 22 below shows the normalized SINR performance vs sample support for *two* randomly placed jammers.

Figure 22: Normalized SINR for two jammers

Though the performance of Fig 22 A.) is superior for higher sample support, both are within the desired range ($\sim 3\text{dB}$) of optimum with approximately 30 samples. Figure 23 shows two cases with *three* jammers

Figure 23: Normalized SINR for three jammers

In this case, the sample support needs to be larger than 80 to achieve desired SINR performance.

Figure 22 and 23 demonstrates the impact of interferers in sample support requirements. Clearly, more samples are needed if more

interferers are present. However this also depends on other factors such as array type, closely spaced signals and etc. Though MWF sample support requirements failed to live up to our hopes in the case of three or more jammers, it is still more favorable than alternatives like PC. Principal component in its original form generally follows the “ $2N$ ” sample rule, while MWF frequently requires only N samples.

4.2 Rank Compression

In this section we will examine multi-stage Wiener filter (MWF) rank compression and factors that affects it. We will first detail few mechanisms that contribute to rank compression then we will describe MWF equation base to understand why its compression is superior to principal component (PC) and cross spectral metric (CSM). Finally, comparisons of PC-SD and MWF rank compression will be simulated.

Consider the unconstrained filter shown in Figure 24.

Figure 24: Classic Wiener filter

The goal of an adaptive filter is to provide the best possible estimate of the desired signal based on the observed data. Thus we are interested in comparing the number of adaptive degrees of freedom (or rank) needed by the MWF and PC. For principal components, the rank of the PC filter is equal to the number of significant eigenvalues of the data covariance matrix that are used in the weight vector calculations. For the MWF, the rank is determined by the number of MWF stages used.

Before we explore MWF rank compression more fully we should first understand a few factors that contribute to rank compression. Some of these factors can be considered as “natural” while others are planned.

4.2.1 Closely spaced signals

We noted earlier that the number of interferers is related to number of significant eigenvalues which in turn is mirrored in the rank of the received signal. However there are many instances that this is not true, and the number of significant eigenvalues of the data covariance matrix can be less than the signal rank. One such case is closely spaced signals.

Consider an $\mathbf{N}$ element array and two jammers arriving from different angles. The resulting jammer covariance matrix $\mathbf{R}_j$ is

$$\mathbf{R}_j = \sigma_1^2 \mathbf{v}_{\theta_1} \mathbf{v}_{\theta_1}^H + \sigma_2^2 \mathbf{v}_{\theta_2} \mathbf{v}_{\theta_2}^H$$

Equation 4.5

The $\mathbf{R}_j$ can be decomposed using the eigenvalue decomposition (EVD)

$$\mathbf{R}_j = \sum_{i=1}^2 \lambda_i \mathbf{v}_i \mathbf{v}_i^H$$

Equation 4.6

The eigenvalues can be expressed in terms of jammer power σ_i^2 and its array manifold vector $\mathbf{v}_{\theta_i}$. A closed form solution was found in reference (70)

$$\lambda_{1(2)} = \frac{1}{2} N (\sigma_1^2 + \sigma_2^2) \left(1 \pm \sqrt{1 - \frac{4\sigma_1^2 \sigma_2^2 (1 - |B_c^{(1,2)}|^2)}{(\sigma_1^2 + \sigma_2^2)^2}} \right)$$

Equation 4.7

is a function of signal separation where $B_c^{(1,2)}$ is the spatial correlation between signal 1 and 2.

$$B_c^{(1,2)} = \frac{\mathbf{v}_{\theta_1}^H \mathbf{v}_{\theta_2}}{N}$$

Equation 4.8

Equation 4.7 showed that when confronting two jammer signals, if the spacing between the signals is wide then there is very little interaction between them. However, if the spacing between the signals is narrow, then the two signals merge into a dominant eigenvalue. This illustrates that closely spaced signals can cause the number of significant eigenvalues to be less than the number of signals received and thus lead to reduced rank data that require fewer adaptive degrees of freedoms to

process. In practice, closely spaced signals or separation that is less than one half power beam-width (HPBW) result in spatial correlation between the signals. This means that if our airborne platform is positioned at the right angle then many received interferers can be condensed into just a few.

4.2.2 Filter constraints

Another factor that contributes to rank compression is filter constraints. We have already discussed one such constraint, the MVDR or $\mathbf{w}^H \mathbf{s} = 1$. Constraints like this provide opportunities to reduce rank for adaptive array processing.

Consider the general side-lobe canceller (GSC) shown in Figure 25. The top path is a physical example of the distortion-less constraint $\mathbf{w}^H \mathbf{s} = 1$. The input data $\mathbf{x}$ is filtered through two blocks $\mathbf{s}$ and $\mathbf{B}$. The block $\mathbf{s}$ corresponds to the desired steering vector hence its output corresponds to the desired main lobe response. Block $\mathbf{B}$ is similar except its output are the side lobe responses. Jammers are suppressed by steering side lobe nulls toward their directions and can be completely annihilated if they align precisely as shown in Chapter 2. In such case, no adaptive stages would be required.

Figure 25: General side-lobe canceller

Unfortunately this is not the case in practice. Nulls cannot be steered to perfection therefore interference energy leaks to all channels. As the spatial separation between the jammers and nulls increase, more MWF and PC adaptive stages would be required to reject the interference.

Figure 26: beam-former with Chebychev taper

4.2.3 Pattern constraints

The MVDR constraint is one of many that can be applied. Another example is the Chebychev side lobe constraint (36). Figure 26 is a demonstration; notice all side lobe levels are capped. As discussed earlier, modern antenna designers favor reduced side lobe levels to a wider main beam. This is due to observations that majority of interference are received by the side lobes, hence if these responses are suppressed than the received interference power are suppressed as well. In practice this translates to rank compression. MWF typically benefit more from this than PC.

4.2.4 Rank compression in MWF

In this section I will detail why MWF offer superior rank compression than PC. Recall that the received data covariance matrix can be decomposed into

$$\mathbf{R}_{\mathbf{x}_0} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^H = \sum_{i=1}^N \lambda_i \mathbf{v}_i \mathbf{v}_i^H$$

Equation 4.8

Principal components algorithm reconstruct the data covariance matrix into a lower rank version by using the eigenvector basis or

$$\mathbf{R}_{\mathbf{x}_0} = \sum_{i=1}^{r^{PC}} \lambda_i \mathbf{v}_i \mathbf{v}_i^H$$

Equation 4.9

$$\mathcal{E}(\mathbf{R}_{\mathbf{x}_0}, r^{PC}) = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{r^{PC}}\}$$

Equation 4.10

where the eigenvalues/eigenvectors are sorted in descending order and $r^{PC} < N$. Therefore only the strong eigenvalues/eigenvectors are retained. But note that the desired steering vector $\mathbf{s}$ is not taken into consideration in the formation of the reduced rank interference covariance. This makes the general PC algorithm known as a *data-dependent* signal-*independent* algorithm.

MWF does not use the eigenvector basis, instead it uses the Krylov basis or

$$\mathcal{E}(\mathbf{R}_{\mathbf{x}_0}, \mathbf{r}_{\mathbf{x}d}, r^{MWF}) = \text{span}\{\mathbf{r}_{\mathbf{x}d}, \mathbf{R}_{\mathbf{x}_0} \mathbf{r}_{\mathbf{x}d}, \mathbf{R}_{\mathbf{x}_0}^2 \mathbf{r}_{\mathbf{x}d}, \dots, \mathbf{R}_{\mathbf{x}_0}^{r^{MWF}-1} \mathbf{r}_{\mathbf{x}d}\}$$

Equation 4.11

where $\mathbf{r}_{\mathbf{x}d}$ is the cross correlation vector between the data $\mathbf{x}$ and desired output d and r^{MWF} is the set rank. Expanding $\mathbf{r}_{\mathbf{x}d}$ we have

$$\mathbf{r}_{\mathbf{x}d} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_N \mathbf{v}_N$$

Equation 4.12

where α_i is the cross correlation coefficient between the desired signal and the eigenvector $\mathbf{v}_i$. $\mathbf{r}_{\mathbf{x}d}$ is the first basis of the Krylov subspace. The second is $\mathbf{R}_{\mathbf{x}_0} \mathbf{r}_{\mathbf{x}d}$. If we expand $\mathbf{R}_{\mathbf{x}_0} \mathbf{r}_{\mathbf{x}d}$ into

$$\mathbf{R}_{\mathbf{x}_0} \mathbf{r}_{\mathbf{x}d} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^H (\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_N \mathbf{v}_N)$$

Equation 4.13

but recall that eigenvectors are orthonormal to each other except with itself, or

$$\mathbf{v}_i^H \mathbf{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Equation 4.14

thus

$$\mathbf{V}^H \mathbf{r}_{xd} = [\alpha_1, \alpha_2, \dots, \alpha_N]^T$$

Equation 4.15

Equation 4.13 can be simplified to

$$\mathbf{R}_{x_0} \mathbf{r}_{xd} = \alpha_1 \lambda_1 \mathbf{v}_1 + \alpha_2 \lambda_2 \mathbf{v}_2 + \dots + \alpha_N \lambda_N \mathbf{v}_N$$

Equation 4.16

Equations 4.12 and 4.16 are the first and second Krylov basis sets. The higher order Krylov basis can be defined as

$$\mathbf{R}_{x_0}^k \mathbf{r}_{xd} = \alpha_1 \lambda_1^k \mathbf{v}_1 + \alpha_2 \lambda_2^k \mathbf{v}_2 + \dots + \alpha_N \lambda_N^k \mathbf{v}_N$$

Equation 4.17

where

$$\mathbf{R}_{x_0}^k = \mathbf{V} \Lambda^k \mathbf{V}^H$$

Equation 4.18

Looking closely at Equation 4.17, we find that each of the Krylov basis vectors is a weighted sum of the eigenvectors. This is similar to principal components (PC). In fact if all $\alpha_i = 1$ the resulting rank compression is same as PC. Since these weight values are the function of both eigenvalue and the cross correlation coefficient then the MWF rank

compression will always be better than or equal to its PC counterpart because $\alpha_i \leq 1$.

Therefore in Krylov subspace, if $r^{MWF} = N$ then all N Krylov basis vectors are kept and the full N dimensional space is spanned. But if $r^{MWF} < N$ then the Krylov subspace dimension can be reduced based on low eigenvalue, low correlation, or a combination of both.

In practice, it is observed that environment with low power interferers are well handled by MWF rank compression due to the low $\alpha_i \lambda_i^k$ product. Environments with closely spaced interference sources are also good candidate for MWF because their close proximity creates a bifurcation into a dominant eigenvector and a weak one. These weaker eigenvectors becomes additional candidates for rank compression by the MWF.

4.2.5 Practical factors to consider

Before we get to our STAP simulations there are more observed factors that impact rank compression. First it is observed that increased sample support does not yield increased rank compression. In some cases, rank increased as sample support increased. The reason for this is that more samples gives us more accurate estimate of basis vectors that prior was considered marginal contributors but are now more significant. Another

factor is that eigenvalues of noise level signals are typically poorly estimated. This inability to accurately define noise subspace is one of the motivations to use reduce rank transformations (RRT). In that note, environments with low power interference typically provide best MWF rank compression. PC-SD algorithm is also a good choice in these environments because it too considers jammer power levels, however much less pronounced than MWF. In practical applications, detrimental effects of covariance matrix tapers (CMT) should not be ignored. They typically do not affect large eigenvalues but can create small ones. This is another reason why MWF is desired because of it's excellence in suppressing these new CMT created eigenvalues.

One of the worst case scenarios in STAP is the sudden appearance of jammers in the environment. This can significantly increase rank because jammers contaminate all doppler bins and infects the sampled data. The typical solution is to estimate the directions of these new jammers and quickly adjust the pattern to nullify them before they cause too much damage. However this solution can be inefficient, as enemies could turn jammers on and off at will, hence why algorithms that can effectively suppress the effects of new jammers draw strong interest. In contrary to sudden appearance of new jammers, the sudden disappearance of jammers does not detriment performance and is of little concern.

4.3 STAP simulation

We can now examine MWF and PC rank compression for space-time adaptive processing (STAP). As mentioned earlier, STAP environment includes three types of undesirable interference signals: jammers, noise, and clutter. Figure 27 shows the eigenspectra of two environments. One environment includes 2 randomly placed jammers of 30dB jammer to noise ratio (JNR), 10dB clutter to noise ratio (CNR), noise at 0dB and ICM effects. The second environment is the same as the first minus the jammers.

Figure 27: Eigenspectra

This eigenspectra reveals the number of significant eigenvalues in the interference covariance matrix. The matrix without the jammers (blue curve) is dominated by clutter and this gives us perspective on the role clutter play. The Brennan's rule or

$$r_c = \mathbf{N} + (\mathbf{M} - 1)\beta$$

Equation 4.19

is an generally accepted guideline when dealing with clutter. It estimates the number of significant eigenvalues created by clutter with only three parameters: the number of elements $\mathbf{N}$, the number of pulses $\mathbf{M}$, and β which was discussed earlier. With our simulating parameters in Table 2,

the calculated clutter rank is 17. This is near the simulated result of 18. The added rank could be the result of covariance matrix tapers (CMT).

The second curve (red) demonstrates the impact of two 30dB JNR jammers on the eigenspectra. As stated earlier jammer signals contaminates all channels. From simulation we see that contamination resulted in many more strong eigenvalues (i.e. rank). Comparing the two curves of Figure 27 we see that adding two jammers have doubled the rank and hence the number of needed adaptive degree of freedoms to cancel out the interference. Therefore it is highly desirable to implement reduce rank transformations (RRT) to lower processing cost.

We can now evaluate the performance of MWF and PC-SD. Our simulation parameters are defined in Table 2.

Table 2: Simulation parameters 2

Parameter	Value
N elements	8
M pulses/CPI	8
d (inter element distance)	$\lambda/2$
Clutter power (CNR)	10dB
# of effective Jammers	2
Jammer power (JNR)	50dB
Noise power	0dB
β (aka DPCA mode)	1
Monte Carlos trials	100

CMT type	ICM
b	5.7
wind speed	10 Mph
PRF	1 KHZ
fc (Carrier Freq)	1 GHZ

Figure 28 shows MWF and PC-SD performance against two jammers located at angles of $(-72\ 23)$ degrees.

Figure 28: MWF vs PC-SD w/ 50dB JNR and 10dB CNR

PC-SD performance reaches lowest MSE at rank of 16, this means it needs 16 adaptive degrees of freedoms (ADoF) to suppress the interference to achieve MVDR. In contrast, MWF only needs 9 ADoF to accomplish the same. Notice that MWF also offers more flexibility in rank selection. As graph shows, MWF's MSE performance of ranks from 5 to 17 are all well

within 3dB range of minimum mean square error (MMSE). This means that the MWF process can stop anywhere within stages 5 to 17 and still yield acceptable result. This type of flexibility is highly desirable.

Now let us vary the JNR. While holding CNR at 10dB we decrease the JNR from 50dB (Fig 29 A.) to 20dB (Fig 29 D.). Figure 29 shows the MSE performances. Rank selection for PC-SD seems unaffected by the JNR changes, however MWF shows dramatic changes.

Figure 29: MWF vs PC-SD for various JNR and 10dB CNR

Table 3 shows the rank selections of MWF and PC-SD save part D. MWF adapts to the interference levels and adjust to its rank selection to

received jammer power while PC-SD makes no adjustments. MWF adaptability in this case is desirable given that in practical situations the environment is constantly changing. In addition, MWF rank selections are less than its PC-SD counterparts which means that it could be done faster.

Table 3: Rank selections for varying JNR

Case	JNR (dB)	CNR (dB)	MWF Rank Selection	PC-SD Rank Selection
A	50	10	5-17	16-30
B	40	10	3-16	16-30
C	30	10	1-12	16-24

Figure 30 and Table 4 shows rank selection for environments where CNR varies from 40dB (Fig 30 A.) to 10dB (Fig 30 D.) while JNR is constant at 50dB. In this case neither rank selection changes much, however MWF still offers lower rank selection.

Figure 30: MWF vs PC-SD w/ 50dB JNR and various CNR

Table 4: Rank selection for varying CNR

Case	JNR (dB)	CNR (dB)	MWF Rank Selection	PC-SD Rank Selection
A	50	20	5-17	16-21
B	50	30	4-20	16-21
C	50	40	5-19	16-21

4.4 Chapter Summary

In this chapter we explored the concepts of rank and sample compressions. Simulation showed that MWF offered superior rank

compression than PC-SD, especially in environments where jammer powers are lowered to 30dB. MWF demonstrated that it can adapt its rank selections to the environment but PC-SD did not. MWF did not significantly reduce sample requirements as hoped. In all instances, $N \times M$ samples were required to have an adequate estimate of the interference covariance matrix.

There are two things we should clarify before we move onto the next chapter. First, generating the MSE performance graphs shown in this chapter are not possible in practice. They are acquired in our simulation because we know exactly what the interference covariance matrix is in our simulated environment, but in practice that would require infinite number of samples which is practically impossible. As a result, our optimum rank selection would be more or less “blind”. Second, the majority of ranks did not achieve our desired minimum variance distortionless response (MVDR). In both PC-SD and MWF, MSE performance degraded further as ranks increased beyond the optimum rank. For case in Figure 28, only 12 out of 64 possible ranks yielded acceptable results. If we blindly select our process rank, the probability of failure would be 81%. Fortunately there are ways to improve our odds. These methods are known as “soft stop” and “hard stop” techniques and they will be explained in the next two chapters.

Chapter 5

Soft stops for MWF

The term “soft stop” is derived from the similar intangible developments yielded the term software. By definition, both terms describes something “that cannot be touched”. The meaning software now broadens to comprise more than just computer programs; in contrast, soft stop is specifically tailored to techniques used to enhance space-time adaptive processing (STAP).

As the name implies, soft stop achieves its goals through manipulation of software, or in our case, data. Recall from earlier that received sample data $\mathbf{x}$ is collected by our array so we can form a sampled interference covariance matrix $\hat{\mathbf{R}}_x = \mathbf{E}[\mathbf{x}\mathbf{x}^H]$ that will have the same statistical properties of the original sample. This sampled covariance matrix can be altered and manipulated to achieve MVDR as shown earlier with PC-SD and MWF. But rather than reconstructing the sampled covariance matrix with its eigenvector basis, a soft stop technique directly alters the content within the covariance matrix itself.

5.1 Diagonal Loading as soft stop

One soft stop technique is known as diagonal loading (DL).

Equation wise it is relatively simple.

$$\tilde{\mathbf{R}}_x = \hat{\mathbf{R}}_x + \sigma_L^2 \mathbf{I}$$

Equation 5.1

where σ_L^2 is the user-defined loading level and $\mathbf{I}$ is the identity matrix of same size as $\hat{\mathbf{R}}_x$. At first glance one might ask the question of how does this loading enhance the performance of MWF and PC-SD? And does this loading contaminate our data? I will attempt to answer both.

Recall that the eigenvector decomposition (EVD) of the covariance matrix is

$$\hat{\mathbf{R}}_x = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^H$$

Equation 5.2

where $\mathbf{V}$ is the unitary matrix of orthonormal eigenvectors and $\mathbf{\Lambda}$ is the diagonal matrix of eigenvalues. Adding a diagonal loading component results to the modified covariance matrix $\tilde{\mathbf{R}}_x$

$$\tilde{\mathbf{R}}_x = \hat{\mathbf{R}}_x + \sigma_L^2 \mathbf{I} = \mathbf{V} \tilde{\mathbf{\Lambda}} \mathbf{V}^H = \mathbf{V} (\mathbf{\Lambda} + \sigma_L^2 \mathbf{I}) \mathbf{V}^H$$

Equation 5.3

This equation shows that diagonal loading increases all eigenvalues equally hence it does not change its statistical properties.

5.2 Diagonal loading examples

To determine the best loading level, we can use a new metric known as the loading to noise ratio (LNR) or

$$\mathbf{LNR} = \frac{\sigma_L^2}{\sigma_n^2}$$

Equation 5.4

where σ_L^2 is the loading level at each element and σ_n^2 is its thermal noise power.

Figure 31 shows PC-SD MSE performance in an environment with 10dB CNR, four 50dB jammers located at (-43, -72, 70, 30) degrees.

Figure 31: PC-SD w/ diagonal loading

Notice that as the loading level increased, MSE performance improved as well. In fact, for DL of 1.8dB, nearly all ranks above the optimum rank (31) yielded acceptable MVDR.

Figure 32 shows a similar behavior for MWF. However its acceptable rank selection is from 12 to 64, which means that if we randomly choose a operating rank, our chance of success would be 81%.

Figure 32: MWF w/ diagonal loading

Diagonal loading (DL) improves MSE performance because it increases contributions of the low eigenvalues for weight vector calculations, and these contributions improves the MSE performance past the optimum rank because those low eigenvalues are typically poorly estimated due to noise, ICM and etc.

5.2.1 Finding the right loading level

Though Figure 31 and 32 seems to suggest that if we continuously increase diagonal loading levels it will yield better MSE performance, this is wrong. If loading level is too high (i.e. over-load), it would make all eigenvalues significant and the MSE performance would be equivalent to a non-adaptive response (i.e. all adaptive degree of freedom are “turned off”). In contrast, if loading is too low (under-load), then MSE performance is near equivalent to a no loading scenario. A common rule of thumb in the signal processing community is to select a loading level that is approximately 5-10 dB above the noise floor.

5.3 Chapter summary

In summary, a properly chosen diagonal loading level will cause MWF to saturate at its peak value over a broad range of ranks, thus giving us more flexibility in rank selection. In addition, diagonal loading provides robustness against a variety of types of mismatch errors including direction-of-arrival mismatch, element perturbations, and “statistical” mismatch errors due to finite sample support (4, 70).

Though diagonal loading is a popular “soft stop” loading technique, it is not the only one. Other methods such as color loading and error loading (58) are also viable.

In this chapter, we've shown a "soft stop" technique that can augment the MWF for better MSE performance. Though this improvement is impressive, it can be further improved by adding a "hard stop" technique.

Chapter 6

Hard stops for MWF

Diagonal loading (DL) may have drawn strong connections with the first part of the name “soft stop” but one might ask how it connects with the latter part “stop”. The answer is not much at all. Though DL does “stop” the degradation of MSE performance past the optimum rank, it still is hardly the “stop” we think of. A true stop is a method that can stop the MWF process at the optimum rank, or the rank that produce optimum MSE performance. Its advantage would be drastically reduced processing time and optimum SINR. As simple and beneficial DL is, it does not offer reduced processing time. For example, with DL we must still run the MWF algorithm to its full rank to guarantee satisfactory result. This means heavy processing cost.

In contrast, the term “hard stop” is true to its name. Hard stop techniques draw strong connection to both “hard” and “stop”. The first part is obvious. The latter part is derived from its ability to literally stop the MWF algorithm on its track. The problem is to know *when* to stop. In this chapter, we will explore three hard stop techniques and reference four others.

6.1 White Noise Gain Constraint

WNGC is a well known approach for selecting the amount of loading in diagonally loaded MVDR-SMI beamformers using a quadratic constraint also known as a “white noise gain constraint” (70). This technique is based on observations that many sources of error in physical systems are approximately uncorrelated from element to element and degrade performance in a manner that is similar to adding white noise to each element (4). In this perspective, array gain (AG) against white noise gives a measure of robustness. Fortunately for us, this technique can also be applied to rank selection for the MWF as well.

First, the “white noise gain” is

$$AG^{white} = \frac{SINR^{output}}{SINR^{input}}$$

Equation 6.1

where $SINR^{output}$ is the signal to interference plus noise ratio at the output of the array,

$$SINR^{output} = \frac{N\sigma_s^2}{\mathbf{w}^H \mathbf{R}_{i,n} \mathbf{w}}$$

Equation 6.2

and $SINR^{input}$ is the element level signal to interference plus noise ratio

$$SINR^{input} = \frac{\sigma_s^2}{\sigma_{i,n}^2}$$

Equation 6.3

Therefore the array gain can be modified to

$$AG^{white} = \frac{N\sigma_{i,n}^2}{\mathbf{w}^H \mathbf{R}_{i,n} \mathbf{w}}$$

Equation 6.4

But since we are only concerned with white noise, the interference covariance matrix is

$$\mathbf{R}_{i,n} = \mathbf{I}\sigma_{i,n}^2$$

Equation 6.5

which further adjust our array gain to

$$AG^{white} = \frac{N}{\|\mathbf{w}\|^2}$$

Equation 6.6

From Equation 6.6, a uniformly weighted (i.e. $\mathbf{w} = (1 \ 1 \ \dots 1)$) beamformer will have the maximum value of array gain. However, if adaptive stages are added, this gain will decrease. In practice, this decrease is at first gradual as the MWF adapts against interference, but once the MWF rank exceeds the signal subspace then the adaptation will be affected by the noise subspace and the array gain will decrease appreciably. This

deterioration will also affect other figure of merits such as the mean square error. We can limit this deterioration by limiting the decrease in AG^{white} . That is, we can *constrain* our white noise gain for the MWF in terms of the white noise gain for a conventional filter, such as

$$10\log(AG_{MWF}^{white}) \geq 10\log(AG_c^{white}) - \eta_{wngc}$$

Equation 6.7

The η_{wngc} is a user defined threshold. If we substitute Equation 6.6 into Equation 6.7 we get

$$10\log N - 10\log(\mathbf{w}^H \mathbf{w}) \geq 10\log N - 10\log(\mathbf{w}_c^H \mathbf{w}_c) - \eta_{wngc}$$

Equation 6.8

where $\mathbf{w}_c$ is the conventional weight vector. This equation can be rewritten to

$$10\log(\mathbf{w}^H \mathbf{w}) \geq 10\log(\mathbf{w}_c^H \mathbf{w}_c) - \eta_{wngc}$$

Equation 6.9

For MWF, a normalized steering vector is used as rank one conventional weight vector or $\mathbf{w}_c^H \mathbf{w}_c = \mathbf{s}^H \mathbf{s} = 1$, therefore

$$10\log(\mathbf{w}^H \mathbf{w}) \leq \eta_{wngc}$$

Equation 6.10

Equation 6.10 is the desired form for white noise gain constraint. By its definition, if we can limit the norm square of the calculated weight vector $\|\mathbf{w}\|^2$ by a user defined threshold, then we can stop the performance

degradation. This stop is translated into MWF rank selection. Please recall that different MWF with different number of stages have different weight vectors or

$$\mathbf{w}^{mwf} = \mathbf{s} - w_1 \mathbf{B}_0^H \mathbf{h}_1 + w_1 w_2 \mathbf{B}_0^H \mathbf{B}_1^H \mathbf{h}_2 - w_1 w_2 w_3 \mathbf{B}_0^H \mathbf{B}_1^H \mathbf{B}_2^H \mathbf{h}_3 + \dots$$

Equation 6.11

therefore if number of stages changes then a new weight vector must be calculated. With the white noise gain constraint, we can stop the MWF process once the norm square of that weight vector supersedes the user threshold.

Figure 33 and 34 shows two examples without DL. Figure 33 shows the first case, there are three 50dB jammers located at angles of (-76, 18, 40) degrees.

Figure 33: WNGC w/ three jammers

From part A.) we see that the calculated norm squared weight vector surpasses the 0.5dB user defined threshold at rank 18. Conversely in part B.) rank 18 offers acceptable (albeit not optimum) performance for the MWF.

In Figure 34 we will try a more difficult scenario with four jammers of similar power located at angles (-71 -53 87 54) degrees. Figure 34 show that norm squared weight vector oscillates about the user threshold of 0.5dB. If we choose the first intersection, it would have resulted to the stopping rank of 14, a rank that fails to meet the desired result. Clearly this example shows that an arbitrary WNGC threshold does not perform well in

all situations, however this performance can improve if we possess prior knowledge of the environment.

Figure 34: WNGC w/ four jammers

6.1.1 The preferred ranks

Figure 35: The preferred ranks

Figure 35 shows an example of 10dB diagonally loaded MWF MSE performance. Its optimum rank is 17. From the perspective of maximizing SINR, it would be desirable to stop the MWF process at rank 17. However this adds unnecessary processing cost that generates little benefits. The MSE difference between ranks 5 and 17 is less than 0.1dB. If we weigh in cost and speed, it's much more economical to select rank 5 instead of rank 17. Figure 35 demonstrates the “preferred” rank range. This range reflects the more economical option in rank selection and will be used as a metric in rank selection performance for the remainder of this thesis.

6.1.2 Histogram of rank selection

Table 5: 10 examples of Optimum thresholds for WNGC

Jmr 1 Angle	Jmr 2 Angle	Avg Jmr Angle	Minimum Preferred Rank	Maximum Preferred Rank	Preferred WNGC threshold

-19.079	12.294	15.686	5	7	0.358
-24.304	11.232	17.768	6	8	0.565
-27.582	16.490	22.036	4	6	0.093
-16.625	32.867	24.746	4	6	0.092
-37.541	12.857	25.199	4	8	0.188
-13.220	37.231	25.226	4	8	0.153
-32.205	21.989	27.097	5	7	0.234
-24.857	29.482	27.170	4	7	0.136
-24.579	30.858	27.719	4	7	0.142
-18.726	36.733	27.730	4	6	0.251
Average of Preferred Thresholds					0.127
Variance of Preferred Thresholds					0.067

Table 5 shows 10 examples from a 100 example study done on the preferred WNGC thresholds generated by MATLAB codes in Appendix E. In this study, two 50dB jammers are randomly positioned between ± 10 to ± 90 degrees. This time we add a 10dB of DL. All other simulation parameters are same as in Table 2 except we run for 30 monte carlo trials. The right-most column in this table shows the WNGC thresholds that correspond to the preferred ranks. The average of these thresholds is calculated from 100 examples.

Figure 36: Example of WNGC rank selection

Figure 36 shows one example from the 100 example study. With threshold set at 0.127 the WNGC selects rank 12 which is only two ranks higher than the preferred rank range of 5 to 10.

Figure 37: WNGC histogram

Figure 37 shows the histogram of rank selection for a similar example using threshold $\|w\|^2 = 0.127$. Seven selections out of thirty resulted inside

the preferred range. The remaining 23 selections are an average of five ranks above the preferred range. Though this shows that we pay some extra processing cost, it is not a serious concern because the selected ranks are well within reason economically.

What is a serious concern is the high variance of the preferred WNGC threshold in Table 5, which means that our average threshold of 0.127 cannot be universally applied.

Figure 38: Example of WNGC over thresholding

Figure 38 shows an example where the threshold of 0.127 overshoots the entire $\|w\|^2$ curve, which results to a full rank process. Figure 39 shows the corresponding histogram of 30 trials which reveals that nearly all selections are extreme over-selections.

Figure 39: WNGC histogram of over-selections

Results similar to Figure 38 and 39 appeared in the WNGC 100 example study with alarming frequency. This concludes that average WNGC thresholds have too much variance to be practical in unknown environments; however if operator possess a-priori knowledge such as interference power and direction then this technique will yield more useful results.

6.2 Modified Hanke-Raus error estimation

The Modified Hanke-Raus error estimation (MHREE) is derived from the original Hanke-Raus error estimate (HREE) stopping algorithm. HREE was developed from conjugate gradient (CG) within the realm of numerical

linear algebra (22, 21, 12). In (65) it is shown that the method of conjugate gradients can be used to implement the MWF, therefore techniques developed in the context of CG could be used in the MWF as well.

MHREE is one such example.

6.2.1 The Hanke-Raus error estimation

Figure 40 shows a filter in its basic form. Input data is $\mathbf{x}$, $\mathbf{w}$ is weight vector, and y is the output.

Figure 40: Basic filter

Within linear algebra this translate to

$$\mathbf{Ax} = \mathbf{b}$$

Equation 6.12

where $\mathbf{A} \in \mathbb{R}^{M \times N}$, $\mathbf{x} \in \mathbb{R}^N$, $\mathbf{b} \in \mathbb{R}^M$. In HREE, a bound is established for the error between the exact solution and the computed solution with conjugate gradients (CG), or

$$\|\mathbf{x}_{exact} - \mathbf{x}_k\| \leq \sqrt{P_k(0)} \|\mathbf{b} - \mathbf{Ax}_k\|$$

Equation 6.13

where $\mathbf{k}$ is the CG iteration, P_k is a CG polynomial of degree $\mathbf{k}-1$ satisfying the recurrence relation (12)

$$P_k(\theta) = (1 - \theta\alpha_k + \frac{\alpha_k\beta_{k-1}}{\alpha_{k-1}})P_{k-1}(\theta) - \frac{\alpha_k\beta_{k-1}}{\alpha_{k-1}}P_{k-2}(\theta) + \alpha_k$$

Equation 6.14

where α_k and β_k are variables used in conjugate gradients least square (CGLS) method. Using CGLS to compute the solution for $\mathbf{x}_k$ yields

$$\mathbf{x}_k = P_k(\mathbf{A}^T \mathbf{A}) \mathbf{A}^T \mathbf{b}$$

Equation 6.15

Therefore HREE is based on treating the bound in Equation 6.13 as equal and interpret the result as an estimate of error $\|\mathbf{x}_{exact} - \mathbf{x}_k\|$. The stopping criteria then select CG iteration (i.e. rank) that minimizes the following function

$$\mathbf{k}^{hree} = \arg \min_{\mathbf{k}} \sqrt{P_k(0)} \|\mathbf{b} - \mathbf{A}\mathbf{x}_k\|$$

Equation 6.16

Equation 6.16 can be translated to the realm of MWF because it is a CG implementation applied to a “normal” equation

$$\mathbf{A}^T \mathbf{A} \mathbf{x} = \mathbf{A}^T \mathbf{b}$$

Equation 6.17

The least squares solution to (6.16) is known to satisfy this normal equation (29). To draw the connection between Equation 6.16 and the MWF begins with the Wiener-Hopf equation $\mathbf{R}_{\mathbf{x}_0} \mathbf{w}_a = \mathbf{r}_{\mathbf{x}_0 d_0}$. Recall that

$$\hat{\mathbf{R}}_{\mathbf{x}_0} = \frac{1}{K} \mathbf{X}_0 \mathbf{X}_0^H$$

Equation 6.18

where $\mathbf{X}_0$ is an $\mathbf{N}$ by K matrix and

$$\hat{\mathbf{r}}_{\mathbf{x}_0 \mathbf{d}_0} = \frac{1}{K} \mathbf{X}_0 \mathbf{d}_0^H$$

Equation 6.19

where $\mathbf{d}_0$ is a 1 by K vector. Substituting these into Wiener-Hopf equation results to

$$\mathbf{X}_0 \mathbf{X}_0^H \mathbf{w}_a = \mathbf{X}_0 \mathbf{d}_0^H$$

Equation 6.20

Notice the similarity in form between Equation 6.20 and Equation 6.17. If we make a substitution of variables as in Table 6

Table 6: translation of filter parameters

<i>Original</i>		<i>Equivalent</i>
l		nt
$\mathbf{A}$	----->	$\mathbf{X}_0^H$
$\mathbf{x}$	----->	$\mathbf{w}_a$
$\mathbf{b}$	----->	$\mathbf{d}_0^H$

we would arrive at the HREE stopping criteria for MWF

$$r^{hree} = \frac{\arg \min}{r} \sqrt{P_r(0)} \|\mathbf{X}_0^H \mathbf{w}_a - \mathbf{d}_0^H\|$$

Equation 6.21

which can be reformed to

$$r^{hree} = \frac{\arg \min}{r} \sqrt{P_r(0)} \sqrt{K * SMSE}$$

Equation 6.22

where r is the rank index and

$$SMSE = \frac{1}{K} \| \mathbf{X}_0^H \mathbf{w}_a - \mathbf{d}_0^H \|^2 = \mathbf{w}^H \hat{\mathbf{R}}_x \mathbf{w}$$

Equation 6.23

The term $P_r(0)$ is known as the Krylov power function and it is computed as follows

$$P_r(0) = \sum_{i=1}^r (\theta_i^r)^{-1}$$

Equation 6.24

The θ_i^r is known as the “Ritz values” associated with r -stage MWF. These Ritz values are the eigenvalues of the MWF tri-diagonal $\mathbf{R}_d$ matrix (12) which is transformed from the data matrix $\mathbf{R}_x$

$$\mathbf{R}_d = \mathbf{L} \mathbf{R}_x \mathbf{L}^H$$

Equation 6.25

with

$$\mathbf{L}^H = [\mathbf{s}, \mathbf{B}_0^H \mathbf{h}_1, \mathbf{B}_0^H \mathbf{B}_1^H \mathbf{h}_2, \mathbf{B}_0^H \mathbf{B}_1^H \mathbf{B}_2^H \mathbf{h}_3, \dots]$$

Equation 6.26

The original HREE did not perform well in rank selection. The *modified* HREE can improve this by adding a threshold to the HREE concept. Let

$$h_r = \sqrt{P_r(0)} \sqrt{K * SMSE}$$

Equation 6.27

and define

$$h_r^{\min} = \min\{h_r\}$$

Equation 6.28

We define the MHREE stopping criteria to be the max rank such that h_r has not significantly exceeded a user-defined minimum

$$r^{MHREE} = \max\{r\} \text{ subject to } 10\log h_r \leq 10\log h_r^{\min} + \eta^{MHREE}$$

Equation 6.29

where η^{MHREE} is another user-defined threshold.

Figure 41 shows the MWF MSE performance with three jammers at (-76, 18, 40) degrees. The corresponding h_r graph details its response for each rank as well as the user threshold η^{MHREE} set at the commonly accepted 5dB.

Figure 41: MHREE w/ three jammers

The max rank that satisfies the stopping criteria is 24, though this rank does provide acceptable MSE performance, its rank selection does not fall within the preferred ranks.

Figure 42 shows another example with four jammers at $(-71 -53 87 54)$ degrees. This time the stopping criteria select rank 33, again the selected rank is outside of the preferred ranks.

Figure 42: MHREE w/ four jammers

By now you may have come to a conclusion that there is no “one size fits all” threshold. You are correct. Modern day STAP systems utilize multiple sensors, a-priori knowledge and extensive testing experience/data to derive these thresholds to improve decision making.

6.3 MHREE with diagonal loading

Figure 43 shows the MSE performance again with three jammers. This time a diagonal loading (DL) of 10dB is included. This will improve MSE performance for ranks pass the optimum rank. User threshold is unchanged. The rank selection (18) is closer to the optimum rank of 14.

Figure 43: MHREE and diagonal loading w/ three jammers

Figure 44 shows the more difficult situation of four jammers. This time the rank selection is 30, but improves over its unloaded counterpart.

Figure 44: MHREE and diagonal loading w/ four jammers

Above two examples demonstrates that the threshold technique in Equation 6.29 performs well in situations when DL is added, but it does not perform well in all cases.

Table 7: The preferred η

Jmr 1 Angle	Jmr 2 Angle	Jmr 3 Angle	Avg Jmr Angle	Minimum Preferred Rank	Maximum Preferred Rank	Preferred η
-54.908	16.566	18.457	16.396	15	19	4.496
-81.550	12.688	15.504	22.370	17	21	2.702
-43.377	31.182	35.446	29.011	13	17	3.178
-75.930	88.584	71.522	59.615	11	18	3.016
-48.556	19.669	79.017	62.442	12	19	3.175
-53.302	67.741	47.879	63.850	10	18	1.294
-66.875	75.232	80.321	66.310	12	18	2.732
-50.152	75.598	67.469	66.819	10	16	2.933
-21.165	76.680	71.508	67.699	11	17	3.034
-57.390	58.877	72.304	70.910	13	18	2.583
Average of Preferred η						3.081
Variance of Preferred η						0.209

Table 7 shows 10 examples from a 50 example study on the best η^{MHREE} value. Like the previous study with WNGC thresholds, the variance of η^{MHREE} for the preferred ranks is also noticeably high. But unlike the WNGC study discussed earlier, the average η^{MHREE} threshold of 3.081 never overshoot the h_r curve and therefore this technique will never result to a time consuming full rank process. This is good news, but the performance of these rank selections is a slightly different story.

Figure 45: MHREE rank selection with $\eta = 3.081$

Figure 46: Histogram of MHREE rank selection with $\eta = 3.081$

Figure 45 and 46 shows four examples of MHREE rank selection and corresponding histogram. They show thirty monte carlos trials using $\eta^{MHREE} = 3.081$. With exception of thirty trials all other simulation parameters are similar to Table 2. These two figures demonstrate this technique's rank selection is generally superior to WNGC but still falls short of the preferred ranks on many occasions. This shortcoming is once again due to the high variance of η^{MHREE} threshold in the preferred ranks.

Another technique for MHREE rank selection is to not use η^{MHREE} threshold but to cap h_r directly. Table 8 shows 10 examples from a 50 example study done on this.

Table 8: The preferred h_r

Jmr 1 Angle	Jmr 2 Angle	Jmr 3 Angle	Avg Jmr Angle	Minimum Preferred Rank	Maximum Preferred Rank	Preferred h_r threshold
-14.17	16.57	18.46	16.40	15	19	9.537
-38.92	12.69	15.50	22.37	17	21	10.197
-20.41	31.18	35.45	29.01	13	17	9.343
-50.83	25.40	19.85	32.03	9	13	9.179
-47.14	45.62	19.92	37.56	6	9	9.147
-45.82	60.39	12.56	39.59	6	10	9.125
-30.70	51.60	37.82	40.04	6	9	9.119
-74.91	33.00	17.29	41.73	5	8	9.080
-15.37	87.82	25.36	42.85	5	9	9.059
-22.65	76.63	30.52	43.27	5	10	9.135
Average of Preferred h_r Thresholds						9.195
Variance of Preferred h_r Thresholds						0.004

Notice this time, the variance of h_r threshold is low or 0.004. This method's rank selection performances are shown in Figure 47 and 48.

Figure 47: MHREE rank selection with $h_r = 9.195$

Figure 48: Histogram of MHREE rank selection with $h_r = 9.195$

Above two figures demonstrate that by capping h_r to 9.195 provides better rank selections than using the technique in Equation 6.29. The low variance of these thresholds certainly made this technique the best performing MWF hard stop method in this thesis.

Table 9 shows the impact of total number of jammers and jammer powers have on the h_r threshold. This supports the general consensus that as the number of jammers multiply and jammer power increases, the h_r threshold needs to be set higher meaning more adaptive stages are needed to suppress the interference.

Table 9: Best h_r thresholds for various numbers of jammers and jammer powers

# of Jmrs	JNR = 30 dB	JNR = 50 dB
2	8.905	9.021

3	8.977	9.195
4	9.221	9.334
5	9.382	9.523

Before we move on it is worth noting that Equation 6.29 can be the more preferable MWF hard stop technique when facing completely *unknown* environments. The h_r cap technique shown here can perform better than the technique in Equation 6.29 if we have some a-priori knowledge on the environment such as number of jammers and their strength; however such knowledge on a constantly changing battlefield could be scarce.

This chapter presented performance comparisons for three different hard stop techniques. These examples stress that different scenarios requires different set of approaches. Therefore having a library of multiple stopping algorithms is desirable and provides flexibility to address different applications and environments.

6.4 Taxonomies of hard stop techniques

Before this chapter is concluded, we should explore some taxonomy of hard stop algorithms. This will detail some of the choices of hard stop techniques through two categorizations.

Figure 49: Basic filter

Please consider once again our basic filter in Figure 49. There are three parameters which can be used to categorize hard stop techniques: the input data $\mathbf{x}$, the weight vector $\mathbf{w}$ and the output y . Table 10 shows taxonomy of algorithms based on these criteria. Several of these are well known in the field.

Table 10: Taxonomy of “hard stop” techniques based on filter parameters

A.) Analyze input data $\mathbf{x}$	B.) Analyze filter weight $\mathbf{w}$	C.) Analyze output $\mathbf{y}$	D.) Hybrid techniques
Eigenvalue Decomposition	Limiting the size of weight vector norm such as WNGC	Based on observable output metric such as Sample	L-Curve is a technique that depends on both SMSE and norm of adaptive weight

		Mean Square Error which is Used in the Generalized Discrepancy Principle (GDP)	vector
Find break point between signal and noise subspace such as Akaike Information Criterion (AIC)		Cross Validation (CV) is a method that depend on Independent Sample Mean Square Error (ISMSE)	MHREE combines use both SMSE and Kyrlov power function
Krylov subspace analysis such as MWF			Input-Output Power Function (IOPF) also use both SMSE and Kyrlov power function

Part A.) describes algorithms that are based on analyzing the input data $\mathbf{x}$. A rank revealing decomposition such as the eigenvalue decomposition (EVD) can be very beneficial. Akaike Information Criterion (AIC) (17) and the Minimum Description Length (MDL) (61, 67) are two such examples. Their rank selection techniques apply to the principal components (PC)

which we already discussed. Other decomposition techniques based on the Krylov subspace also falls into this category.

Part B.) describes techniques based on the weight vector $\mathbf{w}$. White noise gain constraint (WNGC) is one such example. Its development was derived from observations that the norm of the weight vector $\|\mathbf{w}\|$ increases in response to mismatch errors (70), therefore resulted in the notion that constraining $\|\mathbf{w}\|$ via a threshold adds robustness.

Part C.) describes algorithms that are based on the output y . The most ideal metric for this would be the mean square error (MSE), however since MSE is unobservable in practice, we have to use something more practical such as sampled mean square error (SMSE). One approach based on the SMSE is the generalized discrepancy principle (GDP) (12). SMSE monotonically decrease with rank and thus cannot truly function as representation of MSE. GDP makes note of this fact by thresholding SMSE for rank selection. Another approach based on output y is the cross validation (CV) algorithm (20, 72). This algorithm divides the training data into two partitions, one for training and one for testing. The training and testing samples are independent therefore its SMSE could be more faithful to the true MSE curve. This is also known as the independent sample mean square error (ISMSE) technique (47).

Part D.) describes hybrid algorithms. These use multiple criteria for rank selection. The L-curve (14, 6) use both the norm of the adaptive

weight vector $\|\mathbf{w}\|$ and SMSE. Other hybrid algorithms include the modified Hanke-Raus error estimate (MHREE) algorithm and the input-output power function (IOPF). Like the MHREE, IOPF algorithms also use SMSE and the Krylov power function.

Another insightful categorization of stopping criteria algorithms is their dependency on a threshold. By now you would have encountered the term threshold enough times to make you think that it is analogous to hard stop algorithms. This is partially true but there are algorithms that do not depend on thresholds, we refer to these algorithms as “blind”.

Table 11: Taxonomy of “hard stop” techniques based on threshold

<i>Class</i>	<i>With Threshold</i>	<i>Without Threshold</i>
Depend on $\mathbf{x}$	Eigenvalue thresholding	AIC/MDL
Depend on $\mathbf{w}$	WNGC	
Depend on $\mathbf{y}$	GDP	CV
hybrid $\mathbf{wy}$		L-curve
hybrid $\mathbf{xy}$	MHREE	IOPF

Table 11 shows this categorization. The four that does require a threshold are in the middle. Eigenvalue thresholding requires that we know the threshold level separating the signal and noise eigenvalues

(preferably from prior knowledge); WNGC needs a threshold on $\|\mathbf{w}\|^2$; GDP requires a cap on SMSE; and MHREE needs a threshold on its error estimate function.

The four of the right most columns are “blind”. They typically rely on a minimizing a function. AIC and MDL each have embedded penalty functions that forces a minimum for rank selection. CV also generates a minimum for its ISMSE. The L-curve is based upon finding the “knee” of a curve that correlates adaptive weights $\mathbf{w}_a$ and SMSE.

It is the hope of this author that these categorizations can help future system designers to understand the intricacies of different approaches so one might articulate a wide variety of potential stopping criteria, and perhaps after evaluation, determines the most promising method (or combination of methods) for particular application.

6.5 Chapter Summary

In this chapter we described three MWF “hard stop” rank selection techniques and referenced four others. We simulated their performance and found that a rigid user defined threshold yielded respectable but improvable results.

Chapter 7

Thesis summary

This thesis presents a research to advance the understanding and utility of principle component (PC) and multistage Wiener filter (MWF) for space-time adaptive processing (STAP). Both are reduced rank adaptive signal processing algorithms that provide many advantages. These include improved sample support (in MWF) to improved performance over their full rank counterpart. We contrasted MWF and PC, and to a less extent cross spectral metric (CSM) and diagonal loading (DL). We focused on adaptive array processing in space-time adaptive processing (STAP).

In Chapter 4 we discussed rank and sample support compressions. We detailed several mechanisms than can contribute to rank compression such as closely spaced signals and filter constraints. We also explained through eigenvector and Krylov basis analysis why the MWF will have greater rank compression than the principal components. Complex environments with many low power interferers are good candidate for MWF rank compression. However, this rank compression did not translate to an equally impressive sample support compression.

Chapter 5 described a “soft stop” technique called diagonal loading (DL). We discovered that the MWF can saturate at its peak mean square error if the right loading level is applied.

In Chapter 6 we examined “hard stop” algorithms like WNGC and MHREE for MWF rank selection. Multiple simulations and studies are presented to show that one method of using MHREE with DL can yield excellent rank selection performance. However using a combination of these methods from a library of techniques can improve further improve rank selection performance.

Areas for future work include the following: Chapters 5 and 6 only covered a small portion of rank selection techniques for simulation. Plenty of opportunities exist to add other rank selection techniques into the simulation programs in appendices. Their performances can become even more realistic if more interference models are added such as other covariance matrix tapers (CMT) and receiver chain effects like quantization. Additionally, more metrics are needed to assess algorithm performance such as more histogram of rank selection versus the preferred ranks. Lastly, assessing the performance of these algorithms against experimental data would also be an interesting area for future work.

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Appedix A: Matlab codes for MWF vs PC-SD MSE performance

```

%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   Author: Zheng Zhou
%   Email:   zhzhou@calpoly.edu
%   Info:    This Matlab program is created by Zheng Zhou
% in his thesis work for the ongoing Synthetic Aperture Radar
% (SAR) research project with
% Raytheon Space and Airborne System at California Polytechnic
% State University - San Luis Obispo.
%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   Purpose: This program measures the MSE performance of MWF,
%   MWF w/adaptive DL, PC-SD, PC-SD w/ adaptive DL.
%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

clear all;
close all;
tic          % Use tic/toc to time the run

%   Initializations
Nmc=20;      % The number of Monte Carlo trials
N=8;         % The number of array elements
M=8;         % The number of pulses/CPI
Nm1d2=(N-1)/2;% Used for spatial array manifold vectors (AMV)
K=[M*N 5*M*N]; % # of train snapshots/pulse
f_c = 1e9;    % center freq
c = 3e8;
lambda = c/f_c; % wavelength of center freq

Beta = 1;    % DPCA mode
theta = (-89:1:90)*pi/180; % Angle distribution of clutter patches
us = sin(theta); % u-space for spatial AMV
ut = Beta*us; % u-space for temporal AMV

V_spatial = zeros(N,length(theta)); % For spatial AMVs
V_temporal = zeros(M,length(theta)); % For temporal AMVs
s = ones(N*M,1)/norm(ones(N*M,1)); % Broadside steer vec

%%   Forming spatial and temporal steering vectors

for nc=1:length(us) % Forming spatial AMVs
    for n=0:N-1
        V_spatial(n+1,nc)=exp(j*pi*us(nc)*(n-Nm1d2));
    end
    for m=0:M-1
        V_temporal(m+1,nc)=exp(j*pi*ut(nc)*m);
    end
end

%%   Normalize steering vectors
V_spatial = V_spatial/N;
V_temporal = V_temporal/M;

%%   Miscellaneous

% set user defined diagonal loading
User_DL = 0; % not in dB, recommend [0-10]

% Set range of rank to process

```

```

Min_Rank = 1;
Full_Rank = N*M-1;

nK = 1;          % Number of samples

n_MHREE = 5;     % MHREE user set threshold in dB

% Jammer power control parameters
NJP = 2; % total number of jammers in DOA of 0 deg to 90 deg
NJN = 1; % total number of jammers in DOA of -90 deg to 0 deg
Minjmr = 50; % Minimum JNR in dB
Maxjmr = 50; % Maximum JNR in dB
Jmrrange = Maxjmr - Minjmr;
Meanjmrpwr = Minjmr + Jmrrange/2;

% Clutter power control parameters
CNR = 10; % Mean CNR in dB
N_r = 1; % Number of range ambiguities
UWC = 1; % Uniform Weight Control, keep this at 1

% Initializing interference covariance matrix
R_j = 0; % Jammer covariance matrix
R_c = 0; % Clutter covariance matrix
R2 = 0; % Total interference covariance matrix
R_PC_inv = 0; % Inverse PC-SD covariance matrix
R_Eff_I_MWF = 0; % Inverse effective MWF covariance matrix

% Parameters for adaptive diagonal loading
Max_DL = 10; % in dB
Min_DL = 0; % in dB
DL_iteration = 1;
W_DL_PC_H = Min_DL:DL_iteration:Max_DL;
W_DL_PC_H2 = 10.^(W_DL_PC_H/10);

% CMT controlling parameters
b = 5.7;
w = 10; % wind speed in MPH
PRF = 1e3; % PRF in Hz
PRI = 1/PRF;

% Generating AMVs for randomly generated jammer DoAs
theta_deg_neg = ((rand(1,NJN)-1)*80-10); % Randomly
% generate jammer in DOAs of -90 deg to -10 deg
theta_deg_pos = ((rand(1,NJP))*80+10); % Randomly
% generate jammer in DOAs of 10 deg to 90 deg

% Generating AMVs for user generated jammer DoAs
% theta_deg_neg = -76;% Insert user define angles
% theta_deg_pos = [18 40];% Insert user define angles

uj=sin([theta_deg_neg,theta_deg_pos]*(pi/180));
% Jammer DOAs in u-space
V = zeros(N, length(uj));
% Matrix of jammer AMVs

% Generating Jammer powers
Jmrpwr = (((rand(1,length(uj))-.5)*Jmrrange)+Meanjmrpwr)';
% Vector of jammer power

% Forming Jammer Covariance Matrix
for nj=1:length(uj)
for n=0:N-1
    V(n+1,nj)=exp(j*pi*uj(nj)*(n-Nm1d2));
end
end
V = V/N;

for i=1:length(uj)
    R_j = R_j + (10^(Jmrpwr(i)/10)).*kron(eye(M),V(:,i)*V(:,i)');
end

```

```

%% Generating clutter matrix

Clutterpwr = raylrnd(8,N_r,length(theta))+ CNR - 10;
Clutterpwr = Clutterpwr.';

% Forming Clutter Covariance Matrix
for n_r = 1:N_r
    for i=1:length(theta)
        R_c = R_c + Clutterpwr(i,n_r)*kron(V_temporal(:,i)*...
            V_temporal(:,i)',V_spatial(:,i)*V_spatial(:,i)');
    end
end

%% forming ICM CMT
r = 10^((-15.5*log10(w)-12.1*log10(f_c/1e6)+63.2)/10);

Tau = zeros(M);
for i=1:1:M
    for k=1:1:M
        Tau (i,k)=abs(i-k)*PRI;
    end
end

Tau = (4*pi*Tau).^2;
T_ICM = r/(r+1)+(1/(r+1)*(b*lambda)^2./((b*lambda)^2+Tau));
T_ICM = kron(T_ICM,ones(N));

%% Forming total interference matrix

R2 = R_j + R_c + eye(length(R_j));
% Normalized with noise level = unity
R = R_c + eye(length(R_j));
R2 = R2.*T_ICM;
% True interference covariance matrix with jammers (unknown in practice)
R = R.*T_ICM;
% True interference covariance matrix without jammers (unknown in practice)

%%
R_half_power = R2^0.5;
% For generating the snapshot
R_inv = inv(R2);

% Calculating MMSE
W_MVDR_H = (R_inv*s)/(s'*R_inv*s);
% Optimal MVDR wt vector
W_MVDR_H = W_MVDR_H ./ UWC;
MMSE = real(W_MVDR_H'*R2*W_MVDR_H);
% Minimum Mean square error normalized

MSE_PC_SI=zeros(1,N*M);
MSE_PC_SD=zeros(1,N*M);
MSE_MWF=zeros(1,N*M);
MSE_MWF2=zeros(1,N*M);
MSE_PC_SI2=zeros(1,N*M);
MSE_PC_SD2=zeros(1,N*M);
P_r = MSE_PC_SI2;

h_r_MHREE = MSE_PC_SI2;
h_r_MHREE2 = MSE_PC_SI2;
Weight_norm = MSE_PC_SI2;

for Rank = Min_Rank:Full_Rank

    % Initializing MSE variables
    MSE_MWF_Temp = 0;
    MSE_PC_SI_Temp = 0;
    MSE_PC_SD_Temp = 0;
    MSE_MWF_Temp2 = 0;
    MSE_PC_SI_Temp2 = 0;
    MSE_PC_SD_Temp2 = 0;
    x0_Temp = 0;

```

```

SMSE_Temp = 0;

for nmc=1:Nmc % Loop on the number of Monte Carlo trials

    % For data sample
    a = (1/sqrt(2)) * (randn(N*M, K(nK)) + j*randn(N*M, K(nK)));
    % For X
    X = R_half_power*a;
    % data snapshot, Note E[XX']=R
    Block_Matrix = null(s');

    % Initializing variables for MWF
    B_Temp = s;
    B_Save = 1;
    L_k_save = 0;
    a1=X;
    h1 = 1;

    R_hat = (1/K(nK))*(X*X');
    % sampled interference covariance matrix
    R_hat = R_hat + User_DL*eye(length(R_hat));
    % insert user defined diagonal loading

    X0 = Block_Matrix'*X;
    R_X0_X0 = 1/K(nK)*(X0*X0');

    % Initializing variables for PC-SD
    [V_hat,D_hat]=eig(R_hat);
    [V_hat_SD,D_hat_SD]=eig(R_X0_X0);

    s1 = size(D_hat);
    s2 = size(V_hat);

    s1_SD = size(D_hat_SD);
    s2_SD = size(V_hat_SD);

    D_PC = reshape( D_hat(s1(1):-1:1,s1(2):-1:1,:), s1 );
    V_PC = reshape( V_hat(:,s2(2):-1:1,:), s2 );

    D_PC_SD = reshape( D_hat_SD(s1_SD(1):-1:1,...
        s1_SD(2):-1:1,:), s1_SD );
    V_PC_SD = reshape( V_hat_SD(:,s2_SD(2):-1:1,:), s2_SD );

    % Truncate D_PC and V_PC
    D_PC = D_PC(1:Rank,1:Rank);
    V_PC = V_PC(:,1:Rank);

    D_PC_SD = D_PC_SD(1:Rank,1:Rank);
    V_PC_SD = V_PC_SD(:,1:Rank);

    R_PC = 0;
    R_PC_SD = 0;

    % Forming truncated version of Cov matrices

    D = MSE_PC_SI2;
    w_i = MSE_PC_SI2;
    Delta_i = MSE_PC_SI2;
    SMSE = MSE_PC_SI2;

    Krylov_Power_Function = zeros (1,Rank);

    for i=1:Rank
        % PC iteration
        R_PC = R_PC + ((D_PC(i,i)-min(diag(D_hat)))/...
            D_PC(i,i)).*(V_PC(:,i)*V_PC(:,i)');
        R_PC_SD = R_PC_SD + ((D_PC_SD(i,i)-...
            min(diag(D_hat_SD)))/D_PC_SD(i,i)).*...
            (V_PC_SD(:,i)*V_PC_SD(:,i)');

        % MWF iteration

```

```

B = null(B_Temp');
d0 = B_Temp'*a1;
x0 = B'*a1;

r_x0_d0 = (x0*d0')./length(d0);
h1 = r_x0_d0./sqrt(r_x0_d0'*r_x0_d0);

L_k=h1'*B'*B_Save;

if i == 1;
    L_k_save_H = L_k;
    L_k_save_special = s';
else
    L_k_save_H = [L_k_save_H;L_k];
    L_k_save_special = [s';L_k_save_H(1:i-1,:)];
end

% For MHREE calculations
R_d2 = L_k_save_special*R_hat*L_k_save_special';
Krylov_Power_Function(i) = sqrt(trace(inv(R_d2)));
W_MWF = s - L_k_save_H'*(inv(L_k_save_H*R_hat*...
    L_k_save_H'))*L_k_save_H*R_hat*s;
W_MWF = W_MWF/(s'*W_MWF);
W_MWF = W_MWF/UWC;
SMSE(1,i) = real(W_MWF'*R_hat*W_MWF);

% For next iteration
B_Temp=h1;
a1=x0;
B_Save = B'*B_Save;

if i == 1
    D(i) = 1/K(nK)*(d0*d0');
else
    D(i) = (1/K(nK)*(d0*d0'))-Delta_i(i-1)*w_i(i-1);
end

Delta_i(i+1) = sqrt(r_x0_d0'*r_x0_d0);
w_i(i) = Delta_i(i)./D(i);

end

D = diag(D(1:Rank));
L = diag(w_i(2:Rank),-1);
L = L(1:Rank,1:Rank);
L = L + eye(size(L));

for q=1:Rank
    R_d = L(1:q,1:q)*D(1:q,1:q)*L(1:q,1:q)';
    h_r_MHREE(q) = h_r_MHREE(q) + sqrt(K(nK)*...
        SMSE(q))*sqrt(trace(inv(R_d)));
    h_r_MHREE2(q) = h_r_MHREE2(q) + sqrt(K(nK)*...
        SMSE(q))*Krylov_Power_Function(q);
end

% Calculating PC MSE for this trial
R_PC_inv = eye(length(R_PC)) - R_PC;
R_PC_inv2 = eye(length(R_PC)) - R_PC;

R_PC_SD_inv = eye(length(R_PC_SD)) - R_PC_SD;
T = (R_PC_SD*inv(R_X0_X0));

r_z0_d0 = (1/K(nK))*T*Block_Matrix'*R_hat*s;
W_a = R_PC_SD_inv*r_z0_d0;
W_SMI_PC_SD_H = (s - Block_Matrix*W_a);
W_SMI_PC_SD_H = R_PC_inv2*W_SMI_PC_SD_H/...
    (W_SMI_PC_SD_H'*R_PC_inv2*W_SMI_PC_SD_H);
W_SMI_PC_SD_H = W_SMI_PC_SD_H ./ UWC;

R_Eff_I_MWF = s - L_k_save_H'*inv(L_k_save_H*...

```

```

R_hat*L_k_save_H')*L_k_save_H*R_hat*s;

R_PC_inv_Test = zeros(length(R_PC_inv),...
    length(R_PC_inv),length(W_DL_PC_H));
R_MWF_inv_Test = R_PC_inv_Test;

for i=1:length(W_DL_PC_H2)
    R_PC_inv_Test(:, :, i)=R_PC_inv + W_DL_PC_H2(i)*...
        eye(length(R_PC_inv));
    R_MWF_inv_Test(:, :, i)=eye(N*M)-(L_k_save_H'*...
        inv(L_k_save_H*(R_hat + W_DL_PC_H2(i)*...
            eye(length(R_hat))))*L_k_save_H'*...
            (R_hat + W_DL_PC_H2(i)*eye(length(R_hat))));
end

W_PC_Test = zeros(N*M,1,length(W_DL_PC_H2));
W_MWF_Test = zeros(N*M,1,length(W_DL_PC_H2));

for l=1:length(W_DL_PC_H2)
    W_PC_Test(:, :, l) = (R_PC_inv_Test(:, :, l)*s)/...
        (s'*R_PC_inv_Test(:, :, l)*s);
    W_PC_Test(:, :, l) = W_PC_Test(:, :, l) ./ UWC;
    W_MWF_Test(:, :, l) = (R_MWF_inv_Test(:, :, l)*...
        s)/(s'*R_MWF_inv_Test(:, :, l)*s);
    W_MWF_Test(:, :, l) = W_MWF_Test(:, :, l) ./ UWC;
end

MSE_PC_SI_Test = zeros(1,length(W_DL_PC_H));
MSE_MWF_Test = MSE_PC_SI_Test;

for k=1:length(W_DL_PC_H2)
    MSE_PC_SI_Test(k) = real(W_PC_Test(:, :, k) '*...
        R2*W_PC_Test(:, :, k));
    MSE_MWF_Test(k) = real(W_MWF_Test(:, :, k) '*...
        R2*W_MWF_Test(:, :, k));
end

W_SMI_PC_H = (R_PC_inv2*s)/(s'*R_PC_inv2*s);
W_SMI_PC_H = W_SMI_PC_H ./ UWC;

MSE_PC_SI_Temp2 = MSE_PC_SI_Temp2 + 1/Nmc*...
    real(W_SMI_PC_H'*R2*W_SMI_PC_H);
MSE_PC_SD_Temp2 = MSE_PC_SD_Temp2 + 1/Nmc*...
    real(W_SMI_PC_SD_H'*R2*W_SMI_PC_SD_H);
MSE_PC_SI_Temp = MSE_PC_SI_Temp + 1/Nmc*...
    min(MSE_PC_SI_Test);

% MHREE related
x0_Temp = x0_Temp+x0;

% Calculating MWF MSE for this trial
MSE_MWF_Temp2 = MSE_MWF_Temp2 + 1/Nmc*...
    abs(W_MWF'*R2*W_MWF);
MSE_MWF_Temp = MSE_MWF_Temp + 1/Nmc*...
    min(MSE_MWF_Test);

end

Weight_norm(1,Rank) = (W_MWF'*W_MWF);

% MHREE calculations
MSE_PC_SI(1,Rank) = MSE_PC_SI_Temp;
MSE_PC_SI2(1,Rank) = MSE_PC_SI_Temp2;
MSE_PC_SD2(1,Rank) = MSE_PC_SD_Temp2;
MSE_MWF(1,Rank) = MSE_MWF_Temp;
MSE_MWF2(1,Rank) = MSE_MWF_Temp2;

end

h_r_MHREE = (1/Nmc)*h_r_MHREE;
h_r_MHREE2 = (1/Nmc)*h_r_MHREE2;

```

```

h_r_MHREE = [1./((Full_Rank-Min_Rank+1).*ones(1,Min_Rank)),...
1./((Full_Rank-Min_Rank:-1:1),zeros(1,N*M-Full_Rank)).*...
h_r_MHREE;
h_r_MHREE2 = [1./((Full_Rank-Min_Rank+1).*ones(1,Min_Rank)),...
1./((Full_Rank-Min_Rank:-1:1),zeros(1,N*M-Full_Rank)).*...
h_r_MHREE2;
h_r_MHREE = 10*log10(h_r_MHREE. ');
h_r_MHREE2 = 10*log10(h_r_MHREE2. ');

%% Data for Eigenspectra of true interference covariance matrices
[u2 size2 vector2] = svd(R2);
[u3 size3 vector3] = svd(R);

s_size = size(size2);
s_size1 = size(size3);

WNGC_threshold = 0.5;
h_r_threshold = 5;
%% Output Plots

% Output Eigenspectra
figure(1);
plot(1:s_size(1),10*log10(diag(size2)),'-r',...
1:s_size1(1),10*log10(diag(size3)),'-b');
title('Eigenspectra');
% axis([0 150 -10 50]);
xlabel('Eigenvalue index');
ylabel('Eigenvalue(dB)');
legend('Covariance Matrix of Jammer, Clutter, Noise and CMT',...
'Covariance Matrix of Clutter, Noise and CMT',0);
grid on

% Output MWF and PC-SD MSE Performances
figure (2);
plot(1:length(MSE_MWF),10*log10(MSE_MWF),'-or',1:length(MSE_MWF),...
10*log10(MSE_MWF2),'--p',1:length(MSE_PC_SI),10*log10(MSE_PC_SD2),...
'-*y',1:length(MSE_MWF),ones(1,length(MSE_MWF)).*10*log10(MMSE),'-b');
xlabel('Rank');
ylabel('Mean Square Error (dB)');
title('MSE Performance vs Rank');
legend('MWF w/ adaptive DL','MWF w/ User-defined DL',...
'PC-SD w/ User-defined DL','MMSE',0);

% Output for WNGC hard stop technique
figure (3);
subplot(211);
plot(1:length(MSE_MWF), 10*log10(Weight_norm),'-r',...
1:length(MSE_MWF),ones(1,length(MSE_MWF)).*...
WNGC_threshold,'-b');
% axis([0 N*M -10 20]);
xlabel('Rank');
ylabel('Norm Squared Weight Vector (dB)');
title('A.) Weight vector power vs Rank');
legend('||w||^2','WNGC User Threshold',0);

subplot(212);
plot(1:length(MSE_MWF),ones(1,length(MSE_MWF)).*...
10*log10(MMSE),'-b',1:length(MSE_MWF),...
10*log10(MSE_MWF2),'-or');
% axis([0 N*M 0 25]);
xlabel('Rank');
ylabel('Mean Square Error (dB)');
title('B.) MSE Performance vs Rank');
legend('MMSE','MWF w/ User-defined DL', 0);

% Output for MHREE hard stop technique
figure (4);
subplot(211);
plot(1:length(MSE_MWF),real(h_r_MHREE2),'-r',1:length(MSE_MWF),...
ones(1,length(MSE_PC_SI)).*...
h_r_threshold+min(real(h_r_MHREE2(Min_Rank:Full_Rank))), '-b');

```

```

% axis([0 N*M 0 20]);
xlabel('Rank');
ylabel('hr (dB)');
title('A.) 10log(hr) vs Rank');
legend('10log(hr)', 'User Threshold', 0);

subplot(212)
plot(1:length(MSE_MWF), ones(1, length(MSE_MWF)).*...
    10*log10(MMSE), '-b', 1:length(MSE_MWF), 10*log10(MSE_MWF2), '-or');
% axis([0 N*M 0 25]);
xlabel('Rank');
ylabel('Mean Square Error (dB)');
title('B.) MSE Performance vs Rank');
legend('MMSE', 'MWF', 0);

toc

```


Appedix B: Matlab codes for study of SINR vs sample support

```
%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   Author:      Zheng Zhou
%   Email:       zhzhou@calpoly.edu
%   Info:        This Matlab program is created by Zheng Zhou in
% his thesis work for the ongoing Synthetic Aperture Radar (SAR)
% research project with
%               Raytheon Space and Airborne System at California
% Polytechnic State University - San Luis Obispo.
%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   Purpose:     This program calculates SINR performances vs
% available sample support
%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;
close all;
tic

%   Initializations
Nmc=5;
% The number of Monte Carlo trials
N=8;
% The number of array elements
M=8;
% The number of pulses/CPI
Nmld2=(N-1)/2;
% Used for spatial array manifold vectors(AMV)
K=[2*N 4*N 6*N M*N];
% # of train snapshots/pulse
f_c = 1e9;
% center freq
c = 3e8;
lambda = c/f_c;
% wavelength of center freq

Beta = 1;
% DPCA mode
theta = (-89:1:90)*pi/180;
% Angle distribution of clutter patches
us = sin(theta);
% u-space for spatial AMV
ut = Beta*us;
% u-space for temporal AMV

V_spatial = zeros(N,length(theta));
% For spatial AMVs
V_temporal = zeros(M,length(theta));
% For temporal AMVs
s = ones(N*M,1)/norm(ones(N*M,1));
% Broadside steer vec

%% Forming spatial and temporal steering vectors
for nc=1:length(us) % Forming spatial AMVs
    for n=0:N-1
        V_spatial(n+1,nc)=exp(j*pi*us(nc)*(n-Nmld2));
    end
    for m=0:M-1
        V_temporal(m+1,nc)=exp(j*pi*ut(nc)*m);
    end
end

%% Normalize steering vectors
```

```

V_spatial = V_spatial/N;
V_temporal = V_temporal/M;

%% Miscellaneous

% set user defined diagonal loading
User_DL = 10;
% not in dB, recommend [0-10]

% Set range of rank to process
Min_Rank = 1;
Full_Rank = N*M-1;

nK = 1;
% Number of samples

n_MHREE = 5;
% MHREE user set threshold in dB

% Jammer power control parameters
NJP = 2;
% total number of jammers in DOA of 0 deg to 90 deg
NJN = 1;
% total number of jammers in DOA of -90 deg to 0 deg
Minjmr = 50;
% Minimum JNR in dB
Maxjmr = 50;
% Maximum JNR in dB
Jmrrange = Maxjmr - Minjmr;
Meanjmrpwr = Minjmr + Jmrrange/2;

% Clutter power control parameters
CNR = 10;
% Mean CNR in dB
N_r = 1;
% Number of range ambiguities
UWC = 1;
% Uniform Weight Control, keep this at 1

% Initializing interference covariance matrix
R_j = 0;
% Jammer covariance matrix
R_c = 0;
% Clutter covariance matrix
R2 = 0;
% Total interference covariance matrix
R_PC_inv = 0;
% Inverse PC-SD covariance matrix
R_Eff_I_MWF = 0;
% Inverse effective MWF covariance matrix

% Parameters for adaptive diagonal loading
Max_DL = 10;
% in dB
Min_DL = 0;
% in dB
DL_iteration = 1;
W_DL_PC_H = Min_DL:DL_iteration:Max_DL;
W_DL_PC_H2 = 10.^(W_DL_PC_H/10);

% CMT controlling parameters
b = 5.7;
w = 10;
% wind speed in MPH
PRF = 1e3;
% PRF in Hz
PRI = 1/PRF;

% Generating AMVs for randomly generated jammer DoAs
theta_deg_neg=(rand(1,NJN)-1)*80-10);

```

```

% Randomly generate jammer in DOAs of -90 deg to -10 deg
theta_deg_pos=((rand(1,NJP))*80+10);
% Randomly generate jammer in DOAs of 10 deg to 90 deg

% Generating AMVs for user generated jammer DOAs
% theta_deg_neg = -76;
% Insert user define angles
% theta_deg_pos = [18 40];
% Insert user define angles

uj=sin([theta_deg_neg,theta_deg_pos]*(pi/180));
% Jammer DOAs in u-space
V = zeros(N, length(uj));
% Matrix of jammer AMVs

% Generating Jammer powers
Jmrpwr = ((rand(1,length(uj))-0.5)*Jmrrange)+Meanjmrpwr';
% Vector of jammer power

% Forming Jammer Covariance Matrix
for nj=1:length(uj)
for n=0:N-1
    V(n+1,nj)=exp(j*pi*uj(nj)*(n-Nmld2));
end
end
V = V/N;

for i=1:length(uj)
    R_j = R_j + (10^(Jmrpwr(i)/10)).*kron(eye(M),V(:,i)*V(:,i)');
end

%% Generating clutter matrix

Clutterpwr = raylrnd(8,N_r,length(theta))+ CNR - 10;
Clutterpwr = Clutterpwr.';

% Forming Clutter Covariance Matrix
for n_r = 1:N_r
    for i=1:length(theta)
        R_c = R_c + Clutterpwr(i,n_r)*kron(V_temporal(:,i)*...
            V_temporal(:,i)',V_spatial(:,i)*V_spatial(:,i)');
    end
end

%% forming ICM CMT
r = 10^((-15.5*log10(w)-12.1*log10(f_c/1e6)+63.2)/10);

Tau = zeros(M);
for i=1:1:M
    for k=1:1:M
        Tau (i,k)=abs(i-k)*PRI;
    end
end

Tau = (4*pi*Tau).^2;
T_ICM = r/(r+1)+(1/(r+1))*(b*lambda)^2./((b*lambda)^2+Tau);
T_ICM = kron(T_ICM,ones(N));

%% Forming total interference matrix

R2 = R_j + R_c + eye(length(R_j));
% Normalized with noise level = unity
R = R_c + eye(length(R_j));
R2 = R2.*T_ICM;
% True interference covariance matrix with jammers (unknown in practice)
R = R.*T_ICM;
% True interference covariance matrix without jammers (unknown in practice)

%%
R_half_power = R2^0.5;
% For generating the snapshot

```

```

R_inv = inv(R2);

% Calculating MMSE
W_MVDR_H = (R_inv*s)/(s'*R_inv*s);
% Optimal MVDR wt vector
W_MVDR_H = W_MVDR_H ./ UWC;
MMSE = real(W_MVDR_H'*R2*W_MVDR_H);
% Minimum Mean square error normalized

% MSE_PC_SI=zeros(1,N*M);
% MSE_PC_SD=zeros(1,N*M);
% MSE_MWF=zeros(1,N*M);
% MSE_MWF2=zeros(1,N*M);
% MSE_PC_SI2=zeros(1,N*M);
% MSE_PC_SD2=zeros(1,N*M);
% P_r = MSE_PC_SI2;
%
% h_r_MHREE = MSE_PC_SI2;
% h_r_MHREE2 = MSE_PC_SI2;
% Weight_norm = MSE_PC_SI2;

MSE_PC=zeros(1,N*M);
MSE_MWF=zeros(1,N*M);
MSE_MWF2=zeros(1,N*M);
MSE_PC2=zeros(1,N*M);
Rho = zeros(size(K));
Rho1 = zeros(size(K));
Rho2 = zeros(size(K));
Rho3 = zeros(size(K));

for nK = 1:length(K)
    MSE_PC_SI=zeros(1,N*M);
    MSE_PC_SD=zeros(1,N*M);
    MSE_MWF=zeros(1,N*M);
    MSE_MWF2=zeros(1,N*M);
    MSE_PC_SI2=zeros(1,N*M);
    MSE_PC_SD2=zeros(1,N*M);
    P_r = MSE_PC_SI2;

    for Rank = Min_Rank:Full_Rank

        % Initializing MSE variables
        MSE_MWF_Temp = 0;
        MSE_PC_SI_Temp = 0;
        MSE_PC_SD_Temp = 0;
        MSE_MWF_Temp2 = 0;
        MSE_PC_SI_Temp2 = 0;
        MSE_PC_SD_Temp2 = 0;
        x0_Temp = 0;
        SMSE_Temp = 0;

        for nmc=1:Nmc % Loop on the number of Monte Carlo trials

            % For data sample
            a = (1/sqrt(2))*(randn(N*M,K(nK))+j*randn(N*M,K(nK)));
            % For X
            X = R_half_power*a;
            % data snapshot, Note E[XX']=R
            Block_Matrix = null(s');

            % Initializing variables for MWF
            B_Temp = s;
            B_Save = 1;
            L_k_save = 0;
            a1=X;
            h1 = 1;

            R_hat = (1/K(nK))*(X*X');
            % sampled interference covariance matrix
            R_hat = R_hat + User_DL*eye(length(R_hat));
            % insert user defined diagonal loading

```

```

X0 = Block_Matrix'*X;
R_X0_X0 = 1/K(nK)*(X0*X0');

% Initializing variables for PC-SD
[V_hat,D_hat]=eig(R_hat);
[V_hat_SD,D_hat_SD]=eig(R_X0_X0);

s1 = size(D_hat);
s2 = size(V_hat);

s1_SD = size(D_hat_SD);
s2_SD = size(V_hat_SD);

D_PC = reshape( D_hat(s1(1):-1:1,s1(2):-1:1,:),...
    s1 );
V_PC = reshape( V_hat(:,s2(2):-1:1,:), s2 );

D_PC_SD = reshape( D_hat_SD(s1_SD(1):-1:1,...
    s1_SD(2):-1:1,:), s1_SD );
V_PC_SD = reshape( V_hat_SD(:,s2_SD(2):-1:1,:),...
    s2_SD );

% Truncate D_PC and V_PC
D_PC = D_PC(1:Rank,1:Rank);
V_PC = V_PC(:,1:Rank);

D_PC_SD = D_PC_SD(1:Rank,1:Rank);
V_PC_SD = V_PC_SD(:,1:Rank);

R_PC = 0;
R_PC_SD = 0;

for i=1:Rank
    % PC iteration
    R_PC = R_PC + ((D_PC(i,i)-min(diag(D_hat)))/...
        D_PC(i,i)).*(V_PC(:,i)*V_PC(:,i)');
    R_PC_SD = R_PC_SD + ((D_PC_SD(i,i)-...
        min(diag(D_hat_SD)))/D_PC_SD(i,i)).*...
        (V_PC_SD(:,i)*V_PC_SD(:,i)');

    % MWF iteration
    B = null(B_Temp');
    d0 = B_Temp'*a1;
    x0 = B'*a1;

    r_x0_d0 = (x0*d0')./length(d0);
    h1 = r_x0_d0./sqrt(r_x0_d0'*r_x0_d0);

    L_k=h1'*B'*B_Save;

    if i == 1;
        L_k_save_H = L_k;
    else
        L_k_save_H = [L_k_save_H;L_k];
    end

    W_MWF = s - L_k_save_H'*(inv(L_k_save_H*...
        R_hat*L_k_save_H'))*L_k_save_H*R_hat*s;
    W_MWF = W_MWF/(s'*W_MWF);
    W_MWF = W_MWF/UWC;

    % For next iteration
    B_Temp=h1;
    a1=x0;
    B_Save = B'*B_Save;
end

% Calculating PC MSE for this trial
R_PC_inv = eye(length(R_PC)) - R_PC;

```

```

R_PC_inv2 = eye(length(R_PC)) - R_PC;

R_PC_SD_inv = eye(length(R_PC_SD)) - R_PC_SD;
T = (R_PC_SD*inv(R_X0_X0));

r_z0_d0 = (1/K(nK))*T*Block_Matrix'*R_hat*s;
W_a = R_PC_SD_inv*r_z0_d0;
W_SMI_PC_SD_H = (s - Block_Matrix*W_a);
W_SMI_PC_SD_H = R_PC_inv2*W_SMI_PC_SD_H/...
(W_SMI_PC_SD_H'*R_PC_inv2*W_SMI_PC_SD_H);
% W_SMI_PC_SD_H = W_SMI_PC_SD_H ./ UWC;

R_Eff_I_MWF = s - L_k_save_H'*inv(L_k_save_H*...
R_hat*L_k_save_H')*L_k_save_H*R_hat*s;

R_PC_inv_Test = zeros(length(R_PC_inv),...
length(R_PC_inv),length(W_DL_PC_H));
R_MWF_inv_Test = R_PC_inv_Test;

for i=1:length(W_DL_PC_H2)
R_PC_inv_Test(:, :, i)=R_PC_inv +...
W_DL_PC_H2(i)*eye(length(R_PC_inv));
R_MWF_inv_Test(:, :, i)=eye(N*M)-(L_k_save_H'*...
inv(L_k_save_H*(R_hat + W_DL_PC_H2(i)*...
eye(length(R_hat)))*L_k_save_H')*...
L_k_save_H*(R_hat + W_DL_PC_H2(i)*...
eye(length(R_hat))));
end

W_PC_Test = zeros(N*M,1,length(W_DL_PC_H2));
W_MWF_Test = zeros(N*M,1,length(W_DL_PC_H2));

for l=1:length(W_DL_PC_H2)
W_PC_Test(:, :, l) = (R_PC_inv_Test(:, :, l)*s)/...
(s'*R_PC_inv_Test(:, :, l)*s);
W_PC_Test(:, :, l) = W_PC_Test(:, :, l) ./ UWC;
W_MWF_Test(:, :, l) = (R_MWF_inv_Test(:, :, l)*s)/...
(s'*R_MWF_inv_Test(:, :, l)*s);
W_MWF_Test(:, :, l) = W_MWF_Test(:, :, l) ./ UWC;
end

MSE_PC_SI_Test = zeros(1,length(W_DL_PC_H));
MSE_MWF_Test = MSE_PC_SI_Test;

for k=1:length(W_DL_PC_H2)
MSE_PC_SI_Test(k) = real(W_PC_Test(:, :, k) '*...
R2*W_PC_Test(:, :, k));
MSE_MWF_Test(k) = real(W_MWF_Test(:, :, k) '*...
R2*W_MWF_Test(:, :, k));
end

W_SMI_PC_H = (R_PC_inv2*s)/(s'*R_PC_inv2*s);
W_SMI_PC_H = W_SMI_PC_H ./ UWC;

MSE_PC_SI_Temp2 = MSE_PC_SI_Temp2 + 1/Nmc*...
real(W_SMI_PC_H'*R2*W_SMI_PC_H);
MSE_PC_SD_Temp2 = MSE_PC_SD_Temp2 + 1/Nmc*...
real(W_SMI_PC_SD_H'*R2*W_SMI_PC_SD_H);
MSE_PC_SI_Temp = MSE_PC_SI_Temp + 1/Nmc*...
min(MSE_PC_SI_Test);

% Calculating MWF MSE for this trial
MSE_MWF_Temp2 = MSE_MWF_Temp2 + 1/Nmc*...
abs(W_MWF'*R2*W_MWF);
MSE_MWF_Temp = MSE_MWF_Temp + 1/Nmc*min(MSE_MWF_Test);

end

MSE_PC_SI(1,Rank) = MSE_PC_SI_Temp;
MSE_PC_SI2(1,Rank) = MSE_PC_SI_Temp2;
MSE_PC_SD2(1,Rank) = MSE_PC_SD_Temp2;

```

```

MSE_MWF(1,Rank) = MSE_MWF_Temp;
MSE_MWF2(1,Rank) = MSE_MWF_Temp2;

end

Rho(nK) = Rho(nK) + MMSE/min(MSE_MWF(Min_Rank:Full_Rank));
Rho1(nK) = Rho1(nK) + MMSE/min(MSE_MWF2(Min_Rank:Full_Rank));
Rho2(nK) = Rho2(nK) + MMSE/min(MSE_PC_SD2(Min_Rank:Full_Rank));
Rho3(nK) = Rho3(nK) + MMSE/min(MSE_PC_SI(Min_Rank:Full_Rank));

end

plot(K,10*log10(Rho),'-or',K,10*log10(Rho1),'--p',K,...
     10*log10(Rho3),'-og',K,zeros(length(Rho)),'-b');
xlabel('Number of Training Samples');
ylabel('Normalized SINR (dB)');
title('Normalized SINR vs Samples for MWF');
legend('MWF w/ Adaptive DL','MWF','PC-SI w/ Adaptive DL',0);
axis([0 max(K)+2*N 10*log10(min([min(Rho),min(Rho1),...
    min(Rho3)]))-.5 5]);

% plot(K,10*log10(Rho),'-or',K,zeros(length(Rho)),'-b');
% xlabel('Number of Training Samples');
% ylabel('Normalized SINR (dB)');
% title('Normalized SINR vs sample support for MWF');
% legend('MWF w/ 10dB DL',0);
% axis([min(K)-N max(K)+N 10*log10(min([min(Rho),min(Rho1),...
% min(Rho3)]))-.5 1]);

toc

```

Appendix C: Matlab codes for synthesizing SAR

```
%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   Author: Zheng Zhou
%   Email:   zhzhou@calpoly.edu
%   Info:    This Matlab program is created by Zheng Zhou
%            in his thesis work for the ongoing
%            Synthetic Aperture Radar (SAR) research
%            project with Raytheon Space and Airborne
%            System at California Polytechnic State
%            University - San Luis Obispo.
%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   Purpose: The purpose of this program is to synthesize
%            a SAR image using range stacking technique
%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

clear all;
close all;

% Spotlight SAR reconstruction with Range Stacking
c=3e8;           % speed of light
f0=50e6;        % baseband bandwidth is 2*f0
w0=2*pi*f0;
fc=200e6;       % carrier frequency
wc=pi*2*fc;

lambda_min=c/(fc+f0); % Wavelength at highest frequency
lambda_max=c/(fc-f0); % Wavelength at lowest frequency

kc=(pi*2*fc)/c;      % wavenumber at carrier frequency
kmin=(pi*2*(fc-f0))/c; % wavenumber at lowest frequency
kmax=(pi*2*(fc+f0))/c; % wavenumber at highest frequency

% All range are in meters

Xc=2000;           % Range distance to center of
                  % target area
Yc=400;           % Cross-range distance to center
                  % of target area
X0=100;           % target area in slant-range
                  % [Xc-X0,Xc+X0]
Y0=150;           % target area in cross-range
                  % [Yc-Y0,Yc+Y0]

% Case 1: if L (synthetic aperture) < Y0; requires
% zero-padding of SAR signal in synthetic
% aperture domain
L=100;           % synthetic aperture is 2*L

% Case 2: L > Y0; slow-time Doppler subsampling of
% SAR signal spectrum reduces computation
% L=400;

% Initializing recording related parameters
iteration = 2;
M=512;
counter = 1;

Recordimage = zeros(M,M,iteration+1);

Yc_array=zeros(1,iteration+1); % setup array for
                               % storing Yc,dx and dy values
DX_array=zeros(1,iteration+1);
DY_array=zeros(1,iteration+1);
```



```

for Yc=Yc:-L:Yc-iteration*L
    theta_c=atan(Yc/Xc);           % Squint angle
    Rc=sqrt(Xc^2+Yc^2);           % Squint radial range
    L_min=max(Y0,L);              % Zero-padded aperture
                                   % is 2*L_min

    Xcc=Xc/(cos(theta_c)^2);       % replace Xc by Xcc for
                                   % squint processing

    % setup u domain parameters and arrays for
    % compressed SAR signal (the usual case)

    duc=(Xcc*lambda_min)/(4*Y0);   % sample spacing in
                                   % aperture domain for compressed SAR signal
    duc=duc/1.2;                   % 10 percent guard band;
                                   % this guard band would not be
                                   % sufficient for targets
                                   % outside digital spotlight
                                   % filter (use a larger guard
                                   % band, i.e., PRF)

    mc=2*ceil(L_min/duc);          % number of samples on aperture
    uc=duc*(-mc/2:mc/2-1);         % synthetic aperture array
    dkuc=pi*2/(mc*duc);            % sample spacing in ku domain
    kuc=dkuc*(-mc/2:mc/2-1);       % kuc array

    dku=dkuc;                     % sample spacing in ku domain

    % setup u domain parameters and arrays for
    % UNcompressed SAR signal (the unusual case)

    if Yc-Y0-L < 0,                % minimum aspect angle
        theta_min=atan((Yc-Y0-L)/(Xc-X0));
    else
        theta_min=atan((Yc-Y0-L)/(Xc+X0));
    end;
    theta_max=atan((Yc+Y0+L)/(Xc-X0)); % maximum aspect angle

    du=pi/(kmax*(sin(theta_max)-sin(theta_min)));
    % sample spacing in aperture domain
    % for SAR signal

    du=du/1.4;                    % 20% guard band
    m=2*ceil(pi/(du*dku));         % number of samples on aperture
    du=pi*2/(m*dku);              % readjust du
    u=du*(-m/2:m/2-1);            % synthetic aperture array
    ku=dku*(-m/2:m/2-1);          % ku array

    % Fast-time domain parameters and arrays

    Tp=2.5e-7;                    % Chirp pulse duration
    alpha=w0/Tp;                  % Chirp rate
    wcm=wc-alpha*Tp;              % Modified chirp carrier

    if Yc-Y0-L < 0,
        Rmin=Xc-X0;
    else
        Rmin=sqrt((Xc-X0)^2+(Yc-Y0-L)^2);
    end;

    Ts=(2/c)*Rmin;                % start time of sampling
    Rmax=sqrt((Xc+X0)^2+(Yc+Y0+L)^2);
    Tf=(2/c)*Rmax+Tp;             % end time of sampling
    T=Tf-Ts;                      % fast-time interval of measurement
    Ts=Ts-.1*T;                   % 10% guard band in front
    Tf=Tf+.1*T;                   % 10% guard band in rear
    T=Tf-Ts;                      % measurements with guard band

    Tmin=max(T, (4*X0)/(c*cos(theta_max))); % Minimum required T

```

```

dt=1/(4*f0); % Time domain sampling including guard band
n=2*ceil((.5*Tmin)/dt); % number of time samples
t=Ts+(0:n-1)*dt; % time array for data acquisition
dw=pi*2/(n*dt); % Frequency domain sampling
w=wc+dw*(-n/2:n/2-1); % Frequency array (centered at carrier)
k=w/c; % Wavenumber array

% Resolution for Broadside: (x,y) domain rotated by theta_c

DX=c/(4*f0); % range resolution (broadside)
DY=(Xcc*lambda_max)/(4*L); % cross-range resolution (broadside)

% Target model
ntarget=9; % number of targets

xn=zeros(1,ntarget); % range
yn=xn; % cross-range
fn=xn; % reflectivity of target

% Targets within digital spotlight filter

xn(1)=0; yn(1)=0; fn(1)=1;
xn(2)=0; yn(2)=-.6*Y0; fn(2)=1;
xn(3)=0; yn(3)=.6*Y0; fn(3)=1;
xn(4)=-.6*X0; yn(4)=0; fn(4)=1;
xn(5)=-.3*X0; yn(5)=-.3*Y0; fn(5)=1;
xn(6)=-.3*X0; yn(6)=.3*Y0; fn(6)=1;
xn(7)=.3*X0; yn(7)=-.3*Y0; fn(7)=1;
xn(8)=.3*X0; yn(8)=.3*Y0; fn(8)=1;
xn(9)=.6*X0; yn(9)=0; fn(9)=1;

% Possible targets outside digital spotlight filter
% xn(6)=-1.2*X0; yn(6)=.75*Y0; fn(6)=.5;
% xn(7)=.5*X0; yn(7)=1.25*Y0; fn(7)=.3;
% xn(8)=1.1*X0; yn(8)=-1.1*Y0; fn(8)=.2;
% xn(9)=-1.2*X0; yn(9)=-1.75*Y0; fn(9)=.6;
%% Begin simulation

s=zeros(n,mc); % SAR signal array

for i=1:ntarget; % Loop for each target
    td=t(:)*ones(1,mc)-2*ones(n,1)*sqrt((Xc+xn(i)).^2+...
        (Yc+yn(i)-uc).^2)/c;
    s=s+fn(i)*exp(j*(wcm*td+alpha*(td.^2))).*(td >= 0 &...
        td <= Tp & ones(n,1)*abs(uc) <= L & t(:)*ones(1,mc) < Tf);
end;

s=s.*exp(-j*wc*t(:)*ones(1,mc)); % Fast-time baseband conversion

td0=t(:)-2*sqrt(Xc^2+Yc^2)/c; % Time delay of Reference Signal
s0=exp(j*(wcm*td0+alpha*(td0.^2))).*(td0 >= 0 & td0 <= Tp);
s0=s0.*exp(-j*wc*t(:)); % Baseband reference fast-time signal

s=ftx(s).*(conj(ftx(s0))*ones(1,mc)); % Fast-time matched filtering with
    % Reference Signal

tm=(2*Rc/c)+dt*(-n/2:n/2-1); % fast-time array after matched filtering

% Slow-time baseband conversion for squint

kus=2*kc*sin(theta_c)*ones(1,n); % Doppler frequency shift in ku domain
    % due to squint

% Digital Spotlighting and Bandwidth Expansion in ku Domain via Slow-time
% Compression and Decompression

cs=s.*exp(j*2*(k(:)*ones(1,mc)).*(ones(n,1)*sqrt(Xc^2+(Yc-uc).^2))-...
    j*2*k(:)*Rc*ones(1,mc)); % compression
fcs=fty(cs); % Fourier transform of compressed
    % signal w.r.t. u domain

```

```

fp=iftx(fty(cs)); % Narrow-bandwidth Polar Format
% Processed reconstruction
PH=asin(kuc/(2*kc)); % angular Doppler domain
R=(c*tm)/2; % range domain mapped from reference
% fast-time domain

% Full Aperture Digital-Spotlight Filter

W_d=((abs(R(:)*cos(PH+theta_c)-Xc) < X0).*...
(abs(R(:)*sin(PH+theta_c)-Yc) < Y0));

fd=fp.*W_d; % Digital Spotlight Filtering
fcs=ftx(fd); % Transform to (omega,ku) domain

% Zero-padding in ku domain for slow-time upsampling

mz=m-mc; % number of zeros to add in front AND back
fcs=(m/mc)*[zeros(n,mz/2),fcs,zeros(n,mz/2)]; % zero padding

cs=ifty(fcs); % Transform to (omega,u) domain

s=cs.*exp(-j*2*(k(:)*ones(1,m)).*(ones(n,1)*sqrt(Xc^2+(Yc-u).^2))+j*2*k(:)...
*Rc*ones(1,m)); % decompression

% s_ds=s; % Save s(omega,u) array for TDC and
% backprojection algorithms

s=s.*exp(-j*kus(:)*u); % Slow-time baseband conversion for squint
fs=fty(s); % Digitally-spotlighted SAR signal spectrum

% SLOW-TIME DOPPLER SUBSAMPLING

if Y0 < L,
ny=2*ceil(1.2*Y0/du); % Number of samples in y domain with
% 20 percent guard band
ms=floor(m/ny); % subsampling ratio
tt=floor(m/(2*ms));
I=m/2+1-tt*ms:m/2+1+(tt-1)*ms; % subsampled index in ku domain
[tt,ny]=size(I); % number of subsamples
fs=fs(:,I); % subsampled SAR signal spectrum
ky=ku(I); % subsampled ky array
dky=dku*ms; % ky domain sample spacing
else
dky=dku;
ny=m;
ky=ku;
end;

dy=pi*2/(ny*dky); % y domain sample spacing
y=dy*(-ny/2:ny/2-1); % cross-range array

% Reconstruction

ky=ones(n,1)*ky+kus(:)*ones(1,ny); % ky array
kx=(4*k(:).^2)*ones(1,ny)-ky.^2;
kx=sqrt(kx.*(kx > 0)); % kx array

kxmin=min(min(kx));
kxmax=max(max(kx));
dkx=pi/X0; % Nyquist sample spacing in kx domain
nx=2*ceil((.5*(kxmax-kxmin))/dkx);
% Required number of samples in kx
%domain, this value will be increased
% slightly to avoid negative array index

dx=pi*2/(nx*dkx); % range sample spacing in reconstructed image
x=dx*(-nx/2:nx/2-1); % range array

% Range stack wavefront reconstruction

```

```

f_stack=zeros(nx,ny);% Initialize reconstruction
% array in (x,y) domain
for i=1:nx; % Stack's loop for reconstruction at each range
    f_stack(i,:)=ifft(sum(fs.*exp(j*kx*(Xc+x(i))+...
        j*ky*Yc+j*.25*pi-j*2*k(:)*ones(1,ny)*Rc)));
end;

f_stack=f_stack/nx; % Scale it for comparison with Fourier
                    % interpolation

G=abs(f_stack)';
xg=max(max(G)); ng=min(min(G)); cg=255/(xg-ng);
SARImage = 256-cg*(G-ng);
Imagesize=size(SARImage);
CenteredSARImageB=zeros(M,M);

SARImageB=fft2(SARImage,M,M);

for u=1:M
    for v=1:M
        CenteredSARImageB(u,v)=SARImageB(u,v).*...
            exp(-j*2*pi/M*(M-Imagesize(1))*...
                *u/2+(M-Imagesize(2))*v/2));
    end
end

% Recording Yc, dy and dx
Yc_array(1,counter)=Yc;
DX_array(1,counter)=dx;
DY_array(1,counter)=dy;

CenteredSARImage=ifft2(CenteredSARImageB);
Recordimage(:, :, counter)=abs(CenteredSARImage);
counter = counter+1;
end

%% SAR image viewer
% to replay images, highlight codes and hit F9

colormap(gray(256));
for z=1:counter-1
    refreshdata(image(Xc+(DX_array(1,z))*(-M/2:M/2-1)),...
        Yc_array(1,z)+(DY_array(1,z))*(-M/2:M/2-1),Recordimage(:, :, z));
    axis xy; % axis image;
    xlabel('Range X, meters')
    ylabel('Cross-range Y, meters')
    title(['Range Stack Spotlight SAR Reconstruction for Yc = ',...
        num2str(Yc_array(1,z)), ' Meters']);
    drawnow; pause(5); % Pause time between images
end

```

Appendix D: Matlab codes for the studying of CMT effects

```
%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   Author: Zheng Zhou
%   Email:   zhzhou@calpoly.edu
%   Info:    This Matlab program is created by Zheng Zhou
% in his thesis work for the ongoing Synthetic Aperture Radar
% (SAR) research project with
%           Raytheon Space and Airborne System at California
% Polytechnic State University - San Luis Obispo.
% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   Purpose: This program is designed to help users understand
% the effects of clutter and some covariance matrix tapers (CMTs).
% Add new CMTs in here to measure their impacts. Four different
% metrics will be displayed.
% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;
close all;

tic

N = 16;                % number of elements
M = 16;                % number of pulses/CPI
c = 3e8;
CNR = 30;              % clutter power
f_c = 1e9;             % center freq
lambda = c/f_c;
d = lambda/2;
B = 100e6;             % Bandwidth
beta = 1;
theta_a = -89:1:90;    % angle resolution
theta = theta_a*pi/180;
[NN N_bin] = size(theta);

%% Target info
theta_t = 15;
omega_t = -0.2;
%%

SNR = 1;
% assume d=lambda;
% omega_d = beta * omega_s
omega_s = pi*sin(theta);
% doppler shift due to interelement spacing
omega_d = beta*omega_s;
% doppler shift due to aircraft motion and AOI range

aN = zeros(N,N_bin);
bN = zeros(M,N_bin);

aN = exp(-j*(0:N-1)'*omega_s)./sqrt(N);
% Normalised angle bins
bN = exp(-j*(0:M-1)'*omega_d)./sqrt(M);
% Normalised doppler bins

b_test1=exp(j*(0:M-1)'*omega_d);
% Doppler bins
% aN_t = zeros(N,1);
% bN_t = zeros(M,1);

aN_t = exp(-j*pi*(0:N-1)'*sin(theta_t*pi/180))/sqrt(N);
% Normalised target angle
```

```

bN_t = exp(-j*pi*(0:M-1)*omega_t)/sqrt(M);
% Normalised target doppler

% S_t = zeros(M*N,1);
S_t = kron(aN_t,bN_t);

R = zeros(M*N,M*N);
S = zeros(M*N,N_bin);

%% Generating covariance matrix due to clutter only

CNR_matrix = 10^(CNR/10)*(randn(1,N_bin)+j*randn(1,...
    N_bin))/sqrt(2);
for ii = 1:N_bin
    S(:,ii) = kron(aN(:,ii),bN(:,ii));
    R = R + CNR_matrix(ii).*(S(:,ii)*S(:,ii)');
end

%% The Following are CMT models under research

% Uniform decorrelating channel mismatch model. This
% effectively raise the white noise floor but colored
% noise covariance
% rank is unaffected

Delta_e = .02;
% User set - average amplitude change due to channel
% mismatch (angle independent)
Delta_p = 2.5;
% User set - average phase change (in degree) due to
% channel mismatch (angle independent)

T_uni_diag = 1-Delta_e+(Delta_e^3)/3;

% Non-Uniform decorrelating channel mismatch model.
% This effectively increases colored noise covariance rank

e_var = 0.0000001;
% User set - amplitude variance due to channel mismatch
% (angle independent)
p_var = 0.005;
% User set - phase variance (in degree) due to channel
% mismatch (angle independent)

T_nonuni_amp = sqrt(e_var).*randn(N)+Delta_e;
T_nonuni pha = sqrt(p_var*pi/180).*randn(N)+Delta_p;

T_nonuni_offdiag = ((1-T_nonuni_amp./2).^2).*...
    ((sinc(T_nonuni pha.*pi./360)).^2);
T_nonuni_diag = T_uni_diag;
T_nonuni_diag = triu(ones(N),1).*T_nonuni_offdiag+eye(N).*...
    T_nonuni_diag;
% X_c_ai_cor1 = ((triu(ones(N),1)-(X_c_ai_cor_amp./2)).^2).*...
% ((sinc(X_c_ai_cor pha.*pi./360).^2));
% X_c_ai_cor2 = X_c_ai_cor1-eye(size(R)).*X_c_ai_cor1;
% X_c_ai_cor2 = X_c_ai_cor1+(eye(size(R))-eye(size(R)).*...
% X_c_ai_cor_amp+X_c_ai_cor_amp.^3./3);
s = size(T_nonuni_diag);
Y = reshape(T_nonuni_diag(:,s(2):-1:1,:), s );
Y = rot90(Y);

T_nonuni_mismatch_spatial = T_nonuni_diag+tril(ones(N),-1).*Y;
T_nonuni_mismatch = kron(ones(M),T_nonuni_mismatch_spatial);

% Angle independent antenna dispersion due to main beam clutter spread
MB_HPBW = 5*pi/180;
% mainbeam half-power beamwidth in radian
% X_c_ad = sin(theta)'*sin(theta);
% X_c_ad_size = size(X_c_ad);
%
% for i=1:X_c_ad_size(1)

```

```

%     for k=1:1:X_c_ad_size(2)
%         X_c_ad(i,k) = X_c_ad(i,k)*d/c*abs(i-k);
%     end
% end
X_c_ad = zeros(N);
for i=1:1:N
    for k=1:1:N
        X_c_ad (i,k)=abs(i-k);
    end
end

X_c_ad = kron(sinc(B.*X_c_ad.*d./c.*sin(MB_HPBW)),ones(N));

% temp = bN*bN';
% X_c_ad = sinc(abs(kron(temp,X_c_ad))*d/c*B);

% temp = reshape(X_c_ad,N*N,1);
% X_c_ad = kron(temp,sin(theta));

% X_c_ad = X_c_ad*X_c_ad';
% X_c_ad = X_c_ad/N_bin;
%
% X_c_ad = sinc(B.*X_c_ad);
% X_c_ad = S*X_c_ad*S';

% ICM or Billingsley model. User input are operating
% wavelength and wind speed
b = 5.7;
w = 10; % wind speed in MPH
PRF = 1e3; % PRF in Hz
PRI = 1/PRF;

r = 10^((-15.5*log10(w)-12.1*log10(f_c/1e6)+63.2)/10);

Tau = zeros(M);
for i=1:1:M
    for k=1:1:M
        Tau (i,k)=abs(i-k)*PRI;
    end
end

Tau = (4*pi*Tau).^2;
T_ICM = r/(r+1)+(1/(r+1)*(b*lambda)^2./((b*lambda)^2+Tau));
T_ICM = kron(T_ICM,ones(N));

%% Begin Simulations

% Eigenspectra
R2 = R.*T_nonuni_mismatch.*T_ICM.*X_c_ad;
R = R + eye(M*N);
% covariance matrix due to clutter only
R2 = R2 + eye(M*N);
% covariance matrix due to clutter and all CMTs under test

inv_R = inv(R);
inv_R2 = inv(R2);

[u s v] = svd(R);
[u2 s2 v2] = svd(R2);

s_size = size(s);

figure(1);
plot(1:1:s_size(1),10*log10(diag(s)),'--r',1:1:s_size(1),...
    10*log10(diag(s2)),'-g');
title('Eigenspectra');
% axis([0 150 -10 50]);
xlabel('Eigenvalue index');

```

```

ylabel('Eigenvalue(dB)');
legend('w/o CMTs','w/ CMTs',0);
grid on

% Power Spectral Density
% P_f = zeros(N_bin,N_bin);
P_min_var = zeros(N_bin,N_bin);

for ii = 1:N_bin
    for jj = 1:N_bin
        SS = kron(aN(:,ii),bN(:,jj));
        % P_f(ii,jj) = SS'*R2*SS;
        P_min_var(ii,jj) = 1./(SS'*inv_R2*SS);
    end
end

figure(2)
mesh(sin(theta),omega_d/pi,20*log10(abs(P_min_var)));
title('Power Spectral Density');
xlabel('Normalized Angle');
ylabel('Normalized Doppler');
zlabel('Power(dB)');
grid on

% Space-time beam pattern with target shown
w_opt = inv_R*S_t./(S_t'*inv_R*S_t);
% w_opt = inv_R2*S_t./(S_t'*inv_R2*S_t);
res_opt = zeros(N_bin, N_bin);

for ii = 1:N_bin
    for jj = 1:N_bin
        SSS = kron(aN(:,ii),bN(:,jj));
        res_opt(ii,jj) = SSS'*w_opt;
    end
end

figure(3)
%[X,Y]=meshgrid(omega_d/pi,theta_a);
mesh(sin(theta),omega_d/pi,20*log10(abs(res_opt)));
% axis([-1 1 -1 1 -120 10]);
title('Space-Time Beam Pattern w/ Target shown');
xlabel('Normalized Doppler');
ylabel('Normalized Angle');
zlabel('Signal Power(dB)');
grid on

% SINR loss vs Normalized doppler
IF = zeros (N_bin,N_bin);
IF2 = zeros (N_bin,N_bin);

for ii = 1:N_bin
    for jj = 1:N_bin
        SS = kron(aN(:,ii),bN(:,jj));
        % IF(ii,jj) = SS'*inv_R*SS.*trace(R)./(SS'*SS);
        IF(ii,jj) = SS'*inv_R*SS./(SS'*SS);
        IF2(ii,jj) = SS'*inv_R2*SS./(SS'*SS);
    end
end
% IF_temp = 10*log10(abs(IF))+(0-max(max(10*log10(abs(IF)))));
figure(4)
plot(omega_d/pi,10*log10(abs(IF(ceil(N_bin/2),:))),...
    '--r',omega_d/pi,10*log10(abs(IF2(ceil(N_bin/2),:))),'-g');
% axis([-1 1 -40 5]);
title('SINR Loss vs Normalized Doppler');
xlabel('Normalized Doppler');
ylabel('SINR Loss(dB)');
legend('w/o CMTs','w/ CMTs',0);

```



```
grid on
```

```
toc
```

Appendix E: Matlab codes for histogram study of hard stops

```
%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   Author:      Zheng Zhou
%   Email:       zhzhou@calpoly.edu
%   Info:        This Matlab program is created by Zheng Zhou in his
%   thesis work for the ongoing Synthetic Aperture Radar (SAR) research
%   project with Raytheon Space and Airborne System at California
%   Polytechnic State University - San Luis Obispo.
%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   Purpose:     This program shows histogram of rank selection and
%   output data to file "Saved_Data.txt" for analysis
%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

clear all;
close all;
tic          % Use tic/toc to time the run
% for Trials = 1:50
%     keep Trials;
%     close all;

%   Initializations
Nmc=30;
% The number of Monte Carlo trials
N=8;
% The number of array elements
M=8;
% The number of pulses/CPI
Nmld2=(N-1)/2;
% Used for spatial array manifold vectors(AMV)
K=[M*N 5*M*N];
% # of train snapshots/pulse
f_c = 1e9;
% center freq
c = 3e8;
lambda = c/f_c;
% wavelength of center freq

Beta = 1;
% DPCA mode
theta = (-89:1:90)*pi/180;
% Angle distribution of clutter patches
us = sin(theta);
% u-space for spatial AMV
ut = Beta*us;
% u-space for temporal AMV

V_spatial = zeros(N,length(theta));
% For spatial AMVs
V_temporal = zeros(M,length(theta));
% For temporal AMVs
s = ones(N*M,1)/norm(ones(N*M,1));
% Broadside steer vec

%%   Forming spatial and temporal steering vectors

for nc=1:length(us) % Forming spatial AMVs
    for n=0:N-1
        V_spatial(n+1,nc)=exp(j*pi*us(nc)*(n-Nmld2));
    end
    for m=0:M-1
        V_temporal(m+1,nc)=exp(j*pi*ut(nc)*m);
    end
end
```

```

        end
    end

    %% Normalize steering vectors
    V_spatial = V_spatial/N;
    V_temporal = V_temporal/M;

    %% Miscellaneous

    % set user defined diagonal loading
    User_DL = 10;
    % not in dB, recommend [0-10]

    % Set range of rank to process
    Min_Rank = N*M-1;
    Full_Rank = N*M-1;

    nK = 1;
    % Number of samples

    n_MHREE = 5;
    % MHREE user set threshold in dB

    % Jammer power control parameters
    NJP = 1;
    % total number of jammers in DOA of 0 deg to 90 deg
    NJN = 1;
    % total number of jammers in DOA of -90 deg to 0 deg
    Minjmr = 50;
    % Minimum JNR in dB
    Maxjmr = 50;
    % Maximum JNR in dB
    Jmrrange = Maxjmr - Minjmr;
    Meanjmrpwr = Minjmr + Jmrrange/2;

    % Clutter power control parameters
    CNR = 10;
    % Mean CNR in dB
    N_r = 1;
    % Number of range ambiguities
    UWC = 1;
    % Uniform Weight Control, keep this at 1

    % Initializing interference covariance matrix
    R_j = 0;
    % Jammer covariance matrix
    R_c = 0;
    % Clutter covariance matrix
    R2 = 0;
    % Total interference covariance matrix
    R_PC_inv = 0;
    % Inverse PC-SD covariance matrix
    R_Eff_I_MWF = 0;
    % Inverse effective MWF covariance matrix

    % User defined thresholds
    WNGC_threshold = .05;
    h_r_threshold = 3.0;
    delta_MSE = .3;
    slope_threshold = .0105;
    h_r_cap = 9.195;

    % Parameters for adaptive diagonal loading
    Max_DL = 10;
    % in dB
    Min_DL = 0;
    % in dB
    DL_iteration = 1;
    W_DL_PC_H = Min_DL:DL_iteration:Max_DL;
    W_DL_PC_H2 = 10.^(W_DL_PC_H/10);

```

```

% CMT controlling parameters
b = 5.7;
w = 10;
% wind speed in MPH
PRF = 1e3;
% PRF in Hz
PRI = 1/PRF;

% Generating AMVs for randomly generated jammer DOAs
theta_deg_neg=((rand(1,NJN)-1)*80-10);
% Randomly generate jammer in DOAs of -90 deg to -10 deg
theta_deg_pos=((rand(1,NJP))*80+10);
% Randomly generate jammer in DOAs of 10 deg to 90 deg

% Generating AMVs for user generated jammer DOAs
% theta_deg_neg = -83;
% Insert user define angles
% theta_deg_pos = [28 83];
% Insert user define angles

uj=sin([theta_deg_neg,theta_deg_pos]*(pi/180));
% Jammer DOAs in u-space
V = zeros(N, length(uj));
% Matrix of jammer AMVs

% Generating Jammer powers
Jmrpwr = ((rand(1,length(uj))-0.5)*Jmrrange)+Meanjmrpwr';
% Vector of jammer power

% Forming Jammer Covariance Matrix
for nj=1:length(uj)
for n=0:N-1
    V(n+1,nj)=exp(j*pi*uj(nj)*(n-Nmld2));
end
end
V = V/N;

for i=1:length(uj)
    R_j = R_j + (10^(Jmrpwr(i)/10)).*kron(eye(M),V(:,i)*V(:,i)');
end

%% Generating clutter matrix

Clutterpwr = raylrnd(8,N_r,length(theta))+ CNR - 10;
Clutterpwr = Clutterpwr.';

% Forming Clutter Covariance Matrix
for n_r = 1:N_r
    for i=1:length(theta)
        R_c = R_c + Clutterpwr(i,n_r)*kron(V_temporal(:,i)*...
            V_temporal(:,i)',V_spatial(:,i)*V_spatial(:,i)');
    end
end

%% forming ICM CMT
r = 10^((-15.5*log10(w)-12.1*log10(f_c/1e6)+63.2)/10);

Tau = zeros(M);
for i=1:1:M
    for k=1:1:M
        Tau (i,k)=abs(i-k)*PRI;
    end
end

Tau = (4*pi*Tau).^2;
T_ICM = r/(r+1)+(1/(r+1)*(b*lambda)^2./((b*lambda)^2+Tau));
T_ICM = kron(T_ICM,ones(N));

%% Forming total interference matrix

```

```

R2 = R_j + R_c + eye(length(R_j));
% Normalized with noise level = unity
% R = R_c + eye(length(R_j));
R2 = R2.*T_ICM;
% True interference covariance matrix with jammers (unknown in practice)
% R = R.*T_ICM;
% True interference covariance matrix without jammers (unknown in practice)

%%
R_half_power = R2^0.5;
% For generating the snapshot
R_inv = inv(R2);

% Calculating MMSE
W_MVDR_H = (R_inv*s)/(s'*R_inv*s);
% Optimal MVDR wt vector
W_MVDR_H = W_MVDR_H ./ UWC;
MMSE = real(W_MVDR_H'*R2*W_MVDR_H);
% Minimum Mean square error normalized

% MSE_PC_SI=zeros(1,N*M);
% MSE_PC_SD=zeros(1,N*M);
MSE_MWF=zeros(1,N*M);
MSE_MWF2=zeros(1,N*M);
MSE_MWF3=zeros(1,N*M);
% MSE_PC_SI2=zeros(1,N*M);
% MSE_PC_SD2=zeros(1,N*M);
% P_r = zeros(1,N*M);

% h_r_MHREE = zeros(1,N*M);
h_r_MHREE2 = zeros(1,N*M);
Weight_norm = zeros(1,N*M);
Weight_norm3 = zeros(1,N*M);
h_r_MHREE_temp = zeros(1,N*M);

for Rank = Min_Rank:Full_Rank

    % Initializing MSE variables
    MSE_MWF_Temp = 0;
    % MSE_PC_SI_Temp = 0;
    % MSE_PC_SD_Temp = 0;
    MSE_MWF_Temp2 = 0;
    % MSE_PC_SI_Temp2 = 0;
    % MSE_PC_SD_Temp2 = 0;
    x0_Temp = 0;
    SMSE_Temp = 0;
    Optimum_Rank_Matrix = zeros(1,Nmc);
    Optimum_Thresholds = zeros(1,Nmc);
    Selected_Rank_Matrix = zeros(1,Nmc);
    Selected_Rank_Matrix2 = zeros(1,Nmc);
    Weight_norm2 = zeros(1,Nmc);

    for nmc=1:Nmc % Loop on the number of Monte Carlo trials

        % For data sample
        a = (1/sqrt(2))*(randn(N*M,K(nK))+j*randn(N*M,K(nK)));
        % For X
        X = R_half_power*a;
        % data snapshot, Note E[XX']=R
        Block_Matrix = null(s');

        % Initializing variables for MWF
        B_Temp = s;
        B_Save = 1;
        L_k_save = 0;
        a1=X;
        h1 = 1;

        R_hat = (1/K(nK))*(X*X');
        % sampled interference covariance matrix
        R_hat = R_hat + User_DL*eye(length(R_hat));

```

```

% insert user defined diagonal loading

%      X0 = Block_Matrix'*X;
%      R_X0_X0 = 1/K(nK)*(X0*X0');

% Initializing variables for PC-SD
%      [V_hat,D_hat]=eig(R_hat);
%      [V_hat_SD,D_hat_SD]=eig(R_X0_X0);

%      s1 = size(D_hat);
%      s2 = size(V_hat);

%      s1_SD = size(D_hat_SD);
%      s2_SD = size(V_hat_SD);

%      D_PC = reshape( D_hat(s1(1):-1:1,s1(2):-1:1,:), s1 );
%      V_PC = reshape( V_hat(:,s2(2):-1:1,:), s2 );

%      D_PC_SD = reshape( D_hat_SD(s1_SD(1):-1:1,s1_SD(2):-1:1,:), s1_SD );
%      V_PC_SD = reshape( V_hat_SD(:,s2_SD(2):-1:1,:), s2_SD );

% Truncate D_PC and V_PC
%      D_PC = D_PC(1:Rank,1:Rank);
%      V_PC = V_PC(:,1:Rank);

%      D_PC_SD = D_PC_SD(1:Rank,1:Rank);
%      V_PC_SD = V_PC_SD(:,1:Rank);

%      R_PC = 0;
%      R_PC_SD = 0;

% Forming truncated version of Cov matrices

%      D = MSE_PC_SI2;
%      w_i = MSE_PC_SI2;
%      Delta_i = MSE_PC_SI2;
SMSE = zeros(1,N*M);

Krylov_Power_Function = zeros (1,Rank);
MSE_MWF4 = zeros(1,N*M);

for i=1:Rank
    % PC iteration
    %      R_PC = R_PC + ((D_PC(i,i)-min(diag(D_hat)))/...
%      D_PC(i,i)).*(V_PC(:,i)*V_PC(:,i)');
%      R_PC_SD = R_PC_SD + ((D_PC_SD(i,i)-...
% min(diag(D_hat_SD)))/D_PC_SD(i,i)).*(V_PC_SD(:,i)*V_PC_SD(:,i)');

    % MWF iteration
    B = null(B_Temp');
    d0 = B_Temp'*a1;
    x0 = B'*a1;

    r_x0_d0 = (x0*d0')./length(d0);
    h1 = r_x0_d0./sqrt(r_x0_d0'*r_x0_d0);

    L_k=h1'*B'*B_Save;

    if i == 1;
        L_k_save_H = L_k;
        L_k_save_special = s';
    else
        L_k_save_H = [L_k_save_H;L_k];
        L_k_save_special = [s';L_k_save_H(1:i-1,:)];
    end

    % For MHREE calculations
    R_d2 = L_k_save_special*R_hat*L_k_save_special';
    Krylov_Power_Function(i) = sqrt(trace(inv(R_d2)));
    W_MWF = s - L_k_save_H'*(inv(L_k_save_H*...
        R_hat*L_k_save_H'))*L_k_save_H*R_hat*s;

```

```

W_MWF = W_MWF/(s'*W_MWF);

Weight_norm3(i) = Weight_norm3(i)+sqrt(W_MWF'*W_MWF);

if ((sqrt(W_MWF'*W_MWF) >= WNGC_threshold)&&...
    (Weight_norm2(nmc) == 0))
    Weight_norm2(nmc) = i;
end

% W_MWF = W_MWF/UWC;
SMSE(i) = real(W_MWF'*R_hat*W_MWF);

MSE_MWF4(i) = abs(W_MWF'*R2*W_MWF);
% MSE_MWF3(i) = MSE_MWF3(i) + 1/Nmc*abs(W_MWF'*R2*W_MWF);

% For next iteration
B_Temp=h1;
a1=x0;
B_Save = B'*B_Save;

% if i == 1
% D(i) = 1/K(nK)*(d0*d0');
% else
% D(i) = (1/K(nK)*(d0*d0'))-Delta_i(i-1)*w_i(i-1);
% end

% Delta_i(i+1) = sqrt(r_x0_d0'*r_x0_d0);
% w_i(i) = Delta_i(i)./D(i);
end

[r,c] = find(MSE_MWF4(1:Full_Rank) <= delta_MSE+...
    min(MSE_MWF4(1:Full_Rank)));
Optimum_Rank_Matrix(nmc) = c(1);
MSE_MWF3 = MSE_MWF3 + 1/Nmc*MSE_MWF4;

% D = diag(D(1:Rank));
% L = diag(w_i(2:Rank),-1);
% L = L(1:Rank,1:Rank);
% L = L + eye(size(L));

for q=1:Rank
% R_d = L(1:q,1:q)*D(1:q,1:q)*L(1:q,1:q)';
% h_r_MHREE(q) = h_r_MHREE(q) + sqrt(K(nK))*SMSE(q)*...
% sqrt(trace(inv(R_d)));
h_r_MHREE_temp(q) = sqrt(K(nK))*SMSE(q)*...
    Krylov_Power_Function(q);
h_r_MHREE2(q) = h_r_MHREE2(q) + ...
    h_r_MHREE_temp(q);
end

h_r_MHREE_temp = 10*log10(real(h_r_MHREE_temp(1:Full_Rank)));
h_r_MHREE_temp_slope = diff(h_r_MHREE_temp);
h_r_MHREE_temp_slope2 = diff(h_r_MHREE_temp_slope);
h_r_MHREE_temp_slope = (h_r_MHREE_temp_slope <= slope_threshold);
[r_slope c_slope] = find(h_r_MHREE_temp_slope == 1);
% Selected_Rank_Matrix2(nmc) = c_slope(1);

if (sum(size(c_slope)) > 1)
% Test for need of full rank processing
Selected_Rank_Matrix2(nmc) = c_slope(1);
else
Selected_Rank_Matrix2(nmc) = Full_Rank;
end

Optimum_Thresholds(nmc) = h_r_MHREE_temp(c(1))-...
    min(h_r_MHREE_temp(1:Full_Rank));

h_r_MHREE_temp = h_r_MHREE_temp - h_r_cap;
%(h_r_threshold + min(h_r_MHREE_temp(1:Full_Rank)));
[row col] = find(h_r_MHREE_temp >= 0);

```

```

%[row col] = find((h_r_MHREE_temp >= -1)&(h_r_MHREE_temp <= 1));
% Selected_Rank_Matrix(nmc) = col(1);

if (sum(size(col)) > 1)
    % Test for need of full rank processing
    Selected_Rank_Matrix(nmc) = col(1);
else
    Selected_Rank_Matrix(nmc) = Full_Rank;
end

% Calculating PC MSE for this trial
% R_PC_inv = eye(length(R_PC)) - R_PC;
% R_PC_inv2 = eye(length(R_PC)) - R_PC;

% R_PC_SD_inv = eye(length(R_PC_SD)) - R_PC_SD;
% T = (R_PC_SD*inv(R_X0_X0));

% r_z0_d0 = (1/K(nK))*T*Block_Matrix'*R_hat*s;
% W_a = R_PC_SD_inv*r_z0_d0;
% W_SMI_PC_SD_H = (s - Block_Matrix*W_a);
% W_SMI_PC_SD_H = R_PC_inv2*W_SMI_PC_SD_H/...
% (W_SMI_PC_SD_H'*R_PC_inv2*W_SMI_PC_SD_H);
% W_SMI_PC_SD_H = W_SMI_PC_SD_H ./ UWC;

% R_Eff_I_MWF = s - L_k_save_H'*inv(L_k_save_H*...
% R_hat*L_k_save_H')*L_k_save_H*R_hat*s;

% R_PC_inv_Test = zeros(length(R_PC_inv),...
% length(R_PC_inv),length(W_DL_PC_H));
% R_MWF_inv_Test = R_PC_inv_Test;
% R_MWF_inv_Test = zeros(N*M,N*M,length(W_DL_PC_H));

for i=1:length(W_DL_PC_H2)
    R_PC_inv_Test(:, :, i) = R_PC_inv + ...
    W_DL_PC_H2(i)*eye(length(R_PC_inv));
    R_MWF_inv_Test(:, :, i) = eye(N*M) - (L_k_save_H'*...
    inv(L_k_save_H*(R_hat + W_DL_PC_H2(i)*...
    eye(length(R_hat)))*L_k_save_H')*...
    L_k_save_H*(R_hat + W_DL_PC_H2(i)*eye(length(R_hat))));
end

% W_PC_Test = zeros(N*M,1,length(W_DL_PC_H2));
% W_MWF_Test = zeros(N*M,1,length(W_DL_PC_H2));

for l=1:length(W_DL_PC_H2)
    W_PC_Test(:, :, l) = (R_PC_inv_Test(:, :, l)*s)/...
    (s'*R_PC_inv_Test(:, :, l)*s);
    W_PC_Test(:, :, l) = W_PC_Test(:, :, l) ./ UWC;
    W_MWF_Test(:, :, l) = (R_MWF_inv_Test(:, :, l)*s)/...
    (s'*R_MWF_inv_Test(:, :, l)*s);
    W_MWF_Test(:, :, l) = W_MWF_Test(:, :, l) ./ UWC;
end

% MSE_PC_SI_Test = zeros(1,length(W_DL_PC_H));
% MSE_MWF_Test = zeros(1,length(W_DL_PC_H));

for k=1:length(W_DL_PC_H2)
    MSE_PC_SI_Test(k) = real(W_PC_Test(:, :, k)'*R2*W_PC_Test(:, :, k));
    MSE_MWF_Test(k) = real(W_MWF_Test(:, :, k)'*...
    R2*W_MWF_Test(:, :, k));
end

% W_SMI_PC_H = (R_PC_inv2*s)/(s'*R_PC_inv2*s);
% W_SMI_PC_H = W_SMI_PC_H ./ UWC;

%MSE_PC_SI_Temp2 = MSE_PC_SI_Temp2 + 1/Nmc*...
% real(W_SMI_PC_H'*R2*W_SMI_PC_H);
% MSE_PC_SD_Temp2 = MSE_PC_SD_Temp2 + ...
% 1/Nmc*real(W_SMI_PC_SD_H'*R2*W_SMI_PC_SD_H);
% MSE_PC_SI_Temp = MSE_PC_SI_Temp + ...
% 1/Nmc*min(MSE_PC_SI_Test);

```

```

% MHREE related
x0_Temp = x0_Temp+x0;

% Calculating MWF MSE for this trial
MSE_MWF_Temp2 = MSE_MWF_Temp2 + 1/Nmc*...
    abs(W_MWF'*R2*W_MWF);
MSE_MWF_Temp = MSE_MWF_Temp + 1/Nmc*...
    min(MSE_MWF_Test);
end

Weight_norm(1,Rank) = sqrt(W_MWF'*W_MWF);
Weight_norm3 = Weight_norm3/Nmc;

% MHREE calculations
% MSE_PC_SI(1,Rank) = MSE_PC_SI_Temp;
% MSE_PC_SI2(1,Rank) = MSE_PC_SI_Temp2;
% MSE_PC_SD2(1,Rank) = MSE_PC_SD_Temp2;
MSE_MWF(1,Rank) = MSE_MWF_Temp;
MSE_MWF2(1,Rank) = MSE_MWF_Temp2;
end

% Rank_difference = Selected_Rank_Matrix - ...
% Optimum_Rank_Matrix;
Rank_difference = zeros(1,Nmc);
Rank_difference2 = zeros(size(Rank_difference));

for jj=1:length(Rank_difference)
    if ((Selected_Rank_Matrix2(jj) <= max(Optimum_Rank_Matrix))&&...
        (Selected_Rank_Matrix2(jj) >= min(Optimum_Rank_Matrix)))
        Rank_difference2(jj) = 0;
    elseif (Selected_Rank_Matrix2(jj) < min(Optimum_Rank_Matrix))
        Rank_difference2(jj) = Selected_Rank_Matrix2(jj) - ...
            min(Optimum_Rank_Matrix);
    else
        Rank_difference2(jj) = Selected_Rank_Matrix2(jj) - ...
            max(Optimum_Rank_Matrix);
    end

    if ((Selected_Rank_Matrix(jj) <= max(Optimum_Rank_Matrix))&&...
        (Selected_Rank_Matrix(jj) >= min(Optimum_Rank_Matrix)))
        Rank_difference(jj) = 0;
    elseif (Selected_Rank_Matrix(jj) < min(Optimum_Rank_Matrix))
        Rank_difference(jj) = Selected_Rank_Matrix(jj) - ...
            min(Optimum_Rank_Matrix);
    else
        Rank_difference(jj) = Selected_Rank_Matrix(jj) - ...
            max(Optimum_Rank_Matrix);
    end
end

% h_r_MHREE = (1/Nmc)*h_r_MHREE;
h_r_MHREE2 = (1/Nmc)*h_r_MHREE2;
% h_r_MHREE = [1./((Full_Rank-Min_Rank+1).*ones(1,Min_Rank)),...
% 1./((Full_Rank-Min_Rank:-1:1),zeros(1,N*M-Full_Rank)].*h_r_MHREE;
h_r_MHREE2 = [1./((Full_Rank-Min_Rank+1).*ones(1,Min_Rank)),...
    1./((Full_Rank-Min_Rank:-1:1),zeros(1,N*M-Full_Rank)].*h_r_MHREE2;
% h_r_MHREE = 10*log10(h_r_MHREE. ');
h_r_MHREE2 = 10*log10(h_r_MHREE2. ');
h_r_MHREE2(Full_Rank+1)=h_r_MHREE2(Full_Rank);

h_r_slope = diff(h_r_MHREE2);
h_r_slope(Full_Rank) = h_r_slope(Full_Rank-1);
h_r_slope = [h_r_slope;h_r_slope(Full_Rank)];
h_r_slope = real(h_r_slope);%./(max(real(h_r_slope)));

h_r_slope2 = diff(h_r_slope);
% h_r_slope2(Full_Rank) = h_r_slope2(Full_Rank-1);
h_r_slope2 = [h_r_slope2;h_r_slope2(length(h_r_slope2))];
% h_r_slope = real(h_r_slope);%./(max(real(h_r_slope)));

```



```

%% Data for Eigenspectra of true interference covariance matrices
% [u2 size2 vector2] = svd(R2);
% [u3 size3 vector3] = svd(R);
%
% s_size = size(size2);
% s_size1 = size(size3);

%% Output optimum threshold mean and variance

Optimum_Thresholds_Mean = mean(Optimum_Thresholds);
Optimum_Thresholds_Var = var(Optimum_Thresholds);

%% Save to File
Avg_angle = mean([abs(theta_deg_neg),theta_deg_pos]);

Avg_under_selection = ((Rank_difference < 0)*Rank_difference.)/...
    (sum((Rank_difference < 0))+1);
Avg_under_selection2 = ((Rank_difference2 < 0)*Rank_difference2.)/...
    (sum((Rank_difference2 < 0))+1);
Avg_over_selection = ((Rank_difference > 0)*Rank_difference.)/...
    (sum((Rank_difference > 0))+1);
Avg_over_selection2 = ((Rank_difference2 > 0)*Rank_difference2.)/...
    (sum((Rank_difference2 > 0))+1);

Data_to_Record = [theta_deg_neg,theta_deg_pos,Avg_angle,...
    Optimum_Thresholds_Mean,Optimum_Thresholds_Var,min(Optimum_Rank_Matrix),...
    max(Optimum_Rank_Matrix),...
    10*log10(Weight_norm3(round(mean(Optimum_Rank_Matrix)))),...
    Optimum_Thresholds_Mean+min(real(h_r_MHREE2(1:Full_Rank))),...
    mean(abs(Rank_difference)),Avg_under_selection, Avg_over_selection];
%     mean(abs(Rank_difference2)), Avg_under_selection2,
% Avg_over_selection2];

data_to_str = sprintf('%0.5g \t', Data_to_Record);
fid = fopen('Saved_Data2.txt','a');
fprintf(fid, '%s \n', data_to_str);
fclose(fid);
%% Output Plots

% Output Histogram
% figure (1);
% hist(Rank_difference);
% axis([min(Rank_difference)-30 max(Rank_difference)+30 0 20]);
% xlabel('{\Delta} Rank');
% ylabel('Count of occurences');
% title('Histogram of rank selection');

% figure (2);
% hist(Rank_difference2);
% axis([min(Rank_difference2)-10 max(Rank_difference2)+10 0 nmc]);
% xlabel('{\Delta} Rank');
% ylabel('Count of occurences');
% title('Histogram of rank selection');
%

% Output for WNGC hard stop technique
figure (4);
subplot(212);
plot(1:length(MSE_MWF), 10*log10(Weight_norm3),'-r',...
    1:length(MSE_MWF),ones(length(MSE_MWF))*WNGC_threshold,'-b');
% axis([0 N*M -10 20]);
xlabel('Rank');
ylabel('Norm Squared Weight Vector (dB)');
title('A.) Weight vector power vs Rank');
legend('||w||^2','WNGC User Threshold',0);

% subplot(212);
% plot(1:length(MSE_MWF), 10*log10(Weight_norm3),'-r',...
% 1:length(MSE_MWF),ones(length(MSE_MWF))*...
% 10*log10(Weight_norm3(round(mean(Optimum_Rank_Matrix)))),'-b');
% % axis([0 N*M -10 20]);

```

```

% xlabel('Rank');
% ylabel('Norm Squared Weight Vector (dB)');
% title('A.) Weight vector power vs Rank');
% legend('||w||^2', 'WNGC User Threshold', 0);

subplot(211);
plot(1:length(MSE_MWF), ones(1,length(MSE_MWF)).*10*log10(MMSE), ...
    '-b', 1:length(MSE_MWF), 10*log10(MSE_MWF3), '-or');
% axis([0 N*M 0 25]);
xlabel('Rank');
ylabel('Mean Square Error (dB)');
title('B.) MSE Performance vs Rank');
legend('MMSE', 'MWF w/ User-defined DL', 0);

% subplot(211);
% plot(1:length(MSE_MWF), real(h_r_MHREE2), '-r', 1:length(MSE_MWF),
% ones(1,length(MSE_MWF)).*Optimum_Thresholds_Mean+min(real
% (h_r_MHREE2(1:Full_Rank))), '-b');
% % axis([0 N*M 0 20]);
% xlabel('Rank');
% ylabel('hr (dB)');
% title('A.) 10log(hr) vs Rank');
% legend('10log(hr)', 'Optimum Threshold', 0);
%

% Output for MHREE hard stop technique

% figure (3);
% subplot(211);
% plot(1:length(MSE_MWF), real(h_r_MHREE2), '-r', 1:length(MSE_MWF),
% ones(1,length(MSE_MWF)).*Optimum_Thresholds_Mean+min(real
% (h_r_MHREE2(1:Full_Rank))), '-b');
% % axis([0 N*M 0 20]);
% xlabel('Rank');
% ylabel('hr (dB)');
% title('A.) 10log(hr) vs Rank');
% legend('10log(hr)', 'Optimum Threshold', 0);

% subplot(412);
% plot(1:length(MSE_MWF), 1./real(h_r_slope), '-r', 1:length(MSE_MWF),
% ones(1,length(MSE_MWF)).*1./h_r_slope(round(mean(Opti
% mum_Rank_Matrix))), '-b');
% % axis([0 N*M 0 20]);
% xlabel('Rank');
% ylabel('');
% title('Inverse rank slope');
% legend('Inverse rank slope', 'Optimum Threshold', 0);
%

% subplot(413);
% plot(1:length(MSE_MWF), 1./real(h_r_slope2), '-r', 1:length(MSE_MWF),
% ones(1,length(MSE_MWF)).*1./h_r_slope2(round(mean(Optimum_
% Rank_Matrix))), '-b');
% % axis([0 N*M 0 20]);
% xlabel('Rank');
% ylabel('');
% title('Inverse rank slope2');
% legend('Inverse rank slope2', 'Optimum Threshold', 0);

% figure (5);
% subplot(211);
% plot(1:length(MSE_MWF), real(h_r_MHREE2), '-r', 1:length(MSE_M
% WF), ones(1,length(MSE_MWF)).*h_r_cap, '-b');
% % axis([0 N*M 0 20]);
% xlabel('Rank');
% ylabel('hr (dB)');
% title('A.) 10log(hr) vs Rank');
% legend('10log(hr)', 'h_r Cap = 9.195', 0);
%

% subplot(212);
% plot(1:length(MSE_MWF), ones(1,length(MSE_MWF)).*10*log10(M
% MSE), '-b', 1:length(MSE_MWF), 10*log10(MSE_MWF3), '-*b');

```

```

% % axis([0 N*M 0 25]);
% xlabel('Rank');
% ylabel('Mean Square Error (dB)');
% title ('MSE Performance vs Rank');
% legend('MMSE','MWF', 0);

% end
toc

```