Non-Relativistic Limit of Selected Terms from The SME Dirac Lagrangian

A Senior Project

By

Quinn Reece

Advisor, Dr. Thomas D. Gutierrez

Department of Physics, California Polytechnic University SLO

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Author: Quinn Reece

Date Submitted: March, 2024

Senior Project Advisor: Dr. Thomas D. Gutierrez

Signature

Date

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Abstract

We examine a selection of individual CPT/Lorentz violating terms present in the relativistic lagrangian for a free spin-$\frac{1}{2}$ Dirac fermion of mass m in the Standard Model Extension. Euler-Lagrange relations will be applied to give Dirac-like equations including these terms and a novel procedure will be used to generate non-relativistic limits of these equations which are Schrödinger-Pauli-like equations. These equations will be analyzed using classical quantum mechanics toy problems to gain physical intuition for the effects of the CPT violating terms, and the results will be discussed. We will conclude with discussion on future work will include the potential testability of theoretical findings and connections to other current work in physics beyond the Standard Model being done by the CUORE Collaboration.

1 Introduction

1.1 Background

In 1928, Paul Dirac derived his take on a quantum mechanical equation that was compatible with Einstein's special theory of relativity. This long sought after union predicted the existence of antiparticles, as well as showed spin as a consequence of this union. After even further work unifying quantum mechanics with relativity, we arrived at the Standard Model of particle physics, one of humanities great triumphs. But our pursuit of a theory that unifies the four fundamental forces has lead us to arrive at the conclusion Lorentz symmetry, the underlying symmetry of special relativity, may not be unbreakable [1].

As mentioned, attempts to unite the four fundamental forces, such as String Theory, suggest the possibility of a spontaneous breaking of Lorentz Symmetry [2]. A revolutionary new framework called the Standard Model Extension (SME) was proposed in 1998 by Don Colladay and V. Alan Kostelecký. This new framework preserved the Standard Model's gauge invariance, energy-momentum conservation, covariance of observer rotations and boosts, while allowing for broken covariance under particle rotations and boosts [1].

With this new framework in place, the Quantum Electro-Dynamics (QED) sector of the Standard Model will have some additional Lorentz-violating terms. This already has the making for some very interesting physics. The SME QED Lagrangian provides a Lorentz violating description of Quantum Electromagnetic theory, one which is worth exploring at an extremely granular level.

1.2 Why Non-Relativistic?

It is curious to look at a high energy theory, such as String Theory, where these violations of CPT/Lorentz symmetry are predicted at said high energies, and attempt to understand their low energy effects. Why is this useful? While it is possible for one to apply charge conjugation, parity inversion, and time reversal operators to the SME lagrangian in order to see that it is not CPT invariant, this does not give one much physical insight when it comes to the effects of these terms outside of stating "they violate CPT symmetry." In order to understand them, it is best to analyze them in scenarios that provide strong quantum mechanical intuition, such as the classic "particle in a box" that we have all seen in our first course in quantum mechanics. We use particle in a box because it does an excellent job of illustrating principles of quantum mechanics like superposition of states and discretized energy eigenvalues, so why not use it to understand the effects of CPT violating terms? As it turns out, we not only can derive lower energy equations (Schrödinger-Pauli-like equations) which contain these terms (though as we shall soon see is not a trivial task), but also stand to learn a lot about the terms' physical effects in doing so. Additionally, one can (as we shall soon see) find that some terms are capable of producing *detectable* shifts to well established results from quantum mechanics.

1.3 Some Preliminary Notions

As with any sort of science, it is important to establish beforehand what assumptions we shall make and what conventions we shall use. As we shall soon see, the world of particle theory can get very messy, so to aid in cleanliness we shall use the convention of $c = \hbar = 1$ units.

We should also state the lagrangian that we will be working from, which is equation (1) in [3]:

$$\mathcal{L} = \frac{i}{2}\overline{\psi}(\gamma^\mu + c^{\mu\nu}\gamma_\nu + d^{\mu\nu}\gamma_5\gamma_\nu + e^\mu + if^\mu\gamma_5 + \frac{1}{2}g^{\lambda\nu\mu}\sigma_{\lambda\nu})\overleftrightarrow{\partial}_\mu\psi - \overline{\psi}(m + a_\mu\gamma^\mu + b_\mu\gamma_5\gamma^\mu + \frac{1}{2}H^{\nu\mu}\sigma_{\nu\mu})\psi \quad (1)$$

From [3], we are given some conditions for the CPT violating terms. The terms which are responsible for the Lorentz/CPT violation are $a_\mu, b_\mu, c^{\mu\nu}, d^{\mu\nu}, e^\mu, f^\mu, g^{\lambda\nu\mu}, H^{\nu\mu}$. Each of these terms are Lorentz vectors or tensors. Additionally, we are given that $H^{\nu\mu}$ is anti-symmetric, $c^{\mu\nu}, d^{\mu\nu}$ are traceless, and $g^{\lambda\nu\mu}$ is antisymmetric in the first two indices.

We also have some conventions and declarations to establish. The first being γ^μ, where we shall use the convention $\gamma^\mu = (\beta, \beta\vec{\alpha})$, where:

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{2}$$

and:

$$\vec{\alpha} = \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix} \tag{3}$$

Where I is the standard 2×2 unit matrix, and $\vec{\sigma}$ is a vector of the standard 2×2 Pauli spin matrices in x, y, z respectively. Consequently with these conventions we also get:

$$\gamma_5 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{4}$$

Additionally, we have:

$$\sigma_{\lambda\nu} = \frac{i}{2}(\gamma_\lambda\gamma_\nu - \gamma_\nu\gamma_\lambda) \tag{5}$$

which we will note is anti-symmetric ($\sigma_{\lambda\nu} = -\sigma_{\nu\lambda}$), with a diagonal of all zeros. We also have $i\partial_\mu \equiv p_\mu$, and further we will define $p_\mu \equiv (E, -\vec{p})$. This allows us to write the standard free Dirac Equation (again, in our chosen units),

$$i\gamma^\mu\partial_\mu\Psi - m\Psi = 0 \tag{6}$$

in a cleaner form,

$$\gamma^\mu p_\mu\Psi - m\Psi = 0 \tag{7}$$

and then in a very useful matrix form as

$$\begin{bmatrix} E - m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E - m \end{bmatrix} \begin{bmatrix} \Psi_A \\ \Psi_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}$$

which uses conventional upper-lower spinors. We will refer to the matrix acting on Ψ as the Dirac Matrix.

Our main goal throughout is to produce Schrödinger-Pauli-like equations starting from a lagrangians which contain CPT violating terms in (1). We want equations like this so that we can study the terms' effects through quantum mechanical models which provide the best intuition of the theory. The Schrödinger-Pauli equation,

$$\left[\frac{(\vec{p}-q\vec{A})^2}{2m}-\frac{q(\vec{\sigma}\cdot\vec{B})}{2m}+q\phi\right]\Psi=E_{NR}\Psi \tag{9}$$

is a non-relativistic quantum mechanics equation which describes a spin-$\frac{1}{2}$ fermion of mass m in an electromagnetic-potential. This equation can be derived from (8) by applying minimal the minimal coupling procedure, $p_\mu \to p_\mu - qA_\mu$, and taking the non-relativistic limit, $E_{NR}+m \approx m$. This represents the non-relativistic scenario in which we assume mass to be significantly greater than kinetic energy. We arrive at this by choosing to represent our total energy, E, as $E_{NR}+m$, the non-relativistic energy plus the mass. For a complete procedure of deriving the Schrödinger-Pauli equation (9) from the Dirac Equation (6) see [4]. As we shall soon see, we can arrive at a modified version of (9) when we start with lagrangians containing CPT-violating terms from (1).

Another object of note is what is called the Pauli Identity. This will be of great use in our endeavours, as it is an essential step in arriving at the Schrödinger-Pauli Equation from the Dirac Equation. The Pauli identity allows us to write $(\vec{\sigma}\cdot(\vec{p}-q\vec{A}))^2$ in a much more useful form as $(\vec{p}-q\vec{A})^2 - q(\vec{\sigma}\cdot\vec{B})$. It is my personal belief that this identity is the most important thing used in this paper, and that while at times I will nearly gloss over its use by simply writing "... and applying the Pauli Identity..." I assure you that you could teach a whole semester course on what it really means.

Finally, we shall note that as is convention, we will be doing our Euler-Lagrange relations with respect to $\overline{\psi}$ in our lagrangians. An example of this is the standard free Dirac lagrangian,

$$\mathcal{L}=\frac{i}{2}\overline{\psi}\gamma^\mu\partial_\mu\psi-\frac{i}{2}\partial_\mu\overline{\psi}\gamma^\mu\psi-m\overline{\psi}\psi \tag{10}$$

We will use the following Euler-Lagrange relations:

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})}=\frac{\partial\mathcal{L}}{\partial\overline{\psi}} \tag{11}$$

and the result is (6). With all of this in hand, we are ready to begin examining effects of a selection of CPT-violating terms from (1).

2 A Lagrangian with f^μ

2.1 A Dirac-like Equation with f^μ

We shall start by examining the effects of the CPT-violating vector term f^μ by isolating it in a lagrangian:

$$\mathcal{L}=\frac{i}{2}\overline{\Psi}(\gamma^\mu+if^\mu\gamma_5)\partial_\mu\Psi-\frac{i}{2}\partial_\mu\overline{\Psi}(\gamma^\mu+if^\mu\gamma_5)\Psi-m\overline{\Psi}\Psi \tag{12}$$

We apply our established Euler-Lagrange relations to (12) which results in

$$i(\gamma^\mu + if^\mu\gamma_5)\partial_\mu\Psi - m\Psi = 0 \tag{13}$$

This will serve as the basis for our analysis of this term in this section.

2.2 Schrödinger-Pauli-Like Equation for f^μ CPT Violating Dirac Equation

The next useful thing to do is to write (13) in matrix form. Upon inspection, we can see that we should expect (13)'s matrix form to resemble the Dirac matrix from (8) plus another matrix coming from f^μ's addition acting on our spinor, Ψ. This is indeed a useful intuition, as we may write the additional term in the following matrix form:

$$if^\mu\gamma_5 p_\mu = \begin{bmatrix} 0 & i(f_0E - \vec{f}\cdot\vec{p}) \\ i(f_0E - \vec{f}\cdot\vec{p}) & 0 \end{bmatrix} \tag{14}$$

Putting our CPT-violating matrix together with the Dirac matrix and switching to upper-lower spinor form, we arrive at:

$$\left[\begin{bmatrix} E-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E-m \end{bmatrix} + \begin{bmatrix} 0 & i(f_0E - \vec{f}\cdot\vec{p}) \\ i(f_0E - \vec{f}\cdot\vec{p}) & 0 \end{bmatrix}\right]\begin{bmatrix} \Psi_A \\ \Psi_B \end{bmatrix} = 0 \tag{15}$$

Carrying out the matrix multiplication gives the following system:

$$(E-m)\Psi_A + [-\vec{\sigma}\cdot\vec{p} + i(f_0E - \vec{f}\cdot\vec{p})]\Psi_B = 0 \tag{16}$$

and

$$[-\vec{\sigma}\cdot\vec{p} + i(f_0E - \vec{f}\cdot\vec{p})]\Psi_A + (-E-m)\Psi_B = 0 \tag{17}$$

Rearranging (17) results in:

$$\Psi_B = \frac{[\vec{\sigma}\cdot\vec{p} + i(f_0E - \vec{f}\cdot\vec{p})]}{E+m}\Psi_A \tag{18}$$

Put this into (16) and rearrange once more and we find:

$$(E+m)(E-m)\Psi_A = [\vec{\sigma}\cdot\vec{p} + i(f_0E - \vec{f}\cdot\vec{p})][\vec{\sigma}\cdot\vec{p} - i(f_0E - \vec{f}\cdot\vec{p})]\Psi_A \tag{19}$$

Looking at the right hand side of (19) we can simplify it further into:

$$RHS = [(\vec{\sigma} \cdot \vec{p})^2 + (f_0 E - \vec{f} \cdot \vec{p})^2]\Psi_A \tag{20}$$

Now, apply the minimal coupling rule of $p_\mu \rightarrow p_\mu - qA_\mu$ to (20) and we find:

$$RHS = [(\vec{\sigma} \cdot (\vec{p} - q\vec{A}))^2 + (f_0(E - q\phi) - \vec{f} \cdot (\vec{p} - q\vec{A}))^2]\Psi_A \tag{21}$$

Now, we will make the substitution of $E \rightarrow E_{NR} + m$ in (21), resulting in:

$$RHS = [(\vec{\sigma} \cdot (\vec{p} - q\vec{A}))^2 + (f_0(E_{NR} + m - q\phi) - \vec{f} \cdot (\vec{p} - q\vec{A}))^2]\Psi_A \tag{22}$$

Applying the non-relativistic limit of $m >> E_{NR} - q\phi$ to (22) nets us:

$$RHS = [(\vec{\sigma} \cdot (\vec{p} - q\vec{A}))^2 + (f_0 m - \vec{f} \cdot (\vec{p} - q\vec{A}))^2]\Psi_A \tag{23}$$

And applying the Pauli Identity to the first term in (23) we find:

$$RHS = [(\vec{p} - q\vec{A})^2 - q(\vec{\sigma} \cdot \vec{B}) + (f_0 m - \vec{f} \cdot (\vec{p} - q\vec{A}))^2]\Psi_A \tag{24}$$

Going back again to (19), and applying the minimal coupling rule to the left hand side, we find:

$$LHS = (E - q\phi + m)(E - q\phi - m)\Psi_A \tag{25}$$

Making the substitution $E \rightarrow E_{NR} + m$ to (25) and simplifying we find:

$$LHS = (E_{NR} - q\phi + 2m)(E_{NR} - q\phi)\Psi_A \tag{26}$$

Taking the non-relativistic limit of this yields:

$$LHS = 2m(E_{NR} - q\phi)\Psi_A \tag{27}$$

Put together (24) & (27) and then (19) becomes:

$$2m(E_{NR} - q\phi)\Psi_A = [(\vec{p} - q\vec{A})^2 - q(\vec{\sigma} \cdot \vec{B}) + (f_0 m - \vec{f} \cdot (\vec{p} - q\vec{A}))^2]\Psi_A \tag{28}$$

And after a few rearrangements we get the Schrödinger-Pauli like equation for this particular CPT Violating Dirac Equation:

$$\left[\frac{(\vec{p}-q\vec{A})^2}{2m}-\frac{q(\vec{\sigma}\cdot\vec{B})}{2m}+q\phi+\frac{(f_0m-\vec{f}\cdot(\vec{p}-q\vec{A}))^2}{2m}\right]\psi_A=E_{NR}\psi_A \tag{29}$$

This is (by design) written so that you may see the original Schrödinger-Pauli Hamiltonian (first three terms in the brackets) with the additional CPT violating part of the Hamiltonian being the fourth term. Notice that we have kept with the tradition of looking for energy eigenstates, so we have moved to a lowercase ψ in (29), as we are assuming stationary states.

2.3 Toy Examples for CPT Violating Schrödinger-Pauli like Hamiltonian

It is useful to study some toy examples for this new hamiltonian. Elementary quantum mechanics exercises with this new hamiltonian can illuminate the effects of the CPT violating f^μ term.

2.3.1 Free Particle in 1-D

Let us first consider a free particle in one dimension. For this, we will choose $\vec{B}=\phi=\vec{A}=0$. This leaves us with:

$$\left[\frac{\hat{p}^2}{2m}+\frac{(f_0m-f_x\hat{p})^2}{2m}\right]\psi=E_{NR}\psi \tag{30}$$

In our choice of x-basis with $\hat{p}=-i\frac{d}{dx}$, we will consider two scenarios for simplicity: one where $f_0=0$ & $f_x\neq 0$, and the other where $f_0\neq 0$ & $f_x=0$

First consider the case of $f_0=0$ & $f_x\neq 0$. (30) then becomes:

$$\left[-\frac{1}{2m}\frac{d^2}{dx^2}-\frac{f_x^2}{2m}\frac{d^2}{dx^2}\right]\psi=E_{NR}\psi \tag{31}$$

This can be rearranged nicely into:

$$\psi''=-\frac{2mE_{NR}}{1+f_x^2}\psi \tag{32}$$

With the solutions to this clearly being

$$\psi(x)=Ne^{\pm ik_xx} \tag{33}$$

where N is a normalization constant and

$$k_x=\sqrt{\frac{2mE_{NR}}{1+f_x^2}} \tag{34}$$

Comparing this to the value of wavenumber for the original Schrödinger-Pauli equation, $k_x = \sqrt{2mE_{NR}}$, we can see that our new k_x value is essentially being scaled down to be smaller, which corresponds to a lower momentum, or a sort of wind "pushing" against the direction of motion.

Now, let us consider the other case, where $f_0 \neq 0$ & $f_x = 0$. Here, (30) becomes:

$$\left[-\frac{1}{2m}\frac{d^2}{dx^2} + \frac{f_0^2 m^2}{2m}\right]\psi = E_{NR}\psi \tag{35}$$

Which after some rearrangement yields:

$$\psi'' = -(2mE_{NR} - f_0^2 m^2)\psi \tag{36}$$

Where the solutions to this are clearly

$$\psi(x) = Ne^{\pm ik_x x} \tag{37}$$

where N is a normalization constant and

$$k_x = \sqrt{2mE_{NR} - f_0^2 m^2} \tag{38}$$

Here, the new k_x value again corresponds to a lower momentum state than the original Schrödinger-Pauli states. Further free particle study could include examaning a situation where both $f_0, f_x \neq 0$, though it becomes clear quickly that solutions to that differential equation are more complicated, and not as intuitive.

2.3.2 Particle in a 1-D Box

A problem always given in an elementary quantum mechanics course is the particle in an infinite potential well. Let us now see how the same problem is affected with our CPT violating f^μ term. To begin, in (29) we will set $\vec{A} = \vec{B} = 0$, and we will define

$$q\phi = V(x) = \begin{cases} 0, & 0 < x < L \\ \infty, & \text{otherwise} \end{cases} \tag{39}$$

(29) then becomes:

$$\left[\frac{\hat{p}^2}{2m} + V(x) + \frac{(f_0 m - f_x \hat{p})^2}{2m}\right]\psi = E_{NR}\psi \tag{40}$$

Like before, we will first work with the case where $f_0 = 0$ & $f_x \neq 0$. (40) becomes:

$$-\frac{(1+f_x^2)}{2m}\psi'' + V\psi = E_{NR}\psi \tag{41}$$

Evidently outside the box $\psi = 0$ to avoid infinite energy states. Inside the box, (41) becomes:

$$\psi'' = -\frac{2mE_{NR}}{1+f_x^2}\psi = -k_x^2\psi \tag{42}$$

Where in typically Schrödinger mechanics fashion we assume solutions of the form

$$\psi(x) = A\sin(k_x x) + B\cos(k_x x) \tag{43}$$

Applying our boundary conditions of $\psi(0) = 0$, we find that $B = 0$. For the boundary $\psi(L) = 0$, we want to avoid trivial solutions, so we find that for $\psi(L) = 0 = A\sin(k_x L)$ and $A \neq 0$, k_x must be such that

$$k_x = \frac{n\pi}{L} \tag{44}$$

comparing this to what we have in (42), we find

$$k_x = \frac{n\pi}{L} = \sqrt{\frac{2mE_{NR}}{1+f_x^2}} \tag{45}$$

And solving for E_{NR} nets us

$$E_{NR} = \frac{n^2\pi^2}{2mL^2} + f_x^2\frac{n^2\pi^2}{2mL^2} \tag{46}$$

or, written in terms of the standard Schrödinger-Pauli particle in a box energies, $E_{S.P.} = \frac{n^2\pi^2}{2mL^2}$, we get

$$E_{NR} = E_{S.P.} + f_x^2 E_{S.P.} \tag{47}$$

Which corresponds to a *measurable* shift of order f_x^2 to all energies. The reason that this shift is measurable is because it is non-constant: it is dependent on n, so it will not just be some background shift applied uniformly to each energy eigenvalue.

The next case we will analyze is $f_0 \neq 0$ & $f_x = 0$. This results in (29) (while inside the box) becoming:

$$-\frac{1}{2m}\psi'' + \frac{f_0^2 m^2}{2m}\psi = E_{NR}\psi \tag{48}$$

and rearranged

$$\psi'' = -(2mE_{NR} - f_0^2 m^2)\psi \tag{49}$$

To avoid being exhaustive, following the typical particle in a box quantized energies procedure, we find

$$\psi(x) = A\sin(k_x x) \tag{50}$$

with

$$k_x = \frac{n\pi}{L} = \sqrt{2mE_{NR} - f_0^2 m^2} \tag{51}$$

and thus

$$E_{NR} = \frac{n^2\pi^2}{2mL^2} + \frac{f_0^2 m}{2} \tag{52}$$

Which is constant and undetectable shift to the usual energies, as it has no dependence on n.

2.3.3 Particle in $\hat{z}$ Magnetic Field

Another quality of the Schrödinger-Pauli Equation is its use in deriving electron precession in the presence of a magnetic field, a process done often in early quantum mechanics studies as well as observed in undergraduate laboratories.

In a procedure done by Greiner in [5] (not shown here), with $\phi = 0$, $\vec{A} \neq 0$, and $\vec{B} \neq 0$, (29) becomes

$$\left[\frac{\hat{p}^2}{2m} - \frac{q(\hat{L} + 2\hat{S}) \cdot \vec{B}}{2m} + \frac{(f_0 m - \vec{f} \cdot \hat{p})^2}{2m}\right]\psi = E_{NR}\psi \tag{53}$$

Choosing the magnetic field $\vec{B} = B_0\hat{z}$, (53) becomes

$$\left[\frac{\hat{p}^2}{2m} - \frac{qB_0(\hat{L}_z + 2\hat{S}_z)}{2m} + \frac{(f_0 m - \vec{f} \cdot \hat{p})^2}{2m}\right]\psi = E_{NR}\psi \tag{54}$$

And finally making the simplification of a stationary particle, $\hat{p}^2\psi = \hat{p}\psi = \hat{L}_z\psi = 0$, we get the main equation for this problem:

$$\left[-\frac{2qB_0\hat{S}_z}{2m} + \frac{f_0^2 m^2}{2m}\right]\psi = E_{NR}\psi \tag{55}$$

Here, we will convert to a matrix equation with $2\hat{S}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$ Where ψ_1 & ψ_2 are the spin-up and spin-down states respectively. (55) now becomes

$$\left[-\frac{qB_0}{2m}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{f_0^2 m^2}{2m}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} E_{NR_1} & 0 \\ 0 & E_{NR_2} \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{56}$$

Which results in the decoupled equations

$$\begin{cases} \left[-\frac{qB_0}{2m} + \frac{f_0^2 m^2}{2m}\right]\psi_1 = E_{NR_1}\psi_1 \\ \left[\frac{qB_0}{2m} + \frac{f_0^2 m^2}{2m}\right]\psi_2 = E_{NR_2}\psi_2 \end{cases} \tag{57}$$

And clearly

$$\begin{cases} E_{NR_1} = -\frac{qB_0}{2m} + \frac{f_0^2 m^2}{2m} \\ E_{NR_2} = \frac{qB_0}{2m} + \frac{f_0^2 m^2}{2m} \end{cases} \tag{58}$$

Which corresponds to a constant, positive energy shift to both the lower energy spin-up configuration and higher energy spin-down configuration. However, we see that this shift is not measurable, as it is the same shift applied to each energy state.

2.4 Concluding Commentary on f^μ

Pat yourself on the back, you have officially immersed yourself in the world of CPT violating physics. As you will soon find, the joy that you have just experienced will only grow when we analyze CPT-violating matrix and tensor terms.

What can we say about these terms? What effect did they have in the three toy models we analyzed? Starting with the case of the free particle, we should recall the connections between Lorentz and CPT symmetry. Imagine we extend it to 3+1 dimensions, and maintain that $f_0 = f_y = f_z = 0$ and $f_x \neq 0$. Then, it is clear that the y and z directions have unaltered wavenumbers from the standard free particle Schrödinger equation, but x motion is being worked against. This is a direction in space that experiences physics differently: it is inherently Lorentz (and by extension CPT) violating. This is a specific case that was constructed by making the specific choice of $f_0 = f_y = f_z = 0$ and $f_x \neq 0$, but it still shows the effects of CPT violating terms at lower energies.

Our particle in a 1-D box investigation was where we found our most interesting results thus far, where we found a detectable shift in energies in the case of a nonzero f_x. We expect that the size of this correction term is zero, after all experimentally *so far* we have very strong evidence to suggest that CPT symmetry is respected, so we expect all CPT-violating terms in (1) to be zero. So while we expect it to be of zero size, this term does produce something detectable should an experiment be designed to look for it.

The case of a particle in a magnetic field did not turn up any detectable changes to energy eigenvalues, but it did serve as a good exercise in handling the CPT-violating term. As you will see, when doing elementary quantum mechanics that include CPT-violating terms, a lot of times you will not find any measurable change, however, in the case of the free particle, while the change was not measurable it did show what Lorentz symmetry breaking could manifest as on a lower energy scale.

3 A Lagrangian with $c^{\mu\nu}$

3.1 A Dirac-like Equation with $c^{\mu\nu}$

The next term we shall tackle is a bit more complicated than our familiar friend f^{μ}. $c^{\mu\nu}$ is a matrix term, which as mentioned before it traceless. The lagrangian we shall start from is

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}(c^{\mu\nu}\gamma_{\nu} + \gamma^{\mu})\partial_{\mu}\Psi - \frac{i}{2}\partial_{\mu}\overline{\Psi}(c^{\mu\nu}\gamma_{\nu} + \gamma^{\mu})\Psi - m\overline{\Psi}\Psi \tag{59}$$

We apply our usual Euler-Lagrange relations to arrive at our Dirac-like equation,

$$i(c^{\mu\nu}\gamma_{\nu} + \gamma^{\mu})\partial_{\mu}\Psi - m\Psi = 0 \tag{60}$$

3.2 A Schrödinger-Pauli-like equation for Particular $c^{\mu\nu}$ CPT Violating Dirac Equation

As mentioned earlier, the addition of a matrix term proves to be quite complicated. However, we are given one restriction, which is that it is traceless. In order to carry out some meaningful analysis, we will make a very specific choice for what $c^{\mu\nu}$ is:

$$c^{\mu\nu} = \begin{bmatrix} c^{00} & 0 & 0 & 0 \\ 0 & -\frac{c^{00}}{3} & 0 & 0 \\ 0 & 0 & -\frac{c^{00}}{3} & 0 \\ 0 & 0 & 0 & -\frac{c^{00}}{3} \end{bmatrix} \tag{61}$$

This will satisfy our traceless requirement, as well as coming in handy later on. Now, let's put (60) in matrix form. Because of our choice of $c^{\mu\nu}$, the following is true:

$$c^{\mu\nu}\gamma_{\nu} = (c_0\beta, c_i\beta\vec{\alpha}) \tag{62}$$

Because of our choice of $c^{\mu\nu}$, we were able to break it up this way, where $c_i = -\frac{c^{00}}{3}$, and $c_0 = c^{00}$. When coupling it with the 4-momentum operator we get

$$c^{\mu\nu}\gamma_{\nu}p_{\mu} = c_0\beta E - c_i\beta\vec{\alpha}\cdot\vec{p} = \begin{bmatrix} c_0E & -c_i\vec{\sigma}\cdot\vec{p} \\ c_i\vec{\sigma}\cdot\vec{p} & -c_0E \end{bmatrix} \tag{63}$$

and pairing this with the Dirac matrix results in a full Dirac-like matrix equation,

$$\left[\begin{bmatrix} E-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E-m \end{bmatrix} + \begin{bmatrix} c_0E & -c_i\vec{\sigma}\cdot\vec{p} \\ c_i\vec{\sigma}\cdot\vec{p} & -c_0E \end{bmatrix}\right] \begin{bmatrix} \Psi_A \\ \Psi_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{64}$$

Like we did before, we may expand this to get

$$((c_0+1)E-m)\Psi_A - (c_i+1)(\vec{\sigma}\cdot\vec{p})\Psi_B = 0 \tag{65}$$

and

$$(c_i+1)(\vec{\sigma}\cdot\vec{p})\Psi_A + (-(c_0+1)E-m)\Psi_B = 0 \tag{66}$$

We can rearrange (66) and find

$$\Psi_B = \frac{(c_i+1)(\vec{\sigma}\cdot\vec{p})}{((c_0+1)E+m)}\Psi_A \tag{67}$$

Putting (67) into (65) and rearranging yields

$$((c_0+1)E-m)((c_0+1)E+m)\Psi_A = (c_i+1)^2(\vec{\sigma}\cdot\vec{p})^2\Psi_A \tag{68}$$

As with the f^μ term, we will now apply minimal coupling to the left hand side to find

$$LHS = (c_0(E-q\phi)+E-q\phi-m)(c_0(E-q\phi)+E-q\phi-m)\Psi_A \tag{69}$$

Then, apply $E \to E_{NR}+m$ and (69) becomes

$$LHS = (c_0(E_{NR}+m-q\phi)+E_{NR}+m-q\phi-m)(c_0(E_{NR}+m-q\phi)+E_{NR}+m-q\phi-m)\Psi_A \tag{70}$$

Applying the the non-relativistic of $E_{NR}-q\phi+m \approx m$ to (70) we find

$$LHS = (c_0m+E_{NR}-q\phi)(c_0m+2m)\Psi_A \tag{71}$$

Next, we apply minimal coupling to the right hand side of of (68)

$$RHS = (c_i+1)^2(\vec{\sigma}\cdot(\vec{p}-q\vec{A}))^2\Psi_A \tag{72}$$

From here, the Pauli identity may applied, resulting in

$$RHS = (c_i + 1)^2((\vec{p} - q\vec{A})^2 - q(\vec{\sigma} \cdot \vec{B}))\Psi_A \tag{73}$$

It was the specific choice of $c^{\mu\nu}$ that makes it possible for us to use the Pauli identity here. Now, with (71) and (73) we may re-write (68) in our coupled, non-relativistic form:

$$(c_0 m + E_{NR} - q\phi)(c_0 m + 2m)\Psi_A = (c_i + 1)^2((\vec{p} - q\vec{A})^2 - q(\vec{\sigma} \cdot \vec{B}))\Psi_A \tag{74}$$

A rearrangement nets us

$$\left[(1 + c_i)^2\left(\frac{(\vec{p} - q\vec{A})^2}{(c_0 + 2)m} - \frac{q(\vec{\sigma} \cdot \vec{B})}{(c_0 + 2)m}\right) + q\phi - c_0 m\right]\Psi_A = E_{NR}\Psi_A \tag{75}$$

Substitute in $c_i = -\frac{c_0}{3}$ from earlier

$$\left[(1 - \frac{c_0}{3})^2\left(\frac{(\vec{p} - q\vec{A})^2}{(c_0 + 2)m} - \frac{q(\vec{\sigma} \cdot \vec{B})}{(c_0 + 2)m}\right) + q\phi - c_0 m\right]\psi = E_{NR}\psi \tag{76}$$

This is the Schrödinger-Pauli-like equation for this *very* specific case of $c^{\mu\nu}$. Again, keeping with the Schrödinger-Pauli tradition we have transitioned to a lowercase ψ and are assuming energy eigenstates. Examining it, you can see that there is a *small* scalar correction to the original Schrödinger-Pauli hamiltonian in the form of $c_0 m$. Additionally, we see there is a correction to mass in the denominators of the original Schrödinger-Pauli vector part of the equation. Our analysis of some toy models should reveal what the effects of these corrections are, and in turn the effects of the specific, CPT-violating $c^{\mu\nu}$.

3.3 Toy Examples for Schrödinger-Pauli Equation with Effective $c^{\mu\nu}$ contributions

3.3.1 Free Particle in 1-D

As we did with f^μ, for the free particle case we shall assume $\phi = \vec{A} = \vec{B} = 0$. The equation which follows from this is

$$\left[(1 - \frac{c_0}{3})^2\left(\frac{\hat{p}^2}{(2 + c_0)m}\right) - c_0 m\right]\psi = E_{NR}\psi \tag{77}$$

Some rearrangement and use of definitions for $\hat{p}$ gives

$$\psi'' = -\frac{(2 + c_0)m(E_{NR} + c_0 m)}{(1 - \frac{c_0}{3})^2}\psi \tag{78}$$

which we know has solutions of the same form as in (37), with a wavenumber given by

$$k_x = \frac{1}{1 - \frac{c_0}{3}} \sqrt{2(m + \frac{c_0}{2})E_{NR} + c_0 m^2 (2 + c_0)} \tag{79}$$

When compared to the wavenumber of the unmodified Schrödinger-Pauli equation, $k_x = \sqrt{2mE_{NR}}$, we see that there is an overall undetectable scaling applied. Additionally, there is a non-measurable effective mass correction, and a non-measurable positive shift. while in the free particle case, the effects of $c^{\mu\nu}$ are not detectable, they do shed some light on the added complexity at the non-relativistic limit that comes with the inclusion CPT-violating terms.

3.3.2 Particle in a 1-D Box

A particle in a box, while a very idealized model, has a vast amount of beauty to it. It can be done in all three spatial dimensions, with time dependent hamiltonians, degenerate states, and many more variations. It is the Swiss Army Knife of elementary quantum mechanics, and it is very unfortunate that is has earned the reputation among students of being a monotonous procedure of defining a hamiltonian and solving the Schrödinger equation. Here, we will examine (76) through that lens. We start with setting $\vec{A} = \vec{B} = 0$ and

$$q\phi = V(x) = \begin{cases} 0, & 0 < x < L \\ \infty, & \text{otherwise} \end{cases} \tag{80}$$

this gives us

$$\left[(1 - \frac{c_0}{3})^2 \left(\frac{\hat{p}^2}{(2 + c_0)m}\right) + V(x) - c_0 m\right]\psi = E_{NR}\psi \tag{81}$$

which we will then rearrange to

$$-\frac{(1 - \frac{c_0}{3})^2}{(2 + c_0)m}\psi'' + V\psi - c_0 m\psi = E_{NR}\psi \tag{82}$$

In typical particle in a box fashion, to void infinite energies, we assume that outside the box $\psi = 0$. That leaves us with inside the box, where (82) takes on the form

$$\psi'' = -\frac{(2 + c_0)m(E_{NR} + c_0 m)}{(1 - \frac{c_0}{3})^2}\psi = -k_x^2\psi \tag{83}$$

Here, we can see that like in the case of f^μ, we get energy eigenfunctions of the form

$$\psi(x) = A\sin(k_x x) \tag{84}$$

which follows from the standard procedure. As in section 2, because of our boundary conditions, we have

$\frac{n\pi}{L} = k_x$, which results in

$$\frac{n\pi}{L} = \sqrt{\frac{(2m + c_0 m)(E_{NR} + c_0 m)}{(1 - \frac{c_0}{3})^2}} \tag{85}$$

which becomes

$$\frac{n^2\pi^2}{L^2}(1 - \frac{c_0}{3})^2 = (2m + c_0 m)(E_{NR} + c_0 m) \tag{86}$$

From (86), we can determine the energy eigenvalues of this to be

$$E_{NR} = \frac{n^2\pi^2}{(2 + c_0)mL^2} + (\frac{c_0^2}{9} - \frac{2c_0}{3})\frac{n^2\pi^2}{(2 + c_0)mL^2} \tag{87}$$

Compare this to the standard 1-D particle in a box energy eigenvalues of $E_{NR} = \frac{n^2\pi^2}{2mL^2}$: in this new case, we have an undetectable effective mass correction, but a *detectable* shift in energies that scales on the order of c_0 (since $c_0 > c_0^2$ since c_0 is very small)! One should notice a pattern: when CPT violating operators, such as f^μ & $c^{\mu\nu}$, are coupled to the momentum in the lagrangian, when the wavefunctions are subjected to scalar potentials we see measurable shifts in energy eigenvalues.

3.3.3 Particle in $\hat{z}$ Magnetic Field

We conclude our analysis of $c^{\mu\nu}$ with a particle in a uniform $\hat{z}$ magnetic field. We begin by assuming that $\vec{A} = \phi = 0$ & $\vec{B} = B_0\hat{z}$. We will also assume a stationary particle. This all results in

$$\left[\frac{(1 - \frac{c_0}{3})^2}{(c_0 + 2)m}(-q(\vec{\sigma} \cdot \vec{B})) - c_0 m\right]\psi = E_{NR}\psi \tag{88}$$

which we can rearrange using the procedure from [5] to get

$$\left[\frac{(1 - \frac{c_0}{3})^2}{(c_0 + 2)m}(-q(\hat{L} + 2\hat{S}) \cdot \vec{B}) - c_0 m\right]\psi = E_{NR}\psi \tag{89}$$

Since our particle is stationary, $\hat{L}\psi = 0$, and since $\vec{B} = B_0\hat{z}$ we get

$$\left[\frac{(1 - \frac{c_0}{3})^2}{(c_0 + 2)m}(-2qB_0\hat{S}_z) - c_0 m\right]\psi = E_{NR}\psi \tag{90}$$

We will use $2\hat{S}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$ as we did in section 2. In matrix form, (90) becomes

$$\left[-qB_0\frac{(1 - \frac{c_0}{3})^2}{(c_0 + 2)m}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} c_0 m & 0 \\ 0 & c_0 m \end{bmatrix}\right]\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} E_{NR_1} & 0 \\ 0 & E_{NR_2} \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{91}$$

Performing this matrix multiplication gives us

$$\begin{cases} \left[-qB_0\frac{(1-\frac{c_0}{3})^2}{(c_0+2)m} - c_0 m\right]\psi_1 = E_{NR_1}\psi_1 \\ \left[qB_0\frac{(1-\frac{c_0}{3})^2}{(c_0+2)m} - c_0 m\right]\psi_2 = E_{NR_2}\psi_2 \end{cases} \tag{92}$$

and we see that

$$\begin{cases} E_{NR_1} = \left[-qB_0\frac{(1-\frac{c_0}{3})^2}{(c_0+2)m} - c_0 m\right] \\ E_{NR_2} = \left[qB_0\frac{(1-\frac{c_0}{3})^2}{(c_0+2)m} - c_0 m\right] \end{cases} \tag{93}$$

These are non detectable shifts and scalings applied to the energy eigenvalues for the magentic field system.

3.4 Concluding Commentary on $c^{\mu\nu}$

In order to get a Schrödinger-Pauli-like equation, which is how we do all analysis of these terms in this paper, we had to make a very large simplification, which was making $c^{\mu\nu}$ diagonal and maintaining its required traceless-ness. This was done so that the Pauli identity could be used successfully, which is essential for getting a Schrödinger-Pauli-like equation. The presence of off-diagonal terms, while interesting, is not within the scope of this project because of our desire to have Schrödinger-Pauli-like equations for analysis.

We also are now starting to see patterns occuring for particle in a box systems with CPT-violating operators coupled to the momentum in the lagrangian. It should be interesting to see how this pans out when we analyze our most complicated term in the fourth and final section.

4 A Lagrangian with $g^{\lambda\nu\mu}$

4.1 A Dirac-like Equation with $g^{\lambda\nu\mu}$

We now turn our attention to the most complicated term in the the full SME QED lagrangian. We begin with the lagrangian of interest:

$$\mathcal{L} = \frac{i}{2}\overline{\psi}(\gamma^\mu + \frac{1}{2}g^{\lambda\nu\mu}\sigma_{\lambda\nu})\partial_\mu\psi - \frac{i}{2}\partial_\mu\overline{\psi}(\gamma^\mu + \frac{1}{2}g^{\lambda\nu\mu}\sigma_{\lambda\nu})\psi - m\overline{\psi}\psi \tag{94}$$

We then apply Euler-Lagrange with respect to $\overline{\psi}$ which will yield

$$i(\gamma^\mu + \frac{1}{2}g^{\lambda\nu\mu}\sigma_{\lambda\nu})\partial_\mu\psi - m\psi = 0 \tag{95}$$

However, we cannot simply begin our analysis at this point the way we have with the earlier terms. The previous work is built on a more simply defined procedure for how the CPT violating terms will couple to ψ, for example f^μ couples to the wavefunction through γ_5, but one quickly realizes in this case that the coupling through $\sigma_{\lambda\nu}$ is not quite as simple. It will prove useful to utilize the contraction of indices and to define a new term, which we will define in the following way:

$$G^\mu \equiv g^{\lambda\nu\mu}\sigma_{\lambda\nu} \tag{96}$$

We should also recall the following definition:

$$\sigma_{\lambda\nu} = \frac{i}{2}(\gamma^\lambda\gamma^\nu - \gamma^\nu\gamma^\lambda) \tag{97}$$

Note that $\sigma_{\lambda\nu}$ is anti-symmetric. Additionally we are given that $g^{\lambda\nu\mu}$ is anti-symmetric in the first two indices [3]. Using these facts, one can (arduously, see appendix) carry out the computations and arrive at the following:

$$G^\mu = 2\begin{bmatrix} g^{12\mu} & ig^{13\mu}+g^{23\mu} & ig^{03\mu} & ig^{01\mu}+g^{02\mu} \\ -ig^{13\mu}+g^{23\mu} & -g^{12\mu} & ig^{01\mu}-g^{02\mu} & -ig^{03\mu} \\ ig^{03\mu} & ig^{01\mu}+g^{02\mu} & g^{12\mu} & ig^{13\mu}+g^{23\mu} \\ ig^{01\mu}-g^{02\mu} & -ig^{03\mu} & -ig^{13\mu}+g^{23\mu} & -g^{12\mu} \end{bmatrix} \tag{98}$$

We may use the symmetry possessed by this tensor to write it in a way that groups identical 2×2 blocks:

$$G^\mu = 2\begin{bmatrix} \zeta^\mu & \xi^\mu \\ \xi^\mu & \zeta^\mu \end{bmatrix} \tag{99}$$

With this in hand, we write down the equation which shall be used in our analysis:

$$i(\gamma^\mu + \frac{1}{2}G^\mu)\partial_\mu\psi - m\psi = 0 \tag{100}$$

With a term this complicated, there are, of course, things to note. This term is undeniably rich, in fact I would argue one could write a single paper on it alone. The toy models that one can do with it are numerous, as we will discuss once we arrive at that point. For now, let's first notice that ζ^μ is hermitian, while ξ^μ is anti-hermitian. There are a lot of nuances that come with this operator structure. We will solve for a very general Schrödinger-Pauli-like equation, but in the toy model analysis we will pick specific cases of G^μ that avoid complex numbers, as hamiltonians with complex numbers, while of great interest, are slightly beyond the scope of this paper. We will discuss future work with a complex hamiltonian later.

4.2 A Schrödinger-Pauli-like equation for $g^{\lambda\nu\mu}$ CPT Violating Dirac Equation

Using our standard definitions for $i\partial_\mu \equiv p_\mu$, $\gamma^\mu = (\beta, \beta\vec{\alpha})$, $p_\mu = (E, -\vec{p})$, how then should the G^μ make its presence known? 4-vector dot product will tell us that $G^\mu p_\mu = G^0 E - \vec{G}\cdot\vec{p}$, which in matrix form is

$$2\begin{bmatrix}\zeta^\mu & \xi^\mu \\ \xi^\mu & \zeta^\mu\end{bmatrix}\begin{bmatrix}p_\mu & 0 \\ 0 & p_\mu\end{bmatrix} = 2\begin{bmatrix}\zeta^0 E - \vec{\zeta}\cdot\vec{p} & \xi^0 E - \vec{\xi}\cdot\vec{p} \\ \xi^0 E - \vec{\xi}\cdot\vec{p} & \zeta^0 E - \vec{\zeta}\cdot\vec{p}\end{bmatrix} \tag{101}$$

We can insert the Dirac Matrix and (101) into (100) to get the full matrix equation:

$$\left[\begin{bmatrix}E-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E-m\end{bmatrix} + \begin{bmatrix}\zeta^0 E - \vec{\zeta}\cdot\vec{p} & \xi^0 E - \vec{\xi}\cdot\vec{p} \\ \xi^0 E - \vec{\xi}\cdot\vec{p} & \zeta^0 E - \vec{\zeta}\cdot\vec{p}\end{bmatrix}\right]\begin{bmatrix}\Psi_A \\ \Psi_B\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix} \tag{102}$$

Using standard upper-lower spinors. Matrix multiplication results in

$$((E-m) + \zeta^0 E - \vec{\zeta}\cdot\vec{p})\Psi_A + ((-\vec{\sigma}\cdot\vec{p}) + (\xi^0 E - \vec{\xi}\cdot\vec{p}))\Psi_B = 0 \tag{103}$$

and

$$((\vec{\sigma}\cdot\vec{p}) + (\xi^0 E - \vec{\xi}\cdot\vec{p}))\Psi_A + ((-E-m) + \zeta^0 E - \vec{\zeta}\cdot\vec{p})\Psi_B = 0 \tag{104}$$

Rearranging (104) in the way we normally do we find

$$\Psi_B = -\frac{((\vec{\sigma}\cdot\vec{p}) + (\xi^0 E - \vec{\xi}\cdot\vec{p}))}{((-E-m) + \zeta^0 E - \vec{\zeta}\cdot\vec{p})}\Psi_A \tag{105}$$

This will be substituted back into (103) and (non-trivially) rearranged to find

$$(E-m)(E+m)\Psi_A = [(\vec{\sigma}\cdot\vec{p})^2 + (\zeta^0 E - \vec{\zeta}\cdot\vec{p})^2 - 2m(\zeta^0 E - \vec{\zeta}\cdot\vec{p}) - (\xi^0 E - \vec{\xi}\cdot\vec{p})^2]\Psi_A \tag{106}$$

Since we are now experts with these procedures, we can keep these derivations brief, as they require no special choice to be made, unlike in section 3. Like with the previously studied terms, we will apply minimal coupling ($p_\mu \rightarrow p_\mu - qA_\mu$) and then the NR-limit ($E \rightarrow E_{NR} + m$ & $m >> E_{NR} - q\phi$) to each side. Starting with the easier left hand side of (106), just like with the earlier terms , after these treatments will become:

$$LHS = 2m(E_{NR} - q\phi)\Psi_A \tag{107}$$

For the right hand side, after applying both minimal coupling and NR-limit procedures as well as the Pauli identity results in

$$RHS = [(\vec{p}-q\vec{A})^2 - q(\vec{\sigma}\cdot\vec{B}) + (m\zeta^0 - \vec{\zeta}\cdot(\vec{p}-q\vec{A}))^2 - 2m(m\zeta^0 - \vec{\zeta}\cdot(\vec{p}-q\vec{A})) - (m\xi^0 - \vec{\xi}\cdot(\vec{p}-q\vec{A}))^2]\Psi_A \quad (108)$$

Finally, we combine (107) and (108) and rearrange, yielding

$$\left[\frac{(\vec{p}-q\vec{A})^2}{2m} - \frac{q(\vec{\sigma}\cdot\vec{B})}{2m} + q\phi - (m\zeta^0 - \vec{\zeta}\cdot(\vec{p}-q\vec{A})) + \frac{(m\zeta^0 - \vec{\zeta}\cdot(\vec{p}-q\vec{A}))^2 - (m\xi^0 - \vec{\xi}\cdot(\vec{p}-q\vec{A}))^2}{2m}\right]\psi_A = E_{NR}\psi_A \quad (109)$$

Here we see the full complication (and beauty) of CPT violation on display. The inclusion of tensor term $g^{\lambda\nu\mu}$ provides a very rich addition to the ordinary Schrödinger-Pauli equation, and in our analysis of some toy examples we will see exactly what its affects are at the non-relativistic limit.

4.3 Toy Examples for $g^{\lambda\nu\mu}$ Schrödinger-Pauli-like Equation

Before we begin, I must stick true to my promise of giving us a sensible, real-valued hamiltonian for toy model analysis. We will do this by setting $g^{13\mu} = g^{03\mu} = g^{01\mu} = 0$ in (98). This results in

$$\zeta^\mu = \begin{bmatrix} g^{12\mu} & g^{23\mu} \\ g^{23\mu} & -g^{12\mu} \end{bmatrix} \quad (110)$$

and

$$\xi^\mu = \begin{bmatrix} 0 & g^{02\mu} \\ -g^{02\mu} & 0 \end{bmatrix} \quad (111)$$

From here, we may proceed.

4.3.1 Free Particle in 1-D

Our equation for the case of $\vec{A} = \vec{B} = \phi = 0$ in 1-D is given by

$$\left[-\frac{1}{2m}\frac{d^2}{dx^2} - m\zeta^0 - i\zeta^1\frac{d}{dx} + \frac{(m\zeta^0 - i\zeta^1\frac{d}{dx})^2 - (m\xi^0 - i\xi^1\frac{d}{dx})^2}{2m}\right]\psi = E_{NR}\psi \quad (112)$$

Since ζ & ξ are matrices, we ought to write this as a matrix equation, which means we will need to do some matrix multiplication. Firstly, we have that

$$\zeta^\mu\zeta^\nu = \begin{bmatrix} g^{12\mu}g^{12\nu} + g^{23\mu}g^{23\nu} & g^{12\mu}g^{23\mu} - g^{23\mu}g^{12\mu} \\ g^{23\mu}g^{12\nu} - g^{12\mu}g^{23\nu} & g^{23\mu}g^{2}3\nu - g^{12\mu}g^{12\nu} \end{bmatrix} \quad (113)$$

and

$$\xi^\mu \xi^\nu = \begin{bmatrix} -g^{02\mu} g^{02\nu} & 0 \\ 0 & -g^{02\mu} g^{02\nu} \end{bmatrix} \tag{114}$$

These then give us

$$\zeta^0 \zeta^1 + \zeta^1 \zeta^0 = 2 \begin{bmatrix} g^{120} g^{121} + g^{230} g^{231} & 0 \\ 0 & g^{230} g^{231} - g^{120} g^{121} \end{bmatrix} \tag{115}$$

and

$$\xi^0 \xi^1 + \xi^1 \xi^0 = -2 \begin{bmatrix} g^{020} g^{021} & 0 \\ 0 & g^{020} g^{021} \end{bmatrix} \tag{116}$$

The matrix-squared terms follow naturally from (113) and (114). We can roll up our sleeves and put everything together, and we find (112) in full matrix form to be

$$\begin{aligned} &-\frac{1}{2m} \begin{bmatrix} 1 + g^{121^2} + g^{231^2} + g^{021^2} & 0 \\ 0 & 1 + g^{121^2} + g^{231^2} + g^{021^2} \end{bmatrix} \begin{bmatrix} \psi_1'' \\ \psi_2'' \end{bmatrix} \\ &-i \begin{bmatrix} g^{121} - (g^{120} g^{121} + g^{230} g^{231} + g^{020} g^{021}) & g^{231} \\ g^{231} & g^{121} - (-g^{120} g^{121} + g^{230} g^{231} + g^{020} g^{021}) \end{bmatrix} \begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix} \\ &= \begin{bmatrix} E_{NR} + m\left(g^{120} + \frac{g^{120^2} + g^{230^2} + g^{020^2}}{2} \right) & g^{230} \\ g^{230} & E_{NR} + m\left(-g^{120} + \frac{g^{120^2} + g^{230^2} + g^{020^2}}{2} \right) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \end{aligned} \tag{117}$$

Where ψ_1 & ψ_2 denote the spin up and spin down states respectively.

For now, we will not try to solve this; it is far too complicated and any simplification will be overly reductive and not show the important effects provided by this term. This will be solved in detail in a future paper. However, even though we have elected not to solve this, we can still make some very important observations and discuss them, the first of which being that the states are coupled, as some of the matrices have off diagonal terms. Already, we can see that this will result in some *very* interesting dynamics, even if we do not have exact solutions. The second observation we make is that the states will have different dynamics, i.e. they will not behave the same way. This is remarkably interesting to observe in the case of a free particle. How should we interpret the observation that spin up and spin down particles will not evolve the same way in this free system? It is not a large stretch of the imagination to posit that in the case of a free particle $g^{\lambda\nu\mu}$ acts as a quasi-magnetic field, though without further understanding of complete dynamical solutions it is best to not say anything more without substantiated knowledge.

4.3.2 Particle in a 1-D Box

Like with the free particle, for now $g^{\lambda\nu\mu}$'s addition is too complicated to discuss in the context of a particle in a box. The particle in a box equation will look very similar to (117), with the obvious addition of the particle in a box potential from (39) and (80). It will no doubt add some interesting twists to the familiar particle in a box, especially if we allow complex terms in the hamiltonian.

4.3.3 Particle in $\hat{z}$ Magnetic Field

A far more manageable model to study this complicated with term is the stationary particle in a magnetic field. Not only is it more simple because of the stationary particle, but the matrix nature of $g^{\lambda\nu\mu}$'s addition makes it a natural model to consider. However as we shall soon see $g^{\lambda\nu\mu}$'s complexity is still inescapable.

We will again consider the stationary ψ in a uniform, $+\hat{z}$ magnetic field of the form $\vec{B} = B_0\hat{z}$, with $\vec{A} = \phi = 0$. Applying the same procedures as we have in previous magnetic field investigations, from (109) we arrive at

$$\left[-\frac{2qB_0\hat{S}_z}{2m} - m\left(\zeta^0 + \frac{{\zeta^0}^2 - {\xi^0}^2}{2}\right)\right]\psi = E_{NR}\psi \tag{118}$$

Plugging in our matrices and arranging accordingly gives us (118) in vector form to be

$$\left[-\frac{qB_0}{2m}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} - m\begin{bmatrix}g^{120}+k & g^{230}\\ g^{230} & -g^{120}+k\end{bmatrix}\right]\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} = \begin{bmatrix}E_{NR_1} & 0\\ 0 & E_{NR_2}\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \tag{119}$$

Where for convenience we have defined $k \equiv \frac{{g^{120}}^2 + {g^{230}}^2 + {g^{020}}^2}{2}$. When expanded, (119) gives

$$\begin{cases}\left(-\frac{qB_0}{2m} - mg^{120} + k\right)\psi_1 - mg^{230}\psi_2 = E_{NR_1}\psi_1\\ \left(\frac{qB_0}{2m} + mg^{120} + k\right)\psi_2 - mg^{230}\psi_1 = E_{NR_2}\psi_2\end{cases} \tag{120}$$

At my discretion, I will make one further simplification: that $g^{230} = 0$ to decouple the equations. The coupling adds a nice richness to the problem, but it also means that the energy eigenvalues are dependent on one another. What does that mean? I thought about this at length, and I came up with two (in my opinion unsatisfactory) interpretations. The first interpretation is that with the addition of this term there are not the two distinct energy states that we have grown accustomed to with this model. I will discuss ways to tackle this idea more in the conclusion. The second interpretation is that I have made a not-necessarily-true assumption about the structure of the wave functions; perhaps they do not lend themselves to this simple of a treatment with $g^{\lambda\nu\mu}$'s addition. Regardless, setting $g^{230} = 0$ and defining $\kappa \equiv \frac{{g^{120}}^2 + {g^{020}}^2}{2}$ results in

$$\begin{cases} \left(-\frac{qB_0}{2m} - mg^{120} + \kappa\right)\psi_1 = E_{NR_1}\psi_1 \\ \left(\frac{qB_0}{2m} + mg^{120} + \kappa\right)\psi_2 = E_{NR_2}\psi_2 \end{cases} \tag{121}$$

Where clearly

$$\begin{cases} E_{NR_1} = \left(-\frac{qB_0}{2m} - mg^{120} + \kappa\right) \\ E_{NR_2} = \left(\frac{qB_0}{2m} + mg^{120} + \kappa\right) \end{cases} \tag{122}$$

Here, we see something interesting: a *detectable* shift in energies. While κ will act as a uniform positive shift to both energies, the difference in sign of the mg^{120} term means that there is not just a uniform shift, but rather the gap between the energies is larger than without $g^{\lambda\nu\mu}$'s addition. How could this arise? Consider our earlier interpretation of this term acting as a quasi-magnetic field. Looking at this, we see that this could be a reasonable explanation for how this term acts.

4.4 Concluding Commentary on $g^{\lambda\nu\mu}$

This term provided a lot of richness to the problem. Unlike previous terms, further simplification to free particle and particle in a box models will detract from understanding the relevant effects of this term, so we have opted to leave that analysis for another time. However, further work will revolve heavily around the inclusion of complex valued parts of the $g^{\lambda\nu\mu}$ hamiltonian, as those can emulate decay like processes in Schrödinger mechanics.

In the case of the magnetic field model, I mentioned that the coupled energies was an interesting thing worth studying. I believe that we may use the decoupled solutions of (121) and (122), and apply a perturbative procedure to help understand what the couple energies really means, though my authority on the matter is very little, so I shouldn't say too much more about this. However, $g^{\lambda\nu\mu}$'s matrix nature we see when working with these non-relativistic cases does very naturally lend it to being examined with the magnetic field model, and thus far the analytic evidence would suggest that it acts as a quasi-magnetic field of sorts, though there is much more about it to still study.

This term is also greatly interesting because of its connections to neutrinoless double beta decay. CUORE (Cryogenic Underground Observatory for Rare Events) data has been used to assing theoretical upper limits on the size of this term [6]. This term is known to theoretically emmulate Majorana mass/coupling in neutrino systems [6], so by understanding how it acts in general fermion systems in the non-realativistic limit there is hope to understand the connection between CPT violation and neutrinoless double beta decay better as a whole.

5 Conclusion

We have shown several derivations of how to extract non-relativistic equations that contain terms from the CPT violating SME lagrangian, and analyzed those terms through the lens of some useful toy models to better describe the terms' effects. This is very useful for bridging the gap between the complicated topic of CPT violation and relativistic quantum mechanics. In turn, it makes the subject more approachable and can get more students interested in working on these frontier problems. Additionally, we have outlined some more advanced problems to tackle that have spawned from the presence of $g^{\lambda\nu\mu}$, and discussed its relationship with the experimental search for neutrinoless double beta decay by CUORE. The Schrödinger-Pauli-like equations have proven very useful for understanding these terms effects, and will prove useful still in studying the more advanced problems which have been outlined.

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7 References

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8 Appendix

If you were curious about how the term G^μ was derived, this appendix is for you. We start with the claim that $g^{\lambda\nu\mu}$ is anti-symmetric in the first two indices. This means that in general, we may say

$$g^{\lambda\nu\mu} = \begin{bmatrix} 0 & g^{01\mu} & g^{02\mu} & g^{03\mu} \\ -g^{01\mu} & 0 & g^{12\mu} & g^{13\mu} \\ -g^{02\mu} & -g^{12\mu} & 0 & g^{23\mu} \\ -g^{03\mu} & -g^{13\mu} & -g^{23\mu} & 0 \end{bmatrix} \tag{123}$$

We know that $g^{\lambda\nu\mu}$ couples via $\sigma_{\lambda\nu}$, which are defined by the following commutator

$$\sigma_{\lambda\nu} = \frac{i}{2}[\gamma^\lambda, \gamma^\nu] \tag{124}$$

Additionally, these matrices are anti-symmetric, that is $\sigma_{\lambda\nu} = -\sigma_{\nu\lambda}$. These commutators are computed using a computer matrix algebra program, and we find nonzero commutators to be

$$\sigma_{01} = -\sigma_{10} = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \tag{125}$$

$$\sigma_{02} = -\sigma_{20} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{126}$$

$$\sigma_{03} = -\sigma_{30} = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \tag{127}$$

$$\sigma_{12} = -\sigma_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{128}$$

$$\sigma_{13} = -\sigma_{31} = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} \tag{129}$$

$$\sigma_{23} = -\sigma_{32} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{130}$$

Using the summation convention and the facts of anti-symmetry for both $g^{\lambda\nu\mu}$ and $\sigma_{\lambda\nu}$ and zeros on $g^{\lambda\nu\mu}$'s diagonal, we can define G^{μ} to be

$$G^{\mu} = g^{\lambda\nu\mu}\sigma_{\lambda\nu} = 2(g^{01\mu}\sigma_{01} + g^{02\mu}\sigma_{02} + g^{03\mu}\sigma_{03} + g^{12\mu}\sigma_{12} + g^{13\mu}\sigma_{13} + g^{23\mu}\sigma_{23}) \tag{131}$$

We add these matrices up, and the result is

$$G^{\mu} = 2\begin{bmatrix} g^{12\mu} & ig^{13\mu} + g^{23\mu} & ig^{03\mu} & ig^{01\mu} + g^{02\mu} \\ -ig^{13\mu} + g^{23\mu} & -g^{12\mu} & ig^{01\mu} - g^{02\mu} & -ig^{03\mu} \\ ig^{03\mu} & ig^{01\mu} + g^{02\mu} & g^{12\mu} & ig^{13\mu} + g^{23\mu} \\ ig^{01\mu} - g^{02\mu} & -ig^{03\mu} & -ig^{13\mu} + g^{23\mu} & -g^{12\mu} \end{bmatrix} \tag{132}$$