Lorentz Violation in Neutrino Interactions

Pranav Seetharaman

Physics Department, California Polytechnic State Universiy,

San Luis Obispo, California 93407, USA

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Abstract

Both the Standard Model of particle physics and General Relativity require Lorentz symmetry as a fundamental building block. In this paper, we learn about a framework called the Standard Model Extension that allows us to determine how physical phenomenon would change if we deviated from Lorentz invariance in the Standard Model and General Relativity. We use the Standard Model Extension to analyze a specific high-energy, astrophysical neutrino interaction that is only possible if Lorentz symmetry can be broken. The interaction we look at is the decay of a neutrino into an electron-positron pair, which is not possible in conventional physics. The goal of this paper is to determine cutoff energies for where this interaction is possible and use those cutoff energies to better understand Lorentz invariance.

I. INTRODUCTION

After having repeatedly passed experimental tests, the Standard Model and General Relativity are the current working models for physics. These theories, however, are not a complete model for all physical phenomenon. It has been suspected that these theories are low-energy limits of a complete theory of physics [1]. These theories are both based on some shared building blocks, one of which is Lorentz invariance. Before we get into discussing how Lorentz invariance functions, it is useful to describe what a Lorentz transformation is and how it is applied. Two distinct transformations that we will be looking at are observer transformations and particle transformations. Observer transformations occur when the coordinates and reference frames themselves are shifted. Particle transformations occur when we shift just the particles themselves.

A useful example of the difference would be a simple lock and key system. Suppose you tried to put the key in perpendicular to the lock. Obviously this would not fit, and you would not be able to unlock the door. If you performed an observer transformation (a transformation from the perspective of the observer), you would rotate every element of this system, both the lock and the key rotate relative to you. This, however, would not change the fact that the key is still perpendicular to the lock. If, instead, you performed a particle transformation on the key and rotated just the key 90 degrees, you would be able to fit the key into the lock.

Now how does this relate to Lorentz invariance? Lorentz invariance states that an observer transformation will not change the results of a system, and any sensible theory should preserve this. If it did not, then simply shifting your perspective would change the underlying laws of physics. Lorentz invariance also states that particles travelling in a system at different velocities can be related by a particle Lorentz transformation. This second idea is crucial in understanding Lorentz symmetry. Mathematically speaking, equations that are Lorentz invariant to observer Lorentz transformations do not change if a Lorentz transformation is applied to every term in the equation. Equations that are invariant to particle Lorentz transformations, however, do not change when just the particle is transformed using a Lorentz transformation. While this second symmetry is particularly useful and powerful, we still have to question whether or not it is true. Suppose, for example, there exists a universe that is similar to ours in every way except that there are fixed background fields across it. Such a universe would not be invariant under particle Lorentz transformations (but still would be invariant under observer Lorentz transformations). In physics, we have a long standing tradition of questioning fundamental assumptions. In that spirit, Lorentz violations are a field of study that creates the theoretical framework to experimentally verify deviations from Lorentz invariance that may be a part of a unified theory of physics.

Lorentz invariance plays big role in both General Relativity and the Standard Model, so it is useful to study it. Since Lorentz invariance is the one key feature linking the Standard Model and General Relativity, it stands to reason that a more fundamental theory that unites the two would involve Lorentz invariance in some way. In that spirit, studying violations of Lorentz invariance is crucial. If we suppose that the Standard model of particle physics is a low-energy solution of some fundamental theory that includes Lorentz violations, then we can construct a model of physics to test for Lorentz violations. Following that logic, the Standard Model Extension (SME) is the model that we use to test for Lorentz violations [2]. The SME allows us the ability to test how Lorentz-violating phenomenon change the results of our existing and future experiments.

For example, Lorentz violation could imply a new relationship between the energy and momentum of a particle. In the massless case, this might involve changing from $E = |\vec{p}|$ to $E^2 = |\vec{p}|^2 \ + \ \epsilon|\vec{p}|^4$ (in natural units, and epsilon is very small). Changes like this would cause cascading effects that would change how we look at many physical phenomenon. At a higher level, accepting Lorentz violation would cause fundamental changes in the form of the Standard Model of particle physics. The SME shows these changes through breaking the symmetry of the Standard Model and General Relativity. Effectively, the SME treats the Standard Model and General Relativity as a series approximation of a more unified theory of physics that includes Lorentz-violating terms [2]. This is achieved by coupling Lorentz-violating operators with existing field operators. A simple example would be in the QED Lagrangian, which contains the gauge-covariant derivative term D_μ. In the SME, this term would be coupled with Lorentz-violating operators with various components to create a Lorentz-violating theory. The QED Dirac Lagrangian would change from

$$\mathcal{L} = i(D^\nu \bar{\psi}\gamma_\nu\psi - \bar{\psi}\gamma_\nu D^\nu\psi) - \bar{\psi}m\psi \tag{1}$$

to the Lorentz-violating form [1]

$$\mathcal{L} = \frac{-i}{2}(D^\nu \bar{\psi}\Gamma_\nu\psi - \bar{\psi}\Gamma_\nu D^\nu\psi) - \bar{\psi}M\bar{\psi}. \tag{2}$$

This change, while seemingly small, highlights the fundamental difference between the SM and the SME. The terms γ^ν and m are 0th-order series approximations of the Lorentz-violating terms Γ^ν and M. If we expand these terms out, we get

$$\Gamma_\nu = \gamma_\nu + c_{\mu\nu}\gamma^\mu + d_{\mu\nu}\gamma_5\gamma^\mu + \cdots \tag{3}$$

$$M = m + a_\mu\gamma^\mu + b_\mu\gamma_5\gamma^\mu + \cdots \tag{4}$$

Here, the coefficients $a_\mu, b_\mu, c_{\mu\nu}, d_{\mu\nu}$ are the coefficients for Lorentz violation. If they're set to 0, we recover the regular Standard Model and General Relativity. This process is similarly done for QCD, electroweak theory and Gravity [1]. These frameworks together give us the complete SME. In this paper, we will look at electroweak interactions and specifically, neutrino physics.

In recent years high-energy astrophysical neutrinos have been discussed as a feature of great interest [3]. Their nature as light, weakly interacting particles provides ideal probes for distant black holes and star systems. Along with that, astrophysical neutrinos give us the ability to test physics at a level necessary to detect Plank-scale effects. These neutrinos also offer us another interesting avenue to explore; increasingly precise tests of Lorentz invariance. As we discussed earlier, deviations from Lorentz symmetry might be possible at high enough energies. This deviation has been suggested by various unification theories including String theory [4] and Loop Quantum Gravity [3]. The neutrino sector of the SME gives us a framework to discuss these Lorentz-violating phenomenon [2].

Neutrinos are ideal candidates to test Lorentz-violating theories like the SME for many reasons. Alongside low mass and weakly coupled interactions, the decay processes of neutrinos allow us to probe into high-energy, Lorentz-violating phenomenon [3]. Conventional neutrino interactions include various electroweak phenomena. Here we will be discussing Lorentz-violating electroweak phenomena. We highlight a specific neutrino decay that is only possible through the inclusion of Lorentz-violating terms. We look at astrophysical neutrinos decaying into electron-positron pairs through a neutral force carrier. Normally, this interaction is not possible since a particle cannot decay into something that is more massive than itself. However, with the addition of space-time symmetry violating terms, we can allow for these interactions to happen. This effectively "bumps up" the energy depending on the amount of Lorentz violation. Doing this creates new relationships between energy and momentum that apply to Lorentz-violating particles. However, these relationships (due to the nature of the SME) are only applicable at very high energies, hence our necessity for high-energy astrophysical neutrinos. Under these new rules, we can determine what the decay rate (change in energy with respect to time) of a high-energy neutrino undergoing this process is. The goal is to determine cutoff energies for which this interaction is no longer possible. These energies will be dependent on the coefficients for Lorentz violation and will allow us to probe further into Lorentz-violating phenomenon.

II. DEFINING THE INTERACTION

This system is defined by its Feynman diagram in Figure 1, which details how the neutrino decays through a Z^0 boson into an electron-positron pair [5]. This interaction is only possible

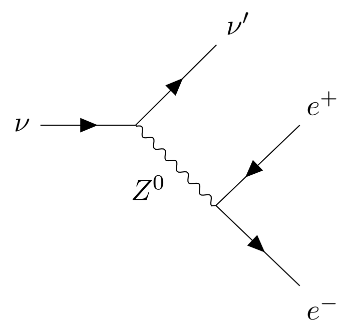

FIG. 1. Feynman diagram detailing a neutrino decaying into an electron-positron pair

with Lorentz violation (as shown in the next section). We make some critical assumptions about the system in order to more easily achieve analytic solutions to the problems we consider. First, we assume that the neutrino mass is approximately 0, or at the very least, much smaller than its energy. Secondly, we assume that the Lorentz violation for the electron or positron is 0, or at least, it is small relative to the energy of the electron or positron.

Now that we have an understanding of how the interaction is defined, let us draw a couple immediate conclusions from this diagram. First, the energy lost from this decay is incredibly large so that tells us that the energy of the incoming neutrino must me much larger than the outgoing neutrino. Second, the last conclusion tells us that this interaction likely only happens once because the outgoing neutrino is not high enough energy to decay like this. With a better understanding of this Lorentz-violating interaction, we can now discuss how to find the cutoff energy for which this interaction happens. The final goal is an analytic expression for the cutoff energy.

III. THE CONVENTIONAL CASE

Before diving into the implications of the Lorentz-violating case, we will discuss the conventional case. Analyzing the system requires an understanding of why the conventional case doesn't allow for this interaction. Consider a particle A that decays into a particle B, following the interaction in the figure below. From the diagram, we can write down the

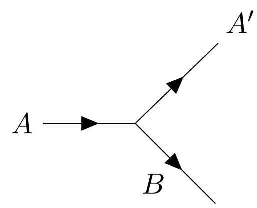

FIG. 2. Model detailing a particle creating a new particle

energy conservation equation in the rest frame of particle A, $m_A = E_{A'} + E_B$. Where m_A is the mass of particle A. Now consider that particle A' must have energy at least equal to it's mass, or substituted in $m_A \geq m_A + E_B$ which implies $0 \geq E_B$. The problem is immediately evident, we cannot have the energy of the decay product be negative. Now that we know that this interaction does not happen in the conventional case, let us figure out what happens in the Lorentz-violating case to allow for this decay to happen.

IV. USEFUL RELATIONS FOR LORENTZ-VIOLATING TERMS

For Lorentz-violating particles, instead of the 4-vector momentum we use the 4-vector, q^μ, where

$$q^\mu = p^\mu + v_L^\mu. \tag{5}$$

This vector is useful for a variety of reasons. The first being that it includes the Lorentz violation term v_L^μ, which conveniently contains all the Lorentz violation inside it. The term v_L^μ is a combination of various Lorentz violating operators and takes the form

$$p_\mu v_L^\mu = \sum_{djm} |\vec{p}|^{d-2} Y_{jm}(\hat{p})((c_L^d)_{jm} - (a_L^d)_{jm}), \tag{6}$$

where Y_{jm} represents the spherical harmonics, and c_L and a_L are coefficients for Lorentz violation. It is useful to define a term to bridge Eq. (6) and the rest of the work in this paper, namely, let us define

$$V = \frac{-p_\mu v_L^\mu}{|\vec{p}|}. \tag{7}$$

Secondly, a useful fact about q^μ is that (for a massless particle) it obeys the relation $q^\mu q_\mu = 0$, effectively replacing the relation $p^\mu p_\mu = m^2$. From these relations, we can derive further useful relations that will be used throughout this paper. The first of these relationships is the new energy-momentum relationship

$$(p^0)^2 = |\vec{p}|^2 + 2|\vec{p}|V - v_L^\mu (v_L)_\mu. \tag{8}$$

Then, applying the assumption that the Lorentz violation of the particle is small compared to the momentum and energy of the particle, we get

$$p^0 = |\vec{p}| + V, \tag{9}$$

which we defined earlier in Eq. (7). These three relations allow us to relate a massless particle's energy to its momentum. We can recover the conventional case by simply setting all of the Lorentz-violating terms to 0. Using these equations, we can define a couple more useful relations.

$$p^\mu p_\mu = 2|\vec{p}|V \tag{10}$$

$$p^\mu p'_\mu = |\vec{p}||\vec{p'}|(1 - \cos\theta) + |\vec{p}|V' + |\vec{p'}|V \tag{11}$$

Where θ is the angle between the three vector momenta of p^μ and p'^μ and V and V' are the Lorentz-violating terms for p^μ and p'^μ respectively. Now that we have these relations in hand, we can apply them to our system and simplify the interaction in Figure 1 that we are trying to describe.

V. ANALYZING THE PHASE SPACE

As stated before, the differences between the conventional and Lorentz-violating case allow for this decay to happen. More specifically, the Lorentz-violating terms open up the phase space of this interaction to enable this decay (represented by a change in the bounds of the integral). To determine this we need two bounds. An upper bound and a lower bound for the outgoing neutrino's momentum. (The reason that these values are useful will be evident when discussing the integral in Section VI.) Let us start by defining κ^μ as the 4-momentum for the virtual Z^0 particle in the Feynman diagram. The lower bound is defined by the minimum energy carried by the Z^0 particle to the electron-positron pair. In this case, that would be just enough energy to produce the particles. From the Feynman diagram, we know that $\kappa^\mu = p^\mu_{e-} + p^\mu_{e+}$, where p^μ_{e-} is the 4-momentum of the electron and p^μ_{e+} is the 4-momentum of the positron. We can square this equation to get the lower bound of

$$\kappa^2 \geq 4m_e^2. \tag{12}$$

We can do something similar, except with $\kappa^\mu = p^\mu_\nu - p^\mu_{\nu'}$ (where p^μ_ν is the 4-momentum of the incoming neutrino and $p^\mu_{\nu'}$ is the 4-momentum of the outgoing neutrino) to achieve

$$-|\vec{p}_\nu||\vec{p}_{\nu'}|(1-\cos\theta) + (|\vec{p}_\nu| - |\vec{p}_{\nu'}|)(V - V') = \frac{\kappa^2}{2} \tag{13}$$

This second equation gives us the upper bound for the phase space. These last two equations give us valuable insight about the phase space that we will use later to discuss appropriate limits for the system we will be working on.

VI. DEFINING THE INTEGRAL

Drawing from the Feynman diagram in Figure 1, the rate of change of energy with respect to distance is given by [5].

$$\frac{dE}{dx} = -\int \mathrm{d}^3k\ \mathrm{d}^3k'\ \mathrm{d}^3p'\ \frac{2G_f^2 M_z^4(1-4s^2+8s^4)(q\cdot k)}{(2\pi)^5\ [(k+k')^2 - M_z^2]^2} \times \frac{(q'\cdot k')(k^0+k'^0)\ \delta^4(p-p'-k-k')}{q^0 q'^0 k^0 k'^0} \tag{14}$$

The variables in equation (14) are defined in Table VI.

Consolidating all the constants into one constant C causes the integral to simplify into

$$-\mathrm{C}\int \mathrm{d}^3k\ \mathrm{d}^3k'\ \mathrm{d}^3p'\ \frac{(q\cdot k)(q'\cdot k')}{[(k+k')^2 - M_z^2]^2}\frac{(k^0+k'^0)\delta^4(p-p'-k-k')}{q^0 q'^0 k^0 k'^0}. \tag{15}$$

With the constants out of the way, it is easier to show the form of the integral. The delta function on the top exists to enforce energy-momentum conservation inside the integral. Note that the $[(k+k')^2 - M_z^2]^2$ term in the bottom arises from the Z propagator and produces

Variable	Definition
M_z	Mass of the Z-boson
q	Lorentz-violating 4-vector for incoming neutrino
q'	Lorentz-violating 4-vector for outgoing neutrino
p	4-momentum for incoming neutrino
p'	4-momentum for outgoing neutrino
k	4-momentum for outgoing electron
k'	4-momentum for outgoing positron
s	The sin of the Weinberg angle
G_f	The Fermi constant

TABLE I. Definitions for the variables in Eq. (14)

a resonance when $(k+k')^2$ is close to M_z^2. From here, we can separate the integral into two parts, one part with dependence on k and k' (the 4 momenta of the outgoing electron and positron respectively) and another part that depends on $\kappa = p - p'$. The resulting two integrals are:

$$-\mathrm{C}\int \mathrm{d}^3p' \; \frac{q^\mu q'^\nu \kappa^0}{q^0 q'^0[\kappa^2 - M_z^2]^2} \times \int \mathrm{d}^3k \; \mathrm{d}^3k' \; \frac{k_\mu k'_\nu}{k_0 k'_0} \; \delta^4(\kappa - k - k') \tag{16}$$

Some key features of the integrals are the Dirac delta functions in the electron integral in Eq. (16) and the resonance term in the neutrino integral of Eq. (16). Along with that, it is important to note that all the Lorentz violation is buried in the neutrino integral in Eq. (16) meaning that the electron integral in Eq. (16) is Lorentz invariant.

VII. SOLVING THE ELECTRON INTEGRALS

Now we will take a look at the electron integrals, shown below.

$$\int \mathrm{d}^3k \; \mathrm{d}^3k' \; \frac{k_\mu k'_\nu}{k_0 k'_0} \; \delta^4(\kappa - k - k') \tag{17}$$

We can make a change in coordinates to the center of mass frame of the electron-positron pair. This system has a net momentum of 0, meaning that $\vec{k} = -\vec{k'}$ and $\vec{\kappa} = 0$. This tells us that the resulting delta functions are

$$\delta^4(\kappa - k - k') = \delta^3(\vec{k'} - (-\vec{k})) \; \delta(\kappa^0 - k_0 - k'_0). \tag{18}$$

From here, we can show that $k'^0 = \sqrt{|\vec{k'}| + m^2} = \sqrt{|\vec{k}| + m^2} = k^0$. This means that Eq. (18) transforms into

$$\delta^3(\vec{k'} - (-\vec{k}))\ \delta(\kappa^0 - k_0 - k'_0) = \delta^3(\vec{k'} - (-\vec{k}))\ \delta(\kappa^0 - 2k_0). \tag{19}$$

From there, we can apply the following property of the Dirac delta function

$$\delta(g(x)) = \frac{\delta(x - x_0)}{|g'(x_0)|}, \tag{20}$$

and use it to change

$$\delta(\kappa^0 - 2k_0) = \delta(\kappa^0 - 2\sqrt{|\vec{k}|^2 + m^2}) = \delta(g(|\vec{k}|)). \tag{21}$$

From here we can transform the final delta function into

$$\frac{\delta\left(|\vec{k}| - \sqrt{(\kappa^0/2)^2 - m^2}\right)}{2|\vec{k}|/k_0} \tag{22}$$

Lastly, we can substitute these back into the original integral in Eq. (17) to achieve

$$\int \mathrm{d}^3k\ \mathrm{d}^3k'\ \frac{k_\mu k'_\nu}{k_0 k'_0}\ \frac{\delta\left(|\vec{k}| - \sqrt{(\kappa^0/2)^2 - m^2}\right)}{2|\vec{k}|/k_0}\ \delta^3(\vec{k'} - (-\vec{k})). \tag{23}$$

Applying all these delta functions gives us

$$\int \mathrm{d}^3k\ \frac{k_\mu(\kappa_\nu - k_\nu)}{k'_0}\ \frac{\delta\left(|\vec{k}| - \sqrt{(\kappa^0/2)^2 - m^2}\right)}{2|\vec{k}|}. \tag{24}$$

Changing into spherical coordinates, applying the last delta function and applying the property $k^0 = k'^0$ gives us

$$\frac{\sqrt{\left(\frac{\kappa^0}{2}\right)^2 - m^2}}{\kappa^0} \iint \mathrm{d}x\mathrm{d}\phi\ k_\mu(\kappa_\nu - k_\nu). \tag{25}$$

We next set the integral above equal to a combination of the metric and $\kappa_\mu\kappa_\nu$. This allows us to turn the solution back into a Lorentz invariant form and not just a form that works for the center of mass reference frame:

$$\frac{\sqrt{\left(\frac{\kappa^0}{2}\right)^2 - m^2}}{\kappa^0} \iint \mathrm{d}x\mathrm{d}\phi\ k_\mu(\kappa_\nu - k_\nu) = \alpha\ \kappa_\mu\kappa_\nu + \beta\ \eta_{\mu\nu}. \tag{26}$$

Now we just need to solve for the coefficients α and β. To do this, we are choosing two situations, the first being $\mu\nu = 0, 0$ and the second being $\mu\nu = 3, 3$. These solutions yield

$$\frac{2\pi}{3\kappa_0}\left(\left(\frac{\kappa_0}{2}\right)^2 - m^2\right)^{1/2}\left[\left(\kappa_0^2 + 2m^2\right)\frac{\kappa_\mu\kappa_\nu}{\kappa_{0^2}} + \left(\frac{\kappa_0^2}{2} - 2m^2\right)\eta_{\mu\nu}\right] \tag{27}$$

Now shifting it out of the center of mass reference frame gives us the following equation:

$$\frac{2\pi}{3\sqrt{\kappa^2}}\left(\frac{\kappa^2}{4}-m^2\right)^{1/2}\left[\left(\kappa^2+2m^2\right)\frac{\kappa_\mu\kappa_\nu}{\kappa^2}+\left(\frac{\kappa^2}{2}-2m^2\right)\eta_{\mu\nu}\right] \tag{28}$$

If we substitute this back into the original integral we get

$$-\frac{2\pi \mathrm{C}}{3}\int \mathrm{d}^3p'\,\frac{q^\mu q'^\nu \kappa_0\left(\frac{1}{4}-\frac{m^2}{\kappa^2}\right)^{1/2}\left[\left(\kappa^2+2m^2\right)\frac{\kappa_\mu\kappa_\nu}{\kappa^2}+\left(\frac{\kappa^2}{2}-2m^2\right)\eta_{\mu\nu}\right]}{q^0 q'^0[\kappa^2-M_z^2]^2}. \tag{29}$$

This gives us an integral that is much simpler than Eq. (14).

VIII. SUMMARY

In summary, we started by describing Lorentz violating systems in general and gives examples of how Lorentz-violating terms could be included into the Standard Model and General Relativity to give the Standard Model Extension. From there, we looked into a specific interaction in the neutrino sector of the Standard Model Extension which follows a neutrino decay into an electron-positron pair. We analyzed the Feynman diagram for the interaction given in Figure 1 and used it to write down the integral in Eq. (14). From there, we used various transformations and techniques to simplify 6 of the integrals (out of 9) into the equation shown in Eq. (29). Future goals of this project include finding a complete analytic solution for this integral and solving it for the charge carrying cases (replacing the Z^0-boson with the $W^{\pm}$-bosons).

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