

Equilibrium Without Statics

The Modern Müller-Breslau Method

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Introduction: Purpose of Project

This senior project is an investigation of the Modern Müller-Breslau Method. The purpose of this senior project was to see how quickly and easily a graduating senior in a structural engineering program could acquaint themselves to the method, as well as create new problems that demonstrate its power in equilibrium without the use of any statics. The focus of this project and paper is determinate structures including beams, axial members, frames, three-hinged arches, and finally three-dimensional trusses. Examples of each type of problem will be shown with explanations along with a reflection on the experience from a student's perspective on learning and applying this method.

Precedent: Classic Müller-Breslau Method

The classic Müller-Breslau method was invented in the 19th century. Commonly known as the influence line, this method of equilibrium without statics is taught in many civil engineering programs today. It is a simple method that can be used to find external reactions or internal forces in a structure subject to gravity loads. This is accomplished by imposing a unit displacement or rotation at the location of the unknown and in the direction of the unknown, while all other boundary conditions must be respected in terms of displacement and/or rotation. The straight line created by the unit displacement and connection to other boundary conditions is the influence line.

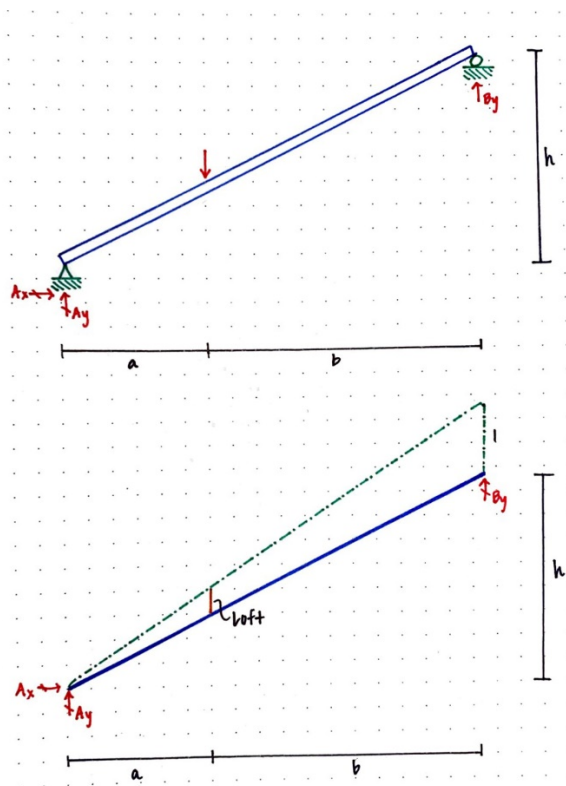


Figure 1

Figure 1 displays a simple statics problem, a simply supported slanted beam with one gravity load, being solved using the classic method. The vertical reaction at the right roller is being sought.

In order to construct the influence line, the right side of the beam is shifted a unit displacement of one upwards, in the direction of the sought reaction. In the classic method it is important to note that the beam does not stay the same length, but rather stretches in order to keep the boundary condition along the same vertical axis. Therefore, the influence line created is longer than the original beam.

Modern Müller-Breslau Method

The Modern Müller-Breslau method was created in the 21st century by Edmond P. Saliklis, PhD, professor in the Architectural Engineering Department of California Polytechnic State University, San Luis Obispo, and appeared briefly in his book, *Structures: A Studio Approach*, published in 2020. This method, though still utilizing the idea of internal and external work for equilibrium, differs from the classic method in significant ways. The most notable difference is that the unit displacement in the classic method becomes a variable displacement in the modern method.

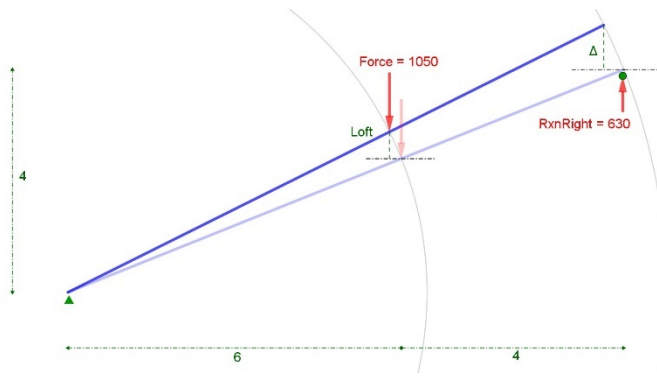


Figure 2

Figure 2 shows the same kind of problem as Figure 1, a simply supported slanted beam with a single gravity load. However, the influence line seen in Figure 1 now becomes the perturbed shape. When the right reaction at the roller is being sought, a displacement is still imposed at the support in the direction of the reaction. However, instead of the beam stretching to create the influence line in the classic method, the actual shape and

length of the beam rotates according to the variable imposed displacement. As shown in the figure, everything now moves along a circle including the applied gravity load, which maintains the original shape of the beam rather than just moving all points of interest vertically. Throughout this document, the loads are assumed to be in their original orientation regardless of the perturbation, instead of remaining at the same angle with the beam throughout. This is shown in Figure 2 where the original load is directly vertical and remains vertical throughout the perturbation.

The capabilities of this new method are powerful because it can be used for the application of both lateral and gravity loads whereas the classic method can only be used for structures with gravity loads only. Whereas the classic method yielded exact answers for a unit displacement or rotation, the new method approaches theoretical values for smaller and smaller perturbations.

This fundamental work equation is the one used for all investigations using the method.

$$Unknown \cdot \Delta + \sum Force_i \cdot Loft_i = 0 \quad \dots\dots\dots (eqn. 1)$$

In equation 1, the *Unknown* is the external reaction, or internal shear or moment of interest. The varying imposed perturbation or rotation Δ is the one applied at the location and in the direction of the *Unknown*. This delta could be a displacement if looking for an external reaction or internal shear, or it could be a rotation if looking for an external moment reaction or internal bending moment. The *Forces* are the loads applied to the structure and the *Loft* is the distance travelled by each load in the direction that load is applied, caused by the imposed perturbation Δ . If multiple forces are applied to the structure or their location varies or changes, a new perturbation

does not have to be created. A perturbation can be used for any loading pattern because it is independent of applied loads. As more forces are applied, they can simply be added into the equation without changing the other terms.

The procedure to apply the Modern Müller-Breslau method is relatively similar for all types of problems. First, remove the capability of the structure to carry the *Unknown*, which could be a reaction or internal shear or moment. At the location of the *Unknown*, apply a perturbation Δ in the direction of the unknown. This could be a displacement if seeking a reaction or internal shear, or a rotation if seeking a moment reaction or internal bending moment. This perturbation can start out large and then reduce to nearly zero as the answer asymptotically approaches the theoretical value. All other boundary conditions must be enforced. The imposed perturbation will create lofts under the original loads that must be measured. These lofts can be closely approximated if using a sketching method on graph paper or can be found exactly if using sines and cosines. Though these are the basic steps, there are some key points to be aware of. If the imposed perturbation causes a kink in the structure, the load has to be broken up into portions on either side of the kink. In a statically determinate structure, which is the focus of this project, all lines will remain straight after the perturbation.

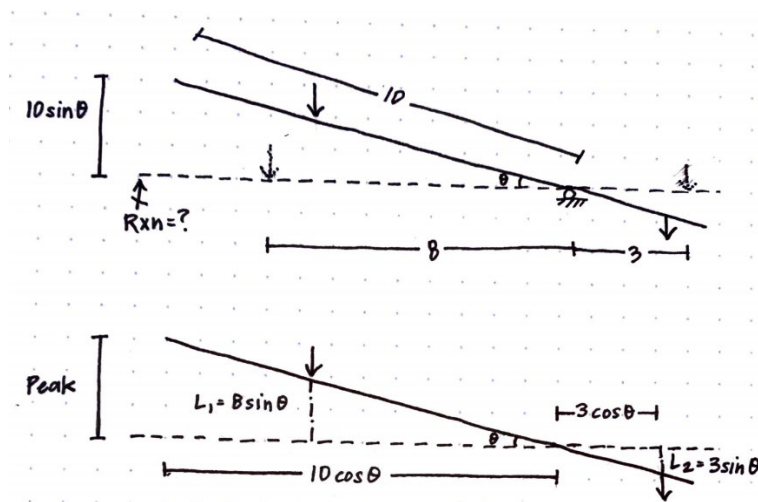


Figure 3

As seen in the sketch above, this method can easily be recreated by hand or on a napkin. For this simply supported beam with an overhang, the lofts can be calculated using simple trigonometry of sines and cosines and can then be put into the fundamental equation. This sketch also illustrates how the beam stays the same length throughout the perturbation and how the loads move inwards. Although not necessary, it is recommended to practice drawing these problems by hand first to get comfortable with the method before drawing in a computer program.

Determinate Beams

Determinate beams are the simplest structure to be investigated and serve as a great starting point to learning the method. The power of the method shows that as problems get more complicated either by varying geometry or loading patterns, the method remains at the same difficulty level whereas the statics would get increasingly more difficult.

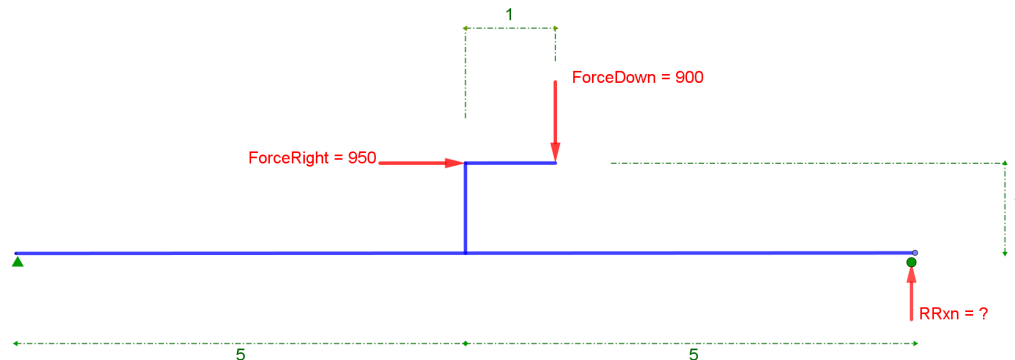


Figure 4

Figure 4 above is a common problem given in statics courses. It tests a student's understanding of the way the forces applied to the handles extended off the beams affect the internal moments and reactions of the main portion of the beam. To solve using statics, one would have to resolve the handle force into a force applied directly to the main beam with an added external moment to account for the eccentricity of the force. With the Modern Müller-Breslau method, no extra steps must be taken in order to solve for the reaction at the right support.

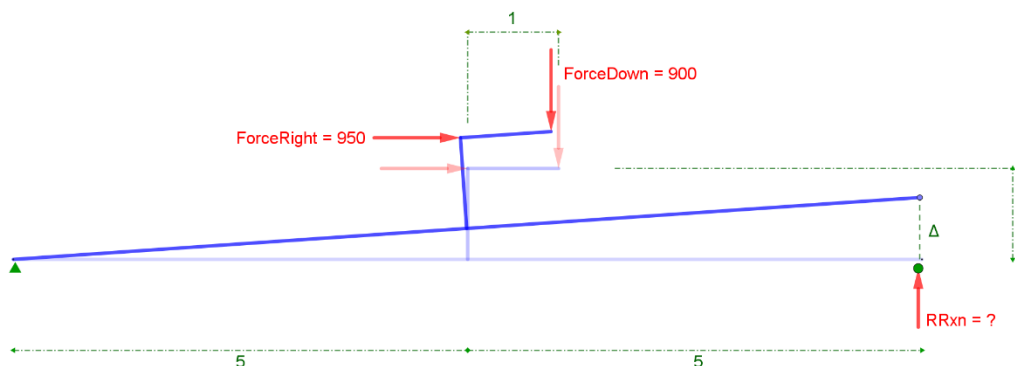


Figure 5

Following the steps of the modern method, if we are seeking the reaction at the right support, perturb the beam at the location of the support in the direction of the reaction. In Figure 5 that can clearly be seen as an upwards displacement of Δ . All other boundary conditions must be enforced, and in this case, this means that the end of the beam supported by the pinned support does not displace. In this figure the perturbation is exaggerated for visibility, but as it is

decreased to nearly zero, the answer will be approached asymptotically. Once again notice that the loads are assumed to be in their original orientation despite the perturbation.

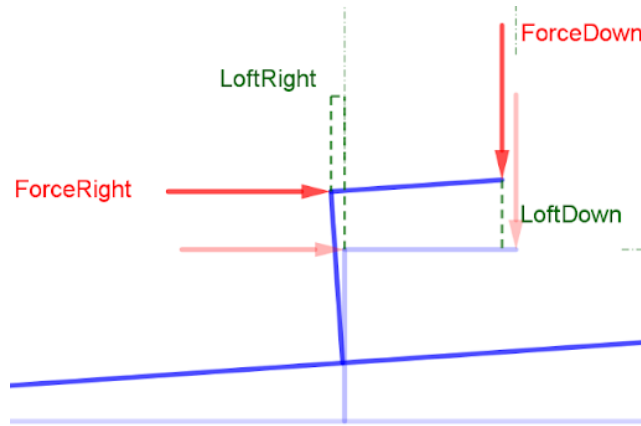


Figure 6

The next step is to measure the lofts of the applied loads, which are shown in Figure 6. For the downward force, the loft should be measured *vertically*. This is because in order to create work, the force and the displacement must be going in the same directions. The signs of the work matter as well. For this problem, as the perturbation Δ increases, the downward force is being applied in the opposite direction as the loft. The force points down, but that point on the beam is being moved up when the perturbation is applied. This means that the work being applied by that force is negative. Similarly, for the force pointing to the right, the loft should be measured *horizontally*. The work done by this force is also negative, because the force is acting to the right while that point on the beam is moving to the left. After these lofts are measured, equation 1 can be assembled and solved for the *Unknown*.

$$RRxn \cdot \Delta - ForceRight \cdot LoftRight - ForceDown \cdot LoftDown = 0$$

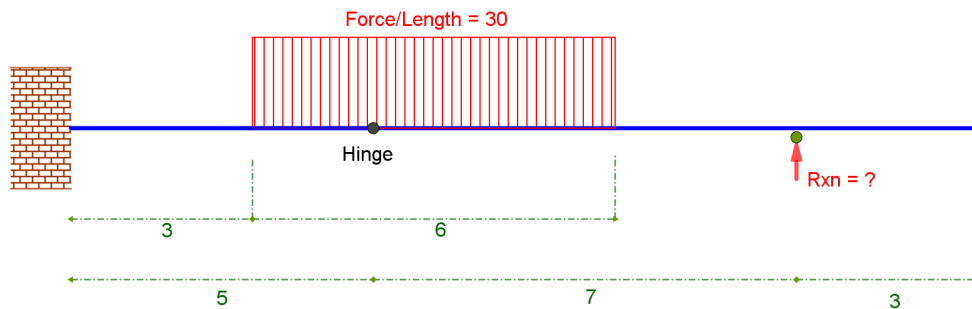


Figure 7

Figure 7 depicts a propped cantilever problem with a hinge. A distributed load is applied in the location shown on either side of the hinge. In order to solve this with statics, the beam must be “broken up” at the hinge and forces at the hinge would be solved one at a time, requiring multiple steps to find the reaction at the roller support. With the Modern Müller-Breslau method, that reaction can be found very quickly and simply using the steps outlined previously.

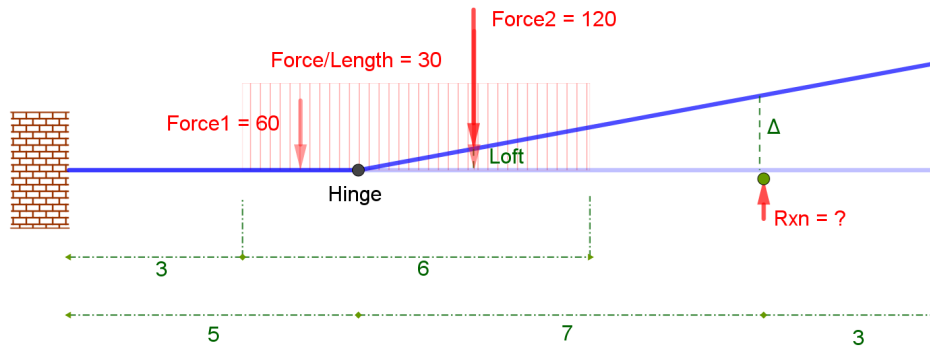


Figure 8

In this problem, the reaction at the roller support is being sought, so a perturbation Δ is applied in its place and in the same direction as the reaction. In this problem Δ is an upwards displacement at the location of the roller support. Remember that all other boundary conditions must be enforced. This means that there is no rotation of the beam when extending out of the fixed support on the left, but it is free to rotate about the hinge. These conditions give us the perturbed shape shown in Figure 8, and a kink in the shape can be clearly seen. Before anything can be done, the applied distributed load must be broken up on either side of the kink. In this example, the force per length amount is split up into Force1 on the left side of the hinge and Force2 on the right side of the hinge. Next, the lofts under the loads can be measured. This problem is especially interesting because it can be clearly seen that only the force on the right side of the hinge experiences a loft, which means that only the force on the right side of the hinge does work. The force on the left stays in the same position and does no work, and so does not contribute to this problem. Finally, equation 1 can be assembled, and the *Unknown* can be solved for.

$$R_{xn} \cdot \Delta - Force1 \cdot 0 - Force2 \cdot Loft = 0$$

For those in engineering programs, internal forces such as internal shear and bending moment are required to complete beam design. Finding these forces can also be completed with the Modern Müller-Breslau method.

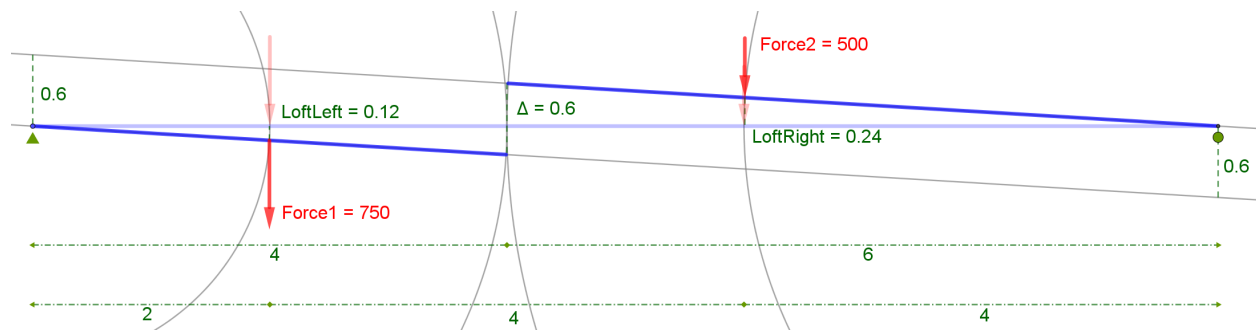


Figure 9

A simply supported beam with two gravity loads is shown in Figure 9, in which the shear at $x = 4$ along the beam, between the two loads is sought. This figure also shows the construction lines that are needed to set up the problem. The first step of the method is to take away the capability of the structure to carry the internal shear of interest. At that location, this would mean that the beam breaks or cracks at that location, so one side goes up and the other side goes down. The perturbation Δ is still variable and occurs at the location of the crack, but it is total perturbation added from both sides of the beam, one side up and the other side down. This is created by the two lines shown in Figure 9 that are along the length of the perturbed beam shapes. In order to perturb the loads, a circle is created at each side of the beam with their centers placed at the supports and intersecting the location of the load. For Force1, the load shifts to the left based on this perturbation and for Force2 shifts to the right. This movement is very subtle but is required in order to maintain all the dimensions of the beam for the method.

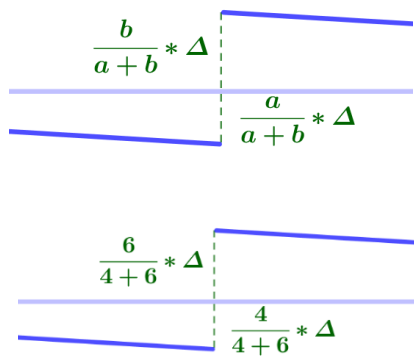


Figure 10

There is also an equation to find the height on each side of the crack based on geometry. In the classic method, the displacement Δ was always equal to 1, so this geometry was very simple. For the modern method since the perturbation Δ is variable and approaches nearly 0, the equation is slightly different.

As can be seen in Figure 10, the height for the side displacing downwards is that side's length divided by the total length of the beam and multiplied by the total desired perturbation Δ . As can be easily observed, the two sides add up to the total perturbation Δ . For this problem, $a = 4$ and $b = 6$, and the geometry is shown in the figure. After these

heights are found, the lofts below the applied loads can be measured, and equation 1 can be set up as shown below. Note that Force1 is creating positive work because it is acting in the same direction as the perturbed shape is moving, and Force2 is creating negative work because it is acting in the opposite direction as the perturbed shape is moving. This time, the *Unknown* is the internal shear instead of a reaction. The work created is still added together and set equal to zero as before, and the shear can be solved for.

$$Shear \cdot \Delta + Force1 \cdot LoftLeft - Force2 \cdot LoftRight = 0$$

Internal bending moment has a similar approach as internal shear but is unique in construction in terms of the method.

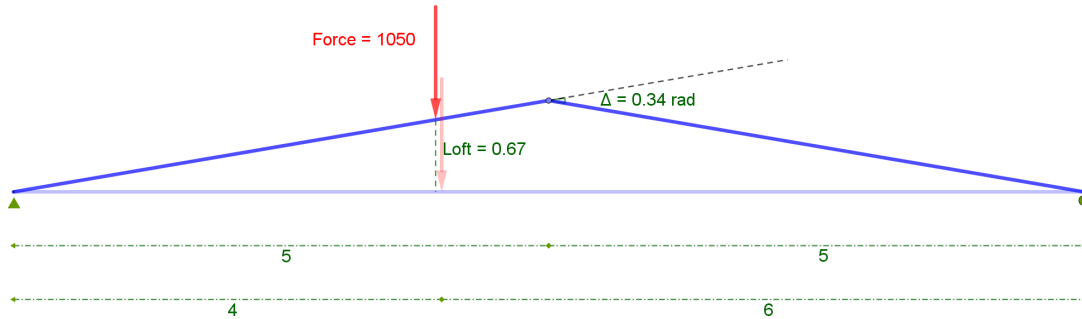


Figure 11

Figure 11 shows a problem depicting a simply supported beam with a single downward load. The moment at midspan is being sought. To find the internal moment, first take away the capability of the beam to carry moment at that location, creating a crack in the beam. This is synonymous with creating a hinge in the beam. The imposed perturbation Δ is a rotation this time, and it must be measured in radians for equation 1 to work. Note the location of the rotation of interest is the one between the two sides of the beam on either side of the crack. Using the correct angle is crucial and can be difficult for a student to understand why it is that angle and not the angle directly between the two pieces. The next step is to measure the vertical loft below the applied load. If using a hand sketching method, the height below the crack will have to be found first.

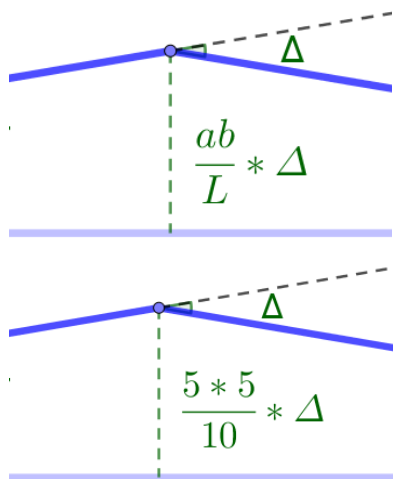


Figure 12

In the classic method, the Δ was a unit rotation of one, so the height below the crack was just the length of the side on the right side of the crack times the length on the left side of the crack divided by the total length of the beam. For the modern method, since the perturbation Δ is variable, this equation must be slightly modified by multiplying by Δ . The height below the crack in the example can be seen in Figure 12. In this example, $a = b = 5$ and $L = 10$. After this is found, the loft below the load can be measured and equation 1 can be assembled as shown below. This can of course be done on a quick sketch using similar triangles or just estimating the loft using relative scale. The force applied is creating negative work because it is being applied in the opposite direction as the loft due to the perturbation of the beam. For equation 1, the *Unknown* is internal bending moment and the perturbation

Δ is an angle in radians. Once put into the equation, the *Unknown* can be solved for.

$$Moment_{crack} \cdot \Delta - Force \cdot Loft = 0$$

Determinate Axial Problems

The modern method can also solve axial problems.

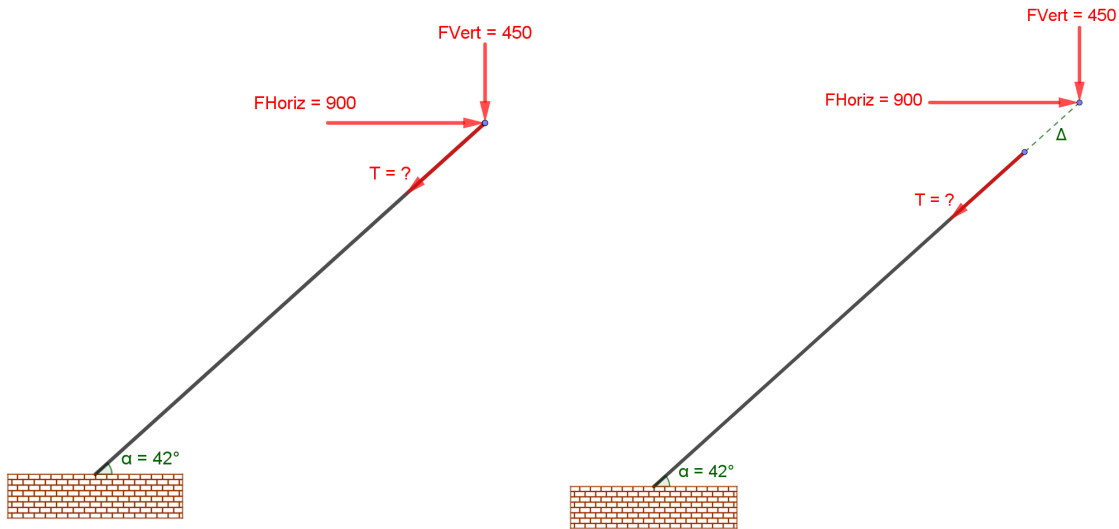


Figure 13

For an axially determinate problem, the steps remain simple. If the internal axial force is sought, then a perturbation Δ must be applied in the opposite direction of that internal force. For this axial problem, tension will be assumed, so a perturbation will be applied along the length of the beam “stretching” it out, as shown in Figure 13.

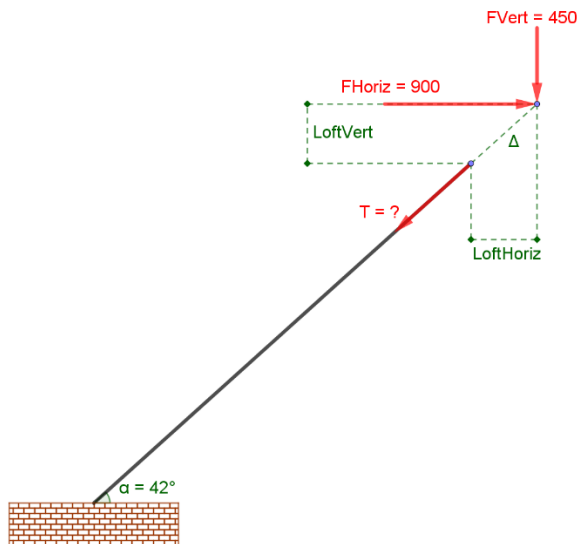


Figure 14

Next, the lofts must be measured. In this problem they are still defined as the vertical or horizontal distance the loads travel when perturbing the member a distance Δ . Note that the work the internal force does is always negative because the perturbation will always be applied in the opposite direction as the internal force. The horizontal force does positive work because it stretches the beam and the vertical force is doing negative work because it compresses the beam. Something to be noted about the perturbation Δ in this problem is that as it decreases the answer stays the same exact as the theoretical answer, instead of approaching it. By observation this can be explained by the fact that the variation

of Δ is along a fixed axis whereas usually the variations of delta are along a circular path, therefore the rise and the run are directly related. Therefore, for any Δ , the answer will be exact. Equation 1 can then be assembled as shown.

$$-T \cdot \Delta + F_{Horiz} \cdot Loft_{Horiz} - F_{Vert} \cdot Loft_{Vert} = 0$$

These steps of applying a variable perturbation along the length of a member to find the axial force can be applied to beam or frame structures that have cables, or trusses. Although those types of problems were not analyzed for the purposes of this project, there will be more on finding internal axial forces in the three-dimensional truss section.

Determinate Frames

The next investigation using this method will be determinate frames. All the steps for finding external reactions and internal forces are the same as for beams, however extra care must be taken in terms of boundary conditions, which requires a good understanding of the method, making it a good next step after beams are mastered.

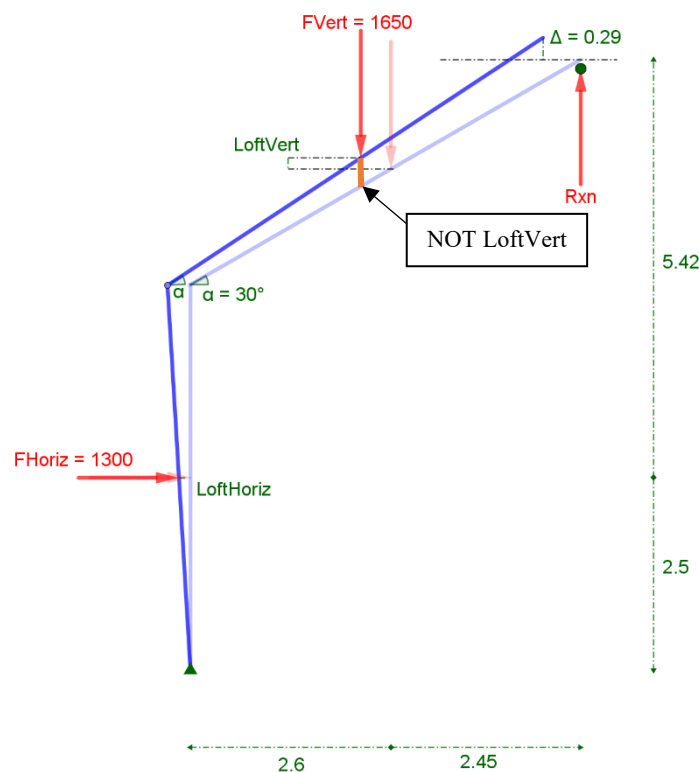


Figure 15

The first example shown in Figure 15 is a two-member frame with a horizontal and vertical load applied. In order to find the upward reaction at the right support, the frame is treated similar to a beam. In the location of the *Unknown*, the reaction, a variable perturbation Δ is applied, in this case a displacement upwards. As can be seen from the perturbed shape in Figure 15, the entire frame is rotated about the left pin, respecting that boundary condition. The angle between the two members that make up the frame, in this case 30 degrees, does not change as it is a rigid connection.

Something that becomes important when examining slanted members is measuring the correct lofts below the loads. The loft is defined as the change in position from the old load to the new load in the direction that

the load is applied. For a vertical load, the loft is measured vertically, meaning the change in vertical position from the original load to the new load on the perturbed shape. A common mistake made by students may be to measure F_{vert} in this example as being the distance directly below the applied load to the old shape when really it is the change in elevation of the applied load. The correct loft to be measured is clearly illustrated in this figure. The perturbation Δ is also measured in this way. The horizontal loft is more straightforward and doesn't require taking extra care. At this time being familiar with the method for beams, it should be clear to the user that both the vertical and horizontal applied force create negative work. Once the lofts are measured equation 1 can be used once again to solve for the *Unknown*.

$$R_{xn} \cdot \Delta - F_{Horiz} \cdot Loft_{Horiz} - F_{Vert} \cdot Loft_{Vert} = 0$$

This is when it begins becoming clear how powerful the method can be. Even though this is now a frame instead of a beam, the process and the fundamental equation does not increase in complexity.

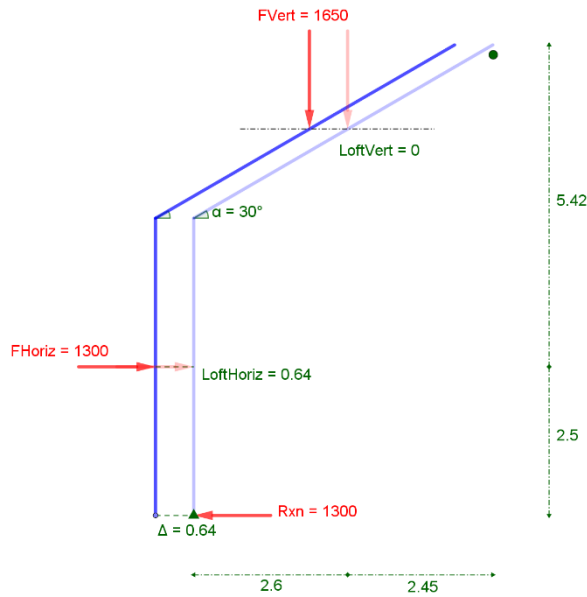


Figure 16

In this same problem, suppose that the thrust at the pin support was the reaction of interest. The first step would be to apply a perturbation Δ in the direction of the thrust. This perturbation moves the entire frame to the left as shown in Figure 16. Even though the entire frame moves from its place, note that the boundary conditions are still respected. The roller support on the right is still enforced because it is restrained vertically but can move horizontally.

If this problem were to be solved using statics, the thrust reaction could be solved by inspection. By just looking at the forces in the horizontal direction the horizontal thrust must equal the applied horizontal force because they are the only forces acting in that direction.

Something similar is shown by the method. In Figure 16 it is clear that the horizontal loft is equal to the perturbation Δ , and that the vertical loft is equal to 0. Since the frame only moves horizontally, the vertical applied force does not experience vertical movement. Since force and distance have to be applied in the same direction to create work, the vertical force does not contribute to the thrust at the pin. The equation for this problem is shown below. Clearly it can be seen that since Δ is equal to $Loft_{Horiz}$, the thrust reaction at the support will be equal to the horizontal applied force.

$$R_{xn} \cdot \Delta - F_{Horiz} \cdot Loft_{Horiz} - F_{Vert} \cdot 0 = 0$$

Finding the internal moment in this frame has a few more steps as it requires two points of movement instead of just one.

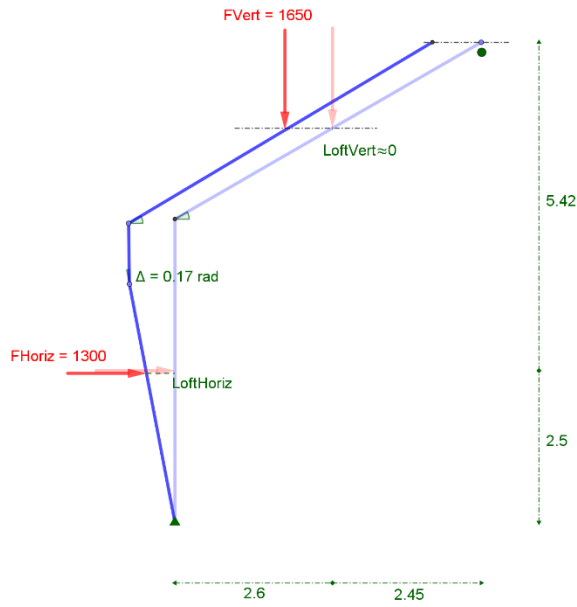


Figure 17

answer will be very far from theoretical answers. Note that the original angle between the two members of the frame remains the same, because that rigid connection is a boundary condition as well. When these boundary conditions and small perturbations are achieved, it can be seen from Figure 17 that the vertical loft is nearly zero which means it does no work and does not contribute to the internal moment at that location. For some students that might be intuitive, but for others the result may be surprising.

$$Moment_{crack} \cdot \Delta - F_{Horiz} \cdot Loft_{Horiz} - F_{Vert} \cdot 0 = 0$$

Note that the Δ in this problem is a rotation and must be input in radians for the equation to work, and the location of this rotation is crucial. Figure 12 for beams can be referenced for any imposed angle Δ for the correct location.

The most difficult part of moving from beams to frames is the increase of moving parts. As a student, it was easy to forget that the loft is measured from the original load to the load on the perturbed shape, and not from the original frame to the perturbed shape. More care had to be taken in respecting boundary conditions. When the boundary condition had to be manually adjusted as in finding the internal bending moment on the frame, this provided the most difficulty in approaching the theoretical answer. Adjusting both the perturbation and the boundary condition at the same time added some checking and re-checking that was not necessary when analyzing beams. Although this was a slightly more time-consuming aspect, the method is still less time consuming than performing statics to determine the internal bending moment at a specific location, especially if the loading patterns are subject to change. The perturbed shapes are independent of the loading pattern, so changing the loads means no redrawing of the perturbed shapes. This is where the method becomes the most powerful because it can be used to analyze multiple loading scenarios quickly and effectively in an extremely simple way.

The moment of interest in this problem is at some point in the vertical member of the frame, above the horizontal applied load. Create a crack in the location of interest, and apply a perturbation Δ , a rotation between the two sides of the member measured in radians, as seen in Figure 17. This is the same process as finding the internal moment in a beam, however there is one extra step to solve this problem. When creating the perturbed shape, there are two points of movement. One is to adjust the size of the perturbation angle, and the other is to adjust the location of the right side of the slanted member. This becomes crucial to finding the answer. If the roller boundary condition isn't respected and the frame isn't adjusted so that the top moves along the horizontal path, the

Three-Hinged Arches

Three-hinged arches are a commonly investigated structure in many engineering courses. These structures usually consist of an arch with two pin supports, with a hinge in between so that it is still statically determinate. In statics, three-hinged arches need to be split up at the hinge, given equal and opposite forces on either side of the hinge, and require evaluating a system of equations in order to solve for reactions. This can be difficult for a lot of students because opening up statics to more complicated algebra can not only be time consuming, but also increase the likelihood of errors. The difficulty of these problems can easily be increased by putting the pin supports at different elevations and making the arch itself asymmetrical. Using the Modern Müller-Breslau method to solve for reactions at the supports of 3-hinged arches drastically cuts down this complexity.

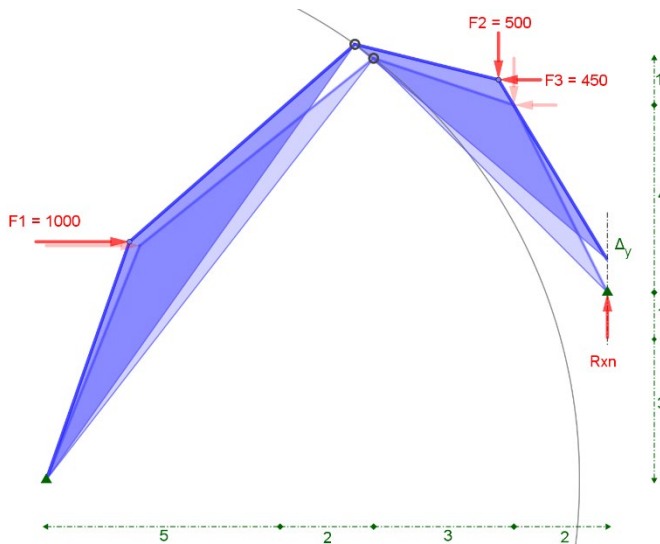


Figure 18

When examining a three-hinged arch, often the reactions at the supports, especially the thrust at the supports, is of interest. This is because as loads are applied to an arch-shaped structure, the sides want to “push out.” It is the thrust at the pinned supports that resist this tendency. A three-hinged arch problem with a combination of horizontal and vertical loads is depicted in Figure 18. This problem implements many of the qualities that would make this difficult to solve using statics. The arch is asymmetrical having the two sides being different lengths, and the supports are at

different elevations. The first reaction of interest is the upwards reaction at the pin on the right side of the arch. The perturbation to be applied to three-hinged arches now has two steps instead of the usual one. Since the sides of the arch are free to rotate about the hinge in the middle, the perturbation has two points of movement that must be adjusted separately. First, the left side of the arch needs to be perturbed. Since we are seeking the upwards reaction on the other side of the arch, perturb the left side upwards about a circular path which is shown in Figure 18. After that, the right side of the arch rotates about the hinge in between and must be adjusted so that the pin boundary condition is respected. In this case, that means that the right side of the arch cannot move horizontally through the perturbation. This can be clearly seen in the figure as the right support moving up and down but not left to right. This is only possible because of the hinge. Similar to finding the internal bending moment at some location on a frame, making the extra adjustment to respect the boundary condition is crucial to finding the right answer. After the lofts are measured equation 1 can be used to solve for the vertical reaction. Note that $F1$ and $F2$ are doing negative work while $F3$ is doing positive work.

$$R_{xn} \cdot \Delta - F1 \cdot Loft1 - F2 \cdot Loft2 + F3 \cdot Loft3 = 0$$

The three-hinged arch problem requires two points of movement for the modern method, just as how it requires two equations to be solved using statics. This is an interesting correlation that was noticed during this project. Similarly, if the internal bending moment at some point in a frame was to be solved using statics, it would also have to be split into two portions with equal and opposite reactions. Solving the problem with the modern method requires two points of movement as well.

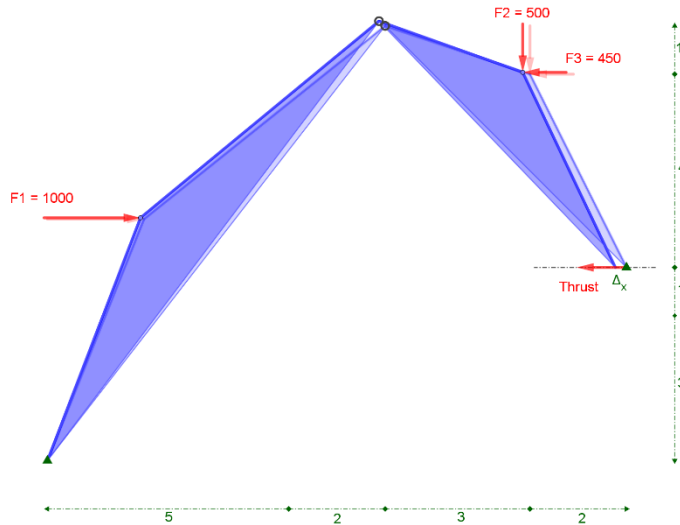


Figure 19

To find the thrust at the same support, a different construction is required, as shown in Figure 19. The first step of the perturbation starts the same as the previous problem, which is that the left side of the arch must be perturbed upwards along a circular path. It may seem counterintuitive to perturb the arch vertically when seeking a horizontal reaction, however, moving the left side of the arch vertically is the only way to move the right side of the arch horizontally since the two sides can rotate about the hinge. The only difference between finding the thrust and the upwards reaction is that

the second step now requires the respect of a different boundary. Since we are seeking the thrust, a horizontal force, we take away the capability of that reaction to resist thrust, however it still retains the capability to resist vertical force. Therefore, the right support can only move horizontally as shown in Figure 19.

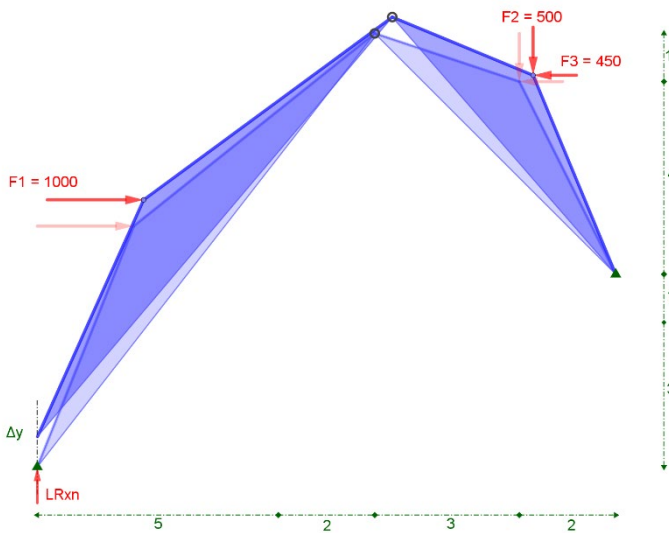


Figure 20

Lastly, if reactions on the left side of the arch are sought, the same perturbation occurs for a vertical reaction but mirrored from the previous example. The perturbed shape for a vertical reaction at the left support can be seen in in Figure 20.

The three-hinged arches discussed so far require accurate adjusting in order to reach exact theoretical answers, and with two points of movement this may be a little tedious because it requires moving back and forth between both points of movement decreasing the perturbation Δ . However, these

examples have been of a small scale, and the accuracy of the method in three-hinged problems increases as the scale does. This is because the perturbation Δ and the measured lofts get smaller and smaller in comparison to the span of the arch so there is less room for error in the equation. From the next example in Figure 21 below, it can be observed that the perturbation gets less obvious as the span increases.

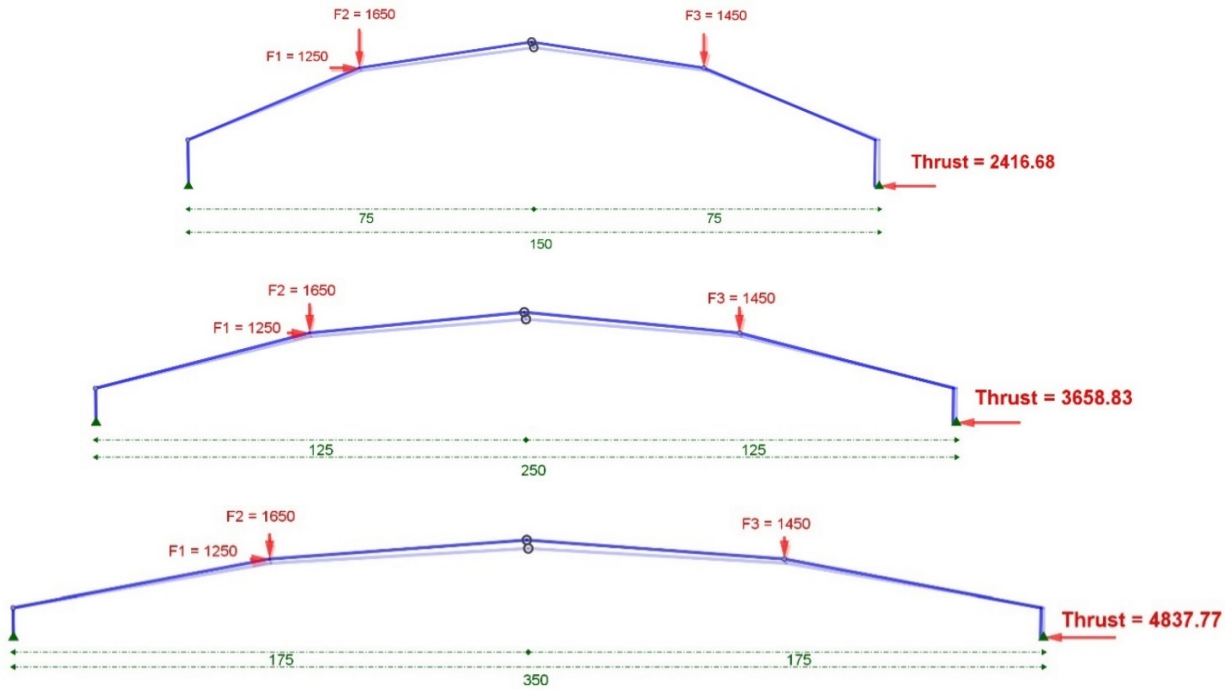


Figure 21

These three-hinged arches with large spans of 150, 250, and 350 in Figure 21 illustrate the speed at which investigations can be made graphically and eloquently, without repetitive statics. By just looking at the figure the observation can be made that as the span of the arch increases, the thrust also increases. Coming to this conclusion using statics for each arch would be very time consuming and this is one of the benefits of the method for students who need a more conceptual understanding of engineering. Furthermore, using this method on such a large scale shows the possibilities for its real-world applications.

Three-Dimensional Problems

All structures in real life are three-dimensional, yet two-dimensional models are appropriate most of the time. Sometimes a two-dimensional model isn't enough, like in the following examples. The statics that go into a three-dimensional problem becomes increasingly complicated because as dimensions increase, the number of equations increase. However, using the modern method for three-dimensional structures remains at the same difficulty level as beams and frames.

The first three-dimensional problem is a simple tripod, as seen in Figure 22. Stick1 and Stick 2 are the same length while Stick3 is longer. The tension in Stick3 is sought in this example. For these problems, the z-axis is drawn in blue, the x-axis is drawn in red, and the y-axis is drawn in green.

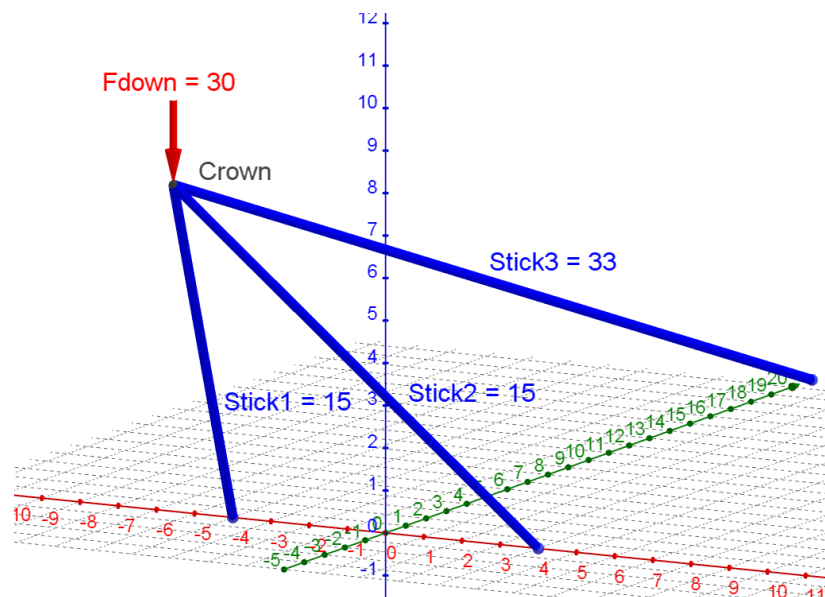


Figure 22

As discussed previously in the axial determinate problem, when an axial force is being sought, the variable perturbation Δ is applied along the length of the member. When applying an axial perturbation to a member that is part of a truss, the perturbation is only applied to that member, while all other members stay the same length. Figure 23 shows the variable perturbation Δ applied along the length of Stick3. This increases the length of Stick3 while Stick1 and Stick2 stay the same length. NewStick3 is the original Stick3 plus Δ .

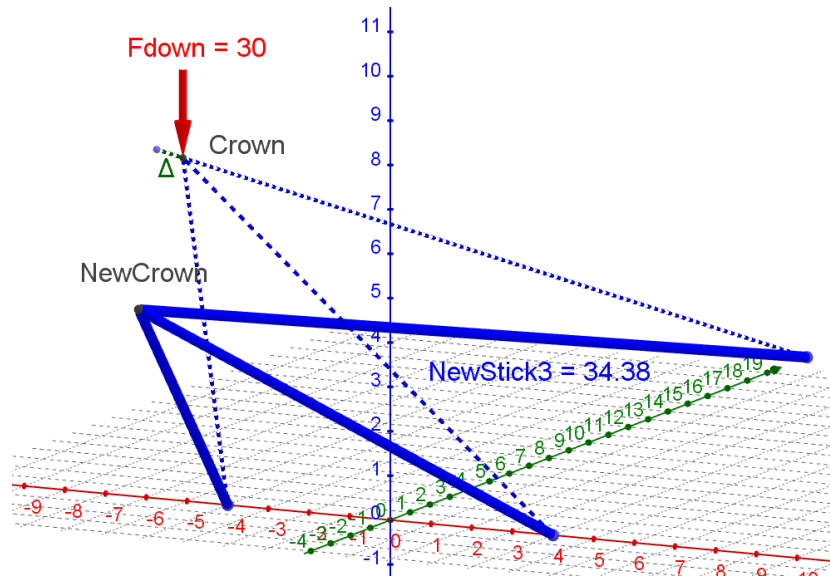


Figure 23

The new perturbed truss can be seen in Figure 23, depicted with thick blue lines. The only point that moves is the original crown, while all the supports stay pinned to the ground. Something to notice is that the new crown created by the perturbation is not in line with Stick3 or the perturbation of Stick3, Δ . This is because since Stick1 and Stick2 must stay the same length, they rotate about their supports to accommodate the stretching of Stick3. Although this makes sense based on the boundary conditions, this result is not intuitive. The force applied is going straight down, so even though the new crown changes position in multiple directions, the only loft to be measured is purely vertical, in the direction of the applied force. The vertical distance from the NewCrown to the original Crown will be the loft for solving the problem. Even though this problem is three-dimensional, the same fundamental equation can be used for this problem. As previously discussed, when seeking an axial force, the work done by the internal force is always negative because it always goes in the opposite direction as the perturbation Δ .

$$-T_{stick3} \cdot \Delta + F_{down} \cdot Loft = 0$$

The next three-dimensional example shown in Figure 24 is slightly more complex and has six sticks with two applied loads. The axial force in Stick4 is being sought.

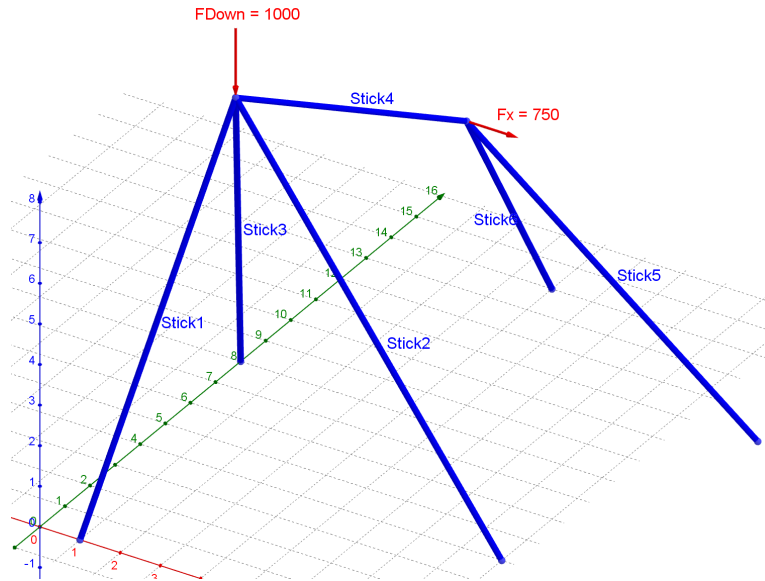


Figure 24

To find the force in Stick4, a variable perturbation Δ must be applied along the length of Stick4. By observation, an intuitive student can see that the point at which F_{Down} is applied is stable. This is because it occurs at a point that is supported in every direction, by the tripod at that location. Therefore, when investigating Stick4, that point does not move. This means the perturbation Δ will be applied along the member starting at the point of F_x , keeping the point of F_{Down} in the same location. This also means that F_{Down} does no work in terms of the internal axial force experienced by Stick4, and that force is only being affected by F_x , which is being applied in the x-direction.

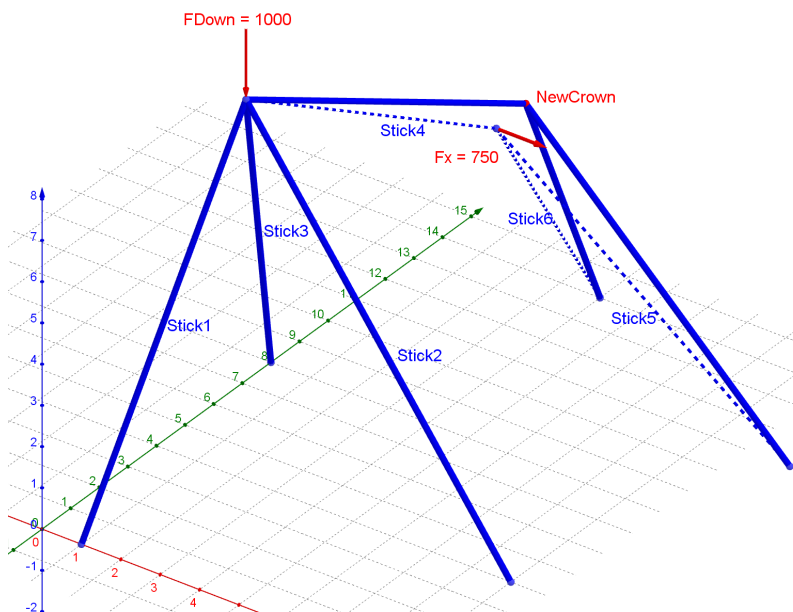


Figure 25

The new perturbed shape of the structure can be seen in Figure 25, in thick blue lines. All supports stay pinned to the ground and the only sticks that move are those at the location of F_x . Stick1, Stick2, and Stick3 do not move because they are connected to a stable point. Stick4 gets stretched through the perturbation while Stick5 and Stick6 stay the same length and rotate about their supports to accommodate the perturbation of Stick4. The only point that moves is the original crown, which becomes the new crown on the perturbed shape. Once again, the new crown is not in line with the original stick or the perturbation Δ . Since the applied load is in the x-direction, the loft to be measured is the change in x dimension from the original crown to the new crown. Then all can be plugged into equation 1 and the tension in Stick4 can be solved for.

$$-T_{Stick4} \cdot \Delta + F_{down} \cdot 0 + F_x \cdot Loft_x = 0$$

These three-dimensional examples are the most impressive and a great ending to analyzing determinate structures using the Modern Müller-Breslau method. These examples show how powerful the method can be, and how easily difficult problems can be solved. At this level of difficulty, most engineers would be creating a digital model in some sort of software. The idea that this method is simpler and more straightforward than creating a digital model is incredible.

Conclusion

As a senior student about to graduate from the ARCE program, I could appreciate the Modern Müller-Breslau method as a complement to my statics courses. I didn't find learning the method to be difficult, but it wasn't trivial either. Most of the things I struggled with were very specific. Although it's still shocking to me that the method works, the process itself is very easy to understand. What I struggled the most with was knowing the correct lofts to be measured, and if the distances were to be measured from the old shape to the new shape, or from the old load to the new load. This confusion is cleared up by always emphasizing that it is the load that performs work, therefore all the lofts measured will be from load to load not beam to beam.

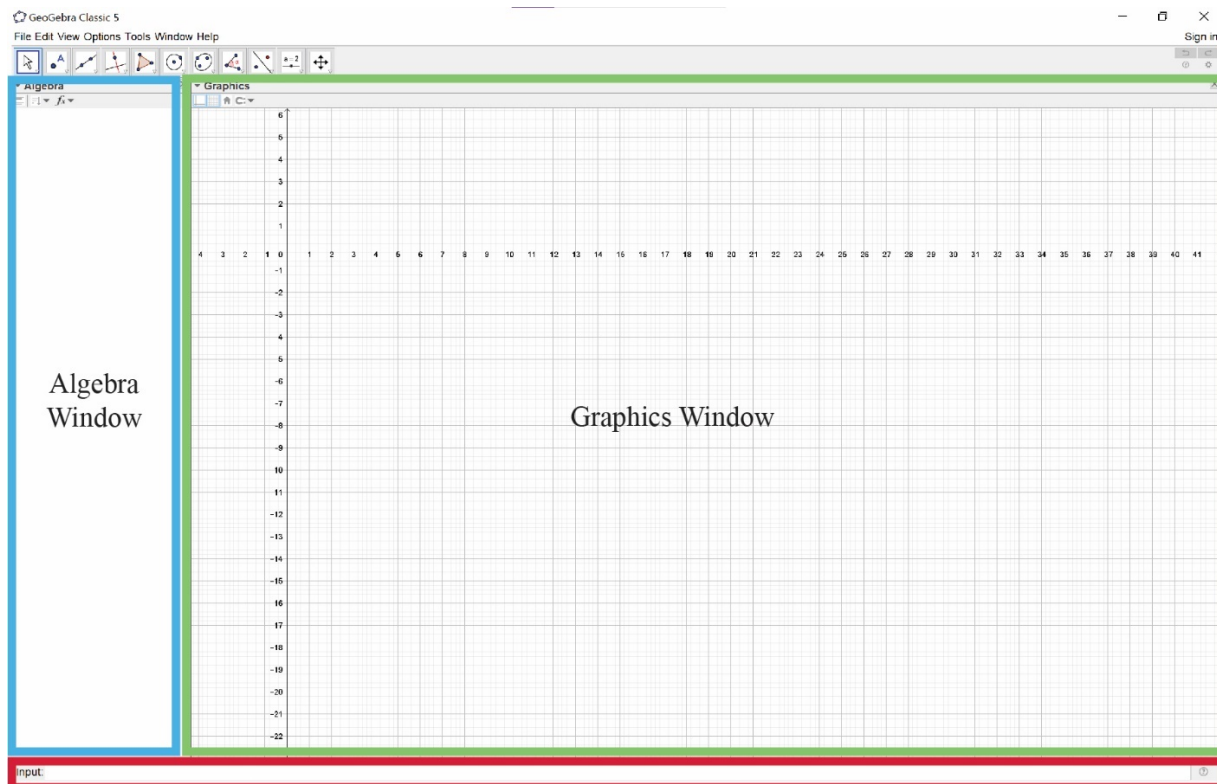
During the discussion of this method, a question came up about the coefficient of $\frac{1}{2}$ that is usually applied to work. This comes from the graph of Force vs. Displacement, in which the area under the curve, the work, is equal to $\frac{1}{2}F\Delta$, as opposed to the modern method being $F\Delta$. This $\frac{1}{2}$ is applied for a gradually increasing load whereas for the modern method the total force is being applied for the entire movement. The load is never changing; therefore, we don't apply the $\frac{1}{2}$ to the fundamental equation.

Learning this method would be beneficial for those in engineering programs. The influence line is widely taught and since this method can cover wider applications it serves as a great replacement. However, I don't think it should replace statics fully because the skills gained from that are still very beneficial to engineers. But I do see it as a replacement for very tedious problems that only cause frustration due to errors with algebra and not errors in conceptual understanding. I could also see this being implemented in practice possibly as a preliminary step to computer modeling when investigating a more geometrically interesting structure.

Introduction to GeoGebra

All the problems in this paper were created in GeoGebra Classic 5, a free program available for download on the internet. As a student who is accustomed to numerous software and coding programs, the learning curve wasn't very steep, however there are some adjustments to the user interface that can make using the method through this program much faster and simpler. Although the method can be completed by hand on a piece of paper, GeoGebra provides solutions faster and easier with almost no math required, as the geometry is performed by the program and answers can just be measured instead of performing geometric calculations like those for similar triangles.

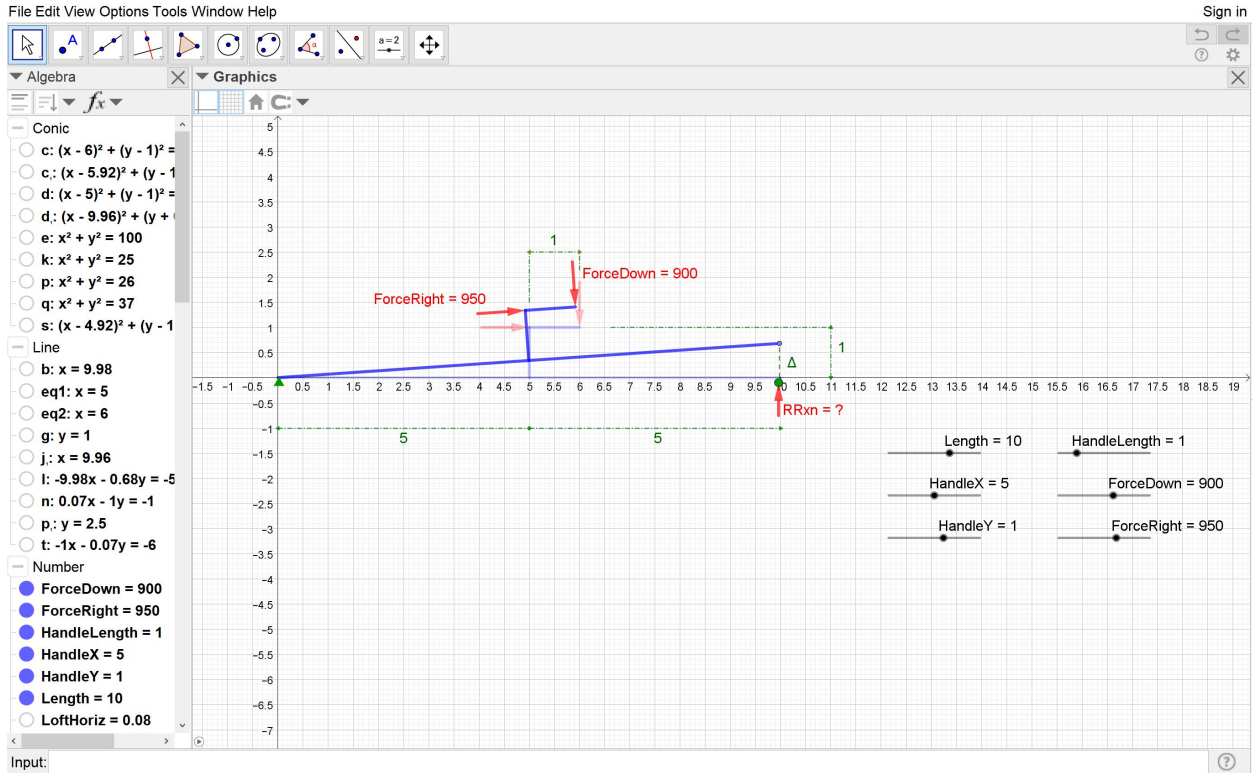
For the examples in this paper, looking online for troubleshooting tips is often futile. The GeoGebra guide provided in this paper will be aimed towards those using it to solve problems using the method described.



To begin, here is an overview of the layout of GeoGebra once it is opened. Highlighted in green below, is the graphics window. This is where all the constructions will be shown and drawn. Click the triangle next to “Graphics” at the top of this window to see toggling options for grid visibility. Press the button with the house to return to a view in which $(0,0)$ is centered on the page. Click the magnet to see a dropdown menu of snapping to grid options, which can aid in drafting if the grid is being used as a guide.

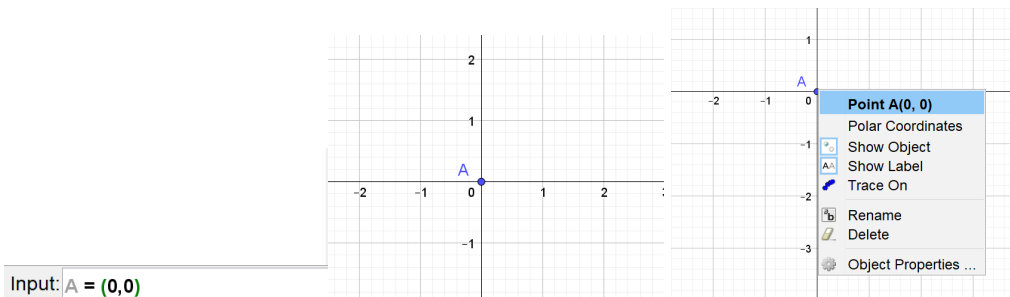
Highlighted in blue is the algebra window, where everything drawn will show up in formulas or numbers. All objects drawn and all numbers defined will be in this window. Click the triangle next to “Algebra” to see options of how the objects in this window can be organized. There are a few ways to organize the information in this window, the easiest of which is arranging the objects by object type. Click the button with a downwards facing arrow to see the options for object arrangement. The default is construction order, but as problems get more complex and there are more and more objects to keep track of, it’s easiest to arrange them by “object type.” The other buttons underneath “Algebra” do not need to be toggled.

At the very bottom of the screen there is an “Input” bar. This is where objects and numbers can be defined. An alternative to typing into the input bar is to instead use the buttons at the top of the screen.

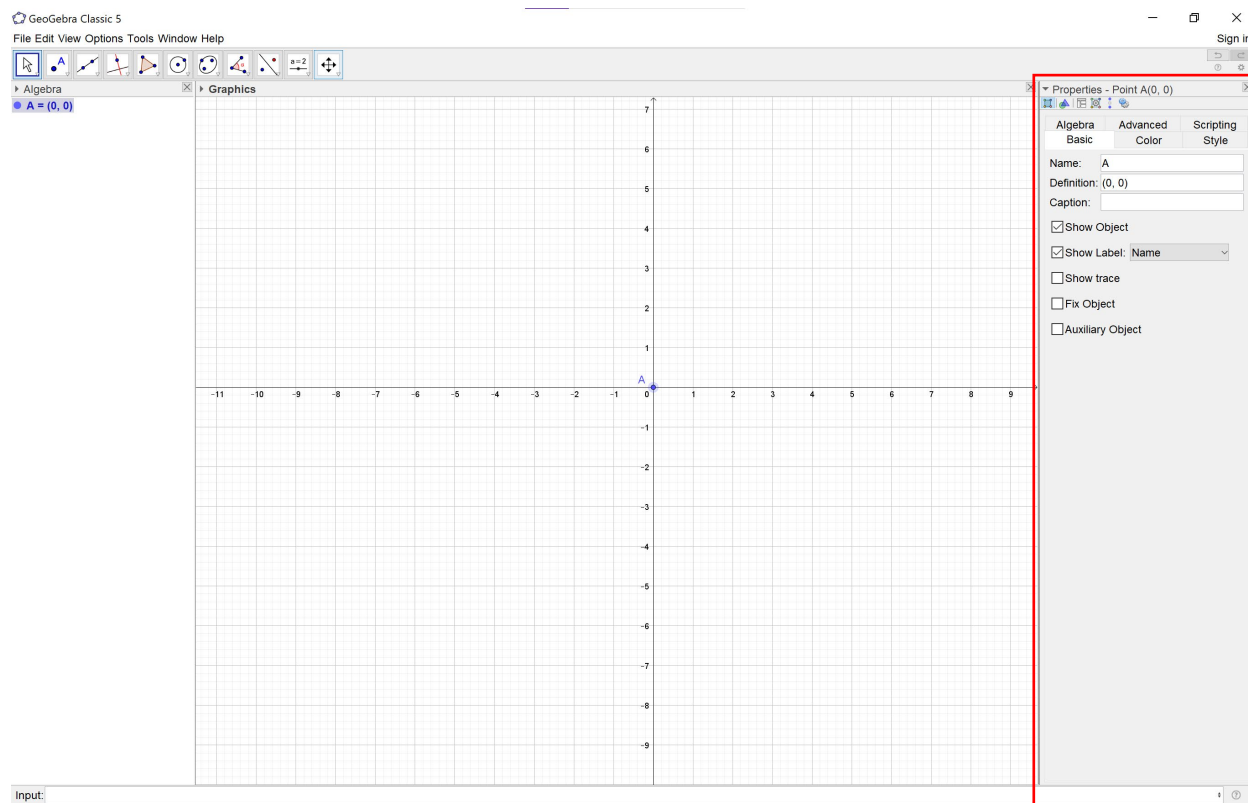


This is what a problem looks like within the GeoGebra program. The picture itself is in the graphics window, while all the shapes defined to make that picture is in the algebra window. The objects in the algebra window are arranged by object type, which are organized as conic, line, number, etc., as shown.

A beam problem in GeoGebra is constructed from points, lines, segments, and circles. First, points need to be made. There are multiple ways to make a point. The first way is to define it in the input window. For the input window, defining points starts with capital letters. For example, $A = (0, 0)$. It can also be a word starting with a capital, such as $Right = (0, 0)$.

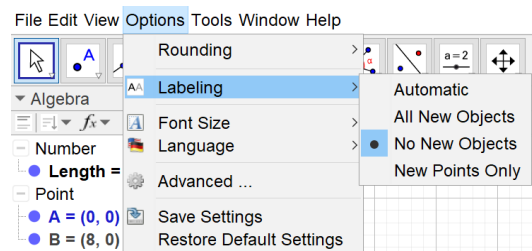
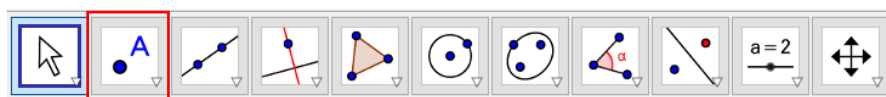


After typing out the point and pressing enter, the point “A” should populate in the graphics window. Right clicking on the object will create a drop-down menu. In this menu the visibility of the object can be adjusted, or it can be deleted. Clicking “Show Label” will remove the label “A” from the point. Clicking into “Object Properties” will open a new window on the side of the screen that looks like this:



In the “Basic” tab of the Object Properties window, the name and the definition of the object can be changed. There is also a toggle for showing the object. Note that choosing not to show an object will not delete it, and it can still always be found in the algebra window. Another way to toggle the visibility of an object is by clicking on the circle next to the object name and definition in the algebra window. Clicking the box to “Fix Object” can be very useful, since the point may be moved by accident if touched later, which could interfere with solving the problem.

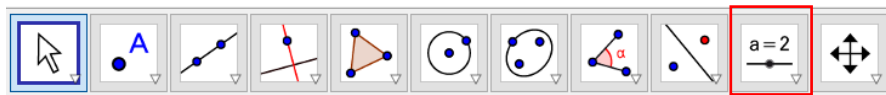
Another way to create a point is by clicking the point tool at the top of the screen. Once the tool is clicked, a point can be created by clicking anywhere inside the graphics window. Snapping to the grid can be useful when using this tool so that coordinates remain integers.



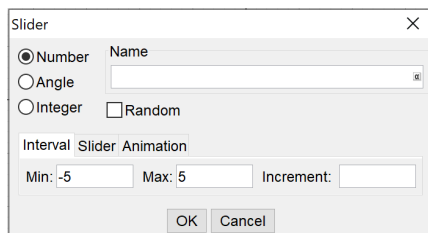
In GeoGebra, all objects are labeled in the graphics window automatically. To turn this off, go to Options, and click on labeling, then set it to “No New Objects.” This will turn off automatic labeling when creating new objects. The names of objects can still be viewed in the algebra window.

Next the second point on the beam must be created, however, part of the power of the method is being able to create variable problems, so the next point will not be defined in the same manner

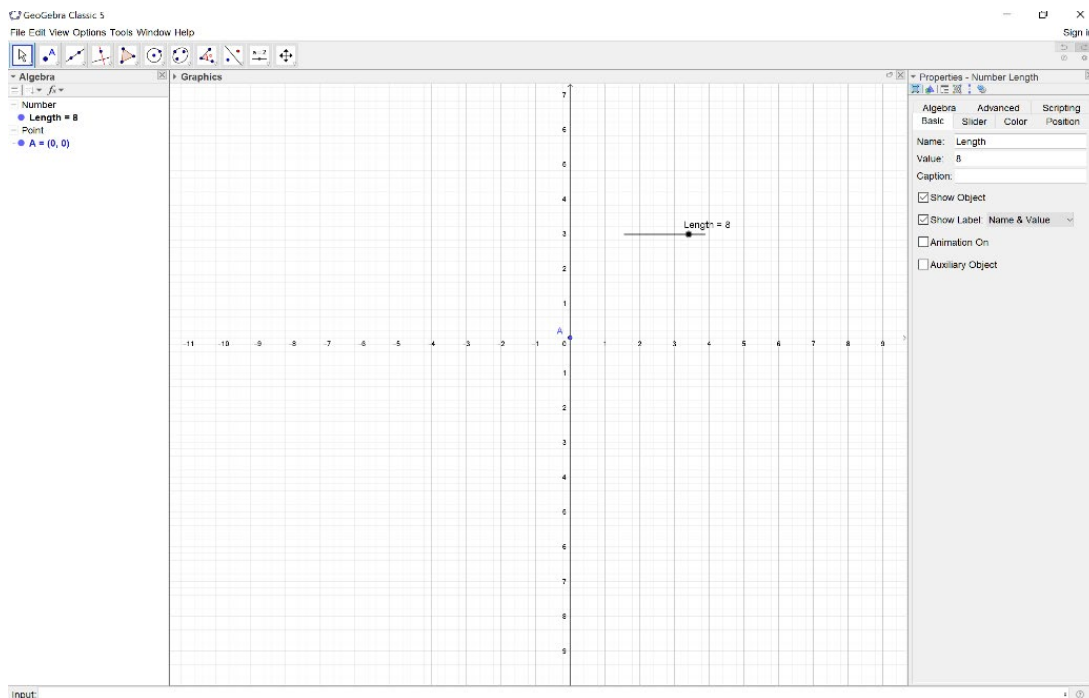
as the point “A.” Utilizing the slider tool, a variable can be created. For any tool, hovering over the tool will give a short description of how to use it.



Click on the slider tool and click anywhere in the graphics window. This window will populate:



In this window the slider can be named and defined. In this case the slider name will be “Length,” set to a number, and the interval will be defined as Min: 0 to Max: 10, but it can be defined as 0 to whatever the maximum length of the beam. The increment is usually set to 1 to keep them as integers but can be any number if a smaller or larger increment is desired.

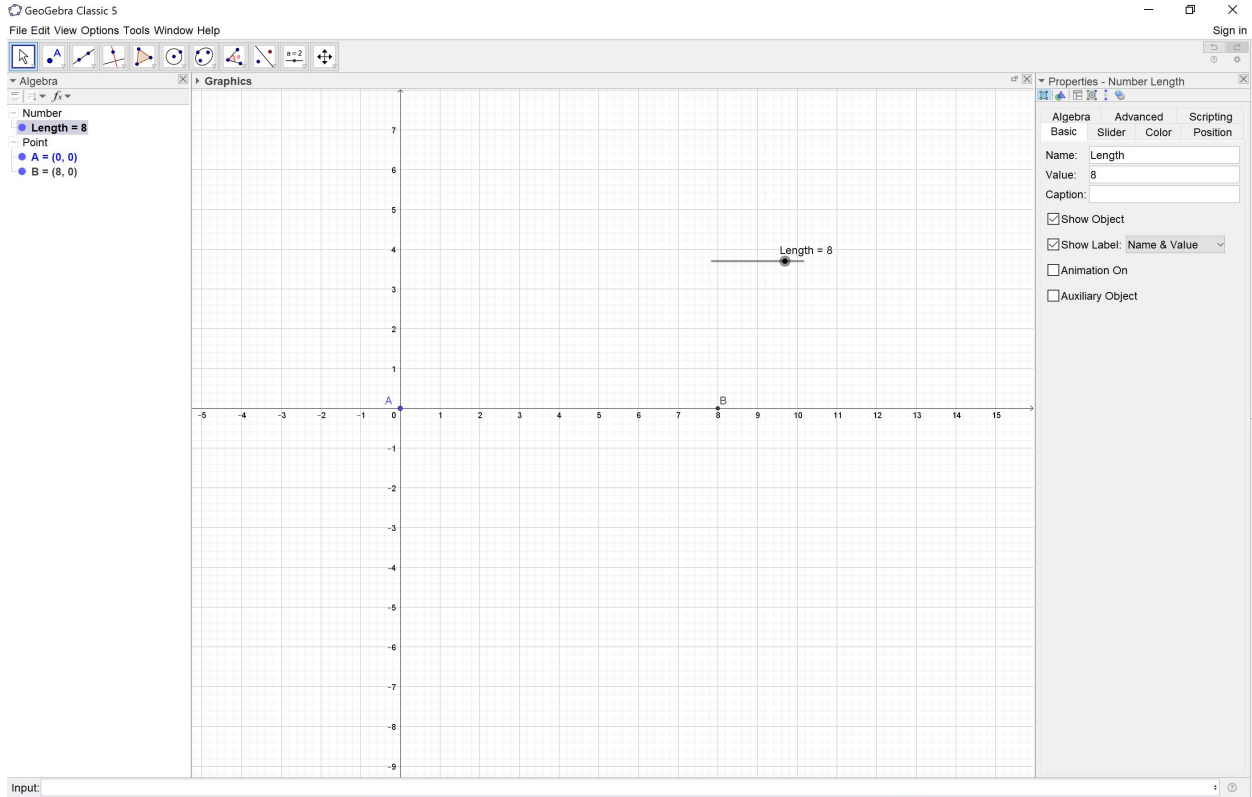


Once the slider “Length” is defined, it populates in the graphics window, and shows up in the algebra window as “Length = (Slider Value).” The dot on the slider can be moved back and forth in order to change the length of the beam.

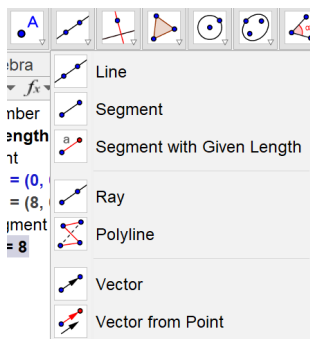
Input: $B = (\text{Length}, 0)$

Now the second point on the beam can be defined. In the input window, type $B = (\text{Length}, 0)$. Note that the variable is case sensitive, so if Length is capitalized on the slider, it must also be capitalized

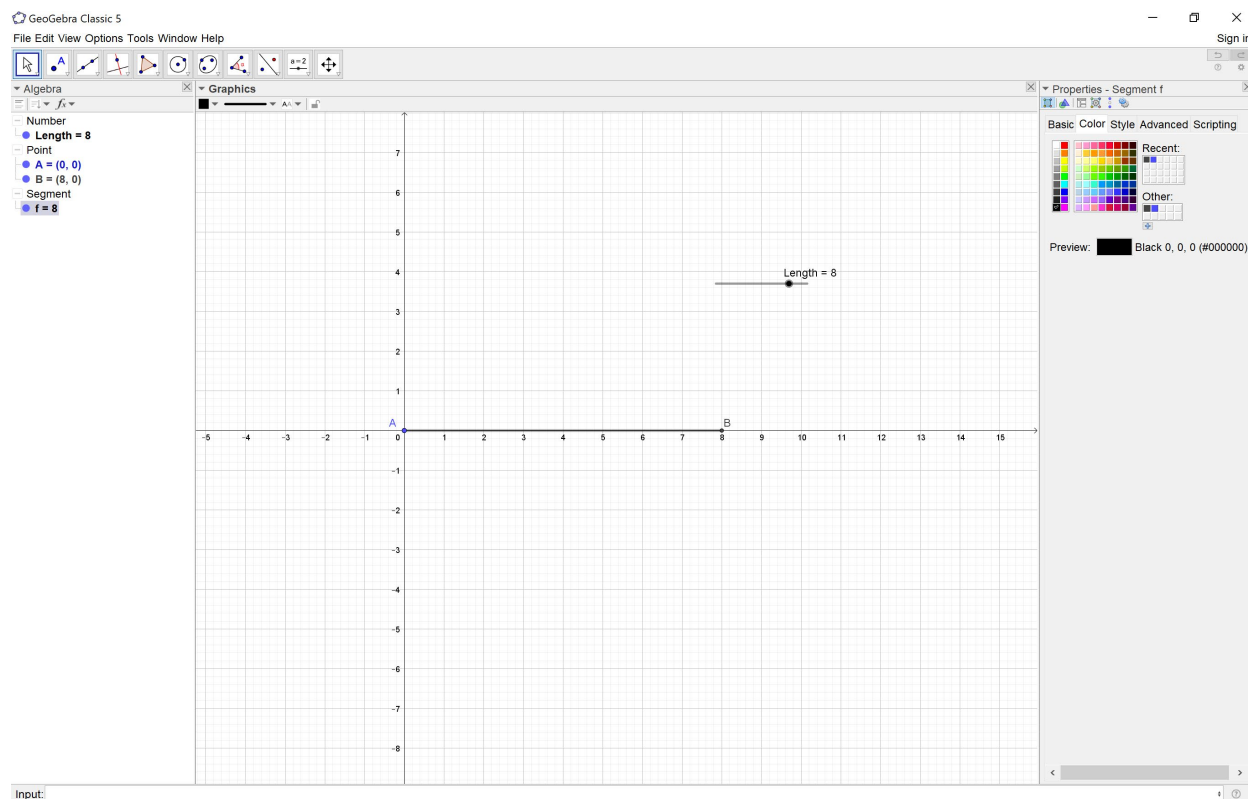
when calling on the variable to define point B. Also notice that when a defined variable is typed into the input bar, it turns a blue color. After pressing enter, two points should now appear in the graphics window. Toggle the “Length” slider and see how point B moves back and forth along the screen. All components of the slider can be edited in the Object Properties window, under the “Slider” tab.



Input: `Segment(A, B)`

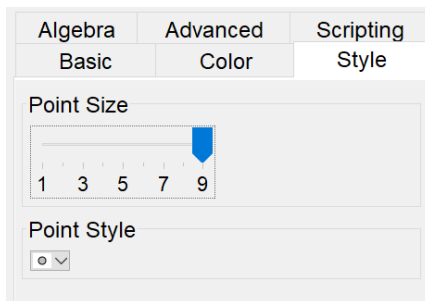
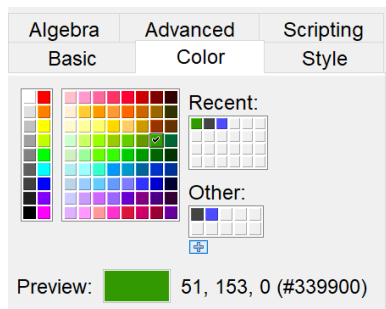


Now, a segment can be created between the two points to draw the beam. In the input window, type “Segment(A,B)” and press enter. This will draw a line between points A and B, and this segment will change lengths according to the “Length” slider value. This is how the length of the beam is adjusted to create variable problems. A segment can also be created using the top buttons, then clicking on point A and point B. The segment will be defined in the exact same way in the algebra window as typing it in the input bar.



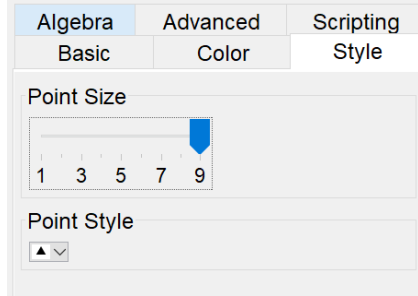
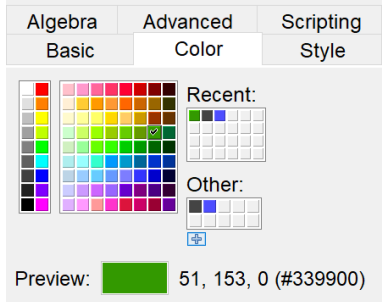
Clicking on the segment that is the “beam” and going into the object properties window will allow options for changing graphics. For the problems in this paper, members drawn have a standard of a thick blue line. In the color tab, the color of the beam can be changed. In the style tab, the thickness and opacity of the line can be changed.

The “supports” for the beams, which could be pins, rollers, or fixed supports can also be drawn in GeoGebra. Pins and rollers can be created graphically by points. The simplest way to do this is to click the point button at the top of the page and then click a location that is under or to the side of the beam where the support should be. Adding supports are for graphical purposes only and won’t be used in measurements so their exact location isn’t critical. Points can be moved after they are placed by simply clicking and dragging. Once the location is set, go to the object properties, and click “Fix Object” to avoid moving it again later.



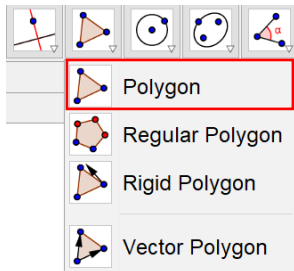
For a roller support, the graphic standard is a solid round point. The graphic standards shown in this paper are shown in the images to the left. For the problems in this paper, they will be the biggest point possible, which

can be adjusted in the style tab and they will be colored green which can be defined in the color tab.

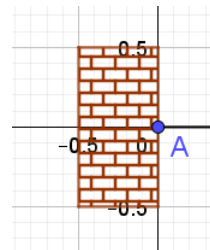
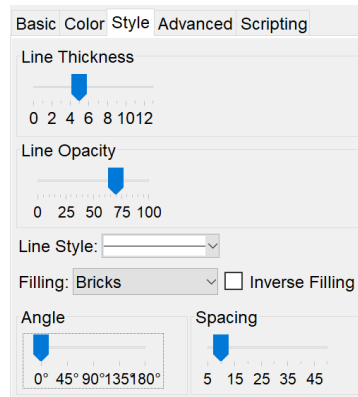
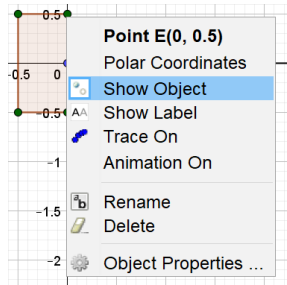
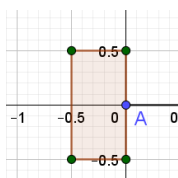


For a pin support, the graphic standard is a solid triangle pointing up. The graphic standards shown in this paper are shown in the images to the left. For the problems in this paper, they will be the biggest point possible, which

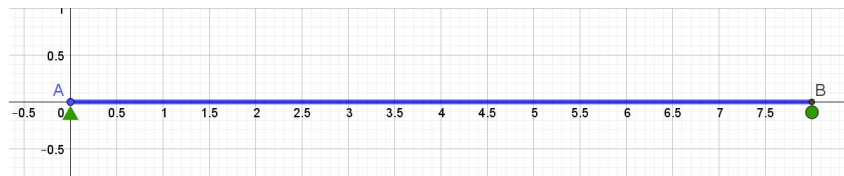
can be adjusted in the style tab. Also in the style tab, change the point style to a solid upwards pointing triangle. Adjust the color in the color tab.



Fixed supports require a few more steps. First click on the polygon button and click “Polygon.” Then, draw a rectangle by clicking on four locations around the end of the beam. In this example the points are placed at 0.5 all around point A. To “close” the rectangle, click the first point placed a second time. The points making up the rectangle can be hidden by right clicking and selecting “show object.” To change the style of the rectangle go to the object properties window, and under the style tab, the filling can be changed to the style “Bricks” and the angle can be changed to 0 degrees. There are many graphics that can be adjusted if desired. This gives the fixed support a “wall” look.

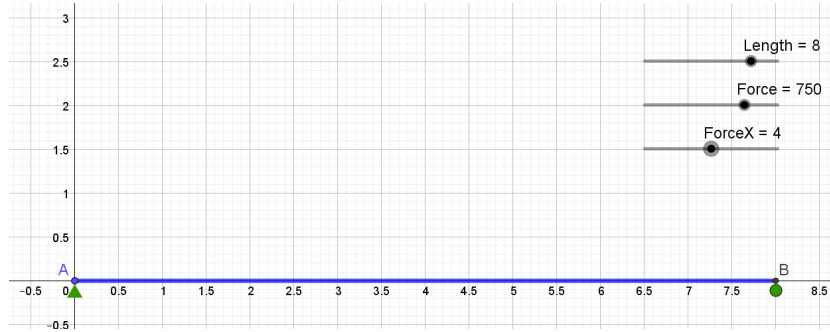


For this example, the beam will be simply supported with a pin support on one side and a roller support on the other side. At this point, the beam should look like this:



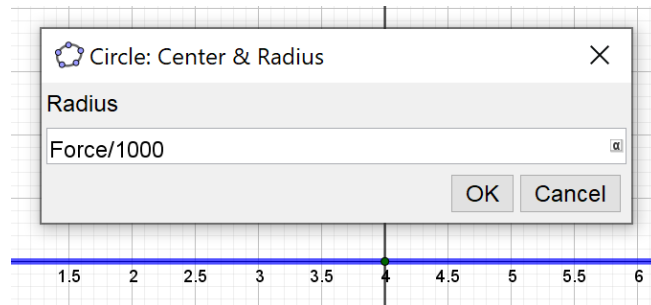
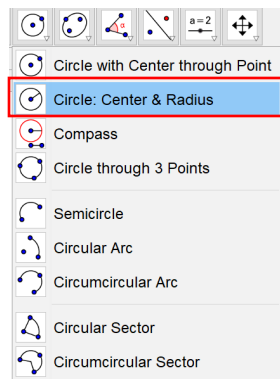
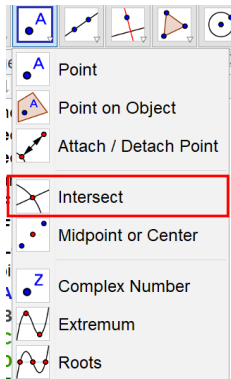
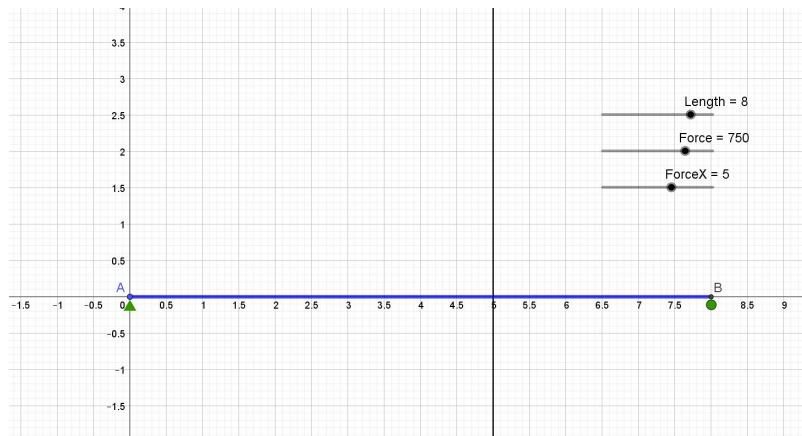
Now that the initial beam shape is drawn, some forces can be defined. This can be done by creating sliders, or by just defining it in the input bar. For example, a slider called “Force” can be

created with limits from 0 to 1000, or a force can be defined by typing “Force = 750” into the input bar. Most of these problems utilize sliders because they allow to demonstrate the power of the method’s variability. A slider is also created to define the point at which the force is acting on the beam, called “ForceX” in this example.

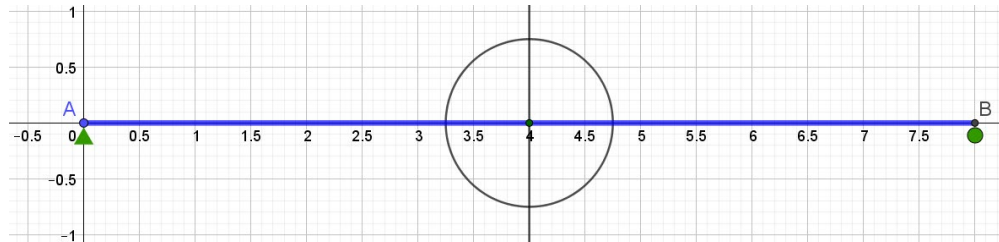


When creating sliders, limits can be set as defined variables. For instance, for the slider named “ForceX,” the minimum can be set to “0” but the maximum can be set to “Length.” This will ensure that a force is not applied beyond the length of the beam.

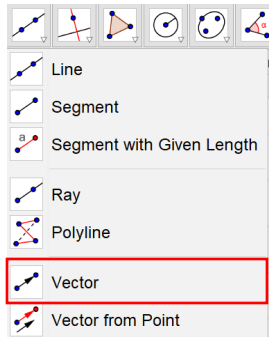
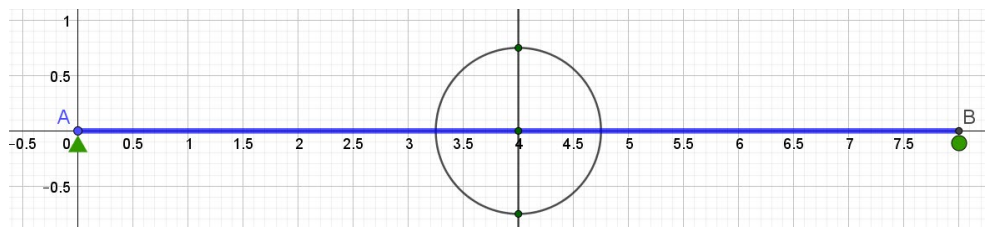
Input: $x = \text{ForceX}$ To graphically show the force, begin by creating a vertical line passing through the point the force is to be applied. This can be achieved by typing “ $x = \text{ForceX}$ ” into the input bar.



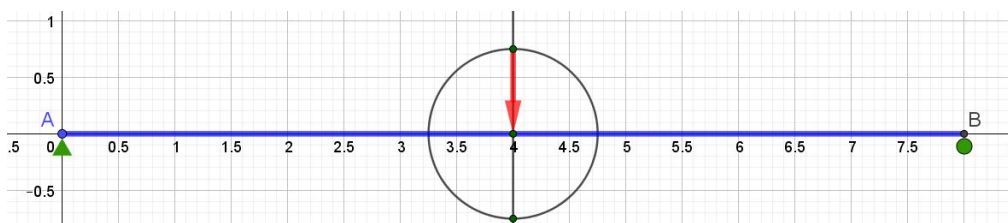
Next a point must be created at the intersection of the vertical line and the beam. Expand the options under the point button at the top of the page and click “Intersect” then select the beam and the vertical line. This will create a point at their intersection. Next create a circle by expanding the options under the circle button and selecting “Circle: Center & Radius.” Then click the point just created by the intersection of the beam and vertical line, which will be the center of the circle. A dialogue will pop up to input the radius. The radius will be the variable defined as force, and then divided by some large number in order to “scale” it. This can be adjusted in the object properties window, but here 1000 will be used to scale the size of the force. This gives all the problems a proportional look.



Next, this circle must be intersected with the vertical line. Using the intersecting tool, as before, click the two objects to intersect, the circle and the vertical lines. Two points should appear.



Next, a vector can be created to graphically show the force. Under the line button, expand the options and click on “Vector.” Then, click on the top point of the circle and the point on the beam. The graphics of the vector created can be changed in the object properties window. For these problems, the graphical standard is the thickest red line.



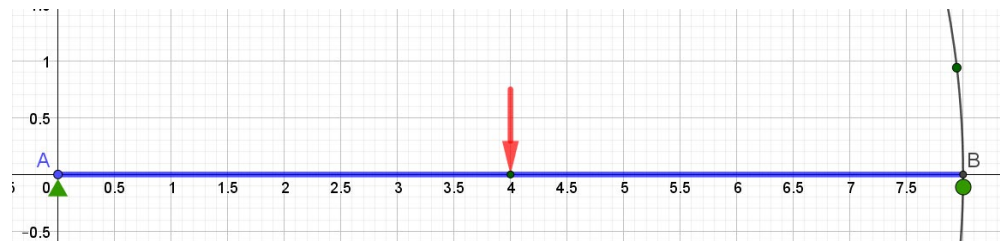
After all these shapes are created, the extra distractions can be removed. The circle, the points, and the lines can all be hidden from view as they will not be used to solve the problem. It is very

important to note that dependent objects cannot be deleted. For example, if the vertical line gets deleted, the points, the circle, and the vector all will be deleted. To avoid this, always hide objects, do not delete them. Even deleting the bottom point on the circle will delete the top point on the circle, and the vector will disappear because it was dependent on those points. To hide an object, right click and select “Show Object.” In order to make it visible again, find the object in the algebra window and either click on the circle beside the name or go to the object properties basic tab and make sure “Show Object” is checked. This can be used to hide objects as well. After hiding the objects, the beam should look like this:

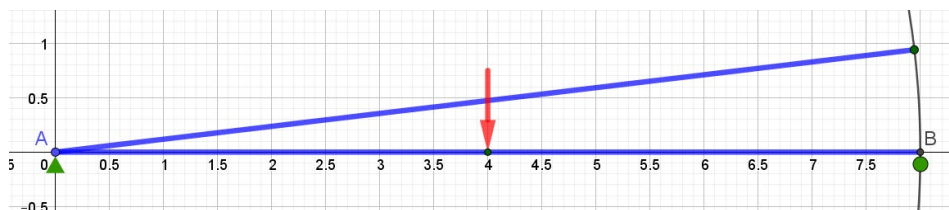


After reaching this step move around the sliders created to make sure everything is working properly. Make sure that the force arrow gets larger as the force increases and that it can move along the length of the beam. If troubleshooting is required, investigate using the object properties window to make sure the correct objects were intersected with each other and that the correct variables were defined.

Finally, the “perturbed shape” can be created. Since all elements remain the same length in the modern method, the perturbed shapes will be mainly constructed from circles. For this example, the reaction of interest will be the upwards reaction at the roller support on the right. In order to keep the beam the same length, draw a circle with the radius “Length.” Once again this can be completed with the “Circle: Center & Radius” tool. It can also be done by clicking the “Circle with Center through Point” tool. Use this by clicking on point A for the center and then point B for the point. Then create a point on the circle. Use the point tool and hover over the circle and click. The point should automatically snap to the circle. To make sure the point is attached to the circle, drag it along the circle’s path.

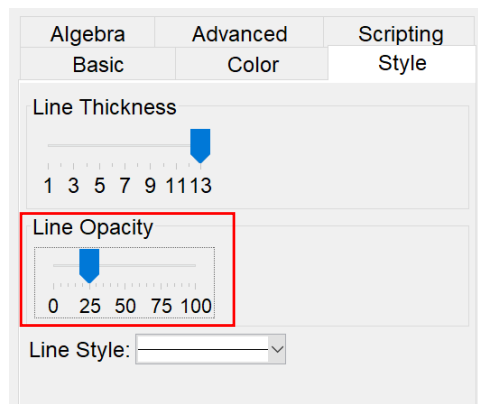
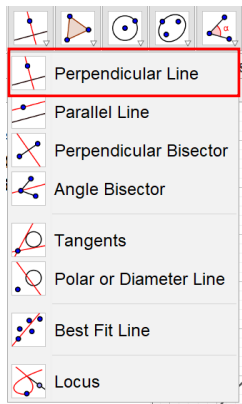
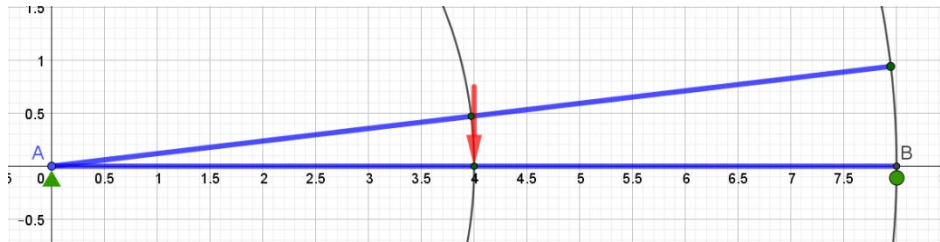


To create the final perturbed shape, draw a segment to connect point A and the point on the circle.



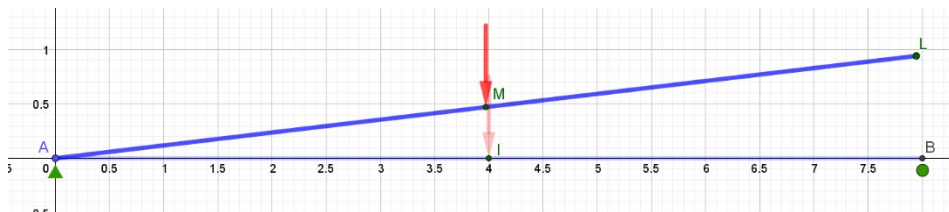
Creating a point that can be dragged along this circle gives us a very simple way of manually changing the perturbation. As these perturbations get smaller and smaller, which is also closer and closer to the original shape, the answer is approached asymptotically.

For the modern method, the loads have to follow the perturbed shape as well. To do this, create another circle that has a radius of “ForceX” or create a circle that passes through the point on the beam at the location of the load. To create a point on the beam for the new perturbed load, intersect the circle passing through the load and the new beam.



Use the same process to create a load at the new point on the perturbed shape. The vertical line can be created by using the “Perpendicular Line” tool, and clicking the beam as the perpendicular line, and then clicking the new point. Then follow the same circle intersection and vector creation process to create the new force. After this is all drawn, the

original shape and loads can be changed to a lighter opacity in the object properties window to achieve better visibility and show that shape as “original.” After all the extra distractions are hidden this is what it should look like.



Note how some points like the points associated with the loads and the points on the end of the new and original beam have not been hidden because they will be used in finding measurements. They have also been edited to show their label so that they can be used in definitions in the input bar. The easiest way to find these measurements is to subtract the point values from each other. The perturbation Δ is the distance between the new beam and the old beam at the location of the support while the loft is the distance traveled by the load. To get these measurements, simply define them in the input bar.

GeoGebra is a very powerful program that can also be used as a calculator. To access the delta symbol, click the little box on the far right of the input bar containing an alpha symbol. This should expand many choices for Greek symbols. The operators are the same as any calculator, (+ addition, - subtraction, * multiplication, / division.) Type “ $\Delta = y(L)-y(B)$ ” into the input bar and press enter. The vertical distance between these two points should populate in the algebra window. Calling out $y(\text{Point})$ or $x(\text{Point})$ will pull the x or y value of that point, and they can be easily used in mathematical operations. Perform the same operation to find the Loft. Next the fundamental equation can be defined in the input bar. Before typing the equation, it will have to be rearranged in terms of the reaction. Type “ $R_{xn} = \text{Force} * \text{Loft} / \Delta$ ” and press enter. The answer to the problem will populate in the algebra window. Since the equation is defined in terms of variables, changing the variables will automatically update the answer. Move around the sliders and make sure everything is working properly, troubleshoot if necessary.

Input: $R_{xn} = \text{Force} * \text{Loft} / \Delta$

Number

- Force = 750
- ForceX = 4
- Length = 8
- Loft = 0.47
- Rxn = 375
- $\Delta = 0.94$

Although this example is a very simple start to drawing problems in GeoGebra, these are the basic tools used in every problem. Most importantly, there are multiple ways to accomplish the same task, so always search for the simplest and easiest way possible. GeoGebra is a very powerful program that can also be used as a calculator. The easiest way to get through the learning curve is to experiment and play with the many tools available, which all have descriptions on how to use them.