An Extremum Principle for Charged Black Holes

A Senior Project Thesis

By

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Abstract

There is a well known static condition that characterizes how two or more charged black holes stay in place: this condition is $|q_i|=m_i$, where a black hole (labeled by i) has electric charge q_i and mass m_i. For nearly half a century, this static condition has eluded a simple physical interpretation, without appealing to forces. I provide the first proof that the static condition, $|q_i|=m_i$, corresponds to an extremum of the black holes' total energy. My approach uses geometry and calculus in the context of general relativity. For the non-specialist, this work draws upon one's intuitive familiarity with static electricity and gravity, and extends these ideas to black holes. For specialists, this work is significant since it uses the preferred approach of energy (not forces) to prove, for the first time, a cornerstone property (maximum or minimum energy) for these black holes, which is an energy property known to hold for other static black holes.

The research summary in the following pages was presented as a student talk at the 2015 CSU Student Research Competition, and won a first prize. A longer version of this work is available as a preprint [5], listed in the references on the last page.

Contents

1 Introduction and Significance

Classical mechanics considers physical interactions in terms of forces that act instantaneously in all reference frames. This assumption of instantaneous action at a distance requires that two separated observers perceive the same physical effects at the same time. Einstein's postulates of Special Relativity take into account the finite, measurable speed of light that accounts for the time dilation and length contraction perceived by reference frames moving with respect to each other. With this framework, the Newtonian (Classical) conception of gravity as an attractive force between masses can be replaced by the more accurate treatment of gravity as the curving effect that mass has on the spacetime around it. Thus, the behavior of bodies like black holes can be understood by a spacetime metric: the description of the geometry of spacetime around these objects that solve Einstein's equations of General Relativity. This thesis will consider the case of N charged black holes, all initially at rest. This means we need to specify the spacetime geometry at an instant of time, and consider only the geometry of space.

There is a well known geometry [1] describing a set of N electrically charged black holes, which remarkably all remain in place, regardless of their separations from each other. In this geometry, each black hole has the same sign of charge with the maximum possible magnitude, $|q_i|$=m_i, where q_i and m_i are the charge and mass of the ith black hole. It is tempting to interpret this situation in terms of balanced forces: the electric repulsion between all of the charges q_i is balanced by the gravitational attraction between all of the masses m_i. However, for black holes, the concept of balanced forces is not precise, since gravity is not a force in general relativity.

Energy is a valid concept in relativity, as in the famous energy-mass relation, $E = Mc^2$. In this thesis, I replace the concept of balanced forces with extremized energy. An "extremum" means a maximum or a minimum. As an analogy, a ball placed at rest on a hill has its potential energy maximized, and thus remains at rest. The "extremization" of energy is the process of finding a maximum or minimum energy of the system; mathematically, this involves using calculus to set the first partial derivatives of the energy function equal to zero. Evaluating the second derivatives then determines if the extremum is a maximum or a minimum. The main result of this thesis is the proof the following extremum principle.

Extremum Principle: *For a set of N arbitrarily charged black holes, all initially at rest, the static condition for each black hole ($|q_i|$=m_i) is the condition for which the total energy E is extremized. This extremum is taken holding the black holes' charges q_i constant and their separation distances r_{ij} constant. I will prove this for large separations r_{ij}, as an expansion in powers of the inverse distances ($1/r_{ij}$).*

My expansion method, in powers of the inverse distances ($1/r_{ij}$), is analogous to how an archeologist uncovers a buried artifact, layer by layer. In this thesis, I will acquire an expression for the total energy E of a configuration of N charged black holes from a known geometry [4], in which the ith black hole has arbitrary mass m_i and arbitrary charge q_i. For this general geometry, I will extremize the energy by setting its appropriate partial derivatives equal to zero, and from this obtain the static condition ($|q_i|$=m_i).

The type of energy extremum property that I prove is one of the cornerstone laws of static black holes. Due to this thesis, this extremum law now includes charged black holes. The particular form of this extremum law, and the particular quantities that are held constant, must be verified separately for each black hole system. The other types of static black holes obeying this law range from an uncharged and nonrotating black hole [2] to theoretical black holes in extra dimensions [3].

I use standard relativistic units, in which the speed of light is $c = 1$, the gravitational constant is $G = 1$, and the electric Coulomb constant is $1/(4\pi\varepsilon_0) = 1$. This means that energy and mass will appear in my equations without the relative factor of c^2 that appears in the energy-mass relation, $E = Mc^2$.

2 Summary of Known Geometry

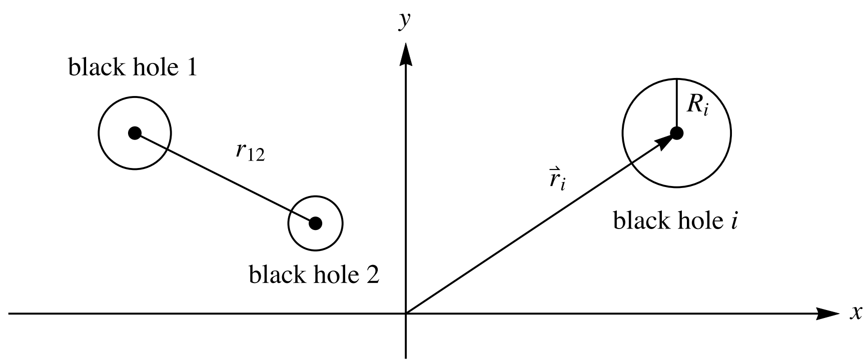

Figure 1: Illustration of three black holes ($N = 3$). Black holes 1 and 2 are separated by distance r_{12} .The ith black hole has position $\vec{r}_i$ and radius R_i.

I consider a set of N charged black holes, all initially at rest. The geometry for this situation was found in [4],

$$ds^2 = f^2 \left(dx^2 + dy^2 + dz^2 \right) \; . \tag{1}$$

Here ds is the distance between two nearby points in space coordinates (x, y, z). The function f is

$$f = \left(1 + \sum_{i=1}^{N} \frac{\alpha_i}{|\vec{r} - \vec{r}_i|} \right) \left(1 + \sum_{i=1}^{N} \frac{\beta_i}{|\vec{r} - \vec{r}_i|} \right) \; . \tag{2}$$

The vector $\vec{r}$ locates any point outside the black holes. As shown in Figure 1, the ith black hole is located at position $\vec{r}_i$, and the distance between the ith and the jth black hole is

$$r_{ij} = |\vec{r}_i - \vec{r}_j| \; . \tag{3}$$

I will consider large values of r_{ij}, and expand all functions in powers of $1/r_{ij}$ (a small quantity) using the following standard approach. I define the order n of the proof as the highest power, $(1/r_{ij})^n$, that I retain in the expansions. To zeroth order, $(1/r_{ij}) \to 0$ which corresponds to $r_{ij} \to \infty$. To first order, I retain all terms proportional to $(1/r_{ij})$. To second order, I retain all terms proportional to $(1/r_{ij})^2$, and so on.

In (2), the parameters α_i and β_i are constants. They are related to physical quantities (energy, mass, and charge) as follows [4]. The ith black hole has mass m_i and charge q_i,

$$m_i = \alpha_i + \beta_i + \sum_{j \neq i} \frac{(\alpha_i \beta_j + \alpha_j \beta_i)}{r_{ij}} \qquad , \qquad q_i = \beta_i - \alpha_i + \sum_{j \neq i} \frac{(\beta_i \alpha_j - \beta_j \alpha_i)}{r_{ij}} \ . \tag{4}$$

As found in [4], the system of N black holes has total energy E and interaction energy E_{int}:

$$E = \sum_{i=1}^{N} (\alpha_i + \beta_i) \qquad , \qquad E_{int} = E - \sum_{i=1}^{N} m_i = - \sum_{i=1}^{N} \sum_{j \neq i} \frac{(\alpha_i \beta_j + \alpha_j \beta_i)}{r_{ij}} \ . \tag{5}$$

For large r_{ij}, useful expressions for α_i and β_i are given in [4] to first order in $1/r_{ij}$:

$$\alpha_i = \frac{(m_i - q_i)}{2} \left[1 - \frac{1}{2} \sum_{j \neq i} \frac{(m_j + q_j)}{r_{ij}} \right] \qquad , \qquad \beta_i = \frac{(m_i + q_i)}{2} \left[1 - \frac{1}{2} \sum_{j \neq i} \frac{(m_j - q_j)}{r_{ij}} \right] \ . \tag{6}$$

Using this, for large r_{ij}, the interaction energy can be written to first order as [4]

$$E_{int} = \sum_{i=1}^{N} \sum_{j > i} \frac{(q_i q_j - m_i m_j)}{r_{ij}} \ . \tag{7}$$

As a check, this is consistent with a Newtonian formulation. Each pair of like charges has a repulsive electrical effect and contributes a positive term $+(q_i q_j)/r_{ij}$ to the potential energy. This opposes the gravitational attraction of the masses and the negative gravitational potential energy, $-(m_i m_j)/r_{ij}$. Although (7) appears Newtonian, black holes require the geometrical approach entailed by general relativity, which I now commence.

3 Proof at Zeroth Order and First Order

The first step in finding the conditions that extremize the energy of a system of charged black holes is to use the geometry (1) to calculate the area of each black hole. Near each black hole, I can use spherical coordinates (r, θ, ϕ) centered at the black hole's position. In general, the black hole is a surface $r(\theta, \phi)$. If the black holes are sufficiently far apart, then each black hole i can be treated as a sphere, $r(\theta,\phi) = R_i =$ constant, as in [4]. From the geometry (1), the differential area of the black hole surface is

$$dA = f^2 {R_i}^2 \sin^2\theta \, d\theta \, d\phi \, . \tag{8}$$

Here the function f in (2) and the sphere radius R_i are [4]

$$f = \left(1 + \frac{\alpha_i}{R_i} + \sum_{j\neq i} \frac{\alpha_j}{r_{ij}}\right)\left(1 + \frac{\beta_i}{R_i} + \sum_{j\neq i} \frac{\beta_j}{r_{ij}}\right) \quad , \quad {R_i}^2 = \frac{\alpha_i \beta_i}{\left(1 + \sum_{j\neq i} \frac{\alpha_j}{r_{ij}}\right)\left(1 + \sum_{j\neq i} \frac{\beta_j}{r_{ij}}\right)} \, . \tag{9}$$

Integrating the differential area dA over the angles θ and ϕ yields the following area formula:

$$A_i = 4\pi {R_i}^2 f^2 = 4\pi {R_i}^2 \left(1 + \frac{\alpha_i}{R_i} + \sum_{j\neq i} \frac{\alpha_j}{r_{ij}}\right)^2 \left(1 + \frac{\beta_i}{R_i} + \sum_{j\neq i} \frac{\beta_j}{r_{ij}}\right)^2 \, . \tag{10}$$

I now prove the extremum principle through first order. Thus, I retain all terms proportional to $(1/r_{ij})$ but I neglect $(1/r_{ij})^2$ and all higher powers when expanding. After doing this, for the area (10) I find

$$\sqrt{\frac{A_i}{4\pi}} = \alpha_i + \beta_i + \sqrt{\alpha_i \beta_i} \left[2 + \sum_{j\neq i} \frac{(\alpha_j + \beta_j)}{r_{ij}}\right] + \sum_{j\neq i} \frac{(\alpha_i \beta_j + \beta_i \alpha_j)}{r_{ij}} \, . \tag{11}$$

I want to express this black hole area A_i in term of physical quantities, the mass m_i and charge q_i, so I substitute α_i and β_i in (6). Then simplifying yields

$$\sqrt{\frac{A_i}{4\pi}} = m_i + \frac{\sqrt{m_i^2 - q_i^2}}{2} \left[1 - \frac{1}{2} \sum_{j\neq i} \frac{m_j}{r_{ij}}\right] \left[2 + \sum_{j\neq i} \frac{m_j}{r_{ij}}\right] \, . \tag{12}$$

Upon expansion, I see that the terms proportional to $1/r_{ij}$ exactly cancel, and I disregard the product proportional to $(1/r_{ij})^2$, so that

$$\sqrt{\frac{A_i}{4\pi}} = m_i + \sqrt{m_i^2 - q_i^2} \, . \tag{13}$$

Solving this for the mass m_i yields:

$$m_i = \sqrt{\frac{\pi}{A_i}} \left(\frac{A_i}{4\pi} + {q_i}^2\right) \, . \tag{14}$$

Due to the cancellation noted above, this mass result m_i contains no terms proportional to $1/r_{ij}$, so this first order result is the same as zeroth order result. I will return to this observation in the next section.

I now find the conditions for extremizing the total energy E that will lead to the static condition, $|q_i| = m_i$. Recalling (5) and (7), the total energy is:

$$E = \sum_{i=1}^{N} m_i + \sum_{i=1}^{N} \sum_{j>i} \frac{(q_i q_j - m_i m_j)}{r_{ij}} \; . \tag{15}$$

Here each mass m_i is given by the result (14). This means we can regard the energy E as a function of several variables: q_i, A_i, and r_{ij}. In order to obtain the static condition, extremizing the energy E could require holding some variables constant. I examined all of the possible choices for holding some variables constant. I found that only the following choice furnishes the static condition, $|q_i| = m_i$: I extremize the total energy E by setting to zero its partial derivatives with respect to each area A_i, while holding the charges q_i and distances r_{ij} constant. This requires:

$$\frac{\partial E}{\partial A_i} = \frac{\partial m_i}{\partial A_i} \left(1 - \sum_{j \neq i} \frac{m_j}{r_{ij}} \right) = 0 \; . \tag{16}$$

The term in parentheses is nonzero for sufficiently large separations r_{ij}, so (16) requires

$$\frac{\partial m_i}{\partial A_i} = 0 \; . \tag{17}$$

Evaluating this condition by differentiating (14) yields

$$|q_i| = \frac{1}{2} \sqrt{\frac{A_i}{\pi}} \; . \tag{18}$$

I now evaluate the mass m_i in (14) for this charge value. Substituting this result for q_i into (14) gives

$$m_i = \frac{1}{2} \sqrt{\frac{A_i}{\pi}} \; . \tag{19}$$

Upon comparing (18) and (19), one sees that $m_i = |q_i|$, which is the static condition. Thus, I have proved the extremum principle through first order. It is also straightforward to see that the energy E is a minimum, by checking that its second partial derivatives with respect to A_i are positive.

4 Higher Orders

Below, I summarize the steps for extending my proof of the extremum principle to second order. At second order, I must treat each black hole as a nonspherical surface $r(\theta, \phi)$. For the simple case of N black holes along a line, this surface simplifies to a function $r(\theta)$. The geometry of this surface is

$$ds^2 = f^2 \left[\left(r^2 + \dot{r}^2 \right) d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \qquad , \qquad \text{where} \qquad \dot{r} = \frac{dr}{d\theta} \; . \tag{20}$$

This is obtained from (1). The expression for f, to be expanded in powers of $1/r_{ij}$, generalizes (9). The differential area element for the geometry (20) is

$$dA = f^2 r \sin\theta \sqrt{r^2 + \dot{r}^2} \, d\theta \, d\phi \; , \tag{21}$$

so the area of the surface is

$$A = \int_0^\pi 2\pi f^2 r \sin\theta \sqrt{r^2 + \dot{r}^2} \, d\theta = \int_0^\pi I \, d\theta \; . \tag{22}$$

This generalizes (10). Given the integrand $I(r, \dot{r}, \theta)$ in (22), it is well known from [4] that the black hole surface $r(\theta)$ is then found by solving the equation

$$\frac{d}{d\theta} \left(\frac{\partial I}{\partial \dot{r}} \right) = \frac{\partial I}{\partial r} \; . \tag{23}$$

This illustrates how the proof at higher orders — involving smaller and smaller separation distances — becomes increasingly more complicated. A simpler alternative (a supporting idea) is suggested by the cancellation that occurred to yield the same mass result (14) at zeroth order and first order. Also note that at first order, the interaction energy E_{int} in (7) vanishes for like-sign charges if $|q_i| = m_i$. The vanishing of E_{int} is a precise statement about energy that appears to support my extremum principle. It is physically reasonable to conjecture that E_{int} vanishes exactly at all orders, if $|q_i| = m_i$. This conjecture does not require calculating black hole area, so I expect to be able to prove it for all orders, which could therefore accommodate smaller black hole separation distances.

5 Conclusion

My proof of the extremum principle at higher orders, as outlined above, is available in [5]. Reference [5] is a detailed preprint article (including some additional comments beyond the scope of this thesis) that has been submitted for publication; the webpage in [5] will list the journal reference for the published version. To my knowledge, this thesis provides the first energy interpretation of the known static condition ($|q_i|$=m_i) for charged black holes. I have addressed fundamental questions about charged black holes' interactions using the preferred approach of energy, not forces. This thesis revisits two separate geometries that have been known for nearly fifty years, and finds a relationship between them: for a set of black holes with arbitrary charges and masses [4], the static configuration [1] is the setup which extremizes the total energy.

References

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