Static Interaction of Black Holes in 1+1 Dimensions

A Senior Project Thesis

By

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Abstract

We consider a version of general relativity in two spacetime dimensions, and study a solution describing two static black holes in the presence of a cosmological constant. We first analytically find an embedding diagram to visualize the geometry outside the black holes. We then examine how the two black holes must be interacting to remain static. Our main result is to show how the black holes behave effectively like two electric charges. This charge model exhibits both attraction and repulsion, which evidently balance and moreover are localized in different regions of space. We also begin an investigation of the black holes' interaction in terms of the gravitational energy localized in a region (similar to a Gauss's law approach). One application of these static black holes is to construct a static wormhole, which was started in a previous thesis. Here we finish this construction by verifying the wormhole's smoothness where the two black hole horizons are matched together.

Contents

1 Introduction

General relativity [1] was first introduced to us by Albert Einstein in 1915 and provided a new way of understanding our universe. General relativity is a geometric theory of gravity. It was originally written in four spacetime dimensions: three spatial dimensions and one time dimension. This number of dimensions describes our observable universe. Einstein's theory has passed several experimental tests. Versions of this theory now exist both in higher dimensions and in lower dimensions, including the lowest number of dimensions which is often called 1+1 dimensions to refer to one spatial dimension and one time dimension. This two-dimensional theory [2] is what will be using in this thesis.

By reducing general relativity to 1+1 dimensions, we greatly simplify the calculations required. It will also allow us to see important physics at work. We will be examining the static interaction that occurs when two black holes are placed near each other. We will do this in two ways: first, by seeing how the system behaves like two charged point sources and second, by examining the gravitational energy [3] contained within a region. This second method is similar to a Gauss's law approach.

We will also perform a calculation (called an embedding) in order to visualize the system of two black holes. These black holes can also be used to construct a wormhole by connecting their horizons together [4]. We verified that this wormhole solution is smooth where we perform the matching. This is similar to a wormhole constructed previously by Misner [5] but has a new feature that it is static.

2 General Relativity and Differential Geometry

2.1 Introduction

General relativity is a geometric theory of gravity that incorporates special relativity as a special case. In special relativity, the speed of light, c, plays a crucial role and it also appears in general relativity. Throughout this thesis, for convenience we will use standard relativistic units in which the speed of light has the value $c = 1$. This means that that quantities such as ct will be written simply as t where t is the time coordinate in spacetime.

Unlike Newton's theory of gravity, general relativity does not treat gravity as a force, but rather as being due to the curvature of spacetime. This means in order to discuss general relativity in any detail, we must discuss some details of geometry and curvature. Therefore in the sections below, we will review the basics of differential geometry. The key quantities we review are tensors, covariant derivatives, and curvature. After this, we will be able to state the field equations of general relativity.

2.2 Tensors

A tensor is defined as a quantity with components having specific transformation properties under a coordinate transformation. The number of components is called the rank of the tensor. The components of a second rank tensor in can be written simply as $M_{\alpha\beta}$ where each index α and β range over the spacetime dimensions: for example in four spacetime dimensions $\alpha = t, x, y, z$ and similarly for β. The notation $M_{\alpha\beta}$ may refer to an individual component of the tensor, or it may be used to summarize the collection of all the components of the tensor: for example if we write the tensor $M_{\alpha\beta}$ in matrix form, we obtain

$$M_{\alpha\beta} = \begin{bmatrix} M_{tt} & M_{tx} & M_{ty} & M_{tz} \\ M_{xt} & M_{xx} & M_{xy} & M_{xz} \\ M_{yt} & M_{yx} & M_{yy} & M_{yz} \\ M_{zt} & M_{zx} & M_{zy} & M_{zz} \end{bmatrix}. \quad (1)$$

Here, the convention is M_{tt} is the time component, M_{xx} is the x spatial component, et cetera. Many (but not all) physical second rank tensors are symmetric. This means in four spacetime dimensions there are 10 unique components. When a tensor of this nature is used in an equation such as $M_{\alpha\beta} = N_{\alpha\beta}$, the result of this is actually 10 unique equations.

If there are repeated upper and lower indices, this indicates the Einstein summation notation, meaning that we must do a sum over the repeated indices. For example:

$$N^{\alpha\beta}M_{\alpha\beta} = \sum_{\alpha=(t,x,y,z)} \sum_{\beta=(t,x,y,z)} N^{\alpha\beta}M_{\alpha\beta} = N^{tt}M_{tt} + N^{xt}M_{xt} + ... + N^{zz}M_{zz}. \quad (2)$$

In this example, the result of this sum is a scalar quantity.

2.3 The Metric Tensor

The geometry of spacetime is described by the metric tensor $g_{\alpha\beta}$ which is a symmetric tensor. In four spacetime dimensions,

$$g_{\alpha\beta} = \begin{bmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ g_{xt} & g_{xx} & g_{xy} & g_{xz} \\ g_{yt} & g_{yx} & g_{yy} & g_{yz} \\ g_{zt} & g_{zx} & g_{zy} & g_{zz} \end{bmatrix}. \tag{3}$$

A simple example is flat space, where

$$g_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

We can also show the information contained within the metric by writing the line element

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{5}$$

This is the four dimensional counterpart to the Pythagorean theorem in three flat spatial dimensions. The differences here are an added time dimension, and an opposite sign between time and space dimensions. The quantity ds^2 is used to define the square of a distance between two nearby events in spacetime. In general, the line element is, using the summation convention,

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta. \tag{6}$$

The metric that we will be using in two spacetime dimensions will be defined later in this thesis.

2.4 Covariant Derivatives

The covariant derivative of a tensor depends on the type of tensor and the placement of its indices. The covariant derivative of a tensor of rank n is defined so that the covariant derivative of the tensor is a tensor of rank $n+1$. When we take the covariant derivative of a rank one tensor v^β with its index upstairs, we receive a second rank tensor. The covariant derivative of v^β is defined as

$$\nabla_\alpha v^\beta = \frac{\partial v^\beta}{\partial x^\alpha} + \Gamma^\beta_{\alpha\gamma} v^\gamma \tag{7}$$

where $\Gamma^\gamma_{\alpha\beta}$ is the Christoffel symbol defined below. Similarly, the covariant derivative of a tensor v_β with its index downstairs is defined as

$$\nabla_\alpha v_\beta = \frac{\partial v_\beta}{\partial x^\alpha} - \Gamma^\gamma_{\alpha\beta} v_\gamma. \tag{8}$$

We can also take the covariant derivative of a tensor of rank two. For example, the covariant derivative of a tensor $t^{\alpha\beta}$ is defined as

$$\nabla_\gamma t^{\alpha\beta} = \frac{\partial t^{\alpha\beta}}{\partial x^\gamma} + \Gamma^\alpha_{\gamma\delta} t^{\delta\beta} + \Gamma^\beta_{\gamma\delta} t^{\alpha\delta}. \tag{9}$$

Similarly to Eqn. 8, the covariant derivative of a tensor $t_{\alpha\beta}$ is defined as

$$\nabla_\gamma t_{\alpha\beta} = \frac{\partial t_{\alpha\beta}}{\partial x^\gamma} - \Gamma^\delta_{\alpha\gamma} t_{\delta\beta} - \Gamma^\delta_{\beta\gamma} t_{\alpha\delta}. \tag{10}$$

The covariant derivative has the property that

$$\nabla_\gamma g_{\alpha\beta} = 0 \tag{11}$$

due to the dependence of the Christoffel symbols' dependence on the metric. This holds true for the inverse metric as well, which we define below.

The Christoffel symbol $\Gamma^\gamma_{\alpha\beta}$ is defined as

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\delta\gamma} \left(\frac{\partial g_{\delta\alpha}}{\partial x^\beta} + \frac{\partial g_{\delta\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\delta} \right). \tag{12}$$

The Christoffel symbols are thus related to first partial derivatives of the metric. Here $g^{\alpha\beta}$ denotes the inverse metric tensor, meaning it is the inverse to the matrix whose components are the components of the metric tensor $g_{\alpha\beta}$. In this case, careful attention to the placement of the indices is very important.

2.5 General Relativity

In general relativity, the geometry of spacetime is described by the metric tensor $g_{\alpha\beta}$. After a choice of coordinates, the metric tensor of four-dimensional spacetime is often a diagonal tensor,

$$g_{\alpha\beta} = \begin{bmatrix} g_{tt} & 0 & 0 & 0 \\ 0 & g_{xx} & 0 & 0 \\ 0 & 0 & g_{yy} & 0 \\ 0 & 0 & 0 & g_{zz} \end{bmatrix}. \tag{13}$$

We can also show this in the line element

$$ds^2 = g_{tt}dt^2 + g_{xx}dx^2 + g_{yy}dy^2 + g_{zz}dz^2. \tag{14}$$

The metric tensor is found by solving the Einstein Equation,

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = 8\pi G T_{\alpha\beta}, \tag{15}$$

where $R_{\alpha\beta}$ is the Ricci curvature tensor (which will be given below) and $R = g^{\alpha\beta}R_{\alpha\beta}$ is the Ricci curvature scalar. Thus the Einstein Equation states that curvature is determined by the stress-energy-momentum tensor $T_{\alpha\beta}$ which describes any sources of matter or energy.

In the Einstein Equation, G is the gravitational constant, the same value as in Newton's theory of gravity. The quantity Λ is a cosmological constant, which physically provides a constant energy density throughout space; in our observed universe with four spacetime dimensions, $\Lambda > 0$ and is the best model to date that explains the accelerating expansion of our universe, as determined by recent experiments.

The Ricci curvature tensor $R_{\alpha\beta}$ is defined as

$$R_{\alpha\beta} = \frac{\partial \Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\gamma}_{\alpha\gamma}}{\partial x^{\beta}} + \Gamma^{\gamma}_{\alpha\beta}\Gamma^{\delta}_{\gamma\delta} - \Gamma^{\gamma}_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} \tag{16}$$

where $\Gamma^{\gamma}_{\alpha\beta}$ are the Christoffel symbols defined above. Since the Ricci tensor contains first derivatives of the Christoffel symbols, the Einstein Equation is a set of second order differential equations for the metric tensor $g_{\alpha\beta}$.

In this thesis, we will consider a version of general relativity in two spacetime dimensions. This will greatly reduce the complexity of the field equations, and will also allow some interesting physics to emerge.

3 Black Holes in 1+1 Dimensions

3.1 General Relativity in 1+1 Dimensions

One aspect of this thesis is to examine a known geometry describing a pair of static black holes, and to study how the black holes must be interacting with each other in order to remain static. In general, the interaction between two black holes can be difficult to determine analytically. Therefore, as a simplification, we will will consider a version of general relativity in two spacetime dimensions, often called 1+1 dimensions. Here we have one time dimension t and one spatial dimension x. With this simplification, our metric becomes

$$g_{\alpha\beta} = \begin{bmatrix} g_{tt} & 0 \\ 0 & g_{xx} \end{bmatrix} \tag{17}$$

and the line element is

$$ds^2 = g_{tt}dt^2 + g_{xx}dx^2 \tag{18}$$

with g_{tt} and g_{xx} defined later in this thesis.

In 1+1 dimensions, an established version of the Einstein Equation is [2]

$$R - \Lambda = 8\pi GT \tag{19}$$

where R is the Ricci scalar and $T = g^{\alpha\beta}T_{\alpha\beta}$ is the trace of the stress-energy-momentum tensor. Here G is the gravitational constant for 1+1 dimensions. The value of G is not specified in this theory, although G is analogous to Newton's gravitational constant in four spacetime dimensions. In this equation, Λ is a cosmological constant, which is analogous to the positive cosmological constant in our observed universe with four spacetime dimensions. We can see that Eqn. 19 now becomes one second order differential equation that will depend on the metric $g_{\alpha\beta}$ due to the Ricci scalar.

We will work with a static spacetime of the form

$$g_{\alpha\beta} = \begin{bmatrix} -\alpha & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix} \tag{20}$$

where α is a function of x only and not a function of t. For this reason it describes a static spacetime. We can also display this information about the geometry with the line element

$$ds^2 = -\alpha\, dt^2 + \frac{dx^2}{\alpha}. \tag{21}$$

For this metric, the nonzero Christoffel symbols are [2]

$$\Gamma^t_{tx} = \frac{1}{2}\alpha^{-1}\frac{d\alpha}{dx} \tag{22}$$

$$\Gamma^x_{xx} = -\frac{1}{2}\alpha^{-1}\frac{d\alpha}{dx} \tag{23}$$

$$\Gamma^x_{tt} = \frac{1}{2}\alpha\frac{d\alpha}{dx}. \tag{24}$$

After evaluating the Ricci scalar R for the metric given in Eqn. 21, the 1+1 dimensional field equation, Eqn. 19, becomes [2]

$$\frac{d^2\alpha}{dx^2} = -8\pi GT - \Lambda. \tag{25}$$

This is the field equation for a general static spacetime. Below, we will consider a specific solution for α which describes static black holes.

3.2 Geometry for Two Static Black Holes

Here we will consider a specific solution to the field equation Eqn. 25 which describes static black holes. A black hole is a region of strong spacetime curvature, so strong that a particle falling inside it cannot escape. The surface of a black hole is called the black hole's event horizon, or simply the horizon. In Eqn. 21, a black hole horizon is where $\alpha = 0$.

A solution to Eqn. 25 that describes N static black holes can be obtained by using a set of N point sources. In this case, the trace T of the stress-energy-momentum tensor is [2]

$$T = -\frac{1}{2\pi G}\sum_{i=1}^{N} M_i \delta(x - x_i) \tag{26}$$

where M_i is a mass parameter for the source located at $x = x_i$. After solving Eqn. 25 for α, the general solution is [2]

$$\alpha = -\frac{\Lambda x^2}{2} - \xi + 2\sum_{i=1}^{N} M_i \left(|x - x_i|\right) \tag{27}$$

where ξ is an arbitrary constant.

For the case of $N = 2$ symmetric point sources located at $x = \pm a$, these results become

$$T = -\frac{m}{2\pi G}\left[\delta(x-a) + \delta(x+a)\right] \tag{28}$$

and

$$\alpha = -\frac{\Lambda x^2}{2} - \xi + 2m\left(|x-a| + |x+a|\right). \tag{29}$$

Looking at T, we notice that it only contributes when we are sitting at $x = -a$ and $x = a$. When we are not at these locations, then T is zero due to the usual properties of delta functions. The points $x = \pm a$ locate a black hole singularity inside each black hole. In general, a black hole singularity is where spacetime curvature is infinite; in this case, from Eqn. 19 we see that each singularity corresponds to a delta function in the value of the Ricci curvature scalar R.

Now let's consider the function α. If we break up space into three regions, $x < -a$ to the left, $-a < x < a$ in between the black hole singularities, and $x > a$ to the right, we can write α as [4]

$$\alpha = -\frac{\Lambda x^2}{2} - \xi + 4m\begin{cases} -x, & \text{if } x < -a \\ a, & \text{if } -a < x < a \\ x, & \text{if } x > a \end{cases} \tag{30}$$

We see that the solution α in Eqn. 30 has four parameters: the mass parameter m, the locations of the singularities $\pm a$, the cosmological constant Λ, and the constant ξ. Arbitrary values of these parameters will not always produce two distinct black hole horizons; often the result will be a single horizon, even when there are N point sources. In order for α in Eqn. 30 to describe two distinct black holes, the values of the parameters must be chosen carefully. For α to describe two distinct black holes, the full set of conditions are given in [4] and include the condition $\Lambda > 0$ Figure 1 shows a plot of α for two distinct black holes.

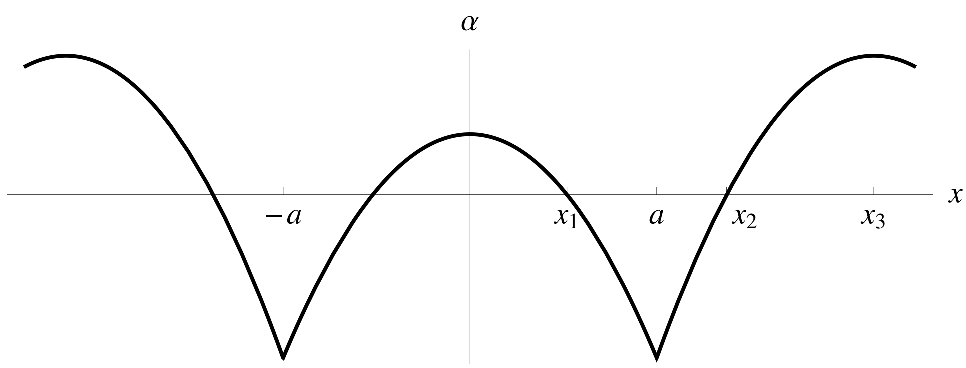

Figure 1: An example plot of $\alpha(x)$ for two symmetric static black holes.

In Figure 1, the points $x = x_1$ and $x = x_2$ are the edges of one black hole horizon. The point $x = a$ is the singularity inside this black hole. The point $x = x_3$ is where α flattens (where $d\alpha/dx = 0$). There is also a location (not shown) at $x_4 > x_3$ where $\alpha = 0$ which is a sometimes called a cosmological horizon, but will not be needed in this thesis. The values for x_1, x_2 and x_3 are given by [4]

$$x_1 = \sqrt{\frac{8ma - 2\xi}{\Lambda}} \tag{31}$$

$$x_2 = \frac{4m - \sqrt{2}\sqrt{8m^2 - \Lambda\xi}}{\Lambda} \tag{32}$$

$$x_3 = \frac{4m}{\Lambda} \tag{33}$$

These values will be used later in this thesis.

3.3 Gravity and Static Test Particles

As summarized above, the spacetime geometry for two static black holes is described by the function α. One aspect of this thesis is to study the interaction between these two black holes. In this section, we will however examine static *test particles*. We will do this to distinguish the simple gravitational rules for *test particles* from the more subtle interaction we expect for static *black holes*.

Recall that in general relativity, gravity is not a force, and is due to spacetime curvature. This can be restated as saying that a small test particle moving only under the influence of gravity will follow a geodesic path. We can quickly obtain the direction a test particle would follow under the influence of gravity, but without the formalism needed for calculating geodesic paths, by considering a test particle that does not move in space at at all: this is a *static* test particle.

The acceleration of a test particle is defined as

$$a_\gamma = u^\beta \nabla_\beta u_\gamma \tag{34}$$

where u_γ is the particle's two-velocity which satisfies $g_{\alpha\beta}u^\alpha u^\beta = -1$. A particle following a geodesic has $a_\gamma = 0$. A static test particle has $u_\gamma = (u_t, u_x) = (u_t, 0)$. Using the inverse metric to raise the indices, we have $u^\gamma = (-\alpha^{-1}u_t, 0)$ and $u^t u_t = -1$. We expand Eqn. 34 using the covariant derivative Eqn. 8 and the appropriate Christoffel symbols from Eqns. 22$-$24, and we find the acceleration to be

$$a_\gamma = (a_t, a_x) = \left(-\alpha^{-1}u_t \frac{\partial u_t}{\partial t}, \frac{\alpha^{-2}u_t^2}{2}\frac{\partial \alpha}{\partial x}\right) = \left(0, \frac{\alpha^{-1}}{2}\frac{d\alpha}{dx}\right). \tag{35}$$

This means a static test particle outside a black hole (where $\alpha > 0$) has acceleration a_x that points in the same direction as $d\alpha/dx$ which is the slope of α in Figure 1. This acceleration is a force per unit mass keeping the particle from moving under the influence of gravity. Thus the gravitational effect alone must point in the direction of $-a_x = -(1/2\alpha)(d\alpha/dx)$, and the slopes in Figure 1 shows this points toward whichever black hole the particle is nearest to. The black holes themselves are not behaving like test particles, since they are not moving toward each other, and we will investigate their interaction later in this thesis.

There are also locations where $d\alpha/dx = 0$ which is where a test particle experiencing only gravity alone would actually be static, if it is placed there at rest. Such a location is an unstable equilibrium, but the fact that a test particle moving under the influence of gravity alone can be static suggests (at least at this location) there is some kind of balance between an attracting effect and a repulsive effect.

If we extend the idea just mentioned to the black holes themselves, we would expect some kind of repulsive effect must be present in how the black holes interact. We might expect the positive cosmological constant Λ would provide a purely repulsive effect between the black holes, based on the experimentally confirmed accelerating expansion of our own universe in four spacetime dimensions. The situation is actually more subtle than this, and we will investigate this issue when we discuss our electric charge model in section 5.4.

4 Embedding in Flat Space

4.1 Introduction

In order to get a visual representation of what our static geometry would look like outside of the black holes, we embed the geometry described by Eqn. 21 in an embedding space. To do this, we take a slice of our geometry by setting $t = constant$, making $dt^2 = 0$. So the slice that we want to embed is

$$ds_s^2 = \frac{dx^2}{\alpha} \tag{36}$$

with ds_s^2 indicating the slice of our geometry. The space that we are going to embed this geometry in is

$$ds_e^2 = g_{xx}dx^2 + g_{zz}dz^2 \tag{37}$$

where ds_e^2 represents our embedding space. Here z is an auxiliary coordinate for visualization purposes. Because Eqn. 36 is one dimensional, this allows us to see the geometry.

We will work in the region with $x > 0$ and break this up into two separate regions, $0 < x < x_1$ and $x > x_2$. Here x_1 and x_2 are given in Eqns. 31 and 32. We can just use $x > 0$ because our geometry is symmetric about the origin $x = 0$. We break up the region $x > 0$ into two regions because from Eqn. 30, we see that α varies over this range. Our geometry will be represented as a surface in the embedding space. This surface will be of the form $z(x)$ which means

$$dz^2 = \left(\frac{dz}{dx}dx\right)^2. \tag{38}$$

When we put Eqn. 38 into Eqn. 37 and collect terms, we get

$$ds_s^2 = \left[g_{xx} + g_{zz}\left(\frac{dz}{dx}\right)^2\right]dx^2. \tag{39}$$

We require that $ds_s^2 = ds_e^2$ because they both represent the same geometry. Setting them equal to each other and rearranging the resulting equation gives us

$$g_{xx} = \frac{1}{\alpha} - g_{zz}\left(\frac{dz}{dx}\right)^2. \tag{40}$$

4.2 Region 1

In Region 1 between the black holes with $0 < x < x_1$, for simplicity we seek a surface $z_1(x)$ that looks like a quarter of an ellipse, so we take

$$z_1(x) = B_1\sqrt{1 - \frac{x^2}{x_1^2}} \tag{41}$$

where B_1 is the ellipse radius in the z direction, and x_1 is a constant that we find by looking at Eqn. 30. If we separate the terms on the right in Eqn. 30 into variables and constants, we see that

$$\alpha_1 = -\frac{\Lambda}{2}x^2 + (4ma - \xi) = \frac{\Lambda}{2}\left(x_1^2 - x^2\right), \tag{42}$$

with α_1 indicating that we are in the region $0 < x < x_1$. In this region, the point $x = x_1$ is one edge of the black hole, where $\alpha_1(x_1) = 0$ and

$$x_1 = \sqrt{\frac{8ma - 2\xi}{\Lambda}}. \tag{43}$$

Differentiating $z(x)$ and putting Eqn. 42 into Eqn. 40, we get

$$g_{xx} = \frac{\left[(2/\Lambda) - \left(g_{zz}B_1^2/x_1^2\right)x^2\right]}{x_1^2 - x^2}. \tag{44}$$

We want to have a smooth embedding space ($g_{xx} = finite$), so to avoid having $g_{xx} \to \infty$ as $x \to x_1$, we set $g_{xx} = C$ where C is a constant and take g_{zz} as a constant to be determined. This gives

$$\left[\frac{2}{\Lambda} - \left(\frac{g_{zz}B_1^2}{x_1^2}\right)x^2\right] = C\left(x_1^2 - x^2\right). \tag{45}$$

Now, we set the coefficients of the separate powers of x equal to each other, in order to solve for C and g_{zz}. Doing so, we find that $g_{xx} = C = 2/\Lambda x_1^2$ and $g_{zz} = 2/\Lambda B_1^2$.

Now that we have found g_{xx} and g_{zz}, we can plug these into Eqn. 37 and we get

$$ds_e^2 = g_{xx}dx^2 + g_{zz}dz^2 \tag{46}$$

$$= \left(\frac{2}{\Lambda x_1^2}\right)dx^2 + \left(\frac{2}{\Lambda B_1^2}\right)dz^2. \tag{47}$$

Since these metric coefficients are constant, this embedding space is flat space, meaning it has zero curvature.

4.3 Region 2

In Region 2 outside the black hole with $x > x_2$, we employ the same technique that we did for Region 1. For simplicity, we choose the surface for this region to be a segment of an ellipse as well. Here, our embedding surface will look like

$$z_2(x) = B_2\sqrt{1 - \frac{(x - x_3)^2}{(x_3 - x_2)^2}} \tag{48}$$

where B_2 is the ellipse radius in the z direction and $(x_3 - x_2)$ is the ellipse radius in the x direction.

The point $x = x_2$ is an edge of the black hole in this region, where $\alpha(x_2) = 0$. The point $x = x_3$ is where $d\alpha/dx = 0$ and the plot of α flattens in Figure 1. For simplicity, we choose this location $x = x_3$ to be where the elliptical embedding surface also flattens.

The values for x_2 and x_3 are given in Eqns. 32 and 33, from which we find the ellipse radius in the x direction is

$$(x_3 - x_2) = \frac{4m}{\Lambda} - \left(\frac{4m}{\Lambda} - \frac{\sqrt{2}\sqrt{8m^2 - \Lambda\xi}}{\Lambda} \right) = \frac{\sqrt{2}}{\Lambda}\sqrt{8m^2 - \Lambda\xi}. \tag{49}$$

We must satisfy the embedding equation Eqn. 40 in Region 2. Using the results from Region 1 for g_{xx} and g_{zz} and differentiating Eqn. 48, after some algebra we are able to obtain the following from Eqn. 40:

$$\frac{B_2^2}{(x_3 - x_2)^2 B^2}\left[x^2 - 2x_3 x + x_3^2\right] = \left[1 - \frac{(x_3 - x_2)^2}{x_1^2} + \frac{x_3^2}{x_1^2}\right] - \left(\frac{2x_3}{x_1^2}\right)x + \frac{x^2}{x_1^2}. \tag{50}$$

From here, we equate the coefficients of separate powers of x. We find

$$\frac{B_2^2}{(x_3 - x_2)^2 B^2} = \frac{1}{x_1^2} \tag{51}$$

from the coefficients of x and x^2 and from the remainder we find that

$$(x_3 - x_2)^2 = x_1^2. \tag{52}$$

This means that both the elliptical embedding surfaces in Regions 1 and 2 have the same radii in the x direction. If we substitute the values of x_1 and x_2 from Eqns. 31 and 32, this condition can be written as the requirement

$$2m = \Lambda a. \tag{53}$$

Thus, this relation among the parameters (m, a, Λ) must be chosen in order to carry out the embedding we have derived. When we put Eqn. 52 into Eqn. 51, we find that

$$B = B_2. \tag{54}$$

This means that both the elliptical embedding surfaces in Regions 1 and 2 have the same radii in the z direction.

4.4 Embedding Diagram

Figure 2 shows an embedding diagram which summarizes the embedding we derived above. The surface $z(x)$ shown in Figure 2 has the same intrinsic geometry as the slice of our spacetime given by $t = constant$. This diagram shows how the geometry for two static black holes would appear, if immersed in the auxiliary space of Eqn. 47. Note that the digram only serves to help visualize the geometry of space outside the black holes.

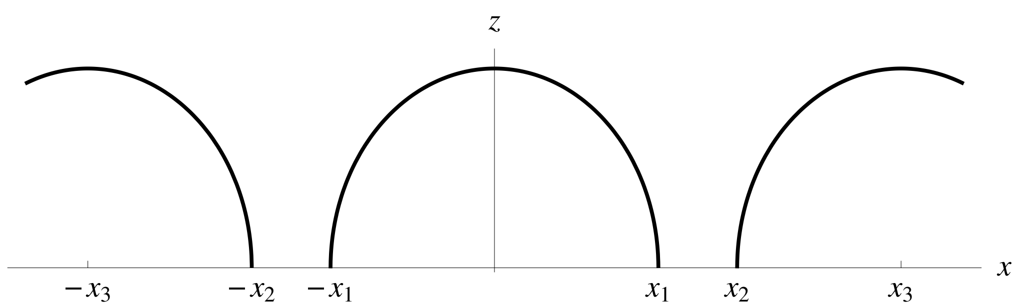

Figure 2: Embedding diagram for two symmetric static black holes.

As a final comment, we note that the flat embedding space in Eqn. 47 has constant coefficients which are not necessarily unity. The conventional representation for flat space is $g_{xx} = 1$ and $g_{zz} = 1$. This can be achieved if we use the values

$$B_1 = x_1 = \sqrt{\frac{\Lambda}{2}}. \tag{55}$$

5 Electric Charge Model

5.1 Introduction

In this section we will show that the geometry of two static black holes with a nonzero cosmological constant Λ is equivalent to the geometry of two black holes produced by two electrically charged point sources with zero cosmological constant Λ. We will consider two regions: outside of the black hole singularities $|x| > a$, and in between them with $|x| < a$. In each region, the charged point sources will reproduce the same geometry provided by a nonzero cosmological constant Λ. To do this, we use Eqn. 25 and set $\Lambda = 0$, but in this case the trace of the stress-energy-momentum tensor is different, as we explain below.

5.2 Charged Point Sources with $\Lambda = 0$

Electromagnetism in 1+1 dimensions is reviewed in [2]. In this case, We find T by computing

$$T = g_{\mu\nu} T^{\mu\nu} \tag{56}$$

where $g_{\mu\nu}$ is the metric and $T^{\mu\nu}$ is the stress-energy-momentum tensor for charged point particles. It is given by

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu} - \epsilon \left(F^{\mu\beta} F^{\nu}{}_{\beta} - \frac{1}{4} g^{\mu\nu} F_{\beta\gamma} F^{\beta\gamma} \right), \tag{57}$$

where u^{μ} is the unit timelike two-velocity for the reference frame, with $g_{\mu\nu} u^{\mu} u^{\nu} = -1$. This gives the trace T as

$$T = -\rho + \epsilon \left(F^{tx} \right)^2 \tag{58}$$

In these expressions, we have

$$\rho = \sum_{i=1}^{N} \frac{M_i}{2\pi G} \delta(x - x_i). \tag{59}$$

For our case of two symmetric point sources,

$$\rho = \frac{m}{2\pi G} \left[\delta(x - a) + \delta(x + a) \right]. \tag{60}$$

Eqn. 60 tells us that ρ only contributes to T when we are directly on top of the point charges, so for our calculations we do not need to include it because our calculations will take place elsewhere.

In Eqn. 57 the sign of the parameter $\epsilon = \pm 1$ must be chosen as appropriate to the situation. The introduction of the positive value $\epsilon = +1$ is discussed further in [2]. This choice of sign for ϵ does not arise in four spacetime dimensions, where $\epsilon = -1$.

In Eqn. 57 the quantity $F^{\mu\nu}$ is the electromagnetic field strength tensor where F^{tx} is proportional to the electric field. The tensor $F^{\mu\nu}$ is antisymmetric which here means its only nonzero components are F^{tx} and F^{xt} with $F^{tx} = -F^{xt}$. We must find $F^{\mu\nu}$ by solving

$$\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}F^{\mu\nu}\right) = \sqrt{-g}J^\mu \tag{61}$$

where $J^\mu = (\rho_Q, 0)$ and for a set of N charged point sources with charge parameters Q_i,

$$\rho_Q = \sum_{i=1}^{N} Q_i \delta(x - x_i). \tag{62}$$

For the case of a single point charge at the origin, the electromagnetic field strength tensor is [2]

$$F^{tx} = \frac{1}{2}Q\left(\theta(x) - \theta(-x)\right) \tag{63}$$

where $\theta(x)$ is the step function. In general, a step function centered at $x = x_0$ is defined by

$$\theta(x - x_0) = \begin{cases} 1 & \text{if} \quad x > x_0 \\ 0 & \text{if} \quad x < x_0 \end{cases} \tag{64}$$

It was observed in [2] that with zero cosmological constant, a single point charge can be regarded as producing an effective cosmological constant in the space surrounding it. Below we will generalize this for our case of two symmetric point sources, each with mass parameter m. It is not immediately clear whether it will be useful to consider two point sources with like charge or opposite charge, so we will consider both cases below.

For two oppositely charged point sources (with equal magnitude of charge), we take

$$F^{tx} = C_1\left[\theta(x - a) - \theta(-(x - a))\right] - C_1\left[\theta(x + a) - \theta(-(x + a))\right] \tag{65}$$

where C_1 is a constant to be determined. The first pair of step functions in Eqn. 65 describes the electric field produced by the point charge located at $x = a$. The last pair of step functions describes the electric field produced by the point charge at $x = -a$. By symmetry, the terms for each point charge have the same coefficient C_1, but with opposite signs since their charges are opposite.

Similarly, for two point sources with the same charge, we take

$$F^{tx} = C_2\left[\theta(x-a) - \theta(-(x-a))\right] + C_2\left[\theta(x+a) - \theta(-(x+a))\right] \tag{66}$$

where C_2 is a constant to be determined. The first pair of step functions in Eqn. 66 describes the electric field produced by the point charge located at $x = a$. The last pair of step functions describes the electric field produced by the point charge at $x = -a$. By symmetry, the terms for each point charge have the same coefficient C_2.

We determine the coefficient C_1 by substituting Eqn. 65 into Eqn. 61. Similarly, we determine the coefficient C_2 by substituting Eqn. 66 into Eqn. 61. In each case, to evaluate Eqn. 61 we use the following math facts. We use the derivative property of the step function,

$$\frac{d}{dx}\theta(x-x_0) = \delta(x-x_0). \tag{67}$$

We also use the chain rule,

$$\frac{d}{dx}\theta\left(-(x-x_0)\right) = -\delta(-(x-x_0)). \tag{68}$$

We also use the following property of the delta function,

$$\delta(-(x-x_0)) = \delta(x-x_0), \tag{69}$$

which is a special case of the following property (for any constant A),

$$\delta(A(x-x_0)) = \frac{1}{|A|}\delta(x-x_0). \tag{70}$$

After we substitute F^{tx} given in Eqns. 65 and 66 into Eqn. 61 and use the math properties in Eqns. 67−69, we find that F^{tx} solves Eqn. 61 for the values

$$C_1 = \frac{Q}{2} \quad , \quad C_2 = \frac{Q}{2}. \tag{71}$$

These coefficients are physically reasonable, based on the form for a single point charge given above in Eqn. 63.

Using the results $C_1 = Q/2$ and $C_2 = Q/2$, we will now summarize our results for F^{tx} using Eqn. 64 which indicates the ranges of x where the step functions involved are either zero or nonzero. Doing this for the case of two oppositely charged point sources, Eqn. 65 becomes

$$F^{tx} = \begin{cases} -Q & \text{if} \quad |x| < a \\ 0 & \text{if} \quad |x| > a \end{cases} \tag{72}$$

Similarly, for the case of two point sources with like charge, Eqn. 66 becomes

$$F^{tx} = \begin{cases} -Q & \text{if} \quad x < -a \\ 0 & \text{if} \quad |x| < a \\ Q & \text{if} \quad x > a \end{cases} \tag{73}$$

Now that we have found the field strength F^{tx} for two different charge models, in the following sections we will determine how to apply them and interpret the results.

Note that the two models above have different physical interactions: the model with two opposite charges exhibits electric attraction between the charges, while the model with two like charges exhibits electric repulsion between the charges. As we indicated in the introduction to this section, it is not immediately clear which of the two charge models above might be more useful, or perhaps whether both models combined together will be useful. We also need to relate the charge parameter Q to our cosmological constant Λ. These are the topics of the next sections.

5.3 Effective Cosmological Constant Λ

In the previous section, we evaluated the field strength F^{tx} for two different models with different physical effects. The model with two opposite charges $\pm Q$ exhibits electric attraction between the charges. The model with two like charges Q exhibits electric repulsion between the charges. In these models based on point charges, we set $\Lambda = 0$.

In order to see how our charge models can in fact reproduce the same effect as a nonzero cosmological constant Λ, we make the following observation: electric sources (with $\Lambda = 0$) can produce an effective nonzero value of Λ, if the electric sources produce a constant value of the field strength F^{tx}. To see this, we examine the field equation Eqn. 25,

$$\frac{d^2\alpha}{dx^2} = -8\pi G T - \Lambda, \tag{74}$$

where for our case of two symmetric uncharged point sources,

$$T = -\rho = -\frac{m}{2\pi G}\left[\delta(x-a) + \delta(x+a)\right]. \tag{75}$$

This indicates that away from the points $x = \pm a$ we have $T = 0$ and the righthand side of the field equation is a constant value of $-\Lambda$.

Now we compare this to our electric charge models with $\Lambda = 0$ given by Eqn. 58,

$$T = -\rho + \epsilon\left(F^{tx}\right)^2. \tag{76}$$

Here ρ is the same expression given above. This field equation will resemble Eqn. 74 if $(F^{tx})^2$ is constant throughout space, except where $|x| = a$.

The attracting charge model of Eqn. 72 has

$$(F^{tx})^2 = \begin{cases} Q^2 & \text{if } \; |x| < a \\ 0 & \text{if } \; |x| > a \end{cases} \tag{77}$$

This shows that in the region $|x| < a$, the field equation with nonzero Λ will be the same as the field equation in the attracting charge model, if $8\pi G(\epsilon Q^2) = \Lambda$.

The repelling charge model of Eqn. 73 has

$$(F^{tx})^2 = \begin{cases} 0 & \text{if } \; |x| < a \\ Q^2 & \text{if } \; |x| > a \end{cases} \tag{78}$$

This shows that in the region $|x| > a$, the field equation with nonzero Λ will be the same as the field equation in the repelling charge model, if $8\pi G(\epsilon Q^2) = \Lambda$.

Thus we can see that the condition of having a constant value of $(F^{tx})^2$ everywhere (except $|x| = a$) is not possible using only the attracting charge model, or only the repelling charge model. But this condition is possible if we effectively use both charge models together. More precisely, recall that solving the field equation determines the function α which determines the spacetime geometry. Thus, if Q and Λ are related by

$$\Lambda = 8\pi G(\epsilon Q^2) \tag{79}$$

then we have the following two results for the geometry: the geometry with nonzero Λ in the region $|x| < a$ is the same geometry produced by two attracting opposite charges $\pm Q$ with zero Λ, and the geometry with nonzero Λ in the region $|x| > a$ is the same geometry produced by two repelling charges Q with zero Λ. Also, we must have $\epsilon > 0$ because we have $\Lambda > 0$ in Eqn. 25. From the restriction placed upon ϵ in Eqn. 57, $|\epsilon| = 1$, we see we must set $\epsilon = 1$. Now we are able to solve for the value of our charge parameter Q and we find

$$|Q| = \sqrt{\frac{\Lambda}{8\pi G}}. \tag{80}$$

5.4 Interpretation and Black Hole Interaction

Taken together, the two charge models above allow us to interpret how two uncharged black holes with nonzero cosmological constant Λ must be interacting, in order to remain static. Since the black holes are static, their interaction evidently includes both attraction and repulsion, which balance. We would expect that two uncharged black holes with nonzero cosmological constant Λ can only interact gravitationally. But in general relativity, gravity is produced by the spacetime geometry: this means the details of the black holes' interaction must reside in the geometry itself.

As we saw above, the geometry between the black holes is the same geometry produced by two attracting opposite charges $\pm Q$, and the geometry outside the black holes is the same geometry produced by two repelling like charges Q, with Q given by Eqn. 80. Thus the charge models exhibit both the attraction and repulsion we anticipated above: attraction between black holes in the geometry between the black holes, and repulsion between black holes in the geometry outside the black holes.

Note that since we must use both charge models to produce an effective cosmological constant throughout space, the cosmological constant Λ is evidently related to both attraction and repulsion. This result is perhaps surprising, since one might have expected that Λ would only provide a repulsive effect, based on the experimentally confirmed accelerating expansion of our own universe in four spacetime dimensions.

6 Localized Mass-Energy

6.1 Introduction

In the previous section, we developed an effective electric charge model for two static black holes in the presence of a cosmological constant Λ. This charge model incorporated two expected features of how the black holes interact: an attraction and a repulsion, which evidently balance.

If there is an effective electric interaction between the black holes, we might consider developing a notion of electric force between the black holes, but there are some reasons to expect this might be neither easy nor useful. First, there is no established framework in general relativity for how to define a force on an extended object. Also, any effective electric force would be a model for how the black holes interact gravitationally, however gravity is simply not treated as a force in general relativity.

For the reasons outlined above, we will begin an investigation as to whether we can describe the black holes' interaction in terms of energy, or mass, as it is usually referred to in general relativity. The idea we are drawing on here is analogous to the procedure in Newtonian physics, where one can take physical laws in terms of forces and essentially rephrase them in terms of work and energy. The difference is that here, we will bypass any description using forces.

In this investigation, it will be interesting to find out whether our static black holes share an interaction energy consistent with the following ideas from nonrelativistic physics. A pair of nonrelativistic particles share a negative gravitational interaction energy. If the particles also have electric charge, they also share an electric interaction energy which is negative for opposite charges, and positive for like charges. This analogy with charged particles would be relevant for our charge model of how the black holes interact.

6.2 Mass Definition

In general relativity, the total mass of a system is measured at infinity, and is a global property of the entire spacetime. Several formulas exist for how to calculate this total mass as a flux integral, similar to a Gauss's law formula in classical electromagnetism. There is also a local definition for the energy or mass M contained inside a finite region of space. We will use this local definition of mass, since our space is not infinite in the x coordinate (meaning there is a cosmological horizon, as mentioned earlier). This local mass definition is [3]

$$M = \frac{1}{4\pi G}(u \cdot \xi)(n^{\beta}\nabla_{\beta}\Psi) - M_0. \tag{81}$$

Here Ψ is an auxiliary field which is present since it is used in the Lagrangian from which our version of general relativity, Eqn. 19, can be derived. The mass M depends on Ψ since the mass is derived from the Lagrangian. We will discuss the field Ψ below.

The quantity M_0 is a reference value that can be chosen, and the other quantities are defined as follows. The vector ξ^{β} is a timelike Killing vector which here means $(\xi^t, \xi^x) = (1, 0)$ using the metric in Eqn. 21. The vector u^{β} is a timeline unit vector for the reference frame, which we take as $(u^t, u^x) = (1/\sqrt{\alpha}, 0)$. This gives $(u \cdot \xi) = g_{\mu\nu}u^{\mu}\xi^{\nu} = -\sqrt{\alpha}$. The vector n^{β} is a unit vector in the x direction which means $(n^t, n^x) = (0, \pm\sqrt{\alpha})$. This means our mass function is

$$M = -\frac{1}{4\pi G}(\pm\alpha)\frac{\partial \Psi}{\partial x} - M_0 \tag{82}$$

For us, this expression is only a preliminary mass formula, since there are two issues we need to consider carefully. First, we need to consider a choice of the the $\pm$ sign, which refers to the choice of direction for the unit normal in the x direction. Second, this mass expression is to be evaluated at a boundary, however as observed in [3] the boundary is either one point or two points (depending on the choice of boundary) which is a subtlety of having one spatial direction x.

We will now evaluate $\partial\Psi/\partial x$. There are two equations for Ψ and they are given in [6]. The first equation for Ψ is

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left[\sqrt{-g}\, g^{\mu\nu}\frac{\partial\Psi}{\partial x^\nu}\right] - R = 0. \tag{83}$$

Here $g^{\mu\nu}$ is the inverse metric. Also, g is the determinant of the metric and R is the Ricci scalar. The second equation for Ψ is more complicated and will be discussed below.

We will now use Eqn. 83 to solve for $\partial\Psi/\partial x$. Since we are considering static black holes, we will take $\Psi(x)$ to be independent of the time coordinate t. We use our metric in Eqn. 21 and we use the notation $' = d/dx$. We will work in regions outside the black holes, so we are away from the black hole singularities $x = \pm a$, so we will not have any contributions from delta functions in $T_{\mu\nu}$ or the Ricci scalar R. In Eqn. 83 we have $g = -1$ and $R = -\alpha''$ and we find that Eqn. 83 becomes

$$\alpha'\Psi' + \alpha\Psi'' = -\alpha''. \tag{84}$$

The lefthand side of this equation is a total derivative. The righthand simplifies, since $\alpha'' = -\Lambda$ from Eqn. 30. Thus we obtain

$$(\alpha\Psi')' = \Lambda. \tag{85}$$

We can immediately solve this for Ψ' which results in

$$\alpha\Psi' = \Lambda x + C \tag{86}$$

where C is an integration constant to be determined. We can solve for the constant C in different regions of space by combining this result with the other equations which Ψ must satisfy, as we describe in the following section. Our preliminary mass formula Eqn. 82 is now

$$M = -\frac{1}{4\pi G}(\pm 1)(\Lambda x + C) - M_0. \tag{87}$$

6.3 Further Details

As mentioned above, Ψ must satisfy two equations. In the previous section, we solved one of these equations to obtain our preliminary mass function, Eqn. 87. The second equation for Ψ is [6]

$$\frac{1}{2}\left[\nabla_\mu\Psi\nabla_\nu\Psi - \frac{1}{2}g_{\mu\nu}\, g^{\alpha\beta}\nabla_\alpha\Psi\nabla_\beta\Psi\right] + g_{\mu\nu}\, g^{\alpha\beta}\nabla_\alpha\nabla_\beta\Psi - \nabla_\mu\nabla_\nu\Psi = 8\pi G T_{\mu\nu} + \frac{\Lambda}{2}g_{\mu\nu}. \quad (88)$$

This is actually a set of equations, one equation for each value of the pair of free indices (μ, ν). For the values $(\mu, \nu) = (t, x)$ or (x, t) we find Eqn. 88 simply states that $0 = 0$. For the values $(\mu, \nu) = (x, x)$ we find Eqn. 88 becomes

$$\alpha'\Psi' + \frac{1}{2}\alpha(\Psi')^2 = \Lambda \quad (89)$$

and for the values $(\mu, \nu) = (t, t)$ we find Eqn. 88 becomes

$$\alpha'\Psi' + 2\alpha\Psi'' - \frac{1}{2}\alpha(\Psi')^2 = \Lambda. \quad (90)$$

The next steps in this investigation will be as follows. We should combine these last two equations for Ψ with Eqn. 86. This should allow us to solve for the constant C in different regions of space. We will need to consider separate regions, due to the form of α in Eqn. 30. We can already see our preliminary mass function in Eqn. 87 depends on position x, and we would like to see if this is consistent with the black holes' sharing an interaction energy, as we described in the introduction above. Part of this will involve the related issues we mentioned earlier regarding the choice of boundary and its unit normal vector. We will pursue all of these steps in a future project.

7 Static Wormhole

7.1 Introduction

An application of our static black holes is to construct a static wormhole, which was started and nearly completed in [4], where the full construction is described in detail. Here, we will finish the construction by just verifying the wormhole's smoothness where the two black hole horizons are patched together. We want to make sure that what we create is smooth and has no discontinuities as we cross from one event horizon to the other. To make sure that our wormhole is in fact smooth there, we must investigate the extrinsic curvature at the horizons.

7.2 Defining a New Time Coordinate

In order to carefully discuss a black hole event horizon we must define new coordinates which are well-behaved on the horizon, since the coordinate system in the line element of Eqn. 21 breaks down at the horizon, where $\alpha = 0$. Following a standard procedure (familiar, for example, from the Schwarzschild black hole spacetime) we define a new time coordinate v,

$$v = t + F(x) \tag{91}$$

where t is the old time coordinate and $F(x)$ allows us to go inside the horizon, if desired. Eqn. 21 depends on the differential dt, so we have

$$dt = dv - \frac{dF}{dx}dx. \tag{92}$$

When we take the square of this and put it in Eqn. 21, we find that

$$ds^2 = -\alpha dv^2 + 2\alpha\frac{dF}{dx}dvdx + \left(\frac{1}{\alpha} - \alpha\left(\frac{dF}{dx}\right)^2\right)dx^2. \tag{93}$$

In Eqn. 93, we require the coefficient of dx^2 to be zero and this gives a differential equation for $F(x)$:

$$\frac{dF}{dx} = \frac{1}{\alpha} \tag{94}$$

where we chose the $+$ sign after taking a square root. This can be solved for $F(x)$ if desired. For example, if we solve this equation in Region 1 with $0 < x < x_1$ and α is given by Eqn. 42, we find

$$F(x) = \frac{2\tanh^-1\left(\frac{x}{x_1}\right)}{\Lambda x_1} + c \tag{95}$$

where c is a constant.

Using the condition imposed by Eqn. 94 we are able to simplify our line element ds^2 to

$$ds^2 = -\alpha\, dv^2 + 2dv\, dx. \tag{96}$$

From Eqn. 96 we are able to determine the metric in the coordinates (v, x) by looking at the coefficients of dv^2 and $dv\, dx$. Because $g_{\mu\nu}$ is symmetric ($g_{vx} = g_{xv}$) we get

$$g_{\alpha\beta} = \begin{pmatrix} -\alpha & 1 \\ 1 & 0 \end{pmatrix}. \tag{97}$$

The inverse metric that we will use to raise indices is

$$g^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix}. \tag{98}$$

7.3 Evaluating the Extrinsic Curvature K

In general relativity, a boundary or surface is smooth if a quantity called extrinsic curvature is continuous as we cross the surface. For our two-dimensional spacetime, we will use the extrinsic curvature defined as [7]

$$K = u^\alpha u^\beta \nabla_\alpha n_\beta \tag{99}$$

where u^α is a tangent vector to the horizon and n_α is a normal vector to the horizon. In [4] a wormhole is constructed by patching the horizons of our two black holes together. In order for this patching to be smooth, K must be continuous as we cross from one horizon to the other. On either side of the horizon, there is a choice of direction for the normal vector, but we won't need to choose this direction since we will show that $K = 0$ on each horizon, which guarantees that K is continuous as we cross from one horizon to the other.

At the horizon, $x = constant$ which means the normal vector n_α will have components $n_v = 0$ and $n_x \neq 0$ giving $n_\alpha = (0, n_x)$. If desired, we can obtain a normal vector with a raised index using the inverse metric, $n^\alpha = g^{\alpha\beta} n_\beta$ which gives $n^\alpha = (n^v, 0) = (n_x, 0)$. The condition that n_α be a null normal vector is:

$$g_{\alpha\beta} n^\alpha n^\beta = 0 \qquad \text{or} \qquad g^{\alpha\beta} n_\alpha n_\beta = 0. \tag{100}$$

If desired, we could solve this for n_x.

In order to find the tangent vector u^α, we use the fact that u^α is orthogonal to n^β so

$$g_{\alpha\beta}u^\alpha n^\beta = 0. \tag{101}$$

By definition, $g_{\alpha\beta}n^\beta = n_\alpha$, so we can rewrite Eqn. 101 as

$$u^\alpha n_\alpha = 0 \tag{102}$$

which can be expanded as

$$u^v n_v + u^x n_x = 0. \tag{103}$$

Because we know $n_\alpha = (0, n_x)$ this equation determines that $u^v \neq 0$ and $u^x = 0$ so we have $u^\alpha = (u^v, 0)$ in order to make Eqn. 103 true and have a nonzero tangent vector.

Now that we have determined the nonzero components of u^α and n_β, we can go on to evaluate Eqn. 99. Upon expanding Eqn. 99, we have

$$K = u^v u^v \nabla_v n_v = u^v u^v \left(\frac{\partial n_v}{\partial v} - \Gamma^\gamma_{vv} n_\gamma \right). \tag{104}$$

Inside the parentheses, we have $n_v = 0$ and n_x is nonzero, so

$$K = -(u^v)^2 \Gamma^x_{vv} n_x = -(u^v)^2 \left(\frac{1}{2}\alpha \frac{d\alpha}{dx} \right) n_x. \tag{105}$$

At the horizon, $\alpha = 0$ which means $K = 0$ at the horizon. This satisfies our requirements for making the wormhole smooth there, as we explained above.

7.4 Discussion of Results

Above, we verified it is possible to patch our two black hole horizons together smoothly to form a wormhole. A wormhole of this type was previously constructed by Misner [5] in four spacetime dimensions with zero cosmological constant, $\Lambda = 0$. Misner's wormhole was constructed to be initially at rest, however it was inherently not static, due to the gravitational attraction between the black holes and the absence of any repulsion between the black holes to balance their attraction. The new feature of our work is that our wormhole is actually static.

8 Conclusions

In a two-dimensional version of general relativity, we investigated the interaction that occurs between two black holes when they are placed near each other in the presence of a cosmological constant Λ. The geometry for these two black holes is determined by a function α and is shown in Figure 1. In this configuration, our two black holes remain static and do not move towards each other as would be expected from their gravitational attraction. To balance the attraction, there there must evidently be repulsion. Before reading this thesis, one might expect this repulsion is due to the positive cosmological constant Λ in our setup, based on the observation that our own four-dimensional universe is expanding with an acceleration caused by $\Lambda > 0$.

To continue this line of thought, as an initial inquiry we determined the acceleration of a static test particle when it is placed in our system of two black holes. When the particle is not inside a black hole ($\alpha > 0$), its acceleration a_x points in the same direction as $d\alpha/dx$. This acceleration keeps the particle static, so it opposes the effect of gravity which is in the direction $-a_x$, towards the nearest black hole. There are also special locations where $d\alpha/dx = 0$, indicating there is a balance between an attractive effect and a repulsive effect. Again, one might expect the repulsive effect to be produced by the constant energy density from $\Lambda > 0$, but we found that the answer for the system of two black holes itself is more subtle than this.

We modeled our system of two static black holes in two ways: as two oppositely charged point sources, and as two like charged point sources (each with $\Lambda = 0$). By combining the solutions from these two models, we were able to reproduce the full geometry of our original system with two black holes and $\Lambda > 0$. Because the geometries are the same, we can interpret the results of our electric charge model for the actual system. We found from our combined electric charge model that there is attraction present in the space between the black holes, and repulsion present is the space outside of both of them. Thus we were able to conclude that the cosmological constant is related to both the attraction and the repulsion of the black holes.

In fact, this localization is one of our more interesting results: the black holes' attraction is localized in the innermost space between them, and their repulsion is localized in the furthermost space outside of them. As an extension of this result, we began a second approach to investigate the black holes' interaction. This method is to examine the gravitational mass-energy localized in a region. We expect this method is well-suited to the localized interaction, although this work is still in progress. In future work, it would be interesting to see if this type of localized interaction occurs in other black hole systems.

As an application of our static black hole system, we completed the wormhole construction in [4] by verifying the smoothness of patching the two horizons together. This wormhole is more general than the similar one constructed by Misner [5] since unlike Misner's original wormhole, our solution is static.

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