Two-Dimensional Pendulum Experiments Using a Spark Generator

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Student understanding of the superposition of perpendicular harmonic oscillators can be enhanced using a spark generator to record position-versus-time data for small-amplitude pendulum motion in two dimensions. Our students have used this arrangement to analyze the motion of a spherical pendulum and a "Y-suspended" (Blackburn) pendulum that has two effective lengths.

Spherical Pendulum

The spherical pendulum is a simple pendulum whose motion is not restricted to a single plane (see Fig. 1a). Our pendulum consists of a metal bob (e.g., a 100-g mass from a standard hanging mass set) suspended by a string (L~1 m) over an air table.[1] Fine magnet wire runs from the high voltage terminal of the spark generator[2] to the bob. A sharp point on the bottom of the bob ensures that all the sparks jump from the same point.[3] Carbon paper[4] is placed on the table and connected to the ground terminal of the spark generator to complete the circuit.

For small-amplitude motion, the height of the bob above the table is approximately constant (gap~0.5 cm) and can be adjusted so that sparks jump between the sharp point on the bob and the carbon paper on the table. With ordinary 8.5 in x 11 in white paper placed over the carbon paper, a record of the bob's position in the horizontal (x-y) plane is obtained on the underside of the paper. If the height of the pendulum bob increases much, sparking to the carbon paper will cease, so recording of small-amplitude motion is guaranteed.

Fig. 1a. Photograph of UPS student Mike Dahl using the apparatus. The spherical pendulum is suspended above the air table and connected by magnet wire to the spark generator (right foreground.) The puck seen in the corner of the table is not mandatory, though it does provide a convenient way to complete the circuit from the carbon paper to the spark generator.

Fig. 1b. Photograph of the Y-suspended pendulum.

For small-amplitude motion in the x-y plane, the pendulum should approximate simple harmonic motion with period $T = 2\pi(L/g)^{1/2}$ in both the x and y directions. In general, $x(t) = A\cos(\omega t)$ and $y(t) = B\cos(\omega t + \phi)$, provided we let $t = 0$ occur when $x = A$. The angle ϕ represents the relative phase difference between the perpendicular oscillations. The position vector $\boldsymbol{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ represents the superposition of these sinusoidal motions and describes elliptical trajectories (including linear and circular paths as special cases).[5] For an ellipse with axes aligned along the coordinate axes, $\phi = \pi/2$ and $x^2/A^2 + y^2/B^2 = 1$.

Spark data for a *single* elliptical orbit are readily obtained for analysis (see Fig. 2a). (Multiple orbits make it difficult to assign times to the data points.) These data are then traced by hand onto graph paper using a light box for illumination. In this way the major and minor axes of the ellipse can be aligned with the coordinate system to simplify the analysis. (Alternatively, one may record the data directly onto graph paper, without controlling the orientation of the

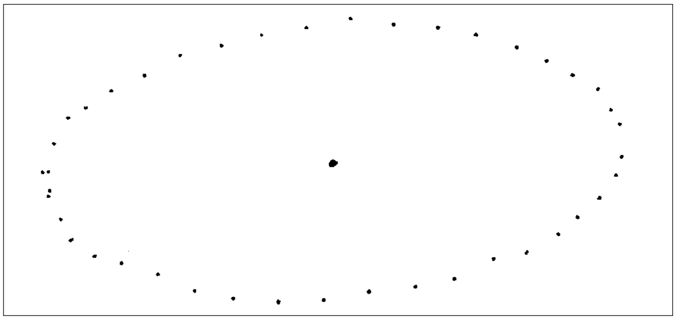

Fig. 2a. Raw data for one elliptical orbit of the spherical pendulum. Spark frequency = 20 Hz.

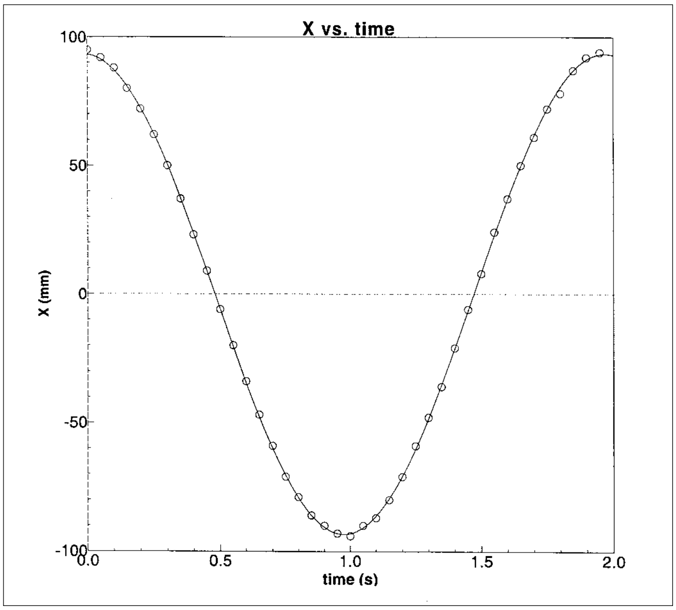

Fig. 2b. $x(t)$ data (dots) plotted along with a sinusoidal curve fit (solid line).

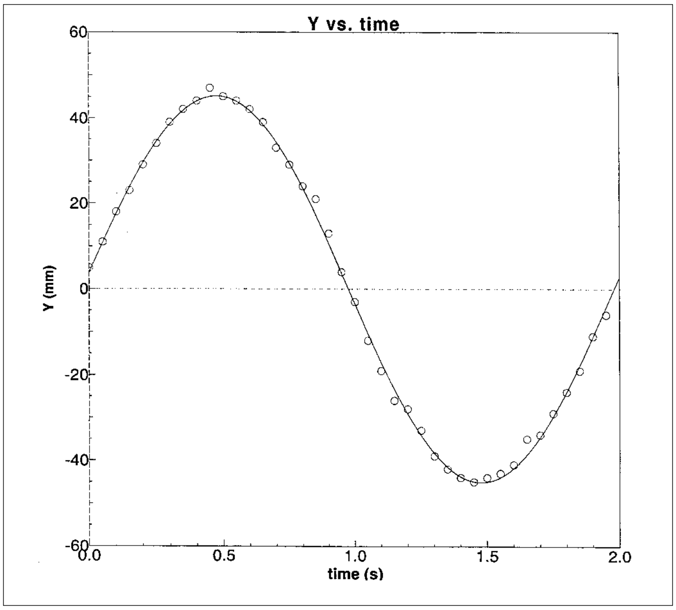

Fig. 2c. $y(t)$ data (dots) plotted along with a sinusoidal curve fit (solid line). Measured pendulum length is $L = 1.015$ m, giving theoretical period $T = 2\pi(L/g)^{1/2} = 2.02$ s. Curve fitting of data yields $T_x = 1.98$ s, $T_y = 2.01$ s, $\phi = 88°$.

major and minor axes.) From the graph paper, $x(t)$ and $y(t)$ values are tabulated and entered into a graphing and curve-fitting program such as *Graphical Analysis.*[6] The x-versus-t and y-versus-t data may then be plotted separately and fit to sinusoidal functions (see Figs. 2b and 2c).

As shown in Fig. 2, students find that the trajectory of the spherical pendulum is indeed composed of perpendicular sinusoidal motions. They see that within the measurement uncertainties, these oscillations appear to have the same period, $T = 2\pi(L/g)^{1/2}$, and are $\pi/2$ out of phase with each other. In addition, the amplitudes (A and B) are easily identified with the semimajor and semiminor axes on the graph paper.

Y – Suspended Pendulum

As an extension of the first exercise, two strings forming a "Y" shape are used to suspend the pendulum bob (see Fig. 1b). This pendulum, sometimes called a *Blackburn pendulum,*[7] has two effective lengths and thus different periods of oscillation in the two perpendicular planes. For small-amplitude motion, complicated trajectories, known as Lissajous figures,[8] can be observed and recorded. With properly chosen lengths, the ratio of the periods can be made rational (i.e., a ratio of two integers), resulting in closed orbits.

Students pick a simple integer period ratio (e.g., 2:1, 3:2, etc.) to investigate and prepare the string lengths based on the relation $T_x/T_y = (L_x/L_y)^{1/2}$. For this exercise, rather than extracting the position-versus-time data, we ask the students to investigate the *shape* (y versus x) of the trajectory (they may record more than one cycle in this case). This record is then compared visually to the corresponding case displayed on an oscilloscope (in "X-Y" mode) whose inputs are sine waves of the appropriate frequency ratio. Unless care is taken, however, the phases of the two frequency generators tend to drift, so that the Lissajous pattern on the oscilloscope is constantly changing. This leads to a discussion of the role of the relative phase between the two oscillators. Students also notice that different pendulum records taken using the spark apparatus may produce patterns that look quite different, depending on how the motion is started. Thus, they are led to think about how releasing the pendulum in different ways determines the initial phase difference.

To sort out these issues, the students prepare a spreadsheet to produce y-versus-x graphs using $x(t) = A\cos(\omega_x t)$ and $y(t) = B\cos(\omega_y t + \phi)$, with ω_x and ω_y corresponding to the period ratio used in the experiment. The resulting plot represents the superposition of perpendicular harmonic oscillators whose frequencies are different. The angle ϕ fixes the relative phase of the two oscillators at $t = 0$. (This does not necessarily represent the release time t_r, however.) This phase angle can be adjusted on the spreadsheet and the resulting patterns observed. In order to compare these patterns to experimental data, students must use techniques for releasing the pendulum bob such that this phase angle is known. Descriptions of the two release methods used by our students follow:

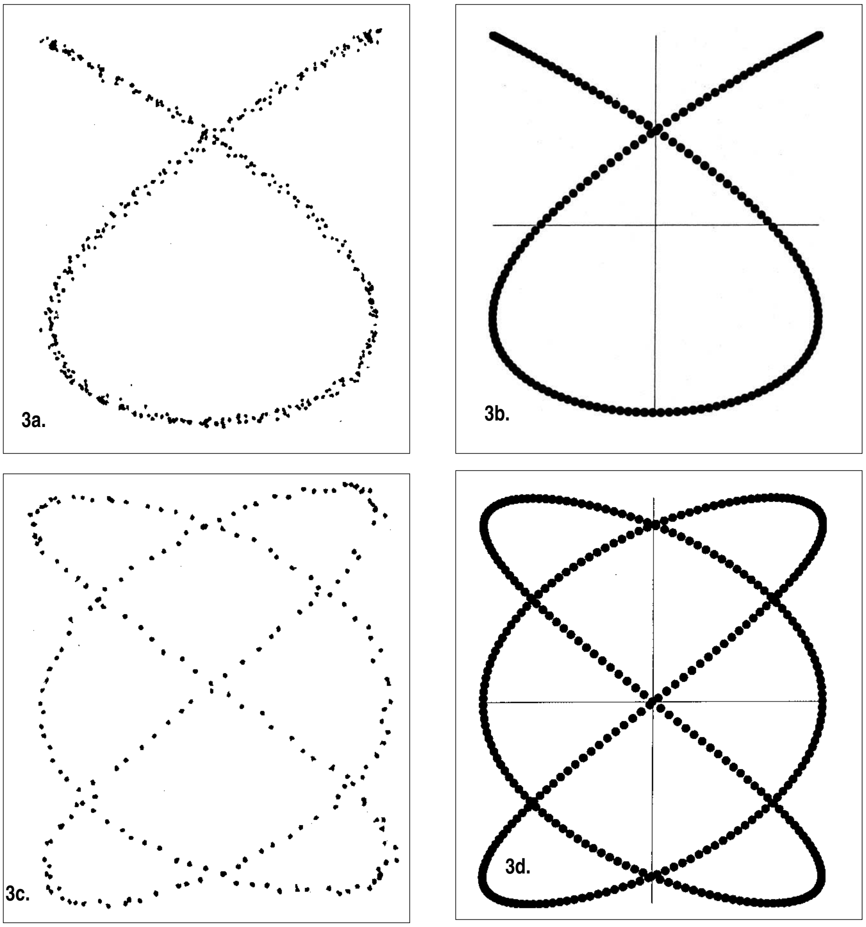

Fig. 3. Data and spreadsheet results for $\omega_x/\omega_y = 3/2$. Note: Spreadsheet aspect ratios (*A/B*) have been adjusted to match the data.
a. Experimental data: pendulum bob released from rest ($\phi = 0$).
b. Spreadsheet plot with $\phi = 0$.
c. Experimental data: pendulum bob released while moving through origin ($\phi = \pi/6$).
d. Spreadsheet plot with $\phi = \pi/6$.

1) The first method involves displacing the bob and releasing it from rest. In this case the phase angle is $\phi = 0$, as both $x(t)$ and $y(t)$ take on their maximum values ($x = A$, $y = B$) at $t = 0$. (In this case $t = 0$ *does* correspond to the moment of release, t_r.)

2) The second release method requires the bob to be released while moving through the origin. With $x = 0$ and $y = 0$ at the same time t_r, the resulting relative phase angle at $t = 0$ can be shown to be $\phi = (\pi/2)(m - n\omega_y/\omega_x)$; m,n = odd integers.[9]

Note that with both release methods, the spark generator is turned on *after* the bob is safely released; with closed orbits, it does not matter when the spark record is started.

The resulting experimental trajectories match the corresponding spreadsheet graphs quite well, as shown in Fig. 3, for both release techniques. This confirms, at least qualitatively, that the complex motion of the Y-suspended pendulum can be represented by perpendicular harmonic oscillators. In this experiment, students produce, manipulate, and analyze the superposition of perpendicular sinusoidal oscillations with integer ratio frequencies in a way that complements the usual “gee whiz” visualization provided by Lissajous figures on the oscilloscope. In addition, students gain insight into the physical significance of the phase angle (ϕ) in terms of how and where the pendulum is released.

Conclusion

Using the spark generator to collect data provides the opportunity to do quantitative investigations of two-dimensional, small-amplitude pendulum motion. The two exercises presented here provide nice additions to the standard simple pendulum labs often used to introduce simple harmonic motion (e.g., period-versus-length measurements). As an added benefit, our students seem to enjoy the noise and sense of danger that are characteristic of the spark apparatus!

Acknowledgment

The authors would like to thank UPS student Mike Dahl, who took the data and prepared the spreadsheets shown in this note.

References

1. For example, the Cenco Precision Air Table, which is typically used to investigate two-dimensional collisions between air-supported pucks. The air supply is not used in our experiments. Although the air table is convenient, any horizontal insulating surface may be used.
2. For example, the Cenco Spark Generator sold for use with the air table. Students should be encouraged to use caution when working with the spark generator.
3. If necessary, a small piece of wire may be attached to the bottom of the bob.
4. The special carbon paper sold by Cenco for use with the air table works well.
5. We neglect the precession that occurs in a spherical pendulum undergoing oscillations of finite amplitude. For a discussion of spherical-pendulum precession, see D.L. Livesey, "The precession of simple pendulum orbits," *Am. J. Phys.* **55**, 618-621 (1987); or M.G. Olsson, "Spherical pendulum revisited," *Am. J. Phys.* **49**, 531-534 (1981); M.G. Olsson, "The precessing spherical pendulum," *Am. J. Phys.* **46**, 1118-1119 (1978).
6. *Graphical Analysis* (available for Windows and Macintosh) from Vernier Software.
7. For an historical account of the Blackburn pendulum, see R.J. Whitaker, "A note on the Blackburn pendulum," *Am. J. Phys*. **59**, 330-333 (1991).
8. Recent discussions of Lissajous figures in this journal include T.B. Greenslade, "All about Lissajous figures," *Phys. Teach.* **31**, 364-370 (1993), and E.Y.C. Tong, "Lissajous figures," *Phys. Teach.* **35**, 491-493 (1997). A good introduction to harmonic oscillations in two dimensions may be found in J.B. Marion and S.T. Thornton, *Classical Dynamics of Particles and Systems*, 4th ed. (Saunders, Fort Worth, 1995), pp. 111-114.
9. The values of m and n depend on *which* $x = A$ point is chosen to correspond to $t = 0$, along with the direction of the orbit. The *shape* of the trajectory does not depend on the specific choices of m and n. When labeling our spreadsheet trajectories, we choose to specify the smallest value of ϕ.