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Riemannian Geometry of Orbifolds

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DEDICATION

To my family

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ABSTRACT OF THE DISSERTATION

Riemannian Geometry of Orbifolds

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We investigate generalizations of many theorems of Riemannian geometry to Riemannian orbifolds. Basic definitions and many examples are given. It is shown that Riemannian orbifolds inherit a natural stratified length space structure. A version of Toponogov's triangle comparison theorem for Riemannian orbifolds is proven. A structure theorem for minimizing curves shows that such curves cannot pass through the singular set. A generalization of the Bishop relative volume comparison theorem is presented. The maximal diameter theorem of Cheng is generalized. A finiteness result and convergence result is proven for good Riemannian orbifolds, and the existence of a closed geodesic is shown for non–simply connected Riemannian orbifolds.

Introduction

The purpose of this investigation is to see to what extent theorems in the Riemannian geometry of manifolds can be proven in the more general category of orbifolds. Roughly speaking a manifold is a topological space locally modelled on Euclidean space $\mathbb{R}^n$. Orbifolds generalize this notion by allowing the space to be modelled on quotients of $\mathbb{R}^n$ by finite group actions. The term *orbifold* was coined by W. Thurston sometime around the year 1976–77. The term is meant to suggest the orbit space of a group action on a manifold. A similar concept was introduced by I. Satake in 1956, where he used the term *V–manifold* (See [S1]). The "V" was meant to suggest a cone–like singularity. Since then, the term orbifold has become the preferred choice probably because V–manifold is misleading in the sense that it seems to describe a type of manifold. We will see, however, many examples where orbifolds are *not* manifolds. In general they can be quite complicated topological spaces. Orbifolds have recently come up in the study of convergence of Riemannian manifolds. See for example [F2] and [A2]. Except for the notes of Thurston [T], there has been very little investigation of orbifolds as a primary object of study. But, even there, Thurston's primary interest is to use the concept of orbifold as a tool for studying 3–manifold topology. What we wish to do is to provide a foundation for the study of the *geometry* of orbifolds and show how many standard and often used results in Riemannian geometry can be carried over to *Riemannian orbifolds*: Orbifolds which are locally modelled on Riemannian manifolds modulo

finite groups of isometries. What allows us to generalize these theorems is that the *singular* points of orbifolds form a *small* set, and that locally orbifolds have a relatively specific structure.

The first section, *Riemannian Orbifolds*, gives all necessary definitions for an orbifold and states various known results (with references) which will help our analysis. In particular, we show that Riemannian orbifolds are naturally stratified length spaces. The second section, *Examples*, is devoted to giving the reader a selection of elementary examples which should guide his intuition as he reads the paper. In the third section, *Toponogov's Theorem for Orbifolds*, we generalize the Toponogov triangle comparison theorem. In particular, we show that orbifolds which are locally covered by Riemannian manifolds with sectional curvatures bounded from below, also have curvature bounded from below in the sense of triangle comparison. The fourth section, *The Structure Theorem for Geodesics in Orbifolds*, provides a fundamental structure theorem for geodesics in orbifolds. We conclude that minimizing segments cannot pass through the singular set and continue to remain distance minimizing. In the fifth section, *Volume Comparison for Orbifolds*, we demonstrate a generalization of the Bishop relative volume comparison theorem to orbifolds which locally satisfy a lower Ricci curvature bound. The sixth section, *Sphere-Like Theorems*, is devoted to a generalization of the maximal diameter theorem of Cheng. Specifically, we show that orbifolds with a lower Ricci curvature bound and maximal diameter have specific representations as suspensions over orbifold space forms of constant curvature. In section seven,

Finiteness Theorems, we generalize a result of M. Anderson to give finiteness result for the isomorphism classes of possible group actions Γ on Riemannian manifolds whose orbit spaces (which are orbifolds) satisfy lower bounds on Ricci curvature and volume, and an upper diameter bound. Moreover, we show how these bounds can be used to prove various precompactness results for orbifolds which arise as global quotients of Riemannian manifolds. The last section, *The Closed Geodesic Problem*, deals with the existence of closed geodesics on compact orbifolds. We show that non–simply connected orbifolds admit at least one closed geodesic.

Riemannian Orbifolds

A C^k Riemannian manifold is a C^∞ differentiable manifold equipped with a C^k metric. Throughout this paper M will denote a C^∞ Riemannian manifold. If the differentiability class of the metric is not C^∞, then this will be explicitly stated.

Length Spaces

The notion of a length space will be fundamental, and so we recall some definitions and related facts concerning them. See [G] for a more detailed discussion.

Definition 1 *Let (X, d) be a metric space and let $\gamma : [a, b] \to X$ be a continuous curve. Then the length of γ, denoted $L(\gamma)$, is defined to be the quantity*

$$\sup\left\{\sum_{i=0}^{n} d\Big(\gamma(t_i), \gamma(t_{i+1})\Big)\right\}$$

where the supremum is taken over all subdivisions $a = t_0 \leq t_1 \leq \ldots \leq t_{n+1} = b$ *of* $[a, b]$.

Remark 2 If (X, d) Is a Riemannian manifold then it can be shown that for a piecewise C^1 curve γ, $L(\gamma) = \int_a^b \|\dot{\gamma}\| dt$. See [R, page 106].

Proposition 3 *The length function* L *is lower–semicontinuous. This means that if* $\{c_n\} : [0,1] \to X$ *is a sequence of continuous maps which converge pointwise to* $c : [0,1] \to X$, *then* $L(c) \leq \liminf L(c_n)$.

Proof: For any fixed partition $0 = t_0 < \ldots < t_k = 1$ we have

$$L(c_n) \geq \sum d\Big(c_n(t_{i-1}), c_n(t_i)\Big) \longrightarrow \sum d\Big(c(t_{i-1}), c(t_i)\Big) \quad \text{as } n \to \infty.$$

Now, for any $\varepsilon > 0$,

$$\liminf L(c_n) \geq \sum d\Big(c(t_{i-1}), c(t_i)\Big) - \varepsilon$$

Thus,

$$\liminf L(c_n) \geq \sum d\Big(c(t_{i-1}), c(t_i)\Big) \quad \text{for any fixed partition}$$

and therefore,

$$\liminf L(c_n) \geq \sup_{\substack{\text{partitions} \\ \text{of } [0,1]}} \sum d\Big(c(t_{i-1}), c(t_i)\Big) = L(c).$$

This completes the proof.

Definition 4 *A metric space (X, d) is a* length space *if the distance between any two points of X is always equal to the infimum of the lengths of continuous curves that join them.*

Example 5 For the induced Euclidean metric, $\mathbf{R}^2 - \{p\}$ is a length space, but $\mathbf{R}^2 - \{\text{line segment}\}$ is not. See Figure 1.

Figure 1

Definition 6 *Let X be a length space. A curve $\gamma : [0,1] \to X$ is called a* minimizing geodesic *or* segment *if $d\big(\gamma(t), \gamma(s)\big) = |t - s| L(\gamma)$, where $t,\ s \in [0,1]$.*

Definition 7 *Let X be a length space. A curve $\gamma : [0,1] \to X$ is a* geodesic *if its restriction to every sufficiently small interval is a minimizing geodesic. If X is a Riemannian manifold then this definition is equivalent to the standard definition where γ is a geodesic if its tangent vector field $\dot{\gamma}$ is self–parallel relative to the Levi–Civita connection $\nabla_W V$. Explicitly in symbols, $\nabla_{\dot{\gamma}} \dot{\gamma} \equiv 0$.*

Definition 8 *A length space X is* geodesically complete *if every geodesic $\gamma : [0,1] \to X$ can be extended to a geodesic $\tilde{\gamma} : \mathbf{R} \to X$.*

The following analogue of the Hopf–Rinow theorem holds for length spaces. A proof can be found in [G].

Theorem 9 *If (X, d) is a locally compact length space then*

(a) The following are equivalent:

(i) (X, d) is complete

(ii) Metric balls are relatively compact

(b) If either (i) or (ii) holds, then any two points can be joined by a minimal geodesic.

For completeness we state the following version of the Arzela–Ascoli theorem.

Theorem 10 *Let $f_i : (X, \rho) \to (Y, d)$ be an equicontinuous family of maps between a separable metric space X, and a locally compact metric space Y. If for all $x \in X$, the set $\{f_i(x)\}$ is bounded, then there exists a subsequence $f_{i_k} \to f$ converging uniformly on compact sets.*

Proof: See [R, page 81].

Group Actions

We will be dealing with the notion of isometry of a metric space X. There is a potential point of confusion that may arise when X is a Riemannian manifold. In this case, there are two competing notions of isometry, one local and the other global.

Definition 11 *Let (M, g) be a Riemannian manifold. A* local *isometry ψ is a map from M to itself which preserves the metric tensor g. This means that $\psi^* g = g$. A* global *isometry is a map from M to itself which preserves the distance function d induced from the metric tensor g. This means that for all $x, y \in M$, we have $d(\psi x, \psi y) = d(x, y)$.*

It is a classical result that in the case of Riemannian manifolds, a global isometry is necessarily a local isometry. See [KN, Theorem 3.10, page 169]. Let Γ be a group of isometries of a metric space X. Then Γ defines a natural group action

$$\Theta : \ \Gamma \times X \longrightarrow X$$

$$(g, p) \mapsto g(p)$$

Proposition 12 *Let M be a C^k, $1 \leq k \leq \infty$ Riemannian manifold. Then the isometry group* $\mathrm{Isom}(M)$ *of M is a Lie group and the mapping Θ above is of class C^k.*

Proof: The isometry group $\mathrm{Isom}(M)$ is locally compact with respect to the compact–open topology [KN, Theorem 4.7]. By a result of Calabi–Hartman [CH] each isometry is of class C^{k+1}. Thus, by a result of Montgomery–Zippin [MZ, page 208], $\mathrm{Isom}(M)$ is a Lie group and the map Θ is of class C^k. See also [SW].

Definition 13 *An action of Γ on X is* effective *if the condition $gx = x \ \forall x \in X$ implies that $g =$* identity. *Said differently, the only element of Γ that fixes everything is the identity.*

Definition 14 *An action of* Γ *on* X *is* discontinuous *if for every* $p \in X$, *and every sequence of elements* $\{g_n\}$ *of* Γ *(where* g_n *are all mutually distinct) the sequence* $\{g_n p\}$ *does not converge to a point in* X.

Definition 15 Γ *acts* properly discontinuously *if*

(1) If $p, p' \in X$ *are not congruent* $\bmod \Gamma$ *(ie.* $gp \neq p'$ *for all* $g \in \Gamma$*) then* p, p' *have neighborhoods* U, U' *such that* $gU \cap U' = \emptyset$ *for all* $g \in \Gamma$.

(2) For each $p \in X$, *the isotropy group* $\Gamma_p = \{g \in \Gamma \mid gp = p\}$ *is finite.*

(3) Each $p \in X$ *has a neighborhood* U *such that* $\Gamma_p U = U$ *and such that* $gU \cap U = \emptyset$ *for all* $g \notin \Gamma_p$.

Proposition 16 *Every discontinuous group* Γ *of isometries of a metric space* X *acts properly discontinuously.*

Proof: (See [KN]) Because the action is discontinuous, for each $x \in X$ the orbit $\Gamma x = \{gx \mid g \in \Gamma\}$ is closed in X. Given a point y outside the orbit Γx, let $r > 0$ be such that $2r$ is less than the distance between y and the orbit Γx. Let U_x and U_y be open balls of radius r centered at x and y respectively. Then $gU_x \cap U_y = \emptyset$ for all $g \in \Gamma$, so condition (1) holds. Condition (2) always holds for a discontinuous action. To prove (3), for each $x \in X$, let $r > 0$ be such that $2r$ is less than the distance between x and the closed set $\Gamma x - \{x\}$. It suffices to take the open ball of radius r and center x as U. This completes the proof.

Definition 17 *Let Γ be a group of isometries acting on a metric space X. Let A be a subset of X, and let $p \in A$. Then the* Dirichlet fundamental domain *centered at p relative to A is the set*

$$\mathcal{D}_p = \{x \in A \mid d(x,p) \leq d(x,gp) \ \ \forall g \in \Gamma\}$$

Orbifolds

Following Thurston [T], (see also [S1]), the formal definition of (topological) orbifold is as follows:

Definition 18 *A (topological)* orbifold *O consists of a Hausdorff space X_O called the* underlying space *together with the following additional structure. We assume X_O has a countable basis of open charts U_i which is closed under finite intersections. To each U_i is associated a finite group Γ_i, an effective action of Γ_i on some open subset $\tilde{U}_i$ of $\mathbf{R}^n$, and a homeomorphism $\phi_i : U_i \to \tilde{U}_i/\Gamma_i$. Whenever $U_i \subset U_j$, there is to be an injective homomorphism*

$$f_{ij} : \Gamma_i \hookrightarrow \Gamma_j$$

and an embedding

$$\tilde{\phi}_{ij} : \tilde{U}_i \to \tilde{U}_j$$

equivariant with respect to f_{ij} (that is, for $\gamma \in \Gamma_i$, $\tilde{\phi}_{ij}(\gamma x) = f_{ij}(\gamma)\tilde{\phi}_{ij}(x)$), such that the diagram below commutes:

$$
\begin{tikzcd}
\tilde{U}_i \arrow[r, "\tilde{\phi}_{ij}"] \arrow[d] & \tilde{U}_j \arrow[d] \\
\tilde{U}_i/\Gamma_i \arrow[r, "\phi_{ij}=\tilde{\phi}_{ij}/\Gamma_i"] & \tilde{U}_j/f_{ij}(\Gamma_i) \arrow[d] \\
 & \tilde{U}_j/\Gamma_j \\
U_i \arrow[uuu, phantom] \arrow[uu, "\phi_i"] \arrow[r, "\subset"] & U_j \arrow[u, "\phi_j"']
\end{tikzcd}
$$

$\tilde{\phi}_{ij}$ is to be regarded as being defined only up to composition with elements of Γ_j, and f_{ij} are defined only up to conjugation by elements of Γ_j. In general, it is not true that $\tilde{\phi}_{ik} = \tilde{\phi}_{jk} \circ \tilde{\phi}_{ij}$ when $U_i \subset U_j \subset U_k$, but there should be an element $\gamma \in \Gamma_k$ such that $\gamma\tilde{\phi}_{ik} = \tilde{\phi}_{jk} \circ \tilde{\phi}_{ij}$ and $\gamma \cdot f_{ik}(g) \cdot \gamma^{-1} = f_{jk} \circ f_{ij}(g)$. Just as in the case for manifolds, the covering $\{U_i\}$ is not an intrinsic part of the structure of an orbifold. We regard two coverings to give the same orbifold structure if they can be combined to give a larger covering still satisfying the definitions. Hence, when we speak of an orbifold, we are speaking of an orbifold with such a maximal cover.

It is clear that orbifolds are locally compact and locally path connected, hence by elementary topology we have:

Proposition 19 *An orbifold O is connected if and only if O is path connected.*

One easy way to get examples of orbifolds is given in the following

Proposition 20 *(Thurston) The quotient space of a manifold M by a group Γ which acts properly discontinuously on M is an orbifold.*

Proof: For any point $x \in M/\Gamma$, choose $\tilde{x} \in M$ projecting to x. Let Γ_x be the isotropy group of $\tilde{x}$. (Γ_x depends on the particular choice of $\tilde{x}$). There is a neighborhood $\tilde{U}_x$ of $\tilde{x}$ invariant by Γ_x and disjoint from its translates by elements of Γ not in Γ_x. The projection of $U_x = \tilde{U}_x/\Gamma_x$ is a homeomorphism. To obtain the required covering of M/Γ, augment some covering $\{U_x\}$ by adjoining finite intersections. Whenever $U_{x_1} \cap \ldots \cap U_{x_k} \neq \emptyset$, this means that some set of translates $\gamma_1 \tilde{U}_{x_1} \cap \ldots \cap \gamma_k \tilde{U}_{x_k}$ has a corresponding non–empty intersection. This intersection may be taken to be $U_{x_1} \cap \ldots \cap U_{x_k}$, with associated group $\gamma_1 \Gamma_{x_1} \gamma_1^{-1} \cap \ldots \cap \gamma_k \Gamma_{x_k} \gamma_k^{-1}$.

Definition 21 *A* Riemannian *orbifold is obtained as above where we require that the $\tilde{U}_i$ are convex, open (possibly non–complete) Riemannian manifolds diffeomorphic to $\mathbb{R}^n$, the Γ_i are finite groups of isometries acting effectively on $\tilde{U}_i$, and the maps $\tilde{\phi}_i$ are isometries. Recall that for a Riemannian manifold to be convex means that there exists a unique minimal geodesic joining any two points.*

We will have the need to distinguish between two types of Riemannian orbifolds.

Definition 22 *A* good Riemannian orbifold *is a pair (M, Γ) where M is a Riemannian manifold and Γ is a (proper) discontinuous group of isometries acting effectively on M. The* underlying space *of the orbifold is M/Γ. A* bad *Riemannian orbifold is a Riemannian orbifold which does not arise as a global quotient.*

The Singular Set and Stratifications

To each point $x \in U_i$ in an orbifold O is associated a group $\Gamma_x^{(i)}$, well-defined up to isomorphism within a local coordinate system: Let $U_i = \tilde{U}_i/\Gamma$ be a local coordinate system. Let $\tilde{x}, \tilde{y}$ be two points which project to x. Let $\Gamma_{\tilde{x}}^{(i)}$ be the isotropy group of $\tilde{x}$. Then if $\gamma \in \Gamma$ is the isometry such that $\gamma\tilde{x} = \tilde{y}$, it is not hard to see that the isotropy group of $\tilde{y}$ must be $\gamma\Gamma_{\tilde{x}}^{(i)}\gamma^{-1}$. Hence, the two isotropy groups are conjugate. Thus, up to isomorphism they can be regarded as the same group. We will denote this group by $\Gamma_x^{(i)}$. The next proposition shows that $\Gamma_x^{(i)}$ up to isomorphism, is also independent of coordinate system U_i.

Proposition 23 *Let O be an orbifold and let $x \in O$. Then the there exists a group Γ_x, called the* isotropy group at *x, which is well-defined. For any coordinate chart $\tilde{U}_i$, $\Gamma_x \cong \Gamma_x^{(i)}$.*

Proof: (See also [S2]) Let $x \in U_i \cap U_j$. Since the cover $\{U_i\}$ is closed under finite intersections, we may assume without loss of generality that $U_i \subset U_j$. We first need to show that if $x \in U_i \subset U_j$, has non-trivial isotropy in U_i, then it also has non-trivial isotropy in U_j. To see this, choose $\gamma \in \Gamma_i$ with $\gamma x = x$, $\gamma \neq$ identity. We have, by definition,

$$\tilde{\phi}_{ij}(x) = \tilde{\phi}_{ij}(\gamma x) = f_{ij}(\gamma)\tilde{\phi}_{ij}(x)$$

Thus, $f_{ij}(\gamma) \in \Gamma_j$ fixes $\tilde{\phi}_{ij}(x)$. Note that since f_{ij} is an injective homomorphism, $f_{ij}(\gamma) \neq$ identity. Hence x, as an element of U_j also has non-trivial isotropy. This

shows that the notion of non-trivial isotropy is well-defined. Next we show that in fact the isotropy group Γ_x is well-defined up to isomorphism. Let $\Gamma_x^{(i)} \subset \Gamma_i$, $\Gamma_x^{(j)} \subset \Gamma_j$ be the corresponding isotropy groups of x relative to the coordinate charts $\tilde{U}_i$, $\tilde{U}_j$. Of course, these groups are only defined up to conjugation by elements in Γ_i, Γ_j, respectively. We will assume in the discussion that follows that the appropriate conjugate of each has been chosen so that all maps make sense. We have already seen that there is an injective map $f_{ij} : \Gamma_x^{(i)} \hookrightarrow \Gamma_x^{(j)}$. Let $\gamma' \in \Gamma_x^{(j)}$. To complete the proof, it suffices to show that $f_{ij}(\gamma) = \gamma'$ for some $\gamma \in \Gamma_i$. To see that this suffices, note that

$$\tilde{\phi}_{ij}(\gamma x) = f_{ij}(\gamma)\tilde{\phi}_{ij}(x) = \gamma'\tilde{\phi}_{ij}(x) = \tilde{\phi}_{ij}(x).$$

The last equality follows from the commutative diagram in Definition 18. Since $\tilde{\phi}_{ij}$ is an embedding, then in fact, we conclude that $\gamma \in \Gamma_x^{(i)}$. Thus, we will have shown that $f_{ij} : \Gamma_x^{(i)} \to \Gamma_x^{(j)}$ is an isomorphism, and the proof will be complete. So, we now focus our attention on the existence of γ. Let $\gamma' \in \Gamma_x^{(j)}$ be arbitrary. Then

$$\gamma'\tilde{\phi}_{ij}(\tilde{U}_i) \cap \tilde{\phi}_{ij}(U_i) \neq \emptyset.$$

Thus, there exists $\tilde{p}$, $\tilde{q} \in \tilde{U}_i$ such that $\gamma'\tilde{\phi}_{ij}(\tilde{p}) = \tilde{\phi}_{ij}(\tilde{q})$. Since $\pi \circ \tilde{\phi}_{ij}(\tilde{p}) = \pi \circ \tilde{\phi}_{ij}(\tilde{q})$, we have, by the commutativity of the diagram in Definition 18, $\pi(\tilde{p}) = \pi(\tilde{q})$. Hence, there exists $\gamma \in \Gamma_i$ with $\gamma(\tilde{p}) = \tilde{q}$. Let $\sigma' = f_{ij}(\gamma)$. Then

$$\gamma'\tilde{\phi}_{ij}(\tilde{p}) = \tilde{\phi}_{ij}(\tilde{q}) = \tilde{\phi}_{ij}(\gamma\tilde{p}) = f_{ij}(\gamma)\tilde{\phi}_{ij}(\tilde{p}) = \sigma'\tilde{\phi}_{ij}(\tilde{p}).$$

Thus, by choosing $\tilde{\phi}_{ij}(\tilde{p})$ not to be in the singular set, which is possible by a result of M.H.A. Newman (see Proposition 26 below), then $\gamma' = \sigma' = f_{ij}(\gamma)$. Hence, we conclude that $\Gamma_x^{(i)} \cong \Gamma_x^{(j)}$. We can now denote this group unambiguously by Γ_x. This completes the proof.

It is worthwhile at this point to observe the following

Proposition 24 *Let O be a Riemannian orbifold. Let $p \in U_i \subset O$. Choose $\tilde{p} \in \tilde{U}_i$ so that it projects to p. Denote the isotropy group of $\tilde{p}$ by Γ_p. Then there exists a neighborhood $U_p \subset U_i$ and corresponding $\tilde{U}_p \subset \tilde{U}_i$ such that $U_p \overset{\text{isom}}{\cong} \tilde{U}_p/\Gamma_p$. The neighborhood U_p will be called a* fundamental neighborhood *of p. The open set $\tilde{U}_p$ will be called a* fundamental chart.

Proof: We have $U_i \overset{\text{isom}}{\cong} \tilde{U}_i/\Gamma$ where Γ is a finite group of isometries. Hence Γ acts discontinuously. Thus, there exists a neighborhood $\tilde{U}_{\tilde{p}}$ of $\tilde{p}$ which is invariant under the action of $\Gamma_{\tilde{p}}$ and disjoint from its translates by elements of Γ not in $\Gamma_{\tilde{p}}$. The projection of $\tilde{U}_{\tilde{p}}/\Gamma_{\tilde{p}}$ is then an isometry onto an open subset $U_p \subset U_i$.

Definition 25 *The* singular set Σ_O *of an orbifold O consists of those points $x \in O$ whose isotropy subgroup Γ_x is non-trivial. We say that O is a manifold when $\Sigma_O = \emptyset$. We may also, by abuse of definition, call points in the local covering $\tilde{U}_i$ with non-trivial isotropy, singular points also. This should cause no confusion since $x \in O$ is singular if and only if a corresponding point $\tilde{x} \in \tilde{U}_i$ is singular.*

Proposition 26 *(M.H.A Newman–Thurston) The singular locus of an orbifold is a closed set with empty interior.*

Proof: For any fundamental neighborhood $U = \tilde{U}/\Gamma$, $\Sigma_O \cap U$ is the image of the union of the fixed point sets in $\tilde{U}$ of elements of Γ. Since Γ is finite, $\Sigma_O \cap U$ is closed, and thus Σ_O is closed. The last statement follows from a result of M.H.A. Newman (see [D]) which states that if a finite group acts effectively on a connected manifold, the the set of points with trivial isotropy group is open and everywhere dense. Thus, locally, the points in $\tilde{U}$ with non-trivial isotropy form a closed set with empty interior. Hence the image $\Sigma_O \cap U$ of these points has empty interior. Since $\Sigma_O = \bigcup_{i=1}^{\infty} \Sigma_O \cap U_i$, and O is a locally compact Hausdorff space, it follows from standard topology that Σ_O must have empty interior. This completes the proof.

Remark 27 It should also be noted that the singular set is not necessarily a submanifold and may have several connected components.

To distinguish certain subsets of the singular set, we make the following definitions.

Definition 28 *Let U be a Riemannian manifold, and let G be a a finite group of isometries acting on U. Let $H \subset G$ be a subgroup of G. The subset*

$$U_H = \left\{ x \in U \mid \Gamma_x = H \right\}$$

is called the stratum of U associated with H. *A* stratification *of* U *is the partitioning of* U *into strata corresponding to every subgroup of* G. *Note that under these hypotheses, any such stratification is the union of a finite number of strata.*

Example 29 Let $U = \mathbf{R}^2$, and let $G \subset \mathbf{O}(2)$ be the group of isometries generated by reflection in the x and y axes. Let $H = \mathbf{Z}_2$, be the group generated by reflection in the x-axis. Then U_H is the x-axis minus the origin. Note that U_H is not a closed submanifold of $\mathbf{R}^2$, but it is a totally geodesic submanifold.

In order to analyze the strata we will need the following theorem contained in the proof of the Soul theorem of Cheeger–Gromoll [CG]:

Lemma 30 *(Cheeger–Gromoll) Let* $C \neq \emptyset$ *be a closed, connected, locally convex subset of a Riemannian manifold* M. *Then* C *carries the structure of an embedded* k*-dimensional submanifold of* M *with smooth totally geodesic interior* $\mathrm{int}(C) = N$ *and (possibly non-smooth) boundary* $\partial C = \overline{N} - N$.

Proof: See [CG, Theorem 1.6].

We have the following structure theorem concerning strata.

Proposition 31 *Any stratum* U_H *associated to a subgroup* $H \subset G$ *is locally convex.*

Proof: Let $x \in U_H$. Let $i(x) = \mathrm{inj}_x M$. Then if $y \in U_H \cap B\big(x, i(x)\big)$, the unique geodesic γ from x to y lies in U_H. To see this, suppose to the contrary that there is a point $z \in \gamma$ such that $\Gamma_z \neq H$. If $H - \Gamma_z \neq \emptyset$, then choose $h \in H - \Gamma_z$. Then

$h\gamma$ is another minimal geodesic from x to y, which is absurd by the choice of y. Thus, $H \subset \Gamma_z$, for all $z \in \gamma$. But, we know by Proposition 24, that if $d(x, z)$ is small enough, then $\Gamma_z \subset \Gamma_x = H$. But, then $\Gamma_z = H$, and we have a contradiction. This completes the proof.

By using the notion of tangent cone, we may in fact, strengthen the previous result.

Proposition 32 *Any stratum U_H associated to a subgroup $H \subset G$ is a totally geodesic submanifold of U.*

Proof: Lemma 30 implies that the connected components of $\overline{U_H}$ are embedded topological submanifolds of U with totally geodesic, connected interior and (possibly) non–smooth boundary. We claim that $\partial\overline{U_H} = \emptyset$. To do this, let $p \in \partial\overline{U_H}$ and form the tangent cone $T_p\overline{U_H}$ at p (see [CG]):

$$T_p\overline{U_H} = \left\{v \in T_pU \mid \exp_p t\frac{v}{||v||} \in N \text{ for some positive } 0 < t < r(p)\right\} \cup \{0_p\}$$

where $r(p)$ denotes the convexity radius at p. It follows, see[CG], that there exists $q \in N$ with the property that if γ is the unit speed segment joining p to q, then $\dot\gamma(0) \in T_p\overline{U_H}$, and $-\dot\gamma(0) \notin T_p\overline{U_H}$. But, since $q \in U_H$, the corresponding H–action on T_pU fixes $\dot\gamma(0)$. Thus, $-\dot\gamma(0)$ is fixed. Hence, p cannot be a boundary point of U_H. This completes the proof.

Remark 33 If we define the subset $U'_H = \left\{x \in U \mid H \subset \Gamma_x\right\} \subset U$ then $U_H \subset U'_H$ and U'_H is a closed totally geodesic submanifold of U. See [Ko]. Thus, although

$\overline{U_H} \subset U'_H$, $\overline{U_H} \neq U'_H$ in general. In the case of Example 29, U'_H is the entire x-axis.

Remark 34 The proof that U'_H is a closed totally geodesic submanifold can be used to show that given any isometry g of a Riemannian manifold U, the fixed point set of g is a closed totally geodesic submanifold of U. See [Ko]. Since Riemannian orbifolds are locally (open) Riemannian manifolds modulo finite group actions, it follows that the singular set, locally, is the image of the union of a finite number of closed submanifolds of U. Since any submanifold of U has empty interior in U, by applying the same reasoning in the proof of Proposition 26, we can conclude that in the case of Riemannian orbifolds, the singular set is closed and has empty interior without reference to Newman's theorem.

Metric Structures

In order to do Riemannian geometry on orbifolds we need to know how to measure the lengths of curves. To do this, we will need a way to lift curves locally, so that we may compute their lengths locally in fundamental neighborhoods. Finally, we will add up these local lengths to get the total length. This will interweave the local geometry of the fundamental neighborhoods with the geometry of the orbifold, which up until this point has not been described. The problem of course, is that locally these lifts are *not* unique. It will turn out, however, that the length of a curve is well–defined. One should keep in mind the techniques of standard covering space theory while reading this section. We adopt the following conventions: O

will denote a Riemannian orbifold. For $p \in O$, $U_p \subset O$ will be a fundamental neighborhood of p. $\pi : \tilde{U}_p \to \tilde{U}_p/\Gamma_p$ will be the natural projection, and we will identify U_p and $\tilde{U}_p/\Gamma_p$ by isometry. Recall that a *lift* of a curve $\gamma \subset U_p$ is a continuous curve $\tilde{\gamma} \subset \tilde{U}_p$ with $\pi(\tilde{\gamma}) = \gamma$. In order to avoid pathological situations we make the following definition.

Definition 35 *A curve $\gamma : [0,1] \to U_p$ is* admissible *if the the interval $[0,1]$ can be decomposed into a countable number of subintervals $[t_i, t_{i+1}]$ so that $\gamma|_{(t_i,t_{i+1})}$ is contained in a single stratum associated to a subgroup $H \subset \Gamma_p$. A curve $\gamma : [0,1] \to O$ is* admissible *if it is admissible in every chart U_p such that $\gamma \cap U_p \neq \emptyset$. Note that this is well-defined since the singular set is well-defined.*

Proposition 36 *Let $\gamma : [0,1] \to U_p$ be an admissible curve. Then there exists a curve $\tilde{\gamma} : [0,1] \to \tilde{U}_p$ which is a lift of γ.*

Proof: Decompose the interval $[0,1]$ into (possibly an infinite number of) subintervals $[t_i, t_{i+1}]$ so that $\gamma_i = \gamma|_{(t_i,t_{i+1})}$ is contained in a single stratum associated to a subgroup $H \subset \Gamma_p$. Note that π restricted to $\tilde{U}_{pH}$ is an m-fold covering map, where $m = (\#\Gamma_p)/(\#H)$. This follows since Γ_p/H is finite and has no fixed points in $\tilde{U}_{pH}$. Let $s_i = \frac{1}{2}(t_{i+1} - t_i)$. Once a preimage $\tilde{\gamma}_i(s_i)$ of $\gamma(s_i)$ is chosen, there is a unique lift $\tilde{\gamma}_i$ of γ_i in $\tilde{U}_{pH}$. Requiring that the lift $\tilde{\gamma}$ be continuous gives a (non-unique) lift of γ. This completes the proof.

Proposition 37 *Let $\gamma : [0,1] \to U_p$ be an admissible curve. Then γ has a well-defined length.*

Proof: As in Proposition 36, decompose the interval $[0,1]$ into (a possibly infinite number of) subintervals $[t_i, t_{i+1}]$ so that $\gamma_i = \gamma|_{(t_i,t_{i+1})}$ is contained in a single stratum associated to a subgroup $H \subset \Gamma_p$. Let $m = (\#\Gamma_p)/(\#H)$. Then γ_i has exactly m lifts $\left\{\tilde{\gamma}_j^{(i)}\right\}_{j=1}^{m}$ in $\tilde{U}_p$ and they are all disjoint, since π restricted to $\tilde{U}_{pH}$ is an m-fold covering map. So define the length of $\gamma|_{(t_i,t_{i+1})} = L\left(\tilde{\gamma}_1^{(i)}\right)$ where the right hand side is computed in $\tilde{U}_p$. Since all other lifts $\tilde{\gamma}_j^{(i)}$ differ from $\tilde{\gamma}_1^{(i)}$ by an isometry, this length is well-defined. To show that this length is independent of the fundamental neighborhood chosen, assume that $\gamma|_{(t_i,t_{i+1})} \subset U_r = U_p \cap U_q$ then by the definition of Riemannian orbifold there are isometric imbeddings of $\tilde{U}_r$ into $\tilde{U}_p$ and into $\tilde{U}_q$ which respect the various group actions. Hence, the length of $\tilde{\gamma}|_{(t_i,t_{i+1})}$ is independent of fundamental neighborhood chosen. Let N be the number of subintervals $[t_i, t_{i+1}]$. $N = \infty$ is possible. Define the length of γ to be

$$L(\gamma) = \sum_{i=1}^{N \text{ or } \infty} L\left(\tilde{\gamma}_1^{(i)}\right).$$

If the sum does not converge, define $L(\gamma) = \infty$.

We are now ready to prove the following theorem:

Theorem 38 *Let $\gamma : [0,1] \to O$ be an admissible curve. Then γ can be assigned an well-defined length $L(\gamma)$.*

Proof: By the Lebesgue number lemma, we can partition $[0,1]$ by $0 = t_0 < t_1 < \ldots < t_n = 1$ so that $\gamma|_{[t_i,t_{i+1}]}$ lies entirely in a fundamental neighborhood of $\gamma(t_i)$.

By the previous proposition, we can assign $\gamma|_{[0,t_1]}$ an unambiguous length. So suppose by induction a unique length L_i can be assigned to $\gamma|_{[0,t_i]}$. The previous proposition assigns a unique length ℓ_{i+1} to $\gamma|_{[t_i,t_{i+1}]}$. Then define the length L_{i+1} of $\gamma|_{[0,t_{i+1}]}$ to be $L_i + \ell_{i+1}$. This finishes the induction and the proof is now complete.

We are now in a position to give a length space structure to any Riemannian orbifold O. Given any two points $x, y \in O$ define the distance $d(x,y)$ between x and y to be

$$d(x,y) = \inf \{L(\gamma) \mid \gamma \text{ is an admissible curve joining } x \text{ to } y\}.$$

Then (O, d) becomes a metric space.

Remark 39 There is no loss in generality by defining d in terms of admissible curves instead of continuous curves. To see this, note that a continuous curve γ joining x to y has local lifts. See for example [B, Theorem II.6.2]. Let $\tilde{\gamma}$ be a particular local lift in some $\tilde{U}_p$. Then since the singular set is locally convex by Proposition 31, if $d\big(\tilde{\gamma}(s), \tilde{\gamma}(t)\big)$ is sufficiently small and $\tilde{\gamma}(s)$, $\tilde{\gamma}(t)$ belong to the same stratum, then we can replace $\tilde{\gamma}|_{[s,t]}$ by a geodesic segment $\tilde{\gamma}'|_{[s,t]}$ entirely contained in a single stratum. Since $L\big(\tilde{\gamma}|_{[s,t]}\big) \geq L\big(\tilde{\gamma}'|_{[s,t]}\big)$, we can without loss of generality use admissible curves when computing $d(x,y)$.

Theorem 40 *With the distance d above, (O,d) becomes a length space, and furthermore, if (O,d) is complete, any two points can be joined by a minimal geodesic realizing the distance $d(x,y)$.*

Proof: (O, d) is a length space by definition and Remark 39. The second statement follows by applying Theorem 9 and noting that orbifolds are locally compact. This completes the proof.

Remark 41 In the case of a good Riemannian orbifold (M, Γ), it follows that for $x, y \in M/\Gamma$,

$$d(x,y) = d_M\Big(\pi^{-1}(x), \pi^{-1}(y)\Big) \stackrel{\text{def}}{=} \inf_{\tilde{x}\in\pi^{-1}(x), \tilde{y}\in\pi^{-1}(y)} d_M(\tilde{x}, \tilde{y}).$$

This is because (M, d_M) is itself a length space. If M is complete, then it follows that x, y can be joined by a minimal geodesic which corresponds the projection of the minimal geodesic realizing the distance $d_M\Big(\pi^{-1}(x), \pi^{-1}(y)\Big)$.

A natural question to ask is whether a good Riemannian orbifold which is complete *as an orbifold* can arise as the quotient of a *non-complete* Riemannian manifold. This is answered in the next proposition.

Proposition 42 *Let $O = (M, \Gamma)$ be a good Riemannian orbifold. Then O is complete if and only if M is complete.*

Proof: By Remark 41, M complete implies that O complete. So suppose, O is complete, but M is not complete. Then there exists a Cauchy sequence $\{\tilde{p}_i\}$ which does not converge to a point of M. Since the projection π to O is distance decreasing, the sequence $\{\pi(\tilde{p}_i)\}$ is Cauchy, and hence by completeness converges to a point $p \in O$. Let $\tilde{p} \in M$ be an element of $\pi^{-1}(p)$, and let $\tilde{U}_p$ be a fundamental

chart containing $\tilde{p}$. Let $S_i = \pi^{-1}\Big(\pi(\tilde{p}_i)\Big) \cap \tilde{U}_p$. Note that each S_i is a finite set. Without loss of generality, we can assume there exists a Cauchy sequence $\{\tilde{s}_i\}$ converging to $\tilde{p}$, with $\tilde{s}_i \in S_i$. For each i, let $g_i \in \Gamma$ be such that $g_i(\tilde{s}_i) = \tilde{p}_i$. Then note that the set $\{g_i\}$ contains only finitely many distinct elements because otherwise the set $\{g_i^{-1}(\tilde{p}_i)\}$ converges to $\tilde{p}$ which contradicts the fact that Γ acts discontinuously. So by passing to a subsequence we may assume that $g_i = g$ for all i. Then

$$d(\tilde{s}_i, \tilde{p}) = d(g\tilde{s}_i, g\tilde{p}) = d(\tilde{p}_i, g\tilde{p})$$

By letting $i \to \infty$, we conclude that $\tilde{p}_i \to g\tilde{p} \in M$. This completes the proof.

We end this section with the following observation:

Proposition 43 *Riemannian orbifolds are locally simply connected.*

Proof: Let $p \in O$ be any point, and let U_p be a fundamental neighborhood of p. Then we have $g\tilde{U}_p \cap \tilde{U}_p = \emptyset$ for $g \in \Gamma - \Gamma_p$. Let $r > 0$ be such that $B(\tilde{p}, r) \subset \tilde{U}_p$ for some lift $\tilde{p}$ of p. Hence if $\gamma : S^1 \to U_p$ is a closed curve based at p of length $< r$, then γ lifts to a closed curve $\tilde{\gamma} : S^1 \to \tilde{U}_p$ based at $\tilde{p}$ with $\tilde{\gamma} \subset B(\tilde{p}, r)$. But, by definition of Riemannian orbifold, $\tilde{U}_p \overset{\text{diffeo}}{\cong} \mathbb{R}^n$ so in fact $\tilde{\gamma}$ is null–homotopic. Since $U_p \overset{\text{isom}}{\cong} \tilde{U}_p/\Gamma_p$, by projecting this homotopy, we conclude that γ is also null–homotopic. This completes the proof.

From now on, unless otherwise stated O will be assumed to be complete Riemannian orbifold.

Examples

Example 1 Let M be any Riemannian manifold, $\tilde{M}$ its universal covering space. Then $\big(M, \pi_1(M)\big)$ is an orbifold with $\pi_1(M)$ acting on M as covering transformations.

Example 2 Let $M = \mathbf{R}^2$ and let Γ be the group generated by the the rotation through angle $2\pi/n$ about 0. Then (M, Γ) is an orbifold whose underlying space is topologically $\mathbf{R}^2$, but metrically is a cone. It is a Riemannian manifold except at the cone point where it has a metric singularity and hence is *not* a manifold.

Example 3 ($\mathbf{Z}_p$–footballs) Let $M = S^2 \subset \mathbf{R}^3$. Define a $\mathbf{Z}_p$–action on S^2 by rotation around z–axis by an angle of $2\pi/p$. Here, the underlying space is (topologically) S^2, but the orbifold is *not* a manifold since Σ consists of the north and south poles.

Example 4 Let M be the 2–sphere as above. Let Γ be the group of order two generated by reflection across the xy–plane. Then M/Γ is topologically a 2–disc (a manifold with non-empty boundary).

Example 5 ($\mathbf{Z}_p$–hemispheres) Let M be the 2–sphere as above. Let Γ be the group generated by reflection across the xy–plane and rotation about the z–axis by an angle of $2\pi/p$. Then M/Γ is again topologically a 2–disc.

Example 6 Let $M = \mathbf{R}^3$ and Γ generated by the antipodal map $x \mapsto -x$. Then M/Γ is topologically a cone over $\mathbf{R}P^2$, which is not a (topological) manifold at the

cone point.

Example 7 By an appropriate quotient of S^4, the suspension over any of the 3–dimensional lens spaces L^3_p can be realized. These spaces are compact and fail to be (topological) manifolds at the suspension points.

Example 8 Let $M = \mathbb{R}^2$. Let p, q, r denote 3 integers so that there is a triangle $\triangle$ with angles $\pi/p, \pi/q, \pi/r$. Thus $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. The only possibilities for p, q, r are $(3,3,3)$, $(2,3,6)$, $(2,4,4)$. The full triangle group $\tilde{\triangle}(p,q,r)$ is the group of isometries generated by reflections in the 3 sides of the triangle. The translates of $\triangle$ tile the plane. Let $\triangle(p,q,r)$ be the subgroup of index 2 of orientation preserving isometries. Then $M/\triangle(p,q,r)$ is a 2–sphere with 3 singular points. Similar constructions can be carried out with quadrilaterals.

All of the orbifolds listed so far are good, we now list two simple cases of bad orbifolds.

Example 9 ($\mathbb{Z}_p$–teardrops) This space is topologically S^2 with a single cone point of order p at the north pole.

Example 10 ($\mathbb{Z}_p$-$\mathbb{Z}_q$–footballs) This space is also topologically S^2 with a cone point of order p at the north pole and another cone point of order q at the south pole.

Most of these examples are illustrated in Figures 2 and 3 on the following pages.

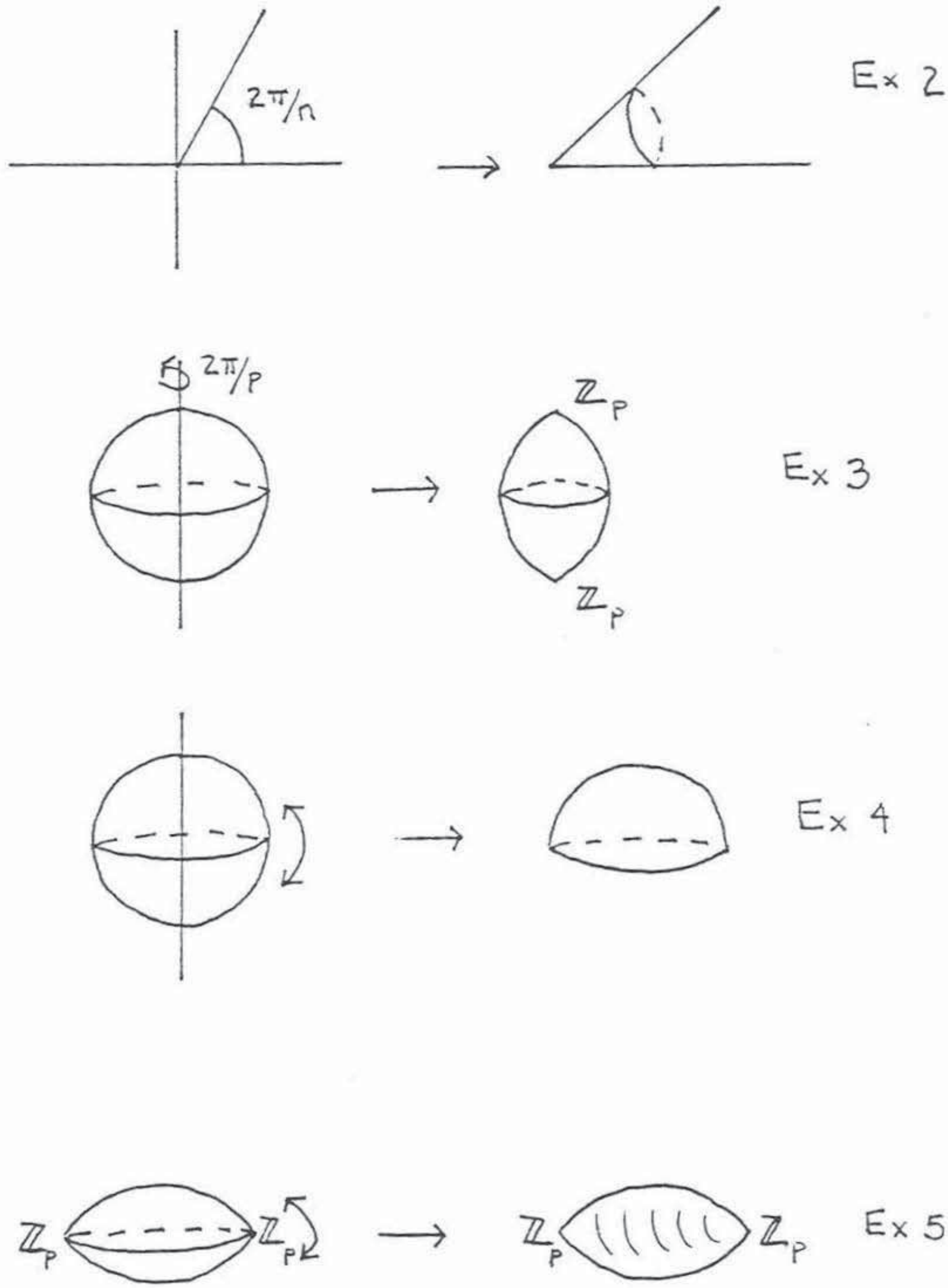
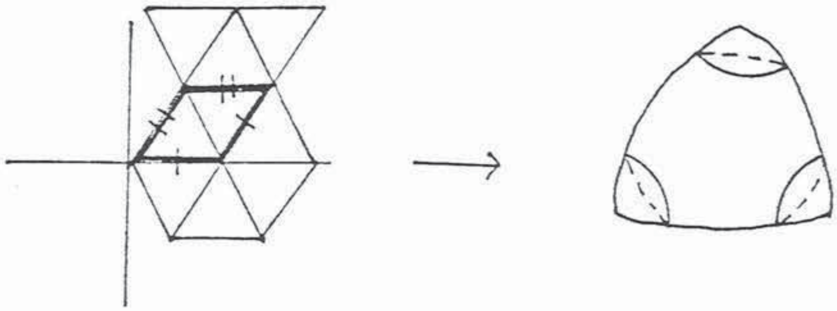
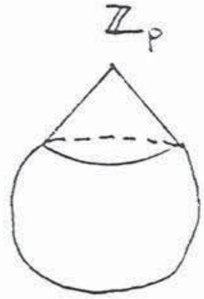


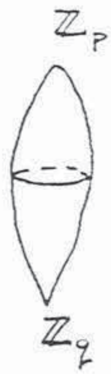
Figure 2



Ex 8



Ex 9



Ex 10

Figure 3

Toponogov's Theorem for Orbifolds

One of the most useful results in Riemannian geometry is the Toponogov Triangle Comparison Theorem. It says roughly that in the presence of a lower sectional curvature bound k, triangles in a Riemannian manifold M may be *compared* to triangles in the two–dimensional simply connected space form of constant curvature k, denoted by S^2_k. When $k > 0$ then S^2_k is the standard sphere S^2 of radius $\frac{1}{\sqrt{k}}$, if $k = 0$ it is the Euclidean plane $\mathbb{R}^2$, and if $k < 0$ it is the hyperbolic planes. The notion of triangle makes sense in any length space, and we say that a length space has (Toponogov) curvature $\geq k$ if it satisfies the conclusion of Toponogov's Theorem. We show that if a Riemannian manifold M has sectional curvature $\geq k$, then the orbifold (M, Γ) has curvature $\geq k$ as a length space.

Theorem 1 (Toponogov's Theorem for Orbifolds) *Let $O = (M, \Gamma)$ be a good Riemannian orbifold such that $K_M \geq k$. Let $\gamma_i : [0,1] \to O$, $i = 1, 2$ be segments with $\gamma_1(0) = \gamma_2(0)$ and $L(\gamma_i) \leq \pi/\sqrt{k}$. Fix $s, t \in (0,1)$. Then choose $\overline{\gamma_i} : [0,1] \to S^2_k$ with the property that $\overline{\gamma_1}(0) = \overline{\gamma_2}(0)$ and $d\big(\gamma_1(s), \gamma_2(t)\big) = d\big(\overline{\gamma_1}(s), \overline{\gamma_2}(t)\big)$. Then*

(i) $d\big(\gamma_1(s'), \gamma_2(t')\big) \leq d\big(\overline{\gamma_1}(s'), \overline{\gamma_2}(t')\big)$ if $s' \geq s$, $t' \geq t$

(ii) $d\big(\gamma_1(s'), \gamma_2(t')\big) \geq d\big(\overline{\gamma_1}(s'), \overline{\gamma_2}(t')\big)$ if $s' \leq s$, $t' \leq t$

Proof: The basic idea is to pull back everything to M, apply the standard Toponogov Theorem there and then push back down. The formal proof goes as follows: Pull back γ_i to minimizing segments $\tilde{\gamma}_i$ from $\pi^{-1}\big(\gamma_i(0)\big)$ to $\pi^{-1}\big(\gamma_i(1)\big)$.

By applying an isometry if necessary we may assume the pull–backs form a hinge in M so that $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0)$. Note that length restriction guarantees that sides of the hinge do not intersect each other. Now fix $s, t \in (0,1)$. To prove (i), assume $s' \geq s$, $t' \geq t$ and let $\ell_1 = d\big(\tilde{\gamma}_1(0), \tilde{\gamma}_1(s)\big)$ and $\ell_2 = d\big(\tilde{\gamma}_2(0), \tilde{\gamma}_2(t)\big)$. By the Toponogov Comparison Theorem there exists a triangle contained inside S_k^2 with sides of length ℓ_1, ℓ_2, and $d\big(\tilde{\gamma}_1(s), \tilde{\gamma}_2(t)\big)$ such that

$$d_M\Big(\tilde{\gamma}_1(s'), \tilde{\gamma}_2(t')\Big) \leq d_{S_k^2}\Big(\overline{\gamma_1}(s'), \overline{\gamma_2}(t')\Big)$$

We are now done with the proof of (i) since $d_O\Big(\gamma_1(s'), \gamma_2(t')\Big) \leq d_M\Big(\tilde{\gamma}_1(s'), \tilde{\gamma}_2(t')\Big)$. To prove (ii) we use (i). Let $s' \leq s$, $t' \leq t$. Consider the following triangles:

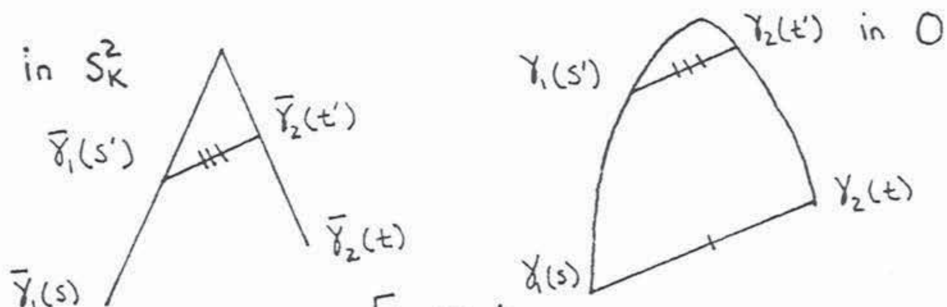


Figure 4

We know by (i) that $d_O\Big(\gamma_1(s), \gamma_2(t)\Big) \leq d_{S_k^2}\Big(\overline{\gamma_1}(s), \overline{\gamma_2}(t)\Big)$. Now in S_k^2 draw the following triangle:

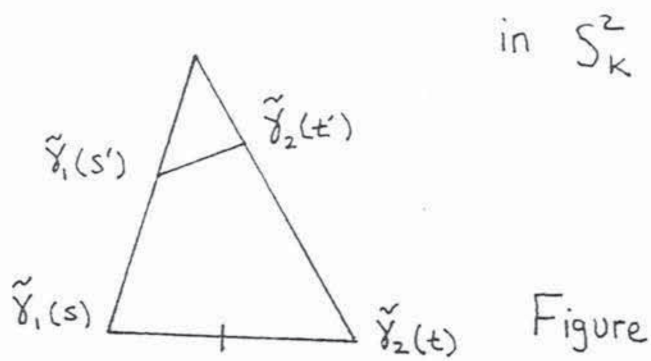


Figure 5

By shrinking $d_{S_k^2}\Big(\overline{\gamma_1}(s), \overline{\gamma_2}(t)\Big)$ to $d_O\Big(\gamma_1(s), \gamma_2(t)\Big)$ in Figure 5 we get that

$$d_{S_k^2}\Big(\overline{\gamma_1}(s'), \overline{\gamma_2}(t')\Big) \geq d_{S_k^2}\Big(\tilde{\gamma}_1(s'), \tilde{\gamma}_2(t')\Big).$$

But since

$$d_{S_k^2}\Big(\overline{\gamma_1}(s'), \overline{\gamma_2}(t')\Big) = d_O\Big(\gamma_1(s'), \gamma_2(t')\Big)$$

we have

$$d_O\Big(\gamma_1(s'), \gamma_2(t')\Big) \geq d_{S_k^2}\Big(\tilde{\gamma}_1(s'), \tilde{\gamma}_2(t')\Big).$$

as was to be shown. The proof of the Toponogov Theorem is now complete.

This result implies that good orbifolds (M, Γ) with $K_M \geq k$ have *Toponogov curvature* $\geq k$ in the sense of length spaces.

Remark 2 In [BGP] it is shown that, for instance, a locally compact length space which has Toponogov curvature $\geq k$ locally, has Toponogov curvature $\geq k$ globally. Combining this result, with the Toponogov theorem above shows that (bad) orbifolds modelled locally on Riemannian manifolds M with $K_M \geq k$ have Toponogov curvature $\geq k$.

The Structure Theorem for Geodesics in Orbifolds

In this section we investigate the behavior of segments in orbifolds. The first result shows that the singular set Σ is locally convex.

Proposition 1 *Let $O = (M, \Gamma)$ be a good Riemannian orbifold, and let Σ be its singular set. Given $p \in \Sigma$ there exists $\varepsilon_p > 0$ such that for all $q \in \Sigma \cap B(p, \varepsilon_p)$ any segment in O between p and q lies in Σ. Thus, Σ is locally convex.*

Proof: Note that the statement is trivial if p is an isolated point of Σ. So assume p is not isolated. Then there exists $\tilde{p} \in \pi^{-1}(p)$ and a neighborhood $\tilde{U}_p$ so that for sufficiently small ε_p, $B(p, \varepsilon_p) \subset U \overset{\text{isom}}{\cong} \tilde{U}_p/\Gamma_p$. If necessary, choose ε_p smaller so that $2\varepsilon_p < \text{inj}_{\tilde{p}} M$. Suppose to the contrary that for some $q \in B(p, \varepsilon_p) \cap \Sigma$, there exists a segment γ from p to q not entirely contained in Σ. Then there exists some point $r \in \gamma$ so that Γ_p does not fix r. By taking a small enough metric ball around q which is contained in $B(p, \varepsilon_p)$, we may assume by Proposition 24 of the first section and definition of orbifold that $\Gamma_q \subset \Gamma_p$. Since $\#\Gamma_q > 1$, pulling γ back to M gives rise to (at least) two segments from $\tilde{p}$ to $\tilde{q}$ in M which is absurd since $\tilde{q} \in B(\tilde{p}, \varepsilon_p)$ and $2\varepsilon_p < \text{inj}_{\tilde{p}} M$. See Figure 6. This completes the proof.

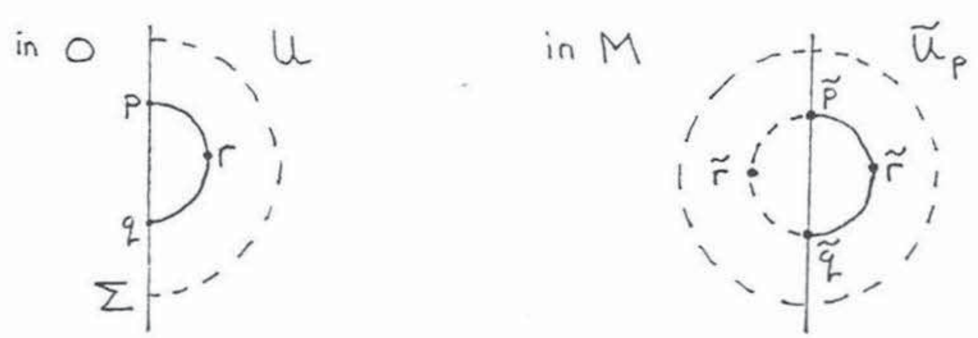


Figure 6

The next lemma assures that a segment in an orbifold minimizes distance between any two of its points.

Proposition 1 *Let $O = (M, \Gamma)$ be a good Riemannian orbifold, and let Σ be its singular set. Given $p \in \Sigma$ there exists $\varepsilon_p > 0$ such that for all $q \in \Sigma \cap B(p, \varepsilon_p)$ any segment in O between p and q lies in Σ. Thus, Σ is locally convex.*

Proof: Note that the statement is trivial if p is an isolated point of Σ. So assume p is not isolated. Then there exists $\tilde{p} \in \pi^{-1}(p)$ and a neighborhood $\tilde{U}_p$ so that for sufficiently small ε_p, $B(p, \varepsilon_p) \subset U \overset{\text{isom}}{\cong} \tilde{U}_p/\Gamma_p$. If necessary, choose ε_p smaller so that $2\varepsilon_p < \text{inj}_{\tilde{p}} M$. Suppose to the contrary that for some $q \in B(p, \varepsilon_p) \cap \Sigma$, there exists a segment γ from p to q not entirely contained in Σ. Then there exists some point $r \in \gamma$ so that Γ_p does not fix r. By taking a small enough metric ball around q which is contained in $B(p, \varepsilon_p)$, we may assume by Proposition 24 of the first section and definition of orbifold that $\Gamma_q \subset \Gamma_p$. Since $\#\Gamma_q > 1$, pulling γ back to M gives rise to (at least) two segments from $\tilde{p}$ to $\tilde{q}$ in M which is absurd since $\tilde{q} \in B(\tilde{p}, \varepsilon_p)$ and $2\varepsilon_p < \text{inj}_{\tilde{p}} M$. See Figure 6. This completes the proof.

The next lemma assures that a segment in an orbifold minimizes distance between any two of its points.

Lemma 2 *Let $\gamma : [0,1] \to O$ be a segment and let $\tilde{\gamma} \subset M$ be a lift of γ such that $L(\tilde{\gamma}) = L(\gamma)$. Then $d_M\big(\tilde{\gamma}(t_1), \tilde{\gamma}(t_2)\big) = d_O\big(\gamma(t_1), \gamma(t_2)\big)$ for all $t_1, t_2 \in [0,1]$.*

Proof: Suppose not. Let $\gamma(t_1) = r$, $\gamma(t_2) = s$, $\tilde{\gamma}(t_1) = \tilde{r}$, $\tilde{\gamma}(t_2) = \tilde{s}$. Then

$$d_O(r,s) = d_M\left(\pi^{-1}(r), \pi^{-1}(s)\right) \neq d_M(\tilde{r}, \tilde{s})$$

by hypothesis. So, suppose that $d_M\left(\pi^{-1}(r), \pi^{-1}(s)\right)$ is realized by $\tilde{r}'$, $\tilde{s}'$ (where $\tilde{r} = \tilde{r}'$ or $\tilde{s} = \tilde{s}'$ is possible, but not both). By applying an isometry taking $\tilde{r}'$ to $\tilde{r}$ we can form a new path $\tilde{\gamma}'$ as follows: $\tilde{\gamma}' = (\tilde{p}, \tilde{r}, g\tilde{s}', h\tilde{q})$ = dotted path shown in Figure 7 with g and h the isometries illustrated. Then $\tilde{\gamma}'$ is a shorter path than γ which projects down to a path from p to q in O, a contradiction since $d_M(\tilde{p}, \tilde{q}) = d_O(p,q)$. This completes the proof.

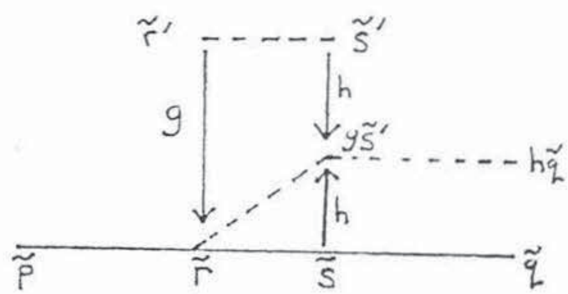


Figure 7

The last theorem of this section shows that in some sense the set Σ forms a geometric barrier to length minimization.

Theorem 3 *Suppose $\gamma : [0,1] \to O$ is a segment. Let $\gamma(0) = p, \gamma(1) = q$. Then either*

(i) $\gamma \subset \Sigma$ or

(ii) $\gamma \cap \Sigma \subset \{p\} \cup \{q\}$

In particular, if $\gamma \not\subset \Sigma$, *then* $\gamma \cap \Sigma = \emptyset$, $\{p\}$, $\{q\}$, or $\{p\} \cup \{q\}$.

Proof: Suppose $\gamma \not\subset \Sigma$ and that $p \notin \Sigma$. Then let $r \in \gamma \cap \Sigma$, $r = \gamma(t_0)$, $t_0 \neq 0$ be the first time γ intersects Σ. Note that such a first time exists, since Σ is closed and $p \notin \Sigma$. If $t_0 = 1$, then $\gamma \cap \Sigma = \{q\}$, which is fine. So assume $t_0 \neq 1$. Now pull γ back to M and observe that there exists an isometry $g \in \Gamma_r$ which must move $\tilde{p}$. But, then we can construct a branching geodesic as follows: Note that the curve $-\tilde{\gamma} = (\tilde{q}, \tilde{r}, \tilde{p})$ has the same length as $-\tilde{\gamma}' = (\tilde{q}, \tilde{r}, g\tilde{p})$. Since γ is a segment we have

$$L(-\tilde{\gamma}') = L(-\tilde{\gamma}) = d(\tilde{p}, \tilde{q}) = d(\Gamma\tilde{p}, \Gamma\tilde{q})$$

We therefore can conclude that $-\tilde{\gamma}'$ realizes the distance between $g\tilde{p}$ and $\tilde{q}$, and thus it is a geodesic. But this situation gives rise to a branching geodesic which is impossible in a Riemannian manifold. See Figure 8.

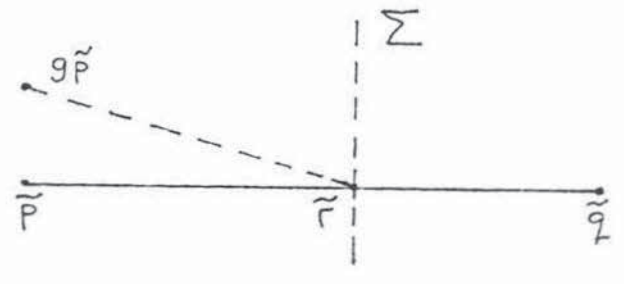


Figure 8

Finally, if $p \in \Sigma$ and $\gamma \not\subset \Sigma$ and γ does not immediately leave Σ, then, by local convexity of Σ (Proposition 1), there exists $\varepsilon > 0$ and $\delta > 0$ so that $\gamma(\varepsilon) \subset \Sigma$ for $0 \leq t \leq \varepsilon < 1$, and $\gamma(\varepsilon + \delta_0) \not\subset \Sigma$ for $0 < \delta_0 < \delta$. Then we have a curve that lies in

Σ, then tries to leave momentarily. This is identical to the situation above. Thus $\gamma \subset \Sigma$ unless no such ε exists. In other words, γ immediately leaves Σ, and we conclude that γ can only intersect Σ again (possibly) at its endpoint q. The proof is now complete.

Remark 4 Since all of the arguments of this section only used the local structure of orbifolds, all of these results hold for general orbifolds.

The significance of the last theorem is apparent. It says that a segment cannot pass through the singular set unless it starts and/or ends there. A trivial consequence of this is that the complement of Σ in O is convex as all points in $O - \Sigma$ can be joined by some segment. Thus, Σ cannot disconnect O. We will next state a criterion to determine when in fact an orbifold is a manifold.

Corollary 5 *A (complete) Riemannian orbifold O is a Riemannian manifold if and only if O is geodesically complete.*

Proof: By the structure theorem, if O is geodesically complete then $\Sigma = \emptyset$. Hence it is a Riemannian manifold. If O is a Riemannian manifold, then the result follows from the Hopf–Rinow theorem. This completes the proof.

Remark 6 It follows that a Riemannian orbifold O is an *almost Riemannian space* if and only if O is a Riemannian manifold. For the definition of almost Riemannian space, see [P].

Volume Comparison for Orbifolds

The Bishop relative volume comparison theorem of Riemannian geometry is

Theorem 1 *Let M be a complete Riemannian manifold. Suppose $\mathrm{Ric}_M \geq (n-1)k$. Then the function*

$$r \mapsto \frac{\mathrm{Vol}\, B(p,r)}{\mathrm{Vol}_k\, B(\overline{p},r)}$$

is non–increasing. $\mathrm{Vol}_k\, B(\overline{p},r)$ *denotes the volume of the metric r-ball in S^n_k. Furthermore, the limit as $r \to 0^+$ is 1.*

Before we define the concept of volume for a Riemannian orbifold, we need to recall the following definitions:

Definition 2 *Let X be a metric space. The σ–algebra generated by the family of open sets in X is called the* Borel *σ–algebra on X and will be denoted by $\mathcal{B}_X$. Given a measure μ on $\mathcal{B}_X$, there is a unique measure $\overline{\mu}$ which is* complete *and extends μ. $\overline{\mu}$ is defined on the new σ–algebra*

$$\overline{\mathcal{B}_X} = \mathcal{B}_X \cup \{F \mid F \subset A,\ A \in \mathcal{B}_X \text{ and } \mu(A) = 0\}$$

and $\overline{\mu}(F) \stackrel{\text{def}}{=} 0$.

By Remark 34 and Remark 33 of the first section, the singular set is covered locally by the union of a finite number of totally geodesic submanifolds. This union thus has measure 0 relative to the canonical Riemannian measure in each $\tilde{U}_p$. Since the natural projection to the orbifold is distance decreasing, it is natural to require

that any measure constructed on the orbifold assign the singular set measure 0. Of course, we also want the orbifold measure to be compatible with the local Riemannian measures that come from the covering. This is the thrust of the next proposition.

Proposition 3 *For any Riemannian orbifold O with singular set Σ, there exists a complete canonical measure $\overline{\mu}$ on $\overline{\mathcal{B}_{O-\Sigma}}$, given by a unique volume form on $O-\Sigma$. Furthermore, $\overline{\mu}$ can be extended to a complete measure $\overline{\nu}$ on $\overline{\mathcal{B}_O}$. Explicitly,*

$$\overline{\nu}(A) = \overline{\mu}(A-\Sigma) = \int_{A-\Sigma} d\mathrm{Vol}$$

for any $A \in \overline{\mathcal{B}_O}$. Here, $d\mathrm{Vol}$ is to be interpreted as $d\overline{\mu}$. In particular, $\overline{\nu}(F) = 0$ for any $F \subset \Sigma$.

Proof: Let $p \in O$, and let $U_p \cong \tilde{U}_p/\Gamma_p$ be a fundamental neighborhood of p. Let $\pi : \tilde{U}_p \to U_p$ be the natural projection. Let $\tilde{\Sigma}_p = \pi^{-1}(\Sigma \cap U_p)$. Then on $\tilde{U}_p - \tilde{\Sigma}$, Γ_p acts properly discontinuously without fixed points. Since the action is by isometries, the canonical Riemannian volume form $\tilde{\Omega}$ on $\tilde{U}_p$ is invariant under the action of Γ_p. Hence it follows that there exists a unique volume form Ω on $U_p - \Sigma$ such that $\pi^*\Omega = \tilde{\Omega}$. See [BG, Lemma 5.3.9]. Since $O-\Sigma$ is connected we conclude that the volume form Ω is unique. Completing the resulting measure gives rise to a complete measure $\overline{\mu}$ on $\overline{\mathcal{B}_{O-\Sigma}}$ which is to be extended to a complete measure $\overline{\nu}$ on $\overline{\mathcal{B}_O}$. The extension is given by the formula

$$\overline{\nu}(A) = \overline{\mu}(A-\Sigma) = \int_{A-\Sigma} d\mathrm{Vol}$$

for $A \in \overline{\mathcal{B}_O}$. Then $\overline{\nu}$ is indeed complete. Note that this definition is compatible with the canonical measure in each $\tilde{U}_p$. For, $\tilde{\Sigma}_p \in \mathcal{B}_{\tilde{U}_p}$ and has measure 0 in $\tilde{U}_p$ since $\tilde{\Sigma}_p$ is the finite union of closed totally geodesic submanifolds of $\tilde{U}_p$. Next since π is distance decreasing it must follow that $\pi(\tilde{\Sigma}_p) = \Sigma \cap U_p \in \mathcal{B}_O$ and has measure 0 in O. This completes the proof.

The geodesic structure theorem of the previous section says that once a geodesic hits the singular set it must stop. Thus, in some sense the domain of the "exponential" map for an orbifold is *smaller* than its counterpart in the local Riemannian covering. Combining this with the fact that the natural projection is distance decreasing gives us, at least intuitively, reason to believe that volume cannot be concentrated *behind* singular points. It is this reasoning that enables us to now extend the Bishop relative volume comparison theorem to orbifolds, but first we need a notion of Ricci curvature.

Definition 4 *A Riemannian orbifold is said to have* $\mathrm{Ric}_O \geq (n-1)k$ *if every point is locally covered by a Riemannian manifold with Ricci curvature* $\geq (n-1)k$.

Theorem 5 *Let O be a complete Riemannian orbifold with singular set Σ. Suppose* $\mathrm{Ric}_O \geq (n-1)k$. *Then the function*

$$r \mapsto \frac{\mathrm{Vol}\, B(p,r)}{\mathrm{Vol}_k\, B(\overline{p},r)}$$

is non-increasing. $\mathrm{Vol}_k\, B(\overline{p},r)$ *denotes the volume of the metric r-ball in S^n_k. Furthermore, the limit as $r \to 0$ is $\frac{1}{\#\Gamma_p}$, where Γ_p is the isotropy subgroup at p.*

Proof: Note that $O - \Sigma$ is a (non-complete) Riemannian manifold. Fix $p \in O$. Let $\varepsilon_i \to 0$ be a sequence of real numbers, and $\{p_i\}$, a sequence of points in O such that $d(p, p_i) < \varepsilon_i$. Then clearly,

$$\lim_{i \to \infty} d_H\Big(B(p_i, r), B(p, r)\Big) = 0$$

where d_H denotes the usual Hausdorff distance between sets in the metric space O. It follows that

$$\operatorname{Vol} B(p_i, r) \longrightarrow \operatorname{Vol} B(p, r).$$

To see this, define the characteristic function $\chi_A : O \to \mathbf{R}$ for a subset $A \subset O$ to be

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

Then we have that

$$\chi_{B(p_i,r)} \to \chi_{B(p,r)}$$

pointwise almost everywhere. For, if $x \in B(p, r)$, then $d(p, x) = r - \delta$, $\delta > 0$, thus by the triangle inequality,

$$d(p_i, x) \leq d(p, p_i) + d(p, x) \leq d(p, p_i) + r - \delta.$$

Hence, if i is chosen so that $d(p, p_i) < \frac{1}{2}\delta$, then $x \in B(p_i, r)$. On the other hand, if $x \notin \overline{B(p, r)}$, then a similar argument shows that $x \notin B(p_i, r)$ for sufficiently large i. Thus, by Lebesgue dominated convergence

$$\operatorname{Vol} B(p_i, r) = \int_{O-\Sigma} \chi_{B(p_i,r)}\, d\mathrm{Vol} \longrightarrow \int_{O-\Sigma} \chi_{B(p,r)}\, d\mathrm{Vol} = \operatorname{Vol} B(p, r)$$

where $d\mathrm{Vol}$ is the Riemannian measure on $O - \Sigma$. Since $O - \Sigma$ is convex, and we have a well-defined exponential map $\exp_{p_i}$ defined on the interior of $\mathrm{Cut}(p_i) - \Sigma \subset \mathrm{Cut}(p_i)$, where $\mathrm{Cut}(p_i)$ denotes the cut locus at p_i, we can apply the standard volume comparison theorem to conclude that

$$\frac{\mathrm{Vol}\, B(p_i, r)}{\mathrm{Vol}\, B(p_i, R)} \geq \frac{\mathrm{Vol}_k\, B(\overline{p}, r)}{\mathrm{Vol}_k\, B(\overline{p}, R)}.$$

Letting $i \to \infty$ gives

$$\frac{\mathrm{Vol}\, B(p, r)}{\mathrm{Vol}\, B(p, R)} \geq \frac{\mathrm{Vol}_k\, B(\overline{p}, r)}{\mathrm{Vol}_k\, B(\overline{p}, R)}.$$

To prove the last statement of the theorem, consider a fundamental neighborhood $U_p \overset{\text{isom}}{=} \tilde{U}_p/\Gamma_p$. Let $r > 0$ be such that $B(\tilde{p}, r) \subset \tilde{U}_p$. Choose a point $\tilde{q}$ not in the fixed point set of Γ_p and choose a Dirichlet domain $\mathcal{D}_r \subset B(\tilde{p}, r)$ centered at $\tilde{q}$. Then the translates of $\mathcal{D}_r$ cover $B(\tilde{p}, r)$ and have volume equal to $\frac{1}{\#\Gamma_p} \cdot \mathrm{Vol}\, B(\tilde{p}, r)$. Since from standard volume comparison we have

$$\lim_{r \to 0^+} \frac{\mathrm{Vol}\, B(\tilde{p}, r)}{\mathrm{Vol}_k\, B(\overline{p}, r)} = 1$$

we conclude

$$\lim_{r \to 0^+} \frac{\mathrm{Vol}\, B(p, r)}{\mathrm{Vol}_k\, B(\overline{p}, r)} = \frac{1}{\#\Gamma_p}.$$

This completes the proof.

Sphere-Like Theorems

The well-known Maximal Diameter Theorem states

Theorem 1 *(Cheng [C]) Let M be a complete n-dimensional Riemannian manifold with $\mathrm{Ric}_M \geq (n-1)$, and $\mathrm{diam}(M) = \pi$. Then M is isometric to S^n with constant curvature 1.*

The following example shows that this theorem cannot be directly generalized.

Example 2 *Let L_p^3 be the three dimensional lens space of order p. Let $O_p = \Sigma L_p^3$ the suspension over L_p^3. Then O_p is an orbifold with Ricci Curvature $\geq (n-1)$ and diameter $= \pi$.* *See Example 7 on page 25.* *However, by the suspension isomorphism*

$$\mathbb{Z}_p \cong H_1(L_p^3, \mathbb{Z}) \cong H_2(O_p, \mathbb{Z})$$

and thus the family $\{O_p\}$ contains infinitely many homotopy types.

In order to prove an orbifold version of Cheng's theorem we will need to recall the following definitions and results.

Definition 3 *A bounded metric space (X, d) is said to have* excess $\leq \varepsilon$ *provided that there are points p, $q \in X$ such that $d(p, x) + d(x, q) \leq d(p, q) + \varepsilon$ for all $x \in X$. The* excess, *denoted $e(X)$, is the infimum over all $\varepsilon \geq 0$ such that X has excess $\leq \varepsilon$.*

Remark 4 If X is compact then there exists p, $q \in X$ such that $d(p, x) + d(q, x) \leq d(p, q) + e(X)$ for all $x \in X$.

The next proposition is a simple generalization to orbifolds of a result in [GP1]. We use the notation there: $B(p, r)$ will denote the closed metric r-ball in O centered at p, and $V(n, r)$ the volume of an r-ball in S^n of constant curvature 1.

Proposition 5 *Let O be a complete Riemannian orbifold with* $\mathrm{Ric}_O \geq (n-1)$ *and* $\mathrm{diam}(O) = D$. *If $p, q \in O$ with $d(p,q) = D$ and $\alpha + \beta = D$ then $O = B(p, \alpha+\varepsilon) \cup B(q, \beta+\varepsilon)$ whenever $V(n,\varepsilon) \geq V(n,D) - 2V(n, \frac{1}{2}D)$. In particular, $e(O) \leq 2\varepsilon$.*

Proof: (See [GP1]) Suppose to the contrary that there is an $x \in O$ with $d(x,p) \geq \alpha + \varepsilon$ and $d(x,q) \geq \beta + \varepsilon$. Then the interiors of the closed balls $B(p,\alpha)$, $B(q,\beta)$, $B(x,\varepsilon)$ are pairwise disjoint. Hence

$$\begin{aligned}\mathrm{Vol}\, O &\geq \mathrm{Vol}\, B(x,\varepsilon) + \mathrm{Vol}\, B(p,\alpha) + \mathrm{Vol}\, B(q,\beta) \\ &\geq \frac{\mathrm{Vol}\, O}{V(n,D)}\big(V(n,\varepsilon) + V(n,\alpha) + V(n,\beta)\big) \\ &\geq \frac{\mathrm{Vol}\, O}{V(n,D)}\big(V(n,\varepsilon) + 2V(n,\frac{1}{2}D)\big).\end{aligned}$$

The second inequality follows from the orbifold volume comparison theorem of the previous section, and the last follows by noticing that the function

$$f(\alpha) = V(n,\alpha) + V(n, D-\alpha) = \mathrm{const}(n)\left(\int_0^{\alpha} \sin^{n-1} t\, dt + \int_0^{D-\alpha} \sin^{n-1} t\, dt\right)$$

has a single critical point in the interval $[0, D]$ at $\alpha = \frac{1}{2}D$ where it is a minimum. To see the last statement, suppose ε is such that $V(n,\varepsilon) \geq V(n,D) - 2V(n,\frac{1}{2}D)$. Fix $x \in O$. Choose α so that $d(p,x) = \alpha + \varepsilon$. Let $x_i \to x$ be a sequence of points with $d(p,x_i) = \alpha + \varepsilon + \delta_i$, $\delta_i \to 0$. Then if $\beta = D - \alpha$ we have, since $O = B(p,\alpha+\varepsilon) \cup B(q,\beta+\varepsilon)$, $d(q,x_i) \leq \beta + \varepsilon$. Thus,

$$d(x_i,p) + d(x_i,q) \leq (\alpha + \varepsilon + \delta_i) + (\beta + \varepsilon) = D + 2\varepsilon + \delta_i$$

Letting $i \to \infty$ we get

$$d(x,p) + d(x,q) - D \leq 2\varepsilon$$

Since x was arbitrary, we conclude that $e(O) \leq 2\varepsilon$. This completes the proof.

Remark 6 It follows that Riemannian orbifolds with $\mathrm{Ric}_O \geq (n-1)$ and diameters close to π have small excess. In particular, if $\mathrm{diam}(O) = \pi$, then $e(O) = 0$.

Definition 7 *Let X be a length space with Toponogov curvature ≥ 1. Then the* sin–suspension, *$\Sigma_{\sin} X$ of X is the topological suspension,*

$$\Sigma X = X \times [0,\pi] \Big/ X \times \{0,\pi\}$$

equipped with the following metric. Let (x,t), (y,s) be two points of ΣX, then

$$d\big((x,t),(y,s)\big) \stackrel{\text{def}}{=} d_{S^2}\big(\gamma_1(t),\gamma_2(s)\big)$$

where γ_i are great circle arcs parametrized by arclength, with $\gamma_1(0) = \gamma_2(0)$ and $\angle\big(\dot\gamma_1(0),\dot\gamma_2(0)\big) = d_X(x,y)$. $\Sigma^m_{\sin} X$ will denote the m–fold sin*–suspension*

$$\Sigma^m_{\sin} X = \underbrace{\Sigma_{\sin}\ldots\Sigma_{\sin}}_{m} X$$

Remark 8 If X is a complete Riemannian manifold with $\mathrm{Ric}_X \geq (n-1)$ then it follows from general formulas for a Riemannian warped products that the radial curvatures of $\Sigma_{\sin} X$ are $\equiv 1$. See [BO] and [GP2]. Also there is a notion of sin–suspensions over general length spaces, but even if the length space has Toponogov curvature $\geq k$, $k < 0$, the resulting suspension will not have Toponogov curvature

$\geq k$ for any $k \in \mathbf{R}$. For example, let $T^2 = S^1 \times S^1$ be the flat torus. Then $\Sigma_{\sin} T^2$ does not have Toponogov curvature $\geq k$ for any $k \in \mathbf{R}$. See [BGP].

Proposition 9 *(Grove–Petersen) Let X be a complete length space with Toponogov curvature ≥ 1 and diameter $= \pi$. Then X contains a π-convex subset E such that X is isometric to $\Sigma_{\sin} E$. Moreover, $e(X) = 0$ and is realized by two points p, q with $d(p,q) = \pi$, and $E = \{x \in X \mid d(p,x) = d(q,x) = \frac{1}{2}\pi\}$.*

Proof: See [GP2].

Definition 10 *A n–*dimensional orbifold space form of constant curvature k *is a good orbifold (M, Γ), where $M \overset{\text{isom}}{=} S^n_k$, the n–dimensional simply connected Riemannian space form of constant curvature k. If $n = 0$, there are exactly two such orbifold space forms, namely, the metric space consisting of exactly two points $\{x, y\}$ with $d(x,y) = \pi/\sqrt{k}$ and the metric space consisting of a single point. Note that technically these two metric spaces can be regarded as 0–dimensional Riemannian space forms.*

The next proposition is a kind of analogue of the Grove–Shiohama [GS] sphere theorem.

Proposition 11 *Let O be an n–dimensional space form of constant curvature 1. If* $\operatorname{diam}(O) < \pi$, *then, in fact,* $\operatorname{diam}(O) \leq \frac{1}{2}\pi$.

Proof: Assume $\frac{1}{2}\pi < \operatorname{diam}(O) < \pi$. Let p, q be such that $d(p,q) = \operatorname{diam}(O)$. Then $d\big(\pi^{-1}(p), \pi^{-1}(q)\big) > \frac{1}{2}\pi$. In particular, the finite set $\pi^{-1}(p) = \{\tilde{p}_1, \ldots, \tilde{p}_m\}$ lies

entirely in an open hemisphere H. We can construct the center of mass $\tilde{p}'_c \in \mathbb{R}^{n+1}$ of the set $\pi^{-1}(p)$, namely,

$$\tilde{p}'_c = \frac{\tilde{p}_1 + \ldots + \tilde{p}_m}{m}$$

Then $\tilde{p}'_c$ lies in the convex hull of $\pi^{-1}(p)$ and hence lies in the open half-space containing H. Thus, $\tilde{p}'_c$ projects to a unique point $\tilde{p}_c \in H \subset S^n$. Since the center of mass $\tilde{p}'_c$ is fixed by Γ, $\tilde{p}_c$ is fixed. Its antipode $-\tilde{p}_c$, must also be fixed. Thus $\operatorname{diam}(O) = \pi$, which is a contradiction. This completes the proof.

Remark 12 For a Riemannian space form the proposition follows easily from the sphere theorem of Grove–Shiohama [GS].

Theorem 13 *Let $O = (M, \Gamma)$ be a complete good n-dimensional Riemannian orbifold with $\operatorname{Ric}_M \geq (n-1)$. If $\operatorname{diam}(O) = \pi$ then $M = S^n$ and $O = S^n/\Gamma$, where $\Gamma \subset \mathrm{O}(n+1)$ is a finite group of isometries of $\mathbb{R}^{n+1}$. Furthermore, either $O = S^n$ or O is a closed hemisphere, or $O = \Sigma_{\sin}^{n-m} X$, for some $1 \leq m < n$, where $X = S^{n-m}/\Gamma$ with $\operatorname{diam}(X) \leq \frac{1}{2}\pi$. In particular, if $n = 2$, then O must be either S^2, a $\mathbb{Z}_p$-football, a closed hemisphere, or a $\mathbb{Z}_p$-hemisphere.*

Proof: By Myers' theorem, $\operatorname{diam}(M) \leq \pi$. Since $\operatorname{diam}(O) = \pi$, there exists a segment in M of length π, so $\operatorname{diam}(M) = \pi$. By Cheng's maximal diameter theorem, it follows that $M \stackrel{\text{isom}}{=} S^n$, the sphere of constant curvature 1. Choose $p, q \in O$ with $d(p,q) = \pi$, and let γ be a segment joining them. This segment then lifts to

a great circle arc on $M = S^n$. Denote the preimages of p, q by $\tilde{p}$, $\tilde{q}$ respectively. Observe that each element of Γ must fix both $\tilde{p}$, $\tilde{q}$. To see this, suppose that $\tilde{p}$ is not fixed by some element $g \in \Gamma$. Let $g\tilde{p} = \tilde{p}'$. Note $\tilde{p}' \neq \tilde{q}$. Thus, the piece of great circle arc joining $\tilde{p}'$ to $\tilde{q}$ which has length $< \pi$, pushes down to a curve in O of length $< \pi$ connecting p to q, which is a contradiction. Thus, every element of Γ must fix $\tilde{p}$ and $\tilde{q}$. Let $N = \{\tilde{x} \in M \mid g\tilde{x} = \tilde{x} \;\; \forall g \in \Gamma\}$. Then $N \subset M$ is a closed totally geodesic submanifold containing $\tilde{p}$ and $\tilde{q}$. Hence $\operatorname{diam}(N) = \pi$ and N satisfies the curvature hypothesis of Cheng's theorem since it is totally geodesic. Thus, $N \overset{\text{isom}}{=} S^k$ for some $0 \leq k < n$. Here we define S^0 of constant curvature 1 to be the two element metric space $\{x, y\}$ with $d(x, y) = \pi$, and S^1 of constant curvature 1 to be the circle of radius 1 contained in $\mathbb{R}^2$. Now, O satisfies the hypothesis of Proposition 9 by applying the Toponogov theorem for orbifolds. Hence, $O = \Sigma_{\sin} E$, where $E = \{x \in O \mid d(p, x) = d(q, x) = \frac{1}{2}\pi\}$. Note that $\pi^{-1}(E) = S^{n-1} \subset S^n$, the equator relative to $\tilde{p}$ and $\tilde{q}$. To see this, suppose $x \in E$. Choose $\tilde{x} \in \pi^{-1}(x)$ so that $d(\tilde{p}, \tilde{x}) = \frac{1}{2}\pi$. But then, since Γ fixes $\tilde{p}$, $d\big(\tilde{p}, \pi^{-1}(x)\big) = \frac{1}{2}\pi$, which implies that $\pi^{-1}(x) \subset S^{n-1}$. Now suppose $\tilde{x} \in S^{n-1}$. Then $\frac{1}{2}\pi = d(\tilde{p}, \tilde{x}) = d(g\tilde{p}, g\tilde{x}) = d(\tilde{p}, g\tilde{x})$ for all $g \in \Gamma$. Thus, $\pi(\tilde{x}) \in E$ and hence $\pi^{-1}(E) = S^{n-1}$. Observe that S^{n-1} is invariant under Γ. The problem now reduces to two cases: (1) $N = S^0$, and (2) $N = S^k$, $0 < k < n$. In case (1), just observe that by definition of N no point of S^{n-1} is fixed by every element of Γ. Hence, $E \overset{\text{isom}}{=} S^{n-1}/\Gamma$ is a $(n-1)$–dimensional orbifold space form of constant curvature 1, and $\operatorname{diam}(E) < \pi$. The argument that the diameter must be less than π is the same as in the beginning of this proof. By

the previous proposition, $\mathrm{diam}(E) \leq \frac{1}{2}\pi$. For case (2), Take $S = S^1 \subset N = S^k$ to be any great circle $\subset N$ containing $\tilde{p}$ and $\tilde{q}$. Then $\{\tilde{x}, \tilde{y}\} = S \cap S^{n-1}$ are fixed by Γ, and hence $E = \pi(S^{n-1})$ has $\mathrm{diam}(E) = \pi$. Finally, since S^{n-1} is invariant under Γ, we can proceed by induction to get the conclusion of the theorem. This completes the proof.

Remark 14 Note the natural inclusion of $\mathrm{O}(n) \subset \mathrm{O}(n+1)$ naturally extends any isometric group action on S^{n-1} to an isometric action on S^n, in which the original action is now an action on an equator of S^n. This induced group action fixes the two antipodal points of S^n which lie on the line in $\mathbb{R}^{n+1}$ perpendicular to this equator. The resulting n–dimensional orbifold space form must be a sin–suspension over E, the equatorial quotient, by Proposition 9. Hence, we can conclude that the sin–suspension of an orbifold space form is again an orbifold space form.

At this point we can only extend Theorem 13 to general orbifolds if we replace the Ricci curvature assumption by a Toponogov curvature assumption.

Theorem 15 *Let O be an n–dimensional Riemannian orbifold with Toponogov curvature ≥ 1 and* $\mathrm{diam}(O) = \pi$. *Then O is a good Riemannian orbifold and hence satisfies the conclusion of Theorem 13.*

Proof: Choose points $p, q \in O$ with $d(p,q) = \pi$. Then by Proposition 9, $O = \Sigma_{\sin} E$, where $E = \{x \in O \mid d(p,x) = d(x,q) = \frac{1}{2}\pi\}$. Choose a fundamental neighborhood U_p for p. Then U_p is a warped product $[0,r) \times_{\sin} E$. But U_p is

isometric to some $\tilde{U}_p/\Gamma_p$, where $\tilde{U}_p$ is a Riemannian manifold. Since Γ_p preserves distance spheres, we can conclude that $\tilde{U}_p$ must be a warped product. Since $\tilde{U}_p$ is a Riemannian manifold, $\tilde{U}_p = [0, r) \times_{\sin} (S^{n-1}, \text{can})$. Thus $\tilde{U}_p$ is an open metric ball in (S^n, can). Since O is a sin-suspension, $E \overset{\text{isom}}{\cong} (S^{n-1}, \text{can})/\Gamma_p$. This shows that O is isometric to a quotient of (S^n, can) by a group of isometries Γ_p and hence O is a good Riemannian orbifold. The proof is now complete.

In the case of a lower Ricci curvature bound we can prove the following

Theorem 16 *Let O be an n-dimensional Riemannian orbifold with* $\text{Ric}_O \geq (n-1)$ *and* $\text{diam}(O) = \pi$. *Then the underlying space of O is homeomorphic to the underlying space of a good topological orbifold.*

Proof: Choose points p, $q \in O$ with $d(p,q) = \pi$. Then by Proposition 5, the excess $e(O) = 0$. By compactness of O, it follows that the Toponogov curvature of O must be bounded from below. Thus, by [GP2, Proposition 2.1], O can be exhibited as a suspension over the set $E = \{x \in O \mid d(p,x) = d(x,q) = \frac{1}{2}\pi\}$. Note that the boundary of a sufficiently small metric ball centered at p is homeomorphic to E. But, by the definition of orbifold, the boundary of this metric ball is homeomorphic to a quotient of S^{n-1} by a finite group Γ. Hence, $E \overset{\text{homeo}}{\cong} S^{n-1}/\Gamma$. Since O is a suspension we can extend the action of Γ continuously via suspension to an action on S^n. Since O is a suspension over E, we have shown that O is homeomorphic to a quotient of S^n. The proof is now complete.

We conjecture that, in fact, Theorem 13 holds without the assumption that the

orbifold is good.

Conjecture 17 *Let O be an n–dimensional Riemannian orbifold with* $\mathrm{Ric}_O \geq (n-1)$ *and* $\mathrm{diam}(O) = \pi$. *Then O is a good orbifold. In particular, it must be of the form described in Theorem 13.*

Remark 18 It should be noted that excess $e(O) = 0$ is not enough to assure that O is a good Riemannian orbifold. For instance, the $\mathbb{Z}_p$–teardrop has such properties but is not good.

Example 19 Consider the following singular space: Let $X = \Sigma_{\sin} S^2(\frac{1}{2})$. Then the Toponogov curvature of X is ≥ 1, and the diameter of X is π. In light of the previous theorems, X is *not* an orbifold. X is an example of a so–called *cone–manifold*. For the definition of cone–manifold, see [HT]. The space X is a counterexample to [HT, Theorem 3] which states that a cone–manifold with $\mathrm{Ric} \geq (n-1)$ and $\mathrm{diam} = \pi$ must have constant curvature 1.

Finiteness Theorems

The following is a generalization of the finiteness theorem stated in [A1].

Theorem 1 *In the class $\mathcal{R}_{k \cdot}{}^{D}{}_{\cdot v}(n)$ of n-dimensional good Riemannian orbifolds (M, Γ) with M simply connected,* $\mathrm{Ric}_M \geq (n\text{-}1)k$, $\mathrm{diam}(O) \leq D$, *and* $\mathrm{Vol}(O) \geq v$, *there are only finitely many isomorphism classes of* Γ.

Proof: In [F1, Prop. 5.1] it is shown that given any point $\tilde{p} \in M$, there exists a set $\{g_1, g_2, \dots\}$ which generates Γ and satisfies relations of the form $g_i g_j = g_k$ and

furthermore, $d(\tilde{p}, g_i\tilde{p}) \leq 13D$. Thus, to prove the theorem, it suffices to show that there is a bound N (depending only on n, k, D, v) on the number of generators in such a set since the isomorphism classes are determined by the number m of generators and a set of relations in $\{1, \dots, m\}^3$. Choose $\tilde{p} \notin \Sigma$. Then the set

$$G = \{g \in \Gamma \mid g\tilde{p} \in B(\tilde{p}, 13D)\}$$

is finite. To see this, suppose this is not the case. Then since $\tilde{p} \notin \Sigma$, there exists a sequence $\{g_i\}$ such that $\{g_i\tilde{p}\}$ is distinct. Hence by compactness of $B(\tilde{p}, 13D)$ we we find a convergent subsequence which contradicts (proper) discontinuity. Thus, $m \leq \#G$ is finite. Let $\{g_1, \dots, g_m\}$ be a generating set. Choose a Dirichlet domain $\tilde{\mathcal{D}} \subset B(\tilde{p}, D)$. Then $\mathrm{Vol}\,\tilde{\mathcal{D}} = \mathrm{Vol}\,O \geq v$. By construction, $g_i\tilde{\mathcal{D}} \cap \tilde{\mathcal{D}}$ has measure zero for all i. Therefore,

$$m \cdot \mathrm{Vol}\,\tilde{\mathcal{D}} \leq \#G \cdot \mathrm{Vol}\,\tilde{\mathcal{D}} \leq \mathrm{Vol}\,B(\tilde{p}, 15D) \leq \mathrm{Vol}_k\,B(\overline{p}, 15D)$$

which implies that

$$\#G \leq \frac{\mathrm{Vol}_k\,B(\overline{p}, 15D)}{\mathrm{Vol}\,\tilde{\mathcal{D}}} \leq \frac{\mathrm{Vol}_k\,B(\overline{p}, 15D)}{v} \stackrel{\mathrm{def}}{=} N$$

N clearly depends only on n, k, D, v and this implies that the cardinality of any generating set of Γ is universally bounded in $\mathcal{R}$ and hence the possibilities for Γ are only finite up to isomorphism. This completes the proof.

In order to prove the next convergence theorem we will need the following results:

Theorem 2 *(Anderson–Cheeger)[AC] The space of complete n–dimensional Riemannian manifolds with* $\mathrm{Ric}_M \geq (n-1)k$, $\mathrm{inj}_M \geq i_0$ *is precompact in the* C^α

topology. In particular, given a sequence $\{M_i\}$ of such manifolds, some subsequence converges to a C^α Riemannian manifold M_∞.

Theorem 3 *(Fukaya–Yamaguchi)[FY] Let $\mathcal{M}$ denote the set of all isometry classes of pointed length spaces (X,p) such that for each R, the metric ball $B(p,R)$ is relatively compact. Let $\mathcal{M}_{\mathrm{eq}}$ be the set of triples (X,Γ,p), where $(X,p)\in\mathcal{M}$ and Γ is a closed group of isometries of X. Let $(X_i,\Gamma_i,p_i)\subset\mathcal{M}_{\mathrm{eq}}$, $(Y,q)\in\mathcal{M}$. Suppose the Hausdorff limit*

$$\lim_{i\to\infty}(X_i,p_i)=(Y,q).$$

Then there exists a group G and a subsequence i_k such that $(Y,G,q)\in\mathcal{M}_{\mathrm{eq}}$ and

$$\lim_{k\to\infty}(X_{i_k},\Gamma_{i_k},p_{i_k})=(Y,G,q)$$

in the equivariant Hausdorff sense.

Theorem 4 *(Fukaya)[F1] Let (X_i,Γ_i,p_i), $(Y,G,q)\in\mathcal{M}_{\mathrm{eq}}$ such that*

$$\lim_{i\to\infty}(X_i,\Gamma_i,p_i)=(Y,G,q)$$

in the equivariant Hausdorff sense. Then

$$\lim_{i\to\infty}(X_i/\Gamma_i,\overline{p_i})=(Y/G,\overline{q})$$

in the ordinary Hausdorff sense.

We will need the following

Lemma 5 *Let (M, Γ) be a good Riemannian orbifold with* $\mathrm{Ric}_M \geq (n-1)k$, $\mathrm{diam}(O) \leq D$, $\mathrm{Vol}(O) \geq v$ *then for any compact subset C of M with* $\mathrm{diam}(C) = R$, *the cardinality of the set*

$$G = \{g \in \Gamma \mid gC \cap C \neq \emptyset\}$$

is bounded above by a constant A which depends only on n, k, D, R, v.

Proof: Since Γ acts properly discontinuously, G is finite. Let $\tilde{p} \in C$. Then $C \subset B(\tilde{p}, R)$. Without loss we may assume $R \geq D$ and it suffices to show that the cardinality of the set

$$G_{\tilde{p}} = \{g \in \Gamma \mid gB(\tilde{p}, R) \cap B(\tilde{p}, R) \neq \emptyset\}$$

is uniformly bounded above. Consider the Dirichlet domain

$$\mathcal{D} = \{\tilde{x} \in B(\tilde{p}, R) \mid d(\tilde{p}, \tilde{x}) \leq d(g\tilde{p}, \tilde{x}) \ \ \forall g \in G_{\tilde{p}}\}.$$

Let $\Gamma_{\tilde{p}}$ be the isotropy subgroup of $\tilde{p}$. Then, if $g \in G_{\tilde{p}} - \Gamma_{\tilde{p}}$ the set $Z = g\mathcal{D} \cap \mathcal{D}$ has measure 0. Since $g\mathcal{D} \subset B(\tilde{p}, 3R)$ for every $g \in G_{\tilde{p}}$ we conclude that

$$\#(G_{\tilde{p}} - \Gamma_{\tilde{p}}) \leq \frac{\mathrm{Vol}\, B(\tilde{p}, 3R)}{\mathrm{Vol}\, \mathcal{D}} \leq \frac{\mathrm{Vol}_k\, B(p, 3R)}{v} \stackrel{\mathrm{def}}{=} A_1.$$

Also

$$\#\Gamma_{\tilde{p}} \leq \frac{\mathrm{Vol}\, B(\tilde{p}, R)}{\mathrm{Vol}\, \mathcal{D}} \leq \frac{\mathrm{Vol}_k\, B(p, R)}{v} \stackrel{\mathrm{def}}{=} A_2.$$

Let $A = A_1 + A_2$. This completes the proof.

We now wish to present the following convergence theorem:

Theorem 6 *Let $(M_i, \Gamma_i) = O_i$ be a sequence of n-dimensional good Riemannian orbifolds with $\mathrm{Ric}_{M_i} \geq (n-1)k$, $\mathrm{inj}_{M_i} \geq i_0$, $\mathrm{diam}(O_i) \leq D$, and $\mathrm{Vol}(O_i) \geq v$. Then a subsequence of the O_i's converges to a n-dimensional C^α orbifold O_∞, $\alpha < 1$. This means $O_\infty = (M_\infty, \Gamma_\infty)$ where M_∞ is a n-dimensional C^α Riemannian manifold and Γ_∞ is a discontinuous group of isometries of M_∞.*

Proof: By the previous theorems we need only to show that Γ_∞ acts discontinuously. Let $p \in M_\infty$, and suppose $g_n p \to q$ with $\{g_n\}_{n=1}^{\infty}$ all mutually distinct elements of Γ_∞. Choose R so that $d(p,q) \leq \frac{1}{3}R$. Let $p_i \to p$, $q_i \to q$ and $g_n^{(i)} \to g_n$. Thus, $g_n^{(i)} p_i \to g_n p$ as $i \to \infty$. Note

$$d(g_n^{(i)} p_i, q_i) \leq d(g_n p, g_n^{(i)} p_i) + d(g_n p, q) + d(q, q_i)$$

which implies that

$$d(g_n^{(i)} p_i, q_i) \leq d(g_n p, q) + 2\varepsilon_i$$

where $\varepsilon_i \to 0$ as $i \to \infty$. Hence for large enough n and i, we have

$$d(g_n^{(i)} p_i, q_i) < 1.$$

Let i be large enough so that $d(p_i, q_i) \leq \frac{2}{3}R$. Then

$$g_n^{(i)} B(p_i, R+1) \cap B(p_i, R+1) \neq \emptyset$$

for large n and i because

$$d(g_n^{(i)} p_i, p_i) \leq d(g_n^{(i)} p_i, q_i) + d(p_i, q_i) \leq \frac{2}{3}R + 1.$$

By the Lemma, however, the number of such $g_n^{(i)}$ is bounded by a constant A independent of i. Let $A_i \leq A$ be the number of distinct elements $g_n^{(i)}$ and relabel them $\left\{g_n^{(i)}\right\}_{n=1}^{A_i}$. The claim is that the number of limit points from the doubly indexed set $\left\{g_n^{(i)}\right\}$ $(n = 1, A_i\,,\ i = 1, \infty)$ is $\leq A$. To see this suppose that there are at least $A+1$ distinct limit points $\{g_1, \dots, g_{A+1}\} \subset \Gamma_\infty$. Then there exists $p_{jk} \in M_\infty$ such that $g_j(p_{jk}) \neq g_k(p_{jk})$ $j \neq k$, $1 \leq j, k \leq A+1$. For each pair (j,k), choose ε_{jk} so that $B(g_j p_{jk}, \varepsilon_{jk}) \cap B(g_k p_{jk}, \varepsilon_{jk}) = \emptyset$. Let $\varepsilon = \min\{\varepsilon_{jk}\}$. Let $p_i^{(jk)} \to p_{jk}$, and choose sequences converging to $g_1, \dots, g_{A+1}$. i.e. $g_j^{(i)} \to g_j$. Then for sufficiently large i, we have

$$d(g_j^{(i)} p_i^{(jk)}, g_j p_{jk}) < \frac{1}{3}\varepsilon$$

for $j = 1, \dots, A+1$. The existence of the bound A guarantees that some $g_j^{(i)} = g_k^{(i)}$, $k \neq j$. But then

$$d(g_j^{(i)} p_i^{(jk)}, g_j p_{jk}) < \frac{1}{3}\varepsilon \ \text{ and } \ d(g_j^{(i)} p_i^{(jk)}, g_k p_{jk}) < \frac{1}{3}\varepsilon$$

but then

$$\varepsilon < d(g_j p_{jk}, g_k p_{jk}) \leq d(g_j^{(i)} p_i^{(jk)}, g_j p_{jk}) + d(g_j^{(i)} p_i^{(jk)}, g_k p_{jk}) = \frac{2}{3}\varepsilon$$

which is absurd. Hence the set of limit points is finite, contradicting the fact that the $\{g_n\}_{n=1}^{\infty}$ were chosen to be mutually distinct. This completes the proof.

Remark 7 Note that the Ricci curvature condition and injectivity radius condition on the M_i's could be replaced with any other set of conditions which guarantee

that the M_i will (sub)-converge to a Riemannian manifold whose metric is of class $C^{k,\beta}$ for some $0 \leq k \leq \infty$ and $0 \leq \beta < 1$. Then the proof above gives a corresponding precompactness result for orbifolds.

In view of this convergence theorem one would hope that a finiteness theorem of some sort would hold. Our intuition is that the presence of singular points *absorbs* volume. Thus, in the presence of a lower volume bound it might be possible to quantify up to finitely many possibilities what kinds of singular points can arise. For example consider the class of compact n-manifolds M with no conjugate points. Then it follows that the universal cover $\tilde{M}$ is diffeomorphic to $\mathbb{R}^n$, and thus $\pi_1(M)$ is torsion free. If we further assume that $\mathrm{Ric}_M \geq (n-1)k$, $\mathrm{diam}(M) \leq D$, and $\mathrm{Vol}(M) \geq v$ then by a paper of Anderson the 1-systole, $\mathrm{sys}_1(M) \geq \delta_0 > 0$. It then follows that the injectivity radius $\mathrm{inj}(M) \geq i_0 > 0$. Hence a subsequence of any sequence $\{M_i\}$ of such manifolds will converge to a C^α Riemannian manifold. Hence no orbifold degeneration can occur. A possible conjecture along these lines might be:

Problem 8 *Let (M_i, Γ_i) be a sequence of orbifolds with M_i as above. Then if the singular set of each of the orbifolds consists entirely of isolated points, is it true that all singular points in the limit orbifold are isolated and that their quantity and type is uniformly bounded?*

The Closed Geodesic Problem

A classical theorem of Lyusternik and Fet states that on every compact Riemannian manifold there exists a closed geodesic. See [Kl]. An obvious question is whether this generalizes to orbifolds. A partial result in this direction is the following:

Proposition 1 *Let O be an n-dimensional, compact Riemannian orbifold. If O is not simply connected, then O contains at least one closed geodesic.*

Proof: Let $\mathcal{C}$ be a non-trivial free homotopy class. Let $\ell = \inf\{L(c) \mid c \in \mathcal{C}\}$. Then $\ell > 0$, for if there exists a sequence $\{c_n\} : [0,1] \to O$ such that $L(c_n) \to 0$ with c_n parametrized proportional to arc length, then by the Arzela–Ascoli theorem some subsequence of $\{c_n\}$ converges to a continuous curve c. Since length is lower–semicontinuous, we have $L(c) = 0$ which implies c is a constant path. But O is locally simply connected, hence $c_n \sim c$ for large n which is a contradiction. Thus, $\ell > 0$. Now choose a sequence $\{c_n\}$ such that $L(c_n) < \ell + \frac{1}{n}$. Then as before, $\{c_n\}$ form an equicontinuous family with $\{c_n(t)\}$ bounded. Hence $c_n \to c$ a continuous curve in $\mathcal{C}$. We have $L(c) \leq \ell$ and hence by definition of ℓ, $L(c) = \ell$. We now show that c is a closed geodesic. If $c \cap \Sigma = \emptyset$, then c is a closed geodesic, for otherwise it could be shortened locally. If $c \cap \Sigma \neq \emptyset$, then $c \subset \Sigma$, for otherwise, by applying the structure theorem for geodesics, we can get a shorter curve $\tilde{c} \sim c$ with $\tilde{c} \cap \Sigma = \emptyset$, which contradicts construction of c. Finally, consider the case where $c \subset \Sigma$. Then c must be entirely contained within a stratum of Σ, and the argument above applies

to this stratum to yield the existence of a closed geodesic. This completes the proof.

I am now working on a generalization of this result, namely:

Conjecture 2 *Let O be an n–dimensional, compact Riemannian orbifold. If there exists $1 \leq k \leq n$ such that $\pi_k(O) \neq 0$, then O contains at least one closed geodesic.*

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