# The Splitting Theorem for Orbifolds

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## Introduction

In this paper we wish to examine a generalization of the splitting theorem of Cheeger–Gromoll [CG] to Riemannian orbifolds. Roughly speaking, a Riemannian orbifold is a metric space locally modelled on quotients of Riemannian manifolds by finite groups of isometries. The term *orbifold* was coined by W. Thurston [T] sometime around the year 1976–77. The term is meant to suggest the orbit space of a group action on a manifold. A similar concept was introduced by I. Satake in 1956, where he used the term *V–manifold* (See [S1]). The "V" was meant to suggest a cone–like singularity. Since then, orbifold has become the preferred terminology.

Recall that if $M$ is a complete connected $n$–dimensional Riemannian manifold with nonnegative Ricci curvature that contains a line, then the Cheeger–Gromoll Splitting Theorem [CG] states that that $M$ is isometric to $N \times \mathbb{R}$. Recall that a line is a unit speed geodesic $\gamma : \mathbb{R} \to M$ such that for any $s,\, t \in \mathbb{R}$, $d(\gamma(s), \gamma(t)) = |s - t|$.

**Theorem 1** *Let $O$ be a complete $n$–dimensional Riemannian orbifold with nonnegative Ricci curvature. If $O$ contains a line, then $O$ splits isometrically as $O = N \times \mathbb{R}$ where $N$ is a complete Riemannian orbifold with nonnegative Ricci curvature.*

**Theorem 2** *Let $O$ be a compact Riemannian orbifold with nonnegative Ricci curvature and let $\tilde{O}$ denote its universal orbifold cover. Then $\tilde{O} = N \times \mathbb{R}^{\ell}$, where $N$ is compact and $\ell \geq 0$. Also, there exists a short exact sequence*

$$1 \to F \to \pi_1^{\mathrm{orb}}(O) \to C \to 1$$

*where $F$ is a finite group and $C$ is a discrete cocompact group of isometries acting on $\mathbb{R}^{\ell}$. That is, $C$ is a crystallographic group.*

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To prove these results we will need several results about orbifolds. All of these results can be found in the first author's Ph.D. thesis [B1]. A basic reference on general orbifolds is [T].

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# Basic Definitions

**Definition 3** *A* Riemannian orbifold *is a metric space $O$ with the following special local structure. For every point $p \in O$, there is a neighborhood $U$ of $p$ which is isometric to an orbit space $\tilde{U}/\Gamma$ where $\tilde{U}$ is a convex, open (possibly non–complete) Riemannian manifold diffeomorphic to $\mathbb{R}^n$, and $\Gamma$ is a finite group of isometries acting effectively on $\tilde{U}$. Recall that for a Riemannian manifold to be convex means that there exists a unique minimal geodesic joining any two points. The collection $\mathcal{U}$ of such $U$'s is an open cover of $O$, and this collection must satisfy the following coherence condition. Namely, if $U, U'$ are two open sets in the collection with $p \in U \cap U'$, then there must exist an isometry $\phi : \tilde{U}_{\tilde{p}} \to \tilde{U}'_{\tilde{p}'}$ from some open neighborhood of $\tilde{p} \in \tilde{U}$ to one of $\tilde{p}' \in \tilde{U}'$ such that $f'\phi = f$, where $f : \tilde{U} \to \tilde{U}/\Gamma \hookrightarrow O$ and $f' : \tilde{U}' \to \tilde{U}'/\Gamma' \hookrightarrow O$ are the natural quotient maps. In addition, the collection $\mathcal{U}$ should be maximal relative to these conditions.*

Each point $x \in U$ in an orbifold $O$ is associated a group $\Gamma_x$, well–defined up to isomorphism: Let $U = \tilde{U}/\Gamma$ be a local coordinate system. Let $\tilde{x}$, $\tilde{y}$ be two points which project to $x$. Let $\Gamma_{\tilde{x}}$ be the isotropy group of $\tilde{x}$. Then if $\gamma \in \Gamma$ is the isometry such that $\gamma\tilde{x} = \tilde{y}$, it is not hard to see that the isotropy group of $\tilde{y}$ must be $\gamma\Gamma_{\tilde{x}}\gamma^{-1}$. Hence, the two isotropy groups are conjugate. Thus, up to isomorphism they can be regarded as the same group. We will denote this group by $\Gamma_x$. It can be shown (see [B1] or [S2]) that $\Gamma_x$ up to isomorphism, is also independent of coordinate system $U$. Let $O$ be a Riemannian orbifold. Let $p \in U \subset O$, where $U \overset{\text{isom}}{\cong} \tilde{U}/\Gamma$ is an open neighborhood of $p$. Choose $\tilde{p} \in \tilde{U}$ so that it projects to $p$. Denote the isotropy group of $\tilde{p}$ by $\Gamma_p$. Since $\Gamma$ is finite, it is easy to see that there exists a neighborhood $U_p \subset U$ and corresponding $\tilde{U}_p \subset \tilde{U}$ such that $U_p \overset{\text{isom}}{\cong} \tilde{U}_p/\Gamma_p$. The neighborhood $U_p$ will be called a *fundamental neighborhood of* $p$. The open set $\tilde{U}_p$ will be called a *fundamental chart*.

**Definition 4** *A Riemannian orbifold $O$ is said to have Ricci curvature* $\mathrm{Ric}(O) \geq (n-1)k$*, if* $\mathrm{Ric}(\tilde{U}_p) \geq (n-1)k$ *for all fundamental charts* $\tilde{U}_p$*.*

**Definition 5** *The* singular set $\Sigma_O$ *of an orbifold $O$ consists of those points $x \in O$ whose isotropy subgroup $\Gamma_x$ is non–trivial. We say that $O$ is a manifold when $\Sigma_O = \emptyset$. We may also, by abuse of definition, call points in the local covering $\tilde{U}$ with non–trivial isotropy, singular points also. This should cause no confusion since $x \in O$ is singular if and only if a corresponding point $\tilde{x} \in \tilde{U}$ is singular.*

**Remark 6** Since Riemannian orbifolds are locally (open) Riemannian manifolds modulo finite group actions, it follows that the singular set, locally, is the image of the union of a finite number of closed totally geodesic submanifolds of $\tilde{U}$. Since any submanifold of $\tilde{U}$ has empty interior in $\tilde{U}$, we can conclude that the singular set is closed and has empty interior.

In order to do Riemannian geometry on orbifolds we need to know how to measure the lengths of curves. To do this, we lift curves locally, so that we may compute their lengths locally in fundamental neighborhoods. Finally, we add up these local lengths to get the total length of the curve. The problem of course, is that locally these lifts are *not* unique. It will turn out, however, that the length of a curve is well–defined. We refer to [B1] for the details. We are now in a position to give a length space structure to any Riemannian orbifold $O$. Given any two points $x, y \in O$ define the distance $d(x, y)$ between $x$ and $y$ to be

$$d(x, y) = \inf \{L(\gamma) \mid \gamma \text{ is a continuous curve joining } x \text{ to } y\}.$$

Then $(O, d)$ becomes a length space. Furthermore if $(O, d)$ is complete, any two points can be joined by a minimal geodesic realizing the distance $d(x, y)$. See [G]. The following structure result for minimizing segments in orbifolds will be of fundamental importance. A proof can be found in [B2].

**Proposition 7** *Let $O$ be a Riemannian orbifold, and let $\gamma : [a, b] \to O$ be a minimizing segment ($a = -\infty$, $b = +\infty$ is permissible). Then the isotropy group is constant along $\gamma|_{(a,b)}$. This means that for any $s, t \in (a, b)$, $\Gamma_{\gamma(s)} = \Gamma_{\gamma(t)}$.*

**Remark 8** Intuitively, this means that a minimizing curve cannot pass through the singular set and remain minimizing. Actually, the proposition says more: in fact, a minimizing curve cannot change *strata* and still remain minimizing. See [B2].

The following observation is clear, but we feel that it should be pointed out explicitly since it will be used throughout the remainder of this paper.

**Observation 9** Let $O$ be a Riemannian orbifold. For $p \in O$, let $U_p = \tilde{U}_p/\Gamma_p$ denote a fundamental neighborhood of $p$. Denote the local projection by $\pi : \tilde{U}_p \to U_p$. Then the map $\pi$ is distance *non–increasing.*

The following example illustrates how Observation 9 is used in the sequel.

**Example 10** Let $\tilde{U}_p = \mathbb{R}^2$, $\tilde{p} = (0,0)$ and let $\Gamma_p$ be the cyclic group of order 4, generated by rotation about $\tilde{p}$ through an angle of $\pi/2$ radians. $U_p$ is then a flat cone. Fix $q \in U_p$ and consider the distance function $\rho : U_p \to \mathbb{R}$ given by $\rho(x) = d(q,x)$. Choose an element $\tilde{q} \in \pi^{-1}(q)$. It follows that the lifted distance function $\tilde{\rho} = \rho \circ \pi : \tilde{U}_p \to \mathbb{R}$ satisfies $\tilde{\rho}(\tilde{x}) \leq d_{\mathbb{R}^2}(\tilde{q},\tilde{x})$. Equality will hold if $\tilde{q}$ is fixed by all of $\Gamma_p$, but in general inequality holds. For instance, in the above situation, take $\tilde{q} = (1,0)$, $\tilde{x} = (0,2)$, then $1 = \rho(x) = \tilde{\rho}(\tilde{x}) < \sqrt{5} = d_{\mathbb{R}^2}(\tilde{q},\tilde{x})$.

The proof of the Splitting theorem will use the notions of weak (super)–harmonicity, so for completeness we recall the following definition.

**Definition 11** *Let $M$ be a Riemannian manifold. A continuous function $f : M \to \mathbb{R}$ is said to satisfy $\Delta f$ (weakly) $\leq \varphi$ for some function $\varphi$, if for each $p \in M$ and $\varepsilon > 0$, there exists a support function $f_{p,\varepsilon}$ defined on a neighborhood $U$ of $p$ with*

*(i)* $f_{p,\varepsilon}(p) = f(p)$

*(ii)* $f_{p,\varepsilon}(q) \geq f(q)$

*(iii)* $\Delta f_{p,\varepsilon}(q) \leq \varphi(q) + \varepsilon$ where $f_{p,\varepsilon}$ is $C^2$ on $U$

**Definition 12** *Let $O$ be a Riemannian orbifold. A continuous function $f : O \to \mathbb{R}$ is said to satisfy $\Delta f$ (weakly) $\leq \varphi$ for some function $\varphi$, if the* pull–back function $\tilde{f} = f \circ \pi : \tilde{U}_p \to \mathbb{R}$ *satisfies* $\Delta \tilde{f}$ (weakly) $\leq \tilde{\varphi}$ *in the sense of the previous definition.*

In the remainder of the paper we will denote functions on the orbifold by standard symbols: $f, g$. The corresponding pull–back functions will be denoted with tildes: $\tilde{f}, \tilde{g}$.

# The Laplacian Comparison Theorem

In this section we generalize the Laplacian Comparison theorem of Calabi [C] to Riemannian orbifolds. Let

$$\mathrm{ct}_k(t) = \begin{cases} \sqrt{k}\cot(\sqrt{k}t) & \text{if } k > 0 \\ t^{-1} & \text{if } k = 0 \\ \sqrt{-k}\coth(\sqrt{-k}t) & \text{if } k < 0 \end{cases}$$

**Proposition 13** *(Laplacian Comparison) Let $O$ be a complete $n$–dimensional Riemannian orbifold with* $\mathrm{Ric}(O) \geq (n-1)k$. *Let $\rho(x)$ be the distance function from any fixed point $p \in O$. Then*

$$\Delta\rho(x)\ (\text{weakly}) \leq (n-1)\,\mathrm{ct}_k\Big(\rho(x)\Big)$$

**Proof:** Fix $x_0 \in O$. Let $\gamma : [0,1] \to O$ be a segment joining $x_0$ to $p$ in $O$. Take a finite covering of $\gamma$ by fundamental neighborhoods $\{U_i\}_{i=0}^N$ such that $U_i \cap U_{i+1} \neq \emptyset$ and $U_i \cap U_{i+2} = \emptyset$. For each $U_i$, choose a Dirichlet fundamental domain $\tilde{D}_i \subset \tilde{U}_i$. Without loss of generality, let $U_0 = U_{x_0}$. Now lift $\gamma$ to $\tilde{U}_0$, denote this lift by $\tilde{\gamma}$. If $x_0 \in \Sigma$, choose the lift that lies in $\tilde{D}_0$. Now construct a Riemannian manifold $M$ as follows:

Let $\pi_i : \tilde{U}_i \to U_i$ denote the local projections. From the orbifold structure, we have isometries

$$\tilde{\phi}_i : \pi_{i+1}^{-1}(U_i \cap U_{i+1}) \cap \tilde{D}_{i+1} \to \pi_i^{-1}(U_i \cap U_{i+1}) \cap \tilde{D}_i.$$

Now form the adjunction space $M_{i+1}$:

$$M_{i+1} = M_i \cup_{\tilde{\phi}_i} \tilde{U}_{i+1}$$

where $M_0 \overset{\text{def}}{=} \tilde{U}_0$. Denote by $M$, the Riemannian manifold $M_N$.

We now lift $\gamma$ to $M$. Pick $t_1 \in [0,1]$ so that $\gamma(t_1) \in U_0 \cap U_1$. Then $\tilde{\gamma}'(t_1)$ defines a tangent vector in $T_{\tilde{\gamma}(t_1)}\tilde{U}_0 = T_{\tilde{\gamma}(t_1)}\tilde{U}_1 \subset TM$. Extend $\tilde{\gamma}$ through $\tilde{U}_1$

by choosing the unique geodesic in $\tilde{U}_1$, with tangent vector $\tilde{\gamma}'(t_1)$. Continuing inductively defines $\tilde{\gamma}$ uniquely in $M$.

Let $\tilde{p} = \tilde{\gamma}(1)$. Let $d_M$ be the distance function on $M$, and let $\rho^M(\tilde{x}) = d_M(\tilde{p}, \tilde{x})$ for $\tilde{x} \in M$. Note that for $\tilde{x}$ in a neighborhood of $\tilde{x}_0$, we have

$$\rho(x_0) = \tilde{\rho}(\tilde{x}_0) = \rho^M(\tilde{x}_0) \text{ and } \tilde{\rho}(\tilde{x}) \overset{\text{Obs. 9}}{\leq} \rho^M(\tilde{x}).$$

The last inequality follows from Observation 9. Now by standard Laplacian comparison

$$\Delta\rho^M \text{ (weakly) } \leq (n-1)\,\mathrm{ct}_k\Big(\rho^M\Big) \leq (n-1)\,\mathrm{ct}_k(\tilde{\rho})$$

This implies that

$$\Delta\tilde{\rho}(\tilde{x}_0) \text{ (weakly) } \leq (n-1)\,\mathrm{ct}_k\Big(\tilde{\rho}(\tilde{x}_0)\Big)$$

since $\rho^M$ is a (continuous) support function for $\tilde{\rho}$ at $\tilde{x}_0$. Since $x_0$ was arbitrary, the proof is now complete.

The Maximum principle of Calabi [C] extends to orbifolds:

**Proposition 14** *(Maximum principle) Let $f : O \to \mathbb{R}$ be continuous. If $\Delta f$ (weakly) $\leq 0$, then either $f$ is constant or attains no global minimum.*

**Proof:** If $f$ has a global minimum, then $\tilde{f} = f \circ \pi$ has a global minimum on $\tilde{U}$. Thus, by standard maximum principle, $\tilde{f} \equiv$ const on $\tilde{U}$ which implies $f \equiv$ const (at least locally). A connectedness argument then gives the desired conclusion. This completes the proof.

# Busemann Functions

The main result of this section is to show that Busemann functions on orbifolds with nonnegative Ricci curvature are (weakly) superharmonic. We first recall the relevant definitions and summarize basic facts.

Let $\gamma : [0, \infty) \to O$ be a unit speed ray. This means that $\gamma|_{[0,t]}$ is a unit speed minimal geodesic for all $t$. Define $b_r(x) = d\Big(x, \gamma(r)\Big) - r$. It then follows that

(1) for fixed $x$, $b_r(x)$ is decreasing and $|b_r(x)|$ is bounded by $d\Big(x, \gamma(0)\Big)$.

(2) $\Big|b_r(x) - b_r(y)\Big| \leq d(x,y)$ for all $r$, $x$, $y$.

(3) If the Ricci curvature of $O$ is nonnegative, then $\Delta b_r$ (weakly) $\leq \dfrac{n-1}{b_r + r}$.

The first two statements are consequences of the triangle inequality, and the third follows from the Laplacian comparison theorem of the previous section. From these properties we see that $\{b_r\}$ converges to a function $b(x) = b_\gamma(x)$ with $\Big|b(x) - b(y)\Big| \leq d(x,y)$, $\Big|b(x)\Big| \leq d\Big(x, \gamma(0)\Big)$, and $b\Big(\gamma(r)\Big) = -r$ for any $r$. $b$ is called the *Busemann* function for the ray $\gamma$. We now construct asymptotes for $\gamma$. Fix $p \in O$, and choose unit speed segments $\sigma_t : [0, \ell_t] \to O$ from $p$ to $\gamma(t)$. The Arzela–Ascoli theorem implies that some subsequence will converge to a ray $\overline{\gamma} : [0, \infty) \to O$ starting at $p$. $\overline{\gamma}$ is called an *asymptote* for $\gamma$ through $p$. We have the following standard results concerning the Busemann function for $\overline{\gamma}$.

(1) $b(x) \leq b(p) + \overline{b}(x)$

(2) $b\Big(\overline{\gamma}(t)\Big) = b(p) + \overline{b}\Big(\overline{\gamma}(t)\Big) = b(p) - t$

We now come to the main result of this section.

**Proposition 15** *If $O$ has nonnegative Ricci curvature, then the Busemann function for any ray is superharmonic.*

**Proof:** We use the notation above. We want to show that for fixed $p \in O$, $\Delta b(p)$ (weakly) $\leq 0$. Note that by our previous considerations for $\tilde{x} \in \tilde{U}_p$ we have

$$\tilde{b}(\tilde{x}) = b(x) \leq b(p) + \overline{b}(x) = \tilde{b}(\tilde{p}) + \tilde{\overline{b}}(\tilde{x})$$

$$\tilde{b}(\tilde{p}) = \tilde{b}(\tilde{p}) + \tilde{\overline{b}}(\tilde{p})$$

Hence it suffices to show that $\Delta\tilde{\overline{b}}(\tilde{p})$ (weakly) $\leq 0$. To do this we construct support functions for $\tilde{\overline{b}}$ at $\tilde{p}$. As in the proof of the Laplacian comparison theorem, construct a Riemannian manifold $M$, by lifting the asymptote $\overline{\gamma}$ through $p$, to $\tilde{\overline{\gamma}}$ in $M$, through $\tilde{p}$. Then for $\tilde{x}$ in a neighborhood of $\tilde{p}$, (namely $\tilde{U}_p$),

$$\tilde{\overline{b}}(\tilde{x}) = \overline{b}(x) \leq d\Big(x, \overline{\gamma}(r)\Big) - r \overset{\text{Obs. 9}}{\leq} d\Big(\tilde{x}, \tilde{\overline{\gamma}}(r)\Big) - r \overset{\text{def}}{=} \hat{b}_r(\tilde{x}).$$

Since $\tilde{p}$ is not in the cut locus for $\tilde{\bar{\gamma}}(r)$, then $\hat{b}_r$ is $C^\infty$ in a (possibly smaller) neighborhood of $\tilde{p}$. Then

$$0 = \tilde{\bar{b}}(\tilde{p}) = \hat{b}_r(\tilde{p})$$

and

$$\tilde{\bar{b}}(\tilde{x}) \leq \hat{b}_r(\tilde{x})$$

for $\tilde{x}$ in a neighborhood of $\tilde{p}$. By Laplacian comparison, we have

$$\Delta \hat{b}_r \leq \frac{n-1}{d(\tilde{x}, \tilde{\bar{\gamma}}(r))} \to 0 \text{ as } r \to \infty.$$

Hence

$$\Delta \tilde{\bar{b}}(\tilde{p}) \text{ (weakly) } \leq 0$$

since $\hat{b}_r(\tilde{x})$ is a support function for $\tilde{\bar{b}}(\tilde{x})$ at $\tilde{p}$. This completes the proof.

# The Splitting Theorem

We are now in a position to prove the Splitting Theorem. The proof will touch on various ideas from [CG], [CGL], [EH]. We use the notation of the previous sections. Assume $O$ is a complete Riemannian orbifold with nonnegative Ricci curvature and that $O$ contains a line $\gamma : \mathbb{R} \to O$. Define $b^+$ as the Busemann function for $\gamma^+ = \gamma|_{[0,\infty)}$ and $b^-$ the Busemann function for $\gamma^- = \gamma|_{(-\infty,0]}$. Thus,

$$b^+(x) = \lim_{t\to\infty} d\Big(x, \gamma(t)\Big) - t$$

$$b^-(x) = \lim_{t\to\infty} d\Big(x, \gamma(-t)\Big) - t$$

The triangle inequality implies that $\Big(b^+ + b^-\Big) \geq 0$ for all $x$. Since $\gamma$ is a line it is easy to see that $\Big(b^+ + b^-\Big)\Big(\gamma(t)\Big) = 0$ for all $t$. Since $b^+$ and $b^-$ are weakly superharmonic, it follows from the Maximum principle that $b^+ + b^- \equiv 0$ on $O$. In particular, $b^+$ and $b^-$ are weakly harmonic and hence by elliptic theory $\tilde{b}^{\pm}$ are $C^\infty$. A standard computation as in [CG, page 121] shows that $\operatorname{Hess} \tilde{b}^+ \equiv 0$.

We next show that through each point $p \in O$, there is a line passing through $p$. To this end, let $\overline{\gamma}^{\pm}$ denote asymptotes to $\gamma^{\pm}$ through $p$, and let $\overline{\gamma}(t) = \overline{\gamma}^{\pm}(\pm t)$, where $\pm$ is chosen according to whether $t$ is positive or negative. Then for any $t_1, t_2 \in \mathbb{R}$ we have

$$d(\overline{\gamma}(t_1), \overline{\gamma}(t_2)) \geq b^+(\overline{\gamma}(t_1)) - b^+(\overline{\gamma}(t_2)) = b^+(\overline{\gamma}(t_1)) + b^-(\overline{\gamma}(t_2))$$

$$= (b^+(p) - t_1) + (b^-(p) - (-t_2)) = t_2 - t_1.$$

Here we have used the identity $b^+ + b^- \equiv 0$. This shows that $\overline{\gamma}$ is a line through $p$. Furthermore, note that for $\tilde{x} \in \tilde{U}_p$

$$\tilde{b}^{\pm}(\tilde{x}) = b^{\pm}(x) \leq b^{\pm}(p) + \overline{b}^{\pm}(x) = \tilde{b}^{\pm}(\tilde{p}) + \tilde{\overline{b}}^{\pm}(\tilde{x})$$

Thus,

$$-\tilde{b}^-(\tilde{p}) - \tilde{\overline{b}}^-(\tilde{x}) \leq -\tilde{b}^-(\tilde{x}) = \tilde{b}^+(\tilde{x}) \leq \tilde{b}^+(\tilde{p}) + \tilde{\overline{b}}^+(\tilde{x}).$$

From this it follows that $(\tilde{\overline{b}}^+ + \tilde{\overline{b}}^-) \geq 0$. Also, $\tilde{\overline{b}}^+(\tilde{p}) + \tilde{\overline{b}}^-(\tilde{p}) = 0$. So by the Maximum principle we can conclude that $(\tilde{\overline{b}}^+ + \tilde{\overline{b}}^-) \equiv 0$, and hence that $\tilde{\overline{b}}^{\pm}$ are $C^\infty$ harmonic functions on $\tilde{U}_p$. This implies that equality holds above and thus

$$\tilde{b}^{\pm}(\tilde{x}) = \tilde{b}^{\pm}(\tilde{p}) + \tilde{\overline{b}}^{\pm}(\tilde{x}).$$

This implies that $\nabla\tilde{b}^{\pm}(\tilde{x}) = \nabla\tilde{\overline{b}}^{\pm}(\tilde{x})$. Thus,

$$\frac{d}{dt}\Big|_{t=0} \tilde{\overline{\gamma}} = -\nabla\tilde{\overline{b}}^+(\tilde{p}) = -\nabla\tilde{b}^+(\tilde{p}) = \nabla\tilde{b}^-(\tilde{p}).$$

We now complete the proof of the Splitting theorem. Let $p \in O$ be arbitrary, and consider a fundamental neighborhood $U_p$ of $p$, and its local chart $\tilde{U}_p$. Assume $p \in {b^+}^{-1}(t_p)$. Then ${\tilde{b}^+}{}^{-1}(t_p) = \tilde{N}_p$ is a totally geodesic submanifold of $\tilde{U}_p$ with unit normal $\nabla\tilde{b}^+$, since $\operatorname{Hess}\tilde{b}^+ \equiv 0$. For each $\tilde{n} \in \tilde{N}_p$, let $\tilde{\overline{\gamma}}_{\tilde{n}}$ be the lift in $\tilde{U}_p$ passing through $\tilde{n}$ of the asymptotic line in $O$ passing through $\pi(\tilde{n})$. Then the map

$$\Phi : \tilde{N}_p \times (-\varepsilon, \varepsilon) \to \tilde{U}_p, \qquad \Phi(\tilde{n}, t) = \exp_{\tilde{n}}(t\nabla\tilde{b}^+) = \tilde{\overline{\gamma}}_{\tilde{n}}(t)$$

is diffeomorphism onto its image. Since $\nabla\tilde{b}^+$ is parallel, it follows that $\Phi$ is an isometry. Since curves of the form $(\tilde{n}, t)$ for $\tilde{n} \in \tilde{N}_p$ fixed, project to asymptotes through $n = \pi(\tilde{n})$, and since these asymptotes are lines, we

conclude by Proposition 7 that $\Gamma_p$ must fix the second factor in the local splitting, and hence

$$U_p \overset{\text{isom}}{=} \tilde{N}_p \Big/ \Gamma_p \times (-\varepsilon, \varepsilon).$$

Let $N_p = \tilde{N}_p/\Gamma_p$. We have shown that $O$ splits as a local product, we now claim that $O$ is a global product.

Let $H = {b^+}^{-1}(0)$. Define a map $\Psi : H \times \mathbb{R} \to O$ by $\Psi(x,t) = \gamma_x(t)$, where $\gamma_x$ is the asymptotic line to $\gamma$ through $x$. Then $\Psi$ is injective since two asymptotic lines cannot intersect, and $\Psi$ is surjective since every point has an asymptote through it and $|\nabla b^+| = 1$. It is clear that $\Psi$ is a homeomorphism. We claim that $\Psi$ is a local (distance) isometry from which it will follow that $\Psi$ is a global (distance) isometry and we will be done. Let $p \in O$, and let $p_0$ be its projection onto $H$. Let $\gamma$ be the subset of the line joining $p_0$ to $p$, say $\gamma(0) = p_0$, $\gamma(t_p) = p$. We can construct product neighborhood $N_\gamma$ of $\gamma$ isometric to $H_{p_0} \times (-\varepsilon, t_p + \varepsilon)$, where $p_0 \in H_{p_0} \subset H$ as follows.

Partition the interval $(-\varepsilon, t_p + \varepsilon)$ into subintervals $I_i$ such that $O$ splits locally along $\gamma|_{I_i}$. This gives rise to a chain of product neighborhoods covering $\gamma$, say $N_i = W_i \times I_i$. Since each of these neighborhoods split off $\gamma$ isometrically, we see that the overlap of any two such adjacent neighborhoods is isometric to a product of the form $V \times J$ where $J$ is an open interval. Using these overlaps we can construct the desired product neighborhood. In particular, we have shown that $O$ splits locally isometrically like $H \times \mathbb{R}$. Thus, since $\Psi$ is a homeomorphism and a local (distance) isometry it is easy to see that $\Psi$ is a global isometry. Explicitly, let $p, q \in H \times \mathbb{R}$, and let $\sigma$ be a minimizing segment joining $p$ to $q$. Then $\Psi(\sigma)$ is a curve in $O$ joining $\Psi(p)$ to $\Psi(q)$. Since $\Psi$ is a local (distance) isometry, it follows that the length of $\Psi(\sigma)$ is the same as that of $\sigma$. (We are using here of course the standard definition of length of curve for an inner metric space). Hence we conclude that $d(p,q) \geq d(\Psi(p), \Psi(q))$. The opposite inequality follows by applying the same argument to $\Psi^{-1}$. The proof is now complete.

## The Universal Orbifold Cover and Fundamental group

We now focus our attention on Theorem 2, but first we recall the following definitions and facts concerning the orbifold fundamental group. See [T] or [Sc].

**Definition 16** *If $X, Y$ are Riemannian orbifolds, then $f : X \to Y$ is an* orbifold covering map *if every point $y \in Y$ has a neighborhood $U$ such that $f^{-1}(U)$ is a disjoint union of open sets $V_\alpha$ and $f|_{V_\alpha} : V_\alpha \to U$ is equivalent to a natural isometric quotient map*

$$V_\alpha = \tilde{V}_\alpha/\Gamma \to \tilde{V}_\alpha/\Gamma' = U \qquad (\Gamma \subset \Gamma')$$

**Example 17** For $p \leq q \in \mathbb{Z}^+$, let $S^2_{p,q}$ denote a $(p,q)$–football. These spaces are (topologically) 2–spheres, but whose north and south poles are modelled metrically on quotients of the 2–disk by cyclic rotation of order $p, q$, respectively. (If $q > p = 1$, then $S^2_{p,q}$ is commonly referred to as a $\mathbb{Z}_q$–teardrop). Then $S^2_{2,3}$ is a two–fold orbifold cover $S^2_{4,6}$.

**Remark 18** This example shows that even though the underlying space ($S^2$, in this case) may be simply connected, there may exist proper orbifold coverings.

**Example 19** Let $M$ be a Riemannian manifold. Let $\Gamma'$ be a proper discontinuous group of isometries acting on $M$, and $\Gamma \subset \Gamma'$ a subgroup. Then the natural quotient map $M/\Gamma \to M/\Gamma'$ is an orbifold covering map. In particular, $\tilde{M} \to M$ is an orbifold covering map, where $\tilde{M}$ denotes the universal cover of $M$.

As in the case of topological spaces, every connected orbifold $O$ has a connected universal covering orbifold $\tilde{O}$ and the same uniqueness holds. It is a regular covering, and the orbifold fundamental group $\pi_1^{\mathrm{orb}}(O)$ is defined to be the group of deck transformations of the cover $p : \tilde{O} \to O$. It follows that $\pi_1^{\mathrm{orb}}(\tilde{O}) = 0$, and that as orbifolds $O = \tilde{O}/\pi_1^{\mathrm{orb}}(O)$.

**Example 20** It can be shown that $S^2_{2,3}$ has no proper orbifold coverings, and hence it follows that $S^2_{2,3}$ is the universal orbifold covering of $S^2_{4,6}$ and that $\pi_1^{\mathrm{orb}}(S^2_{4,6}) = \mathbb{Z}_2$.

We now give the proof of Theorem 2. Although the proof is more or less the same as in the Riemannian manifold case, we give a detailed proof since the arguments used here emphasize the length space structure rather than the Riemannian structure and are somewhat different in flavor than the standard ones. This difference is most notable in the proof that the isometry group of the universal orbifold cover splits.

**Proof of Theorem 2:** Apply Theorem 1 to split the universal orbifold cover as $\tilde{O} = N \times \mathbb{R}^\ell$, where $N$ contains no lines. Then the isometry group of $\tilde{O}$ splits as $\mathrm{Iso}(\tilde{O}) = \mathrm{Iso}(N) \times \mathrm{Iso}(\mathbb{R}^\ell)$. To see this, note that if $\gamma(t) = \big(\gamma_1(t), \gamma_2(t)\big) : \mathbb{R} \to N \times \mathbb{R}^\ell$ is a line, then both $\gamma_1$ and $\gamma_2$ are lines, and hence $\gamma_1 = \text{const}$. Thus all lines $\gamma(t)$ have the form $\gamma(t) = \big(p, \tilde{\gamma}(t)\big)$, where $p \in N$ and $\tilde{\gamma}$ is a line in $\mathbb{R}^\ell$. Consider an isometry $\varphi(p, y) = \big(\varphi_1(p, y), \varphi_2(p, y)\big)$ : $N \times \mathbb{R}^\ell \to N \times \mathbb{R}^\ell$. If $\gamma$ is a line, then $\varphi \circ \gamma$ is a line. The first observation is that for fixed $p \in N$, the function $\varphi_1(p, \cdot) : \mathbb{R}^\ell \to N$ is constant. This follows by considering two intersecting lines $\gamma$, $\tau$ in $\mathbb{R}^\ell$. Let $\gamma(0) = \tau(0)$ and let $\varphi\big(p, \gamma(t)\big) = \big(q, \gamma'(t)\big)$ and $\varphi\big(p, \tau(t)\big) = \big(q', \tau'(t)\big)$. Then $\gamma'$, $\tau'$ are lines and we have

$$0 = d\Big((p, \gamma(0)), (p, \tau(0))\Big) = d\Big(\varphi(p, \gamma(0)), \varphi(p, \tau(0))\Big) = d\Big((q, \gamma'(0)), (q', \tau'(0))\Big)$$

In particular, $q = q'$. Since in $\mathbb{R}^\ell$ given any two lines there is a third line (which may be one of the original lines) intersecting the original two, it follows easily that $\varphi_1\big(p, \gamma(t)\big) \equiv q$ for all lines $\gamma \subset \mathbb{R}^\ell$. This suffices to establish the first observation. We next observe that $\varphi_1(p, y) = \varphi_1(p) : N \to N$ is an isometry. To see this, let $\gamma$ be a line in $\mathbb{R}^\ell$. Then for any $p, q \in N$,

$$\begin{aligned} d^2(p, q) &= d^2\Big((p, \gamma(t)), (q, \gamma(t))\Big) \\ &= d^2\Big((p', \gamma'(t)), (q', \gamma''(t))\Big) = d^2(p', q') + d^2\Big(\gamma'(t), \gamma''(t)\Big) \end{aligned}$$

where $\varphi(p, \gamma) = (p', \gamma')$ and $\varphi(q, \gamma) = (q', \gamma'')$. In particular, it follows that $d^2(p, q) \geq d^2(p', q') = d^2(\varphi_1 p, \varphi_1 q)$. Applying the same argument using $\varphi^{-1}$, $p'$, $q'$ in place of $\varphi$, $p$, $q$ gives the reverse inequality. Hence $\varphi_1$ is an isometry. Furthermore, we can now conclude from the last series of equalities that $\varphi_2\big(p, \gamma(t)\big) = \gamma'(t) = \gamma''(t) = \varphi_2\big(q, \gamma(t)\big)$. This shows that for fixed $y \in \mathbb{R}^\ell$, the function $\varphi_2(\cdot, y) : N \to \mathbb{R}^\ell$ is constant. Finally, since $\varphi$ and $\varphi_1$

are isometries, it follows that $\varphi_2(p,y) = \varphi_2(y) : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is an isometry. The totality of the previous arguments show that all isometries $\varphi$ of $N \times \mathbb{R}^\ell$ are of the form $\varphi(p,y) = \big(\varphi_1(p), \varphi_2(y)\big)$ with $\varphi_1$, $\varphi_2$ isometries of $N$ and $\mathbb{R}^\ell$ respectively. Hence there is a natural isomorphism $\mathrm{Iso}(\tilde{O}) \to \mathrm{Iso}(N) \times \mathrm{Iso}(\mathbb{R}^\ell)$. Let $\mathrm{pr}_1$, $\mathrm{pr}_2$ denote the respective projections.

We now show by contradiction that $N$ is compact. The details for this part are essentially the same as those in [P]. If $N$ were not compact, then a standard application of the Arzela–Ascoli theorem gives the existence of a ray $\gamma : [0,\infty) \to N$. Regarding $\pi_1^{\mathrm{orb}}(O) \subset \mathrm{Iso}(\tilde{O})$ as a group of isometries acting on $\tilde{O}$, we see that as Riemannian orbifolds $O = \tilde{O}/\pi_1^{\mathrm{orb}}(O)$. Note that $\pi_1^{\mathrm{orb}}(O)$ acts (properly) discontinuously on $\tilde{O}$ since $O$ is Hausdorff. By assumption, $O$ is compact, so there exists a compact set $\tilde{K} \subset \tilde{O}$ such that for all $x \in \tilde{O}$, there exists $g \in \pi_1^{\mathrm{orb}}(O)$ so that $gx \in \tilde{K}$. Let $K = \mathrm{pr}_1(\tilde{K})$. Thus, for each $\gamma(t)$, $t > 0$, there exists $g_t \in \mathrm{pr}_1(\pi_1^{\mathrm{orb}}(O))$ so that $g_t(\gamma(t)) \in K$. Extract a convergent subsequence $g_{t_i}(\gamma(t_i)) \to p \in K \subset N$, with $t_i \to \infty$. Define $\gamma_i : [-t_i, \infty) \to N$ by $\gamma_i(t) = g_{t_i}(\gamma(t + t_i))$. The $\gamma_i$'s are rays and by the Arzela–Ascoli theorem a subsequence must converge to a line. This is a contradiction and thus $N$ is compact.

The desired result now follows by considering the kernel of the homomorphism $\mathrm{pr}_2 : \pi_1^{\mathrm{orb}}(O) \to \mathrm{Iso}(\mathbb{R}^\ell)$. The kernel $F$ of $\mathrm{pr}_2$ is finite. For, if not, let $\varphi_n$ be a sequence of mutually distinct isometries of $N$ such that $(\varphi_n, \mathrm{id}) \in F$. Fix $p \in N$. Then $(\varphi_n, \mathrm{id}) \cdot (p, 0) = (\varphi_n(p), 0)$ contains a convergent subsequence, since $N$ is compact. This is a contradiction since $\pi_1^{\mathrm{orb}}(O)$ acts (properly) discontinuously. Hence $F$ is finite and we have the desired exact sequence:

$$1 \to F \xhookrightarrow{\iota} \pi_1^{\mathrm{orb}}(O) \xrightarrow{\mathrm{pr}_2} C \to 1.$$

To verify that $C$ is a crystallographic group, it suffices to check that $C$ acts (properly) discontinuously on $\mathbb{R}^\ell$. Suppose not, then there exists $y \in \mathbb{R}^\ell$ and a sequence of isometries $(\psi_n, \varphi_n) \in \pi_1^{\mathrm{orb}}(O)$ with the $\varphi_n$ mutually distinct and $\varphi_n(y) \to z \in \mathbb{R}^\ell$. Again, since $N$ is compact, we may assume that $\psi_n = \psi$ for all $n$. But then, $(\psi, \varphi_n) \cdot (p, y) \to (\psi(p), z)$ contradicting the fact that $\pi_1^{\mathrm{orb}}(O)$ acts (properly) discontinuously. This completes the proof.

**Remark 21** It is useful to note that if $X$ is any inner metric space such that $X = Y \times \mathbb{R}^\ell$, where $Y$ contains no lines, then the argument in the proof of

Theorem 2 that the isometry group splits remains valid and thus the isometry group of $X$ splits as $\mathrm{Iso}(X) = \mathrm{Iso}(Y) \times \mathrm{Iso}(\mathbb{R}^\ell)$.

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