Asymptotic Functions and the Problem of Multiplication of Distributions*

T. D. Todorov

The asymptotic functions are a new type of generalized functions. But they are not functionals on some space of test-functions as the Schwartz distributions. They are mappings of the set of the asymptotic numbers (1, 3, 5, 6) into itself. On its part, the set of the asymptotic numbers is a totally-ordered set of generalized numbers including the systems of real and complex numbers, as well as infinitesimals and infinitely large numbers. Every two asymptotic functions can be multiplied. On the other hand, the Schwartz distributions have realizations, in a certain sense, as asymptotic functions. The motivations of this work are connected with some physical problems of quantum theory [18, 25].

Introduction

As soon as the book "Theorie des distributions" by L. Schwartz has appeared, it has been realized that the multiplication of distributions is not always possible. For example, the products

$$\delta(x), \delta(x). P\left(\frac{1}{x}\right), \delta(x)P\left(\frac{1}{x}\right), \delta(x)(x), \left[P\left(\frac{1}{x}\right)\right]^n, \text{ etc.,}$$

having significance in the quantum theory [18, 22—26] cannot be satisfactorily defined. What is more, in the rare cases, when the multiplication is possible, it is not associative. The following well known example is given by Schwartz

$$P\left[\frac{1}{x}\right][x\delta(x)] = \left[P\left(\frac{1}{x}\right)\right]x\delta(x).$$

Several papers have been written by mathematicians, as well as by physicists, in order to introduce in some way a natural operation of multiplication of distributions: a) The most popular approach is based on an approximation of distributions by regular sequences [7—12]. The exchange formula relating (formally) the product and the convolution [13, 15], as well as the point of view treating the distributions as boundary values of analytic functions [14, 15, 26] deliver us another base for introducing a multiplication. A typical feature of the approaches from this group is that the product (in the cases it exists) belongs to the space of distributions. Unfortunately, only

* Some of the results of this paper were reported by the author at the conference "Operatoraten — Distributionen und Verwandte Non-Standard Methoden", Oberwolfach, Federal Republic of Germany, 2—8, July 1978.
a rather small number of products can be defined in these ways; b) In another group of works [16—22] an enlargement of the space of distributions \( \mathcal{D}' \) is performed and the product, in general, does belong to this larger space. The methods used here involve either an axiomatic-algebraic technique [16, 17, 26] or non-standard analysis [19—22]. However, the operation of multiplication introduced in these works is not defined on the whole enlargement of \( \mathcal{D}' \), which is the reason why the problem of associativity cannot be even formulated (see, for example, [22]); c) In the third group of approaches the space of distributions is entirely abandoned. It is replaced by another space of generalized functions similar, in a certain sense, to the distributions and a commutative and associative multiplication is introduced for any two generalized functions. These approaches are based usually on a non-archimedean enlargement of the system of real, as well as of complex numbers, and are connected first of all with the non-standard analysis [19—22]. At first sight these methods are more successful, because very strong results about the existence of multiplication are established in their framework. They possess, in a certain sense, a disadvantage, lying in the fact that comparatively few [19—24] numbers of applications to the other branches of mathematics, as well as of physics, have been performed up to now. It is not quite sure whether this kind of generalized functions will turn out to be so interesting and so useful for both mathematics and physics, as the distributions are, although the latter cause well known troubles.

The approach we use belongs to the third group of works. The asymptotic functions are generalized functions similar to the distributions. However, they do not coincide with the distributions, i.e. they are not functions on some space of test-functions. The asymptotic functions are mappings of the set of asymptotic numbers \( A \) into itself. On its part, \( A[1, 3, 5, 6] \) is a totally-ordered set of generalized numbers, which includes isomorphically the field of the real numbers \( \mathbb{R} \), as well as infinitely small (infinitesimals) and infinitely large numbers. Every two asymptotic functions can be multiplied (because every two asymptotic numbers can be multiplied) and the distributions have realizations, in a certain sense, as asymptotic functions.

The motivations of this work are, in fact, connected with some problems in quantum theory [1, 2, 3, 18, 23, 24, 25]. In a further work we intend to probe the asymptotic functions instead of the distributions in some topics of quantum theory and to make use of the existence of multiplication between the asymptotic functions.

The present paper is organized as follows.

(i) In Sec. 1 the most general notion of asymptotic function is introduced (Definition (1.1)). This type of asymptotic functions are different from the asymptotic functions introduced in [2] and [4], although the notion of asymptotic numbers \( (1, 3, 5, 6) \) is the common base idea for both types of asymptotic functions (see Remark (1.13)). (ii) In Sec. 2 a particular type of sets of asymptotic numbers (subsets of \( A \)), called extended sets, are separated and their properties are studied. These sets (of asymptotic numbers) will play the role of domains of asymptotic functions of a particular type defined in Sec. 3. (iii) In Sec. 3 a particular type of asymptotic functions, called extended asymptotic functions, are considered.

Our plan for the future is the following.

In a series of papers we are going to define two particular classes of asymptotic functions: the class \( F \) of quasi-extended asymptotic functions and the class \( \Phi_{er} \) of the so-called quasi-distributions. The asymptotic functions of these two classes are very similar to the Schwartz distributions and at the
ed in these ways; b) In another
he space of distributions \( \mathcal{D} \) is
long to this larger space. The
-algebraic technique [16, 17, 26]
formation of multiplication in
whole enlargement of \( \mathcal{D} \), which
ity cannot be even formulated
of approaches the space of dis-
ary space of generaliz. 
e distributions and a commuta.
for any two generalized func.
non-archimedean enlargement
bers, and are connected first
are multiplied (because
ctions). The asymptotic functions are different from the
although the latter cause well
group of works. The asympto-
to the distributions. However,
e, they are not functionals on
ctions are mappings of the
A[1, 3, 5, 6] is a totally-
cludes isomorphically the field of
( (infinitesimals) and infinitely
ns can be multiplied (because
) and the distributions have
ctions, called extended asym-

1. Asymptotic Functions

(1.1) Definition (Asymptotic Function). Every mapping of the type

\[ f : D \rightarrow A^* \]
where $D \subseteq A$ will be called an asymptotic function of one variable and if $D \subseteq A \times A \times \cdots \times A$ (n-times), $f$ will be called an asymptotic function of n variables. As usually, we shall often write $f(a)$, $a \in D$ or $f(a_1, a_2, \ldots, a_n)$, $(a_1, a_2, \ldots, a_n) \in D$ respectively, instead of (1.2).

(1.3) **Remark.** Corresponding to the above definition, the values of the asymptotic functions are asymptotic numbers. Let $\mu(a)$, $\nu(a)$ and $\lambda(a)$ be the power, the order and the relative order of $f(a)$, respectively for some $a \in D$. It is clear that $\mu$, $\nu$ and $\lambda$ are mappings of the following type:

\[ (1.4) \quad \mu : D \to \mathbb{Z} \cup \{\infty\}, \]
\[ (1.5) \quad \nu : D \to \mathbb{Z} \cup \{\infty\}, \]
\[ (1.6) \quad \lambda : D \to \mathbb{N}_0 \cup \{\infty\} = \{0, 1, 2, \ldots, \infty\}. \]

(1.7) **Definition (Algebraic Operations).** The algebraic operations: addition, subtraction, multiplication and division between two asymptotic functions $f$ and $g$ defined on $D$ and $E$, respectively (where $D \subseteq A$ or $E \subseteq A \times A \times \cdots \times A$) will be introduced by means of their values (i.e. just like the ordinary functions are added, subtracted, multiplied and divided). In other words

\[ (1.8) \quad (f \pm g)(a) = f(a) \pm g(a), \quad a \in D \cap E, \]
\[ (1.9) \quad (f . g)(a) = f(a) . g(a), \quad a \in D \cap E, \]
\[ (1.10) \quad (f / g)(a) = f(a) / g(a), \quad a \in D \cap E \/ E_0, \]

where

\[ (1.11) \quad E_0 = \{a : a \in E, g(a) \neq 0\}. \]

(1.12) **Remark.** It is clear that every two asymptotic functions (for which $D \cap E \neq \emptyset$) can be added (subtracted) and multiplied (since every two asymptotic numbers can be added and multiplied). As we know, the set of the real numbers $\mathbb{R}$ is isomorphically embedded in the set of real asymptotic numbers $A$ [5, Theorem 20] and the set of the complex numbers $C$ is isomorphically embedded in the set of the complex asymptotic numbers $A^*$. In other words, $\mathbb{R} \subset A$ and $C \subset A^*$ (and $\mathbb{R} \subset C$, as well as $A \subset A^*$, of course) Consequently, the set of all ordinary functions (we mean the complex-valued functions of real variables) is isomorphically embedded in the set of all asymptotic functions (of the above-mentioned type).

The notion of asymptotic functions was introduced for the first time by Chri sto v as equivalence classes of sequences of ordinary functions [2, 4]. The notion of asymptotic function just introduced (1.1) is obviously different from that one given in [2] and [4]. There exists a connection between these two types of asymptotic functions, but this well become clear only when analytic operations are introduced (which will be done in a next paper).

(1.13) **Lemma.** Let $M$ be a set (a subset) of asymptotic functions and let $M$ be closed with respect to the algebraic operations addition, multiplication, or addition and multiplication. Then $M$ has the same algebraic properties as $A$ and $A^*$ have (we mean the identities [5, Theorem 6], which are valid in $A$ and $A^*$ are valid in $M$, too). In particular, the set of all asymptotic functions has the same algebraic structure as $A$ and $A^*$ have [5].

**Proof.** The theorem follows directly from the fact that the asymptotic functions are $A^*$-valued functions.
2. Asymptotic Extension of Subsets of \( R \).

As it is known, the ordinary functions (as well as the distributions) are interesting first of all because of their analytic operations differentiation, integration, Fourier-transformation, convolution and so on. That is why we must introduce somehow the analytic operations for just defined asymptotic functions. But there are some difficulties connected with this problem since \( A \) (as well as \( A^* \)) is a non-archimedean set (containing infinitesimals) and consequently \( A \) (and \( A^* \)) is not Dedekind completed. What is more, \( A \) (and \( A^* \)) is also disconnected [6, Theorem 44]. These features are typical not only for the asymptotic numbers but also for any non-archimedean extensions of the real or complex numbers [19]. So, we cannot introduce the analytic operations by the standard way (by a given measure, the set of the measurable functions and so on). We shall introduce the analytic operations in another way: We are going to separate some very special classes of asymptotic function (close­ly connected with the ordinary functions) in which classes the analytic operations can be naturally defined. The domains of the asymptotic functions of these particular classes cannot be arbitrary subsets of \( A \). This section is devoted to these rather special subsets of \( A \), called extended sets, because they are obtained as asymptotic extension of ordinary sets of real numbers.

(2.1) Definition (Asymptotic Extension of Open Subsets). Let \( X \) be an open subset of \( R \). The set of all real asymptotic numbers \( a \in A \) for which

\[
\{ s : a(s) \in X \} \in \mathcal{E}
\]

for all \( a \in A \) where \( \mathcal{E} \) is the filter defined in [6] will be called asymptotic extension of \( X \) and will be denoted by \( X_{as} \).

(ii) Let \( X_{i}, i = 1, 2, \ldots, n (n \in \mathbb{N}) \) be open subsets of \( R \) and let

\[
X = X_{1} \times X_{2} \times \ldots \times X_{n}.
\]

The set of all \( n \)-tuples \( (a_{1}, a_{2}, \ldots, a_{n}) \) where \( a_{k} \in A \), \( k = 1, 2, \ldots, n \) for which

\[
\{ s : (a_{1}(s), a_{2}(s), \ldots, a_{n}(s)) \in X \} \in \mathcal{E}
\]

for all \( a_{k} \in A \), \( k = 1, 2, \ldots, n \) will be called an asymptotic extension of \( X \) and will be denoted also by \( X_{as} \); (iii) A subset \( D \) of \( A \) (or of \( A^{*} \)) for which there exists an open subset \( X \) of \( R \) (or \( X \) is of the type of (2.3)) such that \( D = X_{as} \) will be called an extended set.

(2.5) Remark. For the sake of convenience we shall remind the definition of the filter \( \mathcal{E} \) [6, Definition 2]: \( \mathcal{E} \) is the set of all subsets \( E \) of \( \{0, 1\} \), which contains an interval of the type \( (0, \varepsilon) \), where \( \varepsilon \in \mathbb{R}, \varepsilon > 0 \). As any filter, \( \mathcal{E} \) possesses the following (filter) properties:

\[
\emptyset \notin \mathcal{E},
\]

\[
E, F \in \mathcal{E} \implies E \cap F \in \mathcal{E},
\]

\[
E \subseteq F \subseteq \{0, 1\} \text{ and } E \notin \mathcal{E} \implies F \notin \mathcal{E}.
\]

(2.9) Remark. It is easy to see that

\[
(X_{1} \times X_{2} \times \ldots \times X_{n})_{as} = X_{1_{as}} \times X_{2_{as}} \times \ldots \times X_{n_{as}}.
\]

That is why we shall concentrate our attention only on the case \( n = 1 \). The generalization to the case \( n > 1 \) is done immediately by means of (2.10).
The following Lemma replaces the expression “for all $aEa$” in the above definition by the more simple one “there exists $aEa$”. It will help us to construct the asymptotic extensions of concrete open subsets of $R$.

(2.11) Lemma. Let $X$ be an open subset of $R$. Then: (i) $\varnothing_{as} = \varnothing$ and $X_{as} = X$ if and only if $X = \varnothing$; (ii) $\theta_{as} = A$ and $X_{as} = A$ if and only if $X = R$; (iii) $O^{-\infty}X_{as}$ for some $n(E)N$ if and only if $X = \varnothing$ (and consequently, $X_{as} = A$); (iv) Let $X = R$ and $aEa$. Then $aE X_{as}$, if and only if the following two conditions (denoted by $(a)$ and $(b)$) are valid:

a) $a-xE0$ for all $x \notin X \setminus X$, where $X$ is the closure of $X$;

b) There exists $aEa$ for which (2.2) holds.

(2.12) Remark. The conditions a) are equivalent to the following two conditions (denoted by $(a)$* and $(a)$**, respectively):

\begin{align*}
(a)^* & \quad a \notin \{0^{-n}: n \in \mathbb{N}\}; \\
(a)^{**} & \quad a \notin \{x + 0^{-n}: x \notin X \setminus X, \quad n = 0, 1, \ldots, \infty\}.
\end{align*}

Proof. (i) and (ii) are obvious. Let us consider (iii). If $X = \varnothing$, then $0^{-\infty}X_{as}$, of course, since $X_{as} = A$, corresponding to (ii). Let $a \notin \{0^{-n}: n \in \mathbb{N}\}$, i.e. (2.2) holds for all $aEa$. In particular, for $a(s) = x$ where $x$ is any real number (we mean $\lim_{s \to 0} x = 0$ for any $x \in X$), (2.2) implies $x \notin X$ for any $x \notin X$, i.e. $X = \varnothing$; (iv) $a \notin \{x \in \mathbb{R}: x \not\in X \setminus X\}$ and $a \in \{x \in \mathbb{R}: x \not\in X \setminus X\}$, corresponding to (iii). If $a$ is infinitely large, i.e. $a \in \Omega_\infty$, then $a \notin \{x \in \mathbb{R}: x \not\in X \setminus X\}$. Let $a \in \Omega_\infty$, i.e. corresponding to [6, Theorem 36, (iv)], $a$ can be represented in the form $a = x_0 + h$ where $x_0 \in \mathbb{R}$ and $h \notin \Omega_\infty$ is uniquely determined by $a = x_0 + h$ [6, Definition 10]; So, (2.2) reduces to

\begin{align*}
\{s: x_0 + x(s) \in X\} \subset & \quad \mathbb{E}
\end{align*}

for all $x \in h$. If $h \notin \mathbb{E}$, then $(a)^{**}$ holds obviously. Let $h \notin \mathbb{E}$. Then we must show that $x_0 \notin X \setminus X$. Indeed, we can put $x = x_0$ in (2.13), where $x_0 \in \mathbb{E}$ and we shall obtain $x_0 \notin X$. Let now $a$ and $b$ be valid (or, which is the same, let $(a)^*$, $(a)^{**}$ and $b$ be valid); It is necessary to consider the cases $a \in \Omega_\infty$, $a \in \Omega^+_\infty$ and $a \in \Omega^-_\infty$ separately, where $\Omega^-_\infty$ and $\Omega^+_\infty$ are the sets of infinitely large negative and positive asymptotic numbers, resp. Let $a \in \Omega_\infty$ and let $a = x_0 + h$ be the representation we have talked above; In this case $b$) reduces to

\begin{align*}
\{s: x_0 + x(s) \in X\} \subset & \quad \mathbb{E}
\end{align*}

for some $x \in h$ and consequently, $x_0 \notin X$ (since $\lim_{s \to 0} x_0(s) = 0$). If $x \notin X$, then (2.10) will be valid for all $x \notin h$ (since $X$ is open), i.e. $a = x_0 + h \notin X_{as}$. If $x \notin X$ (i.e. $x \notin X \setminus X$), then the condition $a$) (for $x = x_0$) will imply $h \notin \mathbb{E}$, i.e. $h$ can be represented in the form $h = r^n + ds^n$, where $r \notin \mathbb{N}$, $r \in \mathbb{R}$, $r \neq 0$ and $A$ is an infinitesimal. In the case $r < 0$, (2.13) will imply $X \not\supset (x_0 - \varepsilon, x_0)$ for some $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ and in the case $r > 0$, (2.13) implies $X \not\supset (x_0, x_0 + \varepsilon)$ for some $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. In both cases we obtain:

\begin{align*}
\{s: x_0 + x(s) \in X\} \subset & \quad \mathbb{E}
\end{align*}

for all $x \in h$, i.e. $a = x_0 + h \notin X_{as}$. Let $a \in \Omega^-_\infty$, i.e. $a$ can be represented in the form $a = r^n + ds^n$ for some $r \in \mathbb{R}$, $r < 0$ and some infinitesimal $A$. In the same way
consider the cases $a(x,\varepsilon)$, i.e., (2.2) holds for all $a\in A$. 
If its part, is equivalent to $a^*\equiv \varepsilon$. In particular, for $a(x)\equiv x$, $x_0=0$ for any $x\in \mathbb{R}$, (2.2) implies

$$a \equiv x_0+h(n)\in X, \quad x_0 \in \mathbb{R},$$

we get $x_0=0$ and $h(n)$ be the resolute $b$ reduces to

$$x_0+\varepsilon$$

and $h(n)$ is non-empty open subset of $\mathbb{R}$. Then $C\in X$ contains $X$ properly in the sense of the isomorphisms $\mathbb{R}=\mathbb{R}$ and $
\mathbb{R}=\mathbb{R}$. Then Theorem (2.20) can be formulated as follows. If $X$ is a non-empty open subset of $\mathbb{R}$, then $X\subseteq X_0$. Moreover, $\mathbb{R} \cap X_0$ implies $x_0=X$. 

(2.24) Theorem (Some properties of $X_0$). Let $X$ and $Y$ be two open subsets of $\mathbb{R}$. Then: (i) $X\subseteq Y$ implies $X_0\subseteq Y_0$; (ii) $(X\cap Y)_0=\cap Y_0$. 

Proof. (i) Let $X\subseteq Y$ and $x_0\in X_0$, i.e., (2.2) holds for all $a\in A$. We let

$$E(a)=\{s: a(s)\in X\}, \quad a\in A,$$

which implies $a\in (X\cap Y)_0$. The theorem is proved.

(2.25) Remark. The facts worth keeping in mind from this section are: the following: $\mathbb{R}_0=\mathbb{R}$; $(X\cap Y)_0=\cap Y_0$; all examples (2.15-2.19). More specially, we are going to use in future the fact that $(0,1)$ contains all positive infinitesimals $\varepsilon$ which are not asymptotic zeros. In particular,

$$r^0+\varepsilon(0,1)\in \mathbb{R}, \quad n\in N, \quad v\in N\cup\{\infty\}, \quad n\leq r, \quad r\in \mathbb{R}, \quad r\neq 0.$$
to a certain degree, to the so-called "non-standard extension \(X^*\) of a set \(X\)" in the framework of the non-standard analysis. The lemma (2.11), as well as the theorems (2.20) and (2.24) have also counterparts in the nonstandard analysis. This analogy will continue in the future developments, too. For example, we are going to introduce the notions of "asymptotic extension of a given ordinary function" and "quasi-extended asymptotic functions" which will have analogs in the non-standard analysis.

3. Extended Asymptotic Functions

In this section we are going to consider a very special type of asymptotic functions which are obtained as an extension (continuation) of continuous ordinary functions (we mean complex-valued functions of real variables). These functions are called extended asymptotic functions.

(3.1) Definition (Extended Asymptotic Functions). (I) Let \(X\) be an open subset of \(R\) and \(\varphi: X \rightarrow C\) be a continuous ordinary function defined on \(X\). We shall say that an asymptotic function of the type \(\varphi_{as}: D \rightarrow A^*\) is an asymptotic extension of \(\varphi\) if the following two conditions are valid: a) \(X \subseteq D \subseteq X_{as}\) (2.1) and for every \(a \in D\) the set (of functions):

\[
\alpha^* = \{\varphi(a): a \in \alpha[a]\}
\]

possesses an asymptotic cover \(asa^*\) [5, Definition 7]; b) If \(a \in D\), then \(\varphi_{as}(a)\) is (by definition) the asymptotic cover of \(a^*\), i.e.

\[
\varphi_{as}(a) = asa^*, \quad a \in D;
\]

(ii) Let \(X_n, k = 1, 2, \ldots, n\) be open subsets of \(R\) and let \(\varphi: X \rightarrow C\) be a continuous ordinary function defined on \(X\), where

\[
X = X_1 \times X_2 \times \ldots \times X_n.
\]

We shall say that an asymptotic function of the type \(\varphi_{as}: D \rightarrow A^*\) is an asymptotic extension of \(\varphi\) if the following two conditions are valid: a) \(X \subseteq D \subseteq X_{as}\) (2.10) and for every point \((a_1, a_2, \ldots, a_n)\in D\) the set (of functions):

\[
a^* = \{\varphi(a_1, a_2, \ldots, a_n): a_k \in \alpha[a_k], \quad k = 1, 2, \ldots, n\}
\]

possesses an asymptotic cover \(asa^*\) [5, Definition 7]; b) If \((a_1, a_2, \ldots, a_n)\in D\), then (by definition)

\[
\varphi_{as}(a_1, a_2, \ldots, a_n) = asa^*;
\]

(iii) The asymptotic functions obtained as an asymptotic extension of some continuous ordinary functions will be called extended asymptotic functions.

(3.5) Remark. It is clear that (i)-part of the above definition is a particular case of its (ii)-part for \(n = l\). On this point onwards, we shall consider the case \(n = l\) only, but all results established further can be easily generalized for \(n > l\).

(3.7) Remark. Let us remind [5, Definition 7], that the asymptotic cover \(asa^*\) of a subset \(a^*\) of \(A^*\) [5, Sec. 2] is an asymptotic number, i.e. \(asa^*(A^*)\) which contains \(a^*\), i.e. \(a^* \subseteq asa^*\) and such that there is no other (different from \(asa^*\)) asymptotic number \(a'\) for which \(a^* \subseteq a' \subseteq asa^*\). Let us recall fur-
standard extension \( X \) of a set \( X \)

The lemma (2.11), as well as parts in the nonstandard ana-

developments, too. For example, asymptotic extension of a given

functions" which will have

very special type of asympto-

tion (continuation) of continuous

functions). (I) Let \( X \) be an open

function defined on \( X \).

if the type \( \varphi_{ax}: D \to A^a \) is an

conditions are valid: a) \( X \subseteq D \)

(b) \( \varphi(a) \) is a con-

X.

type \( \varphi_{ax}: D \to A^a \) is an asymp-
totic extension of some extended asymptotic functions.

the above definition is a particular

onwards, we shall consider the

rather can be easily generalized

on 7], that the asymptotic cover

asymptotic number, i.e. \( \text{asa}^*(A^a, A^a) \),

at there is no other (different

\( a^a \subseteq a^* \subseteq \text{asa}^* \). Let us recall fur-

used by the author at the Conference

"Methoden", Oberwolfach, 2-8 July,

ther [5, Theorem 5], that \( a^* \) has an unique asymptotic cover if and only if there exists \( a^* \subseteq A^* \) such that \( a^* \subseteq a^* \).

The following three lemmas will help us to construct the asymptotic ex-
tensions of some ordinary functions.

(3.8) Lemma. Let \( \varphi(x), x \in X \) be an ordinary function differentiable any

time of times defined on the open subset \( X \) of \( R \). Then \( \varphi_{ax} \) exists for

every finite asymptotic number of the type \( a = x + h \) where \( x \in X \) and \( h \) is an

infinitesimal. Moreover,

(3.9)

\[
\varphi_{ax}(x+h) = \sum_{k=0}^{\infty} \frac{1}{k!} \varphi^{(k)}(x) h^k, \quad x \in X, \quad h \in \Omega_0.
\]

(3.10) Remark. The series on the right hand side of (3.9) is convergent

with respect to the interval topology of \( A \) [6, Sec. 5].

Proof. (3.9) is, in fact, a translation of the well-known formula for the

asymptotic expansion of \( \varphi(x+\epsilon) \)

(3.11)

\[
\varphi(x+\epsilon) \sim \sum_{r=0}^{\infty} \frac{\varphi^{(r)}(x)}{r!} \epsilon^r
\]

in terms of the asymptotic number terminology. Notice that the series on the right hand side of (3.11) is divergent (in general) with respect to the ordinary

topology of \( R \), in contrast to the series in (3.9) which is convergent

with respect to the topology of \( A \).

(3.12) Lemma. Let \( \varphi(x), x \in X \) be a continuous ordinary function defined on

the open subset \( X \) of \( R \) and let \( \varphi \in X \), where \( X \) be the closure of \( X \).

(II) If there exists \( m \in \mathbb{Z} \) for which

(3.13)

\[
\lim_{t \to 0} t^{-m} \varphi(x+t) = 0,
\]

then the value \( \varphi_{ax} \) exists for all infinitesimals \( h \) for which \( x + h \in X_{ax} \) and the

inequality (an estimate for \( \varphi_{ax} \))

(3.14)

\[
\varphi_{ax}(x+h) \leq o_{m^x}, \quad x + h \in X_{ax}
\]

holds, where \( \mu \) is the power of \( h \); (II) Let, moreover, \( \varphi \) have an asymptotic

expansion of the type

(3.15)

\[
\varphi(x+t) = \sum_{k=0}^{n} c_k t^k + \Delta(t), \quad t \in \Omega, \quad x + t \in X
\]

for all \( n \in \mathbb{Z}, n \leq v \), where \( \mu, \nu \in \mathbb{Z} \cup \{\infty\}, \mu \leq v, c_k \in C \) and

(3.16)

\[
\lim_{t \to 0} \Delta(t)/t^\mu = 0.
\]

And let, finally, \( \varphi \) does not have an asymptotic expansion of the type (3.15)

by a higher order than \( v \). (The last is not a restriction in the case \( v = \infty \).

Then

(3.17)

\[
\varphi_{ax}(x+h) = \sum_{k=\mu}^{v} c_k h^k + o_{r^x h}, \quad x \in X, \quad h \in \Omega_0, \quad x + h \in X_{ax}.
\]
Proof. The above lemma follows immediately from [6, Theorem 41]. It is, in fact, a periphrasis of the notion of asymptotic expansion of a given function in our “asymptotic number language”.

(3.18) Remark. The above Lemma (3.12) reduces to Lemma (3.8) in the following special case: \( x \in X \) and \( q^{(k)}(x) \), \( k = 0, 1, \ldots \) exist. In this case we have \( q^{(k)}(x) = c_k \). We wrote out Lemma (3.8) only for the sake of convenience. We are going to use Lemma (3.12) (but not Lemma (3.8)) only in the case \( x \in X \setminus X \).

(3.19) Lemma. Let \( q(x) \), \( x \in X \) be a continuous ordinary function defined on \( X \), where \( X \) is an open subset of \( R \). Let \( X \) contain intervals of the type \((t, \infty)\) or \((-\infty, t)\) for some \( t \in R \) (the case \( X = R \) is included here). If there exists \( m_\pm \in Z \) (respectively) such that

\[
\lim_{x \to \pm \infty} x^{-m_\pm} q(x) = 0
\]

then \( q_{as}(a) \) exists for every positive (or negative, resp.) infinitely large asymptotic numbers \( a \) and

\[
q_{as}(a) \subseteq o^{a \cdot m_\pm}
\]

holds, where \( \mu \) is the power of \( a \).

Proof. Elementary; it is sufficient to put \( a = \pm s^\mu + s^\nu h \) for some \( \mu \in Z \), \( \mu < 0 \), \( r \in R \), \( r > 0 \) and some infinitesimal \( h \), to replace any \( a \alpha a \) (which can be represented in the form \( a(s) = \pm r^\mu + s^\nu \chi(s) \) for \( \lim \chi(s) = 0 \)) in (3.20) and use the definition (3.1).

Recall the definition of the asymptotic numbers [5, Definition 5]. The following lemma establishes a connection between this definition and the notion of asymptotic extension of a function.

(3.22) Lemma. Let \( q(x) \), \( x \in X \) be a continuous ordinary function defined on the open subset \( X \) of \( R \) and let \( q_{as}(a) \), \( a \in D \) be an asymptotic extension of \( q \). Then

\[
q_{as}(a) = a s \{q(a)\} + o^{a(a)}, \quad a \in D
\]

for any \( a \alpha a \), where \( \nu(a) \) is the order of \( q_{as}(a) \).

(3.24) Corollary. Let \( a \in A^* \). Then

\[
a = q_{as}(s + o^\nu) + o^r
\]

for any \( a \alpha a \), where \( \nu \) is the order of \( a \).

Proof. (3.23) follows immediately from [5, Theorem 3].

(3.26) Remark. Recall [5, Sec. 6] that \( s + o^\nu \) is the asymptotic number defined as follows:

\[
s + o^\nu = \{a: a \in A_n, a(s) = s + A(s), \lim_{s \to 0} A(s)/s^n = 0 \text{ for all } n \in Z\}.
\]

Recall, as well, that \( s + o^\nu \cdot l^\infty \). Moreover, corresponding to [5, Definition 12], the number \( s + o^\nu = s l^\infty \) can be denoted simply as \( r \), i.e.

\[
s + o^\nu = s l^\infty = s.
\]

(3.29) Remark. In the \( n \)-dimensional case (3.23) could be replaced by

\[
q_{as}(a_1, \ldots, a_n) = a s \{q(a_1, \ldots, a_n)\} + o^{a_1, \ldots, a_n}(a_1, \ldots, a_n)D
\]

for any \( a_k \in A_n \), \( k = 1, 2, \ldots, n \).
from [6, Theorem 41]. It is, expansion of a given function to Lemma (3.8) in the case 3.32). This ordinary function defined contains intervals of the type $\mathbb{A}$ is included here). If there

\[ a = \pm r s^n + s^h k \]

place any $a = a$ (which can be \( x(s) = 0 \)) in (3.20) and use $s \to 0^-$ [5, Definition 5]). The following definition and the notion of an ordinary function defined on the open subset $X$ of $\mathbb{R}$ is included here). If there

\[ \lim_{s \to 0^-} \frac{x(s)}{s^n} = 0 \] for all $n \in \mathbb{Z}$. Corresponding to [5, Definition 12], as $r$, i.e.

(3.23) could be replaced by

\[ \ldots a_1, (a_2, \ldots, a_n) \in D \]

$\Delta D$

Corresponding to [5, Definition 5]). The following definition and the notion of an ordinary function defined on the open subset $X$ of $\mathbb{R}$ is included here). If there

\[ \lim_{s \to 0^-} \frac{x(s)}{s^n} = 0 \] for all $n \in \mathbb{Z}$. Corresponding to [5, Definition 12], as $r$, i.e.

(3.23) could be replaced by

\[ \ldots a_1, (a_2, \ldots, a_n) \in D \]

