The Products $\delta^2(x)$, $\delta(x).X^{-n}$, $\Theta(x).X^{-n}$, etc. in the Class of the Asymptotic Functions

T. D. Todorov

Several products like $\delta^n(x)$, $\delta(x)\Theta(x)$, $\delta^{(m)}(x).X^{-n}$, $\Theta(x).X^{-n}$, etc., where $\delta(x)$, $\Theta(x)$, X^{-n}, etc., are kernels of the corresponding Schwartz distributions, are studied in the framework of the class of the asymptotic functions F_0 introduced in a previous paper [11]. In some particular cases many formulae are derived and several examples are presented. The work is of mathematical type but its motivations lie in some problems in quantum theory. It is closely connected with a series of previous works [1—11] and first of all with [11].

Introduction

In ([11], (4.20)) we showed that every Schwartz distribution possesses kernels in the class of the asymptotic functions F_0 ([11], (2.1)), i. e. for every $T \in \mathscr{D}'$ there exists $f \in F_0$ such that the infinitesimal equality

$$\langle T, \varphi\rangle \approx \int_{-\infty}^{\infty} f(x)\,\varphi(x)\mathrm{d}x, \quad \varphi \in \mathscr{D},$$

is valid. On the other hand, the class F_0 is a ring and, in particular, any two asymptotic functions from F_0 can be correctly multiplied. So, we are able to construct products like $\delta^2(x)$, $\Theta(x).X^{-n}$, $\delta(x).X^{-n}$, etc., where $\delta(x)$, X^{-n}, $\Theta(x)$, etc. are kernels of the corresponding Schwartz distributions.

The purpose of the present paper is to study the properties of the products of the above mentioned type and to offer several examples of them. In particular, a series of formulae of "asymptotic functionals" ([11], (4.1)) determined by the products $\delta^2(x)$, $\delta(x).X^{-n}$, $\Theta(x).X^{-n}$, etc. is derived and their dependence on the choice of the kernels δ, $\Theta(x)$, X^{-n}, etc., is established. Our interest in the products of that kind is connected with some problems in quantum field theory where expressions like $\delta^2(x)$, $\delta(x).X^{-n}$, $\Theta(x).X^{-n}$, etc. appear in some calculations and cause disadvantages of the mathematical basis of this theory.

The paper is closely connected with the work [11]; we presuppose the knowledge at least of the definitions of the class F_0 and some of its properties. We advise the reader as well to have a look at the last two pages of [11] where some useful remarks are given ([11], (4.30), (4.31)). And finally, we would like to note that for the sake of simplicity we are going to choose our examples first of all among the so called "standard asymptotic functions" ([11], (2.28)) since their properties can be more easily investigated.

In this paper the asymptotic functions, respectively the Schwartz distributions, of one variable only are considered but most of the results can be easily generalized to the case of asymptotic functions and distributions of more than one variables.

1. Asymptotic Delta-Functions

Under the term "asymptotic delta-function" or equivalently "asymptotic Dirac's function" we shall understand any kernel of the Dirac's distribution in the sense of ([11], (4.9)), i. e. any asymptotic function $\delta \in F_0$ ([11, (2.1)) for which the infinitesimal equality

(1.1) $$\int_{-\infty}^{\infty} \delta(x)\varphi(x)\mathrm{d}x \approx \varphi(0), \quad \varphi \in \mathscr{D}$$

is valid, where $\mathscr{D}$ is the well-known class of test-functions with compact supports used in the Schwartz theory of distributions and the integral in (1.1) is defined in ([11], (2.3)). Recall that the integrals of the asymptotic functions are, in general, asymptotic numbers [1—5]. On the other hand, two asymptotic numbers a and b are called to be "infinitely close" and we write this as $a \approx b$, if $a-b$ is an infinitely small number (infinitesimal) [6]. Besides that we are going to use the same or nearly the same notations for a given distribution and its kernels in F_0 as we just did in (1.1). And finally, let us recall once again [11] that a given Schwartz distribution possesses not one but infinitely many kernels in the class of asymptotic functions F_0 so, in particular, we have not one but infinitely many asymptotic delta functious in F_0.

Here are some examples of asymptotic delta-functions:

(1.2) Example: Let us put

(1.3) $$\Delta(x, \varepsilon) = \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right), \quad x \in \mathscr{R}, \quad \varepsilon \in (0, 1),$$

where $\rho \in \mathscr{D}$ and

(1.4) $$\int_{-\infty}^{\infty} \rho(x)\mathrm{d}x = 1.$$

Let $\delta \in F_0$ be the asymptotic function determined by Δ, i. e.

(1.5) $$\delta = [\Delta],$$

where [ ] is the corresponding equivalence class ([11], (2.1)). Now, it is evident that Δ is a delta-sequence, i. e.

(1.6) $$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \Delta(x, \varepsilon)\varphi(x)\mathrm{d}x = \varphi(0), \quad \varphi \in \mathscr{D}.$$

Hence we obtain (1.1) having in mind ([6], Theorem 30, (i)) and, of course, the definition of the notion of "integral" in F_0 ([11], (2.3)).

(1.7) Remark: The above example offers, in fact, a family of kernels of the Dirac's distribution where the function ρ plays the role of a parameter of the family. In other words, the different ρ in example (1.2) gives different asymptotic delta-functions δ; more precisely, we should have written $\Delta\rho$ and $\delta\rho$ instead of Δ and δ, respectively, but we preferred the latter notations for the sake of convenience.

(1.8) Example: Let the asymptotic function δ be determined by the well-known delta-sequence

(1.9) $$\Delta(x, \varepsilon) = \frac{1}{\pi}\,\frac{\varepsilon}{x^2+\varepsilon^2}, \quad x \in \mathscr{R}, \quad \varepsilon \in (0, 1),$$

i. e. $\delta=[\Delta]$. Then δ is another example of a kernel of the Dirac's distribution.
(1.10) Example: Let

(1.11) $$\sigma: C \to C$$

be any entire function and let us set

(1.12) $$\Delta_\sigma(x, \varepsilon)=\Delta(x, \varepsilon)+\sigma(x+i\varepsilon)-\sigma(x-i\varepsilon), \quad x \in \mathscr{R}, \quad \varepsilon \in (0, 1),$$

where Δ is determined by (1.9). It is easy to verify that the asymptotic functions $\delta_\sigma=[\Delta_\sigma]$ are also kernels of the Dirac's distribution. Moreover, the different σ determined different kernels δ_σ.
(1.13) Remark (Standard asymptotic functions): In ([11], (2.28)) we introduced the notion of a "standard asymptotic function" which will be important for the rest of the paper. Notice that the asymptotic delta-functions exposed in example (1.2) and this one from example (1.8) are standard asymptotic functions while the asymptotic delta-functions from example (1.10) by $\sigma \neq 0$ are not standard.

2. Powers of δ

It is well-known that the n-th power of the Dirac's distribution does not exist for $n \geq 2$ in the Schwartz theory of distributions. But the n-th power δ^n of any kernel δ of the Dirac's distribution exists since, as we mentioned above, the class of the asymptotic functions F_0 is a ring ([11], (2.19)) and, consequently, every two asymptotic functions can be correctly multiplied. Our purpose in this section is to study the properties of δ^n for different choice of the kernels δ of the Dirac's distribution. To this end we are going to use some results of ([11], Section 4) and, in particular, the asymptotic distributions generated by the above mentioned power δ^n.
(2.1) Theorem (The n-th Power of δ): Let $\delta \in F_0$ be any standard ([11], (2.28)) kernel of the Dirac's distribution (i. e. Dirac's delta-function) for which the integrals

(2.2) $$\int_{-\infty}^{\infty} \delta^n(x) X^k \mathrm{d}x, \quad k=0, 1, \ldots, n-1,$$

are convergent (n is a natural number) and δ^n is the n-power of δ ([11], (2.19)). Then the infinitesimal equality

(2.3) $$\int_{-\infty}^{\infty} \delta^n(x)\varphi(x)\mathrm{d}x \approx \sum_{k=0}^{n-1} M_{nk}(-1)^k \varphi^{(k)}(0), \quad \varphi \in \mathscr{D}$$

is valid where the coefficients M_{nk} (which are asymptotic numbers in general) are determined by the formula

(2.4) $$M_{nk}=\frac{(-1)^k}{k!} \int_{-\infty}^{\infty} \delta^n(x) X^k \mathrm{d}x, \quad k=0, 1, \ldots, n-1.$$

(2.5) Remark: The asymptotic functions presented in example (1.2) and example (1.8) satisfy the assumption of the above theorem in contrast to the asymptotic functions from example (1.10) which do not satisfy it for $\sigma \neq 0$, since they do not possess the property to be "standard" ones.

Proof: (i) The assumption for δ "to be standard" means that δ is the equivalence class determined by some function $v \in V$ ([11], (1.1)) of the type

$$v(x, \varepsilon) = \varepsilon^{-m} \rho\left(\frac{x - x_0}{\varepsilon}\right), \quad x \in \mathscr{R}, \quad \varepsilon \in (0, 1)$$

for some bounded smooth function ρ, some $x_0 \in \mathscr{R}$ and some $m \in Z$ ([11], (1.19)).

(ii) The assumption for δ "to be a kernel of the Dirac's distributions" implies

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} v(x, \varepsilon) \varphi(x) \mathrm{d}x = \varphi(0), \quad \varphi \in \mathscr{D}$$

which implies $m = 1$, $x_0 = 0$ and

$$\int_{-\infty}^{\infty} \rho(x) \mathrm{d}x = 1, \tag{2.6}$$

i. e. v is, in fact, of the type

$$v(x, \varepsilon) = \frac{1}{\varepsilon} \rho\left(\frac{x}{\varepsilon}\right), \quad x \in \mathscr{R}, \quad \varepsilon \in (0, 1), \tag{2.7}$$

for some bounded function $\rho \in C^\infty$ satisfying (2.6);

(iii) From the assumption "the integrals (2.2) to be convergent" follows the convergence of the integrals

$$\int_{-\infty}^{\infty} \rho^n(x) X^k \mathrm{d}x, \quad k = 0, 1, \ldots, n-1;$$

(iv) Finally, the formulae (2.3) and (2.4) are obtained as a particular case of ([11], Lemma (2.36)). The proof is completed.

(2.8) Remark: (The square of δ) We shall especially pay attention to the case $n = 2$ for the significance of the square $\delta^2(x)$ in quantum field theory. In this case condition (2.2) reduces to the convergence of the integrals

$$\int_{-\infty}^{\infty} \delta^2(x) X^k \mathrm{d}x, \quad k = 0, 1, \tag{2.9}$$

and the formulae (2.3) and (2.4) go to

$$\int_{-\infty}^{\infty} \delta^2(x) \varphi(x) \mathrm{d}x \approx M_{20} \varphi(0) - M_{21} \varphi'(0), \quad \varphi \in \mathscr{D}, \tag{2.10}$$

and

$$M_{20} = \int_{-\infty}^{\infty} \delta^2(x) \mathrm{d}x, \tag{2.11}$$

and

$$M_{21} = -\int_{-\infty}^{\infty} \delta^2(x) . x \mathrm{d}x, \tag{2.12}$$

respectively.

(2.13) Example: Let δ be the asymptotic delta-function from example (1,2) Then (2.4) gives

$$M_{nk}=s^{k-n+1}\frac{(-1)^k}{k!}\int_{-\infty}^{\infty}\rho^n(x)X^k\mathrm{d}x,\quad k=0,\ 1,\ldots,\ n-1, \tag{2.14}$$

so, we see that the calculation of the asymptotic coefficients M_{nk} reduces to the calculation of common Riemann's integrals. Recall [5] that s is a (fixed) positive infinitesimal defined by the formula

$$s=\{\alpha\mid\alpha(\varepsilon)=\varepsilon+0(\varepsilon),\ \lim_{\varepsilon\to 0}\varepsilon^{-k}0(\varepsilon)=0,\ \ k\in Z\}. \tag{2.15}$$

We see as well that all coefficients M_{nk} are either infinitely large numbers or zero excepting $M_{n,n-1}$ which is a usual complex number.

(2.16) Example: Let δ be the asymptotic delta-function of example (1.8). Then we obtain

$$M_{n.2k}=s^{2k-n+1}\frac{2}{\pi^n(2k)!}\int_{-\infty}^{\infty}\frac{x^{2k}\mathrm{d}x}{(x^2+1)^n},\ \ k=0,\ 1,\ldots,\ \left[\frac{n-1}{2}\right], \tag{2.17}$$

$$M_{n.2k+1}=0,\quad k=0,\ 1,\ldots,\left[\frac{n-1}{2}\right]. \tag{2.18}$$

In particular, in the case $n=2$ we have

$$\int_{-\infty}^{\infty}\delta^2(x)\varphi(x)\mathrm{d}x\approx\frac{1}{2\pi s}\varphi(0),\quad \varphi\in\mathcal{D}. \tag{2.19}$$

Notice that

$$\int_{-\infty}^{\infty}\delta^2(x)\,\mathrm{d}x=\frac{1}{2\pi s};\quad \delta(0)=\frac{1}{\pi s}, \tag{2.20}$$

so, we have in this case

$$\int_{-\infty}^{\infty}\delta^2(x)\mathrm{d}x\neq\delta(0)$$

as opposed to the use of some formal expressions in quantum field theory.

(2.21) Remark: A formula which is similar to (2.19) is derived in [15] in the framework of another formalism.

In some particular cases the asymptotic functional in the left side of (2.3) could be calculated exactly in the form of infinite series.

(2.22) Theorem: Let $\delta\in F_0$ be a standard kernel of the Dirac's distribution for which all the integrals

$$\int_{-\infty}^{\infty}\delta^n(x)X^k\mathrm{d}x,\quad k=0,\ 1,\ldots \tag{2.23}$$

are convergent (n is a natural number). Then we have

$$\int_{-\infty}^{\infty}\delta^n(x)\varphi(x)\mathrm{d}x=\sum_{k=0}^{\infty}M_{nk}(-1)^k\varphi^{(k)}(0),\quad \varphi\in\mathcal{D}, \tag{2.24}$$

where

$$(2.25) \qquad M_{nk}=\frac{(-1)^k}{k!}\int_{-\infty}^{\infty}\delta^n(x)X^k\mathrm{d}x, \quad k=0,\ 1,\ldots$$

and the series in (2.24) is convergent in the interval topology of the system of the asymptotic numbers [6].

(2.26) Remark: Notice that the asymptotic delta-function from example (1.2) satisfies the assumptions of the above theorem.

Proof: Just as in (i) and (ii) of the proof of Theorem (2.1) we conclude that δ is determined by function of the type (2.7). The assumption "all integrals (2.23) to be convergent" implies the convergence of the integrals

$$\int_{-\infty}^{\infty}\rho^n(x)X^k\mathrm{d}x, \quad k=0,\ 1,\ldots$$

So, we have the asymptotic expansion

$$\int_{-\infty}^{\infty}v^n(x,\ \varepsilon)\varphi(x)\mathrm{d}x\underset{\varepsilon\to 0}{\sim}\sum_{k=0}^{\infty}\varepsilon^{k-n+1}c_{nk}(-1)^k\varphi^{(k)}(0),$$

where the complex coefficients c_{nk} are determined by the formula

$$c_{nk}=\frac{(-1)^k}{k!}\int_{-\infty}^{\infty}\rho^n(x)X^k\mathrm{d}x.$$

Corresponding to ([11], Lemma (2.29)), we obtain

$$\int_{-\infty}^{\infty}\delta^n(x)\varphi(x)\mathrm{d}x=\sum_{k=0}^{\infty}s^{k-n+1}c_{nk}(-1)^k\varphi^{(k)}(0), \quad \varphi\in\mathcal{D}.$$

Notice that the convergence of the above series follows directly from ([6], Theorem 41). On the other hand, we easily establish the connection between c_{nk} and M_{nk}

$$M_{nk}=s^{k-n+1}c_{nk}$$

and the last formula but one reduces to (2.24). The proof is completed.

3. Asymptotic Heaviside's Functions

Corresponding to the terminology introduced in [11], "asymptotic Heaviside's function" or "asymptotic Θ-function" will be called any of the kernels of the Heaviside's distribution, i. e. any $\Theta\in F_0$ for which

$$(3.1) \qquad \int_{-\infty}^{\infty}\Theta(x)\varphi(x)\,\mathrm{d}x\approx\int_{0}^{\infty}\varphi(x)\mathrm{d}x, \quad \varphi\in\mathcal{D}$$

holds. Notice that there are many Θ for which (3.1) holds and we shall offer some examples of them.

(3.2) Example: Let $\rho\in\mathcal{D}$ and

$$\int_{-\infty}^{\infty} \rho(x)\mathrm{d}x = 1 \tag{3.3}$$

and let us put

$$t(x, \varepsilon) = \int_{-\infty}^{x/\varepsilon} \rho(y)\mathrm{d}y, \quad x \in \mathscr{R}, \quad \varepsilon \in (0, 1). \tag{3.4}$$

Then the asymptototic function Θ determined by t, i. e. $\Theta = [t]$, offers an example of an asymptotic Heaviside's function; in fact, for the different choice of ρ we obtain different Θ so, we have not one but a family of asymptotic Heaviside's functions. It is easy to verify that the derivative Θ' of Θ is asymptotic delta-function of the type described in example (1.2).

(3.5) Example: Let us put

$$\tau(x, \varepsilon) = \begin{cases} -\frac{1}{\pi} \operatorname{arc\,tg}(\varepsilon/x), & x < 0, \\ 1/2, & x = 0, \\ 1 - \frac{1}{\pi} \operatorname{arc\,tg}(\varepsilon/x), & x > 0, \quad x \in \mathscr{R}, \quad \varepsilon \in (0, 1) \end{cases} \tag{3.6}$$

and let Θ be the asymptotic function determined by τ, i. e.

$$\Theta = [\tau]. \tag{3.7}$$

Then Θ is another (different from (3.2)) example of a kernel of the Heaviside's distribution. The derivative Θ' of Θ coincides with the asymptotic delta-function of example (1.8).

(3.8) Theorem: If Θ is an asymptotic Heaviside's function, then its derivative Θ' is an asymptotic delta-function.

Proof: The above theorem is a direct corollary of ([11], Theorem (4.14), (iii)).

4. The Product $\delta(x).\Theta(x)$

Recall that the product $\delta(x).\Theta(x)$ has no sense in the Schwartz theory of distributions. If $\delta(x)$ and $\Theta(x)$, however, are not distributions themselves but corresponding kernels in F_0 of the delta-function and Heaviside's distributions, respectively, the product $\delta(x).\Theta(x)$ is correctly defined and we are able to study its properties. Provided some additional assumptions the following theorem takes place

(4.1) Theorem (The product $\delta(x).\Theta(x)$): Let δ and Θ be two standard kernels ([11], (2.28)) from the class F_0 of the delta-distribution and the Heaviside's distribution, respectively. Then: (i) The infinitesimal equality

$$\int_{-\infty}^{\infty} \delta(x)\Theta(x)\mathrm{d}x \approx I\varphi(0), \quad \varphi \in \mathscr{D} \tag{4.2}$$

is valid where the coefficient I is a complex number (but not an asymptotic one) determined by the formula

$$I = \int_{-\infty}^{\infty} \delta(x)\Theta(x)\mathrm{d}x; \tag{4.3}$$

(ii) In the particular case when $\delta=\Theta'$ the coefficient I is equal to $1/2$, i, e.

(4.4)
$$\int_{-\infty}^{\infty}\delta(x)\Theta(x)\,\varphi(x)\,\mathrm{d}x\approx\frac{1}{2}\,\varphi(0),\quad \varphi\in\mathscr{D}.$$

(4.5) Remark: Notice that $\delta=\Theta'$ is not valid for every asymptotic delta-function δ and for every asymptotic Heaviside's function Θ (compare with Theorem (3.8)).

Proof: (i) The assumption for δ and Θ to be "standard asymptotic functions" leads to the conclusion that δ and Θ are determined by functions of the type

$$\Delta(x,\,\varepsilon)=\frac{1}{\varepsilon}\,\rho\left(\frac{x}{\varepsilon}\right),\quad x\in\mathscr{R},\quad \varepsilon\in(0,\,1)$$

and

$$\tau(x,\,\varepsilon)=\chi\left(\frac{x}{\varepsilon}\right),\quad x\in\mathscr{R},\quad \varepsilon\in(0,\,1),$$

respectively, for some bounded smooth functions ρ and χ provided as well condition (2.6) for ρ (see the (i)-part of the proof of Theorem (2.6)). From this point onward the proof is elementary. Indeed, we can easily calculate

$$\int_{-\infty}^{\infty}\Delta(x,\,\varepsilon)\tau(x,\,\varepsilon)\varphi(x)\mathrm{d}x=\int_{-\infty}^{\infty}\rho(y)\chi(y)\varphi(\varepsilon y)\mathrm{d}y$$

$$=\varphi(0)\int_{-\infty}^{\infty}\rho(y)\chi(y)\mathrm{d}y+\int_{-\infty}^{\infty}\rho(y)\chi(y)\,[\varphi(\varepsilon y)-\varphi(0)]\mathrm{d}y.$$

Notice now that

(4.6)
$$\int_{-\infty}^{\infty}\rho(y)\chi(y)\mathrm{d}y=\int_{-\infty}^{\infty}\delta(x)\Theta(x)\,\mathrm{d}x=I$$

and

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\rho(y)\chi(y)[\varphi(\varepsilon y)-\varphi(0)]\mathrm{d}y=0$$

which follows to

$$\int_{-\infty}^{\infty}\delta(x)\Theta(x)[\varphi(x)-\varphi(0)]\mathrm{d}x\approx 0.$$

In order to obtain (4.2) we have to apply the definition of "integral" in the class of the asymptotic functions F_0 ([11], (2.3)). As to the constant I the formula (4.6) reduces its calculation to the calculation of a usual Riemann's integral of the usual smooth function $\rho\,.\,\chi\in C^{\infty}$ so, we have $I\in C$;

(ii) And finally, $\delta=\Theta'$ implies $\rho=\chi'$ and the formula (4.6) gives $I=1/2$. The proof is completed.

(4.7) Example: Let δ and Θ be chosen as in example (1.8) and example (3.5), respectively. In this particular case we obtain

(4.8)
$$I=\int_{-\infty}^{\infty}\delta(x)\Theta(x)\mathrm{d}x=\frac{1}{2},$$

so, the formula (4.4) holds for the above mentioned choice of δ and Θ.
(4.9) Example: Let δ be an asymptotic delta-function of the type described in example (1.2) for

$$\rho(x)=\frac{1}{\sqrt{\pi}}e^{-x^2}, \quad x\in\mathscr{R}. \tag{4.10}$$

Let Θ be the asymptotic Θ-function of example (3.5). Then

$$I=\int_{-\infty}^{\infty}\delta(x)\Theta(x)dx=-\frac{1}{\pi\sqrt{\pi}}\int_{-\infty}^{0}e^{-x^2}\operatorname{arc\,tg}(1/x)dx \tag{4.11}$$

$$+\frac{1}{\sqrt{\pi}}\int_{0}^{\infty}e^{-x^2}\left(1-\frac{1}{\pi}\operatorname{arc\,tg}(1/x)\right)dx=1$$

and the formula (4.2) reduces to

$$\int_{-\infty}^{\infty}\delta(x)\Theta(x)\varphi(x)dx\approx\varphi(0), \quad \varphi\in\mathscr{D}, \tag{4.12}$$

in the particular case under consideration.

5. The Powers of X^{-n}

Recall that the distribution $P(X^{-n})$ (where n is a natural number) named "the principal value of X^{-n}" is defined by means of the following formula:

$$\langle P(X^{-n}),\varphi\rangle=\lim_{\varepsilon\to 0}\frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{1}{(t+i\varepsilon)^n}+\frac{1}{(t-i\varepsilon)^n}\right]\varphi(t)\,dt, \quad \varphi\in\mathscr{D}. \tag{5.1}$$

Recall, moreover, that if m is a natural number bigger than 1, then the powel

$$[P(x^{-n})]^m$$

has no sense in the conventional theory of distributions. But if $X^{-n}\in F_0$ is any kernel of $P(x^{-n})$ (recall that every distribution possesses kernels in the class of the asymptotic functions F_0 ([11], Theorem (4.20)), then the power $(X^{-n})^m$ has sense in F_0 and we are going to study some of its properties.
(5.2) Theorem: Let $X^{-n}\in F_0$ be a standard kernel of the distribution $P(X^{-n})\in\mathscr{D}'$ (5.1) and let m be a natural number. Then infinitesimal equality

$$\int_{-\infty}^{\infty}[X^{-n}(t)]^m\varphi(t)dt\approx\langle P(x^{-nm}),\varphi\rangle+\sum_{k=0}^{nm-2}M_{n.m.k}(-1)^k\varphi^{(k)}(0), \quad \varphi\in\mathscr{D}, \tag{5.3}$$

is valid where the coefficients $M_{n.m.k}$ (which are asymptotic numbers in general) are determined by the formula

$$M_{n.m.k}=\frac{(-1)^k}{k!}\int_{-\infty}^{\infty}[X^{-n}(t)]^m t^k\,dt, \quad k=0,1,\ldots,nm-2. \tag{5.4}$$

We shall omit the proof of the above theorem since it is very similar to the proofs of the theorems exposed so far.

(5.5) Corollary: By the assumption of the above theorem the formula

$$\int_{-\infty}^{\infty} [X^{-n}(t)]^m \varphi(t) dt \approx \int_{-\infty}^{\infty} X^{-nm}(t)\varphi(t)\, dt + \sum_{k=0}^{nm-2} M_{n.m.k}(-1)^k \varphi^{(k)}(0), \quad \varphi \in \mathscr{D} \tag{5.6}$$

is valid.
Proof: (5.3) implies directly (5.6) having in mind the infinitesimal equality:

$$\int_{-\infty}^{\infty} X^{-nm}(t)\varphi(t)\, dt \approx \langle P(X^{-nm}), \varphi\rangle, \quad \varphi \in \mathscr{D}$$

which holds by assumption.
(5.7) Example: Let r_n be the smooth function defined as follows:

$$r_n(t) = (t^2+1)^{-n} \sum_{k=0}^{[n/2]} (-1)^k t^{n-2k}, \quad t \in \mathscr{R}, \tag{5.8}$$

and let us put

$$v_n(t, \varepsilon) = \varepsilon^{-n} r_n\left(\frac{t}{\varepsilon}\right), \quad t \in \mathscr{R}, \quad \varepsilon \in (0, 1). \tag{5.9}$$

Comparing with (5.1) we can verify that

$$v_n(t, \varepsilon) \xrightarrow[\varepsilon \to 0]{\mathscr{D}'} P(X^{-n}),$$

so, the asymptotic fnnction defined by

$$X^{-n}[v_n] \tag{5.10}$$

is an example of a standard kernel of the distribution $P(X^{-n})$.
(5.11) Example: Let us put $n=1$ and $m=2$ in (5.3) and (5.4) and let us choose for X^{-1} the asymptotic function defined in example (5.7) for $n=1$ Then, formula (5.4) gives

$$\overset{*}{M}_{1,2,0} = \int_{-\infty}^{\infty} [X^{-1}(t)]^2\, dt = \frac{\pi}{2s} \tag{5.12}$$

which is obviously an infinitely large asymptotic number (2.15). So, we have.

$$\int_{-\infty}^{\infty} [X^{-1}(t)]^2 \varphi(t)\, dt \approx \langle P(X^{-2}), \varphi\rangle + \frac{\pi}{2s}\varphi(0), \quad \varphi \in \mathscr{D}. \tag{5.13}$$

(5.14) Remark: Having in mind formula (2.20) we can write (5.13) in the following way:

$$\int_{-\infty}^{\infty} [X^{-1}(t)]^2 \varphi(t)\, dt \approx \langle P(X^{-2}), \varphi\rangle + \pi^2 \int_{-\infty}^{\infty} \delta^2(t)\varphi(t)\, dt, \quad \varphi \in \mathscr{D}, \tag{5.15}$$

where, let us recall once again, δ and X^{-1} are the asymptotic functions defined in example (1.8) and example (5.7), respectively, being some specially chosen kernels of the Dirac's distribution and Θ-distribution, respectively. In fact, it is easy to verify that we have

$$(5.16) \qquad (X^{-1})^2 = X^{-2} + \pi^2\delta^2$$

which implies, of course, the strict (but not infinitesimal) equality

$$(5.17) \qquad \int_{-\infty}^{\infty} [X^{-1}(t)]^2\varphi(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} X^{-2}(t)\varphi(t)\,\mathrm{d}t + \pi^2 \int_{-\infty}^{\infty} \delta^2(t)\varphi(t)\,\mathrm{d}t, \quad \varphi \in \mathcal{D}.$$

(5.18) Remark: Formula (5.16) is very similar to the formula

$$[P(X^{-1})]^2 = P(X^{-2}) + \pi^2\delta^2(x)$$

derived in the framework of the Mikusinski's sequential approach to the theory of distributions ([14], p. 286).

6. The Product $\delta^{(m)}(x).X^{-n}$

(6.1) Theorem: Let δ and X^{-n} be two standard kernels of the Dirac's distribution and distribution $P(X^{-n})$ (5.1), respectively, and let m be a non-negative integer. Then the infinitesimal equality

$$(6.2) \qquad \int_{-\infty}^{\infty} \delta^{(m)}(t)X^{-n}(t)\,\varphi(t)\,\mathrm{d}t \approx \sum_{k=0}^{m+n} d_{m.n.k}(-1)^k\varphi^{(k)}(0), \quad \varphi \in \mathcal{D}$$

is valid where the coefficients $d_{m.n.k}$ (which are asymptotic numbers in general) are determined by the formula

$$(6.3) \qquad d_{m.n.k} = \frac{(-1)^k}{k!} \int_{-\infty}^{\infty} \delta^{(m)}(t)X^{-n}(t)t^k\,\mathrm{d}t, \quad k=0,\ 1, \ldots, \ m+n.$$

The proof is very similar to the proofs of the previous theorems and we shall omit it.

(6.4) Example: We shall calculate the coefficients $d_{m.n.k}$ (6.3) for the asymptotic delta-function δ defined in example (1.8) and for the asymptotic function X^{-n} defined in example (5.7). In this case we have

$$(6.5) \qquad d_{m.n.k} = \frac{(-1)^k}{k!} s^{k-m-n} \int_{-\infty}^{\infty} \rho^{(m)}(t)r_n(t)t^k\,\mathrm{d}t,$$

where

$$(6.6) \qquad \rho(t) = \frac{1}{\pi}\,\frac{1}{1+x^2}, \quad x \in \mathcal{R},$$

and r_n is defined by (5.8). So, the calculation of $d_{m.n.k}$ is reduced to the calculation of some common Riemann's integrals. We obtain easily

(6.7) $d_{m.n.k}=0$ when $m+n$ and k are both odd or even which implies

$$(6.8) \quad \int_{-\infty}^{\infty} \delta^{(m)}(t)X^{-n}(t)\,\varphi(t)\,\mathrm{d}t \approx \sum_{k=0}^{[\frac{m+n}{2}]} d_{m.n.\ m+n-2k}\cdot(-1)^{m+n-2k}\varphi^{(m+n-2k)}(0), \quad \varphi \in \mathcal{D}.$$

Notice that a formula similar to (6.8) is obtained in [12] in the framework of another formalism.

Here are some particular cases of (6.8):

(i) The case $m=n-1$: we have

$$d_{n-1,n,k}=0, \quad k=0, 1, \ldots, 2n-2$$

and

$$d_{n-1,n,2n-1}=\frac{(-1)^n(n-1)!}{2(2n-1)!}$$

and (6.8) reduces to

$$\int_{-\infty}^{\infty} \delta^{(n-1)}(t)X^{-n}(t)\varphi(t)\,dt \approx \frac{(-1)^{n-1}(n-1)!}{2(2n-1)!}\varphi^{(2n-1)}(0), \quad \varphi \in \mathscr{D}. \tag{6.9}$$

Notice that formula (6.9) is very similar to a formula obtained by B. Fisher [13] in the framework of the theory of distributions.

(ii) The case $m=0$, $n=2$:

$$\int_{-\infty}^{\infty} \delta(t)\, X^{-2}(t)\varphi(t)\,dt \approx -\frac{1}{4s^2}\varphi(0)+\frac{1}{8}\varphi''(0), \quad \varphi \in \mathscr{D}; \tag{6.10}$$

(iii) The case $m=n=1$:

$$\int_{-\infty}^{\infty} \delta'(t)X^{-1}(t)\varphi(t)\,dt \approx -\frac{1}{4s^2}\varphi(0)-\frac{3}{8}\varphi''(0), \quad \varphi \in \mathscr{D}, \tag{6.11}$$

etc.

7. The Product $\Theta(x).X^{-n}$

Recall that the product $\Theta(x).P(X^{-n})$, where Θ is the Heaviside's distribution and $P(X^{-n})$ is the principal value of X^{-n} (5.1) has no sense in the conventional theory of distributions. On the other hand, it is known [15] that this product has a significance in quantum field theory. So, our purpose here will be to study some properties of this product in cases when Θ and $P(X^{-n})$ are replaced by their kernels in the class of the asymptotic functions F_0.

(7.1) Theorem (The product $\Theta(x).X^{-n}$): Let Θ, $X^{-n} \in F_0$ be two standard kernels of the Heaviside's distribution (Heaviside's function) and $P(X^{-n})$-distribution (5.1), respectively (n is a natural number). Then the following infinitesimal equality

$$\int_{-\infty}^{\infty} \Theta(t)X^{-n}(t)\varphi(t)\,dt-\sum_{k=0}^{n-2} M_{nk}(-1)^k\varphi^{(k)}(0) = \begin{cases} \langle P_+(X^{-n}), \varphi\rangle, & \varphi^{(n-1)}(0)=0, \\ 0^{-1} & \varphi^{(n-1)}(0)\neq 0, \quad \varphi \in \mathscr{D} \end{cases} \tag{7.2}$$

is valid where the coefficients M_{nk} (which are asymptotic numbers in general) are determined by the formula

$$M_{nk}=\frac{(-1)^k}{k!}\int_{-\infty}^{\infty} \Theta(t)X^{-n}(t)t^k\,dt, \quad k=0, 1, \ldots, n-2, \tag{7.3}$$

and the distribution $P_+(X^{-n})\in\mathscr{D}'$ is defined as follows:

$$(7.4)\qquad \langle P_+(X^{-n}),\ \varphi\rangle=\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}[w_n(t+i\varepsilon)-w_n(t-i\varepsilon)]\varphi(t)\,\mathrm{d}t,\quad \varphi\in\mathscr{D},$$

for

$$(7.5)\qquad w_n(z)=-\frac{1}{2\pi i}\,\frac{\ln z}{z^n}+\frac{1}{2z^n},\quad z\in C\setminus\mathscr{R}.$$

(7.6) Remark: Recall ([5], Definition 5, (v)) that 0^{-1} is an infinitely large asymptotic number defined by the formula

$$(7.7)\qquad 0^{-1}=\{\alpha\mid\alpha:(0,\,1)\to C,\ \ \lim_{\varepsilon\to 0}\varepsilon\alpha(\varepsilon)=0\}.$$

We shall omit the proof of the above theorem because it is very similar to the proofs of Theorem (2.1) and Theorem (4.1).

(7.8) Definition (The asymptotic function X_+^{-n}): The asymptotic function $X_+^{-n}\in F_0$ will be defined as an equivalence class determined by the function

$$(7.9)\qquad v_n^+(x,\,\varepsilon)=w_n(x+i\varepsilon)-w_n(x-i\varepsilon),\quad x\in\mathscr{R},\quad \varepsilon\in(0,1),$$

where w is defined by (7.5). In other words, we have

$$(7.10)\qquad X_+^{-n}=[v_n^+].$$

Having in mind (7.4) we conclude that X_+^{-n} is a kernel of the distribution $P_+(X^{-n})$, i. e.

$$(7.11)\qquad \int_{-\infty}^{\infty}X_+^{-n}(t)\varphi(t)\,\mathrm{d}t\approx\langle P_+(X^{-n}),\ \varphi\rangle,\quad \varphi\in\mathscr{D}.$$

(7.12) Corollarly: By the assumption of Theorem (7.1) the following infinitesimal equality is valid:

$$(7.13)\quad \int_{-\infty}^{\infty}\Theta(t)X^{-n}(t)\varphi(t)\,\mathrm{d}t\approx\int_{-\infty}^{\infty}X_+^{-n}(t)\varphi(t)\,\mathrm{d}t+\sum_{k=0}^{n-2}M_{nk}(-1)^k\varphi^{(k)}(0),\quad \varphi\in\mathscr{D},$$

where M_{nk} are determined by formula (7.3).
Proof: Formula (7.13) follows directly from (7.2) and (7.11).

8. Some Concluding Remarks

1. The reader has probably noted that most of the results presented in this paper are obtained by the assumption that the asymptotic functions under consideration are "standard" ones, i. e. they are a very particular kind of asymptotic functions from the class F_0 described in ([11], (2.28)). This assumption is an essential one; many of the theorems presented so far fail to be true if the corresponding asymptotic functions are not standard. Here is an example:
(8.1) Example: Let δ be the asymptotic delta function defined in example (1.2) and let δ^* be the asymptotic function defined by the equivalence class

$$(8.2)\qquad \delta^*=[\Delta^*],$$

where

$$\Delta^*(x,\varepsilon)=\frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right)+\frac{1}{\varepsilon^2}\rho''\left(\frac{x}{\varepsilon}\right),\quad x\in\mathscr{R},\quad \varepsilon\in(0,1), \tag{8.3}$$

and ρ is the smooth function taken from example (1.2). It is easy to verify that δ^* is another (different from δ) kernel of the Dirac's distribution, i. e.

$$\int_{-\infty}^{\infty}\delta^*(x)\,\varphi(x)\,dx\approx\int_{-\infty}^{\infty}\delta(x)\varphi(x)\,dx\approx\varphi(0),\quad \varphi\in\mathscr{D}. \tag{8.4}$$

On the other hand, function (8.3) is not obviously of the type ([11], (1.20)) which means that δ^* is not a standard asymptotic function in contrast to δ which is a standard one. It is not difficult to verify as well that formula (2.10) is not true for δ^*, i. e. for every two asymptotic numbers M_{20} and M_{21} there exists $\varphi\in\mathscr{D}$ for which the difference

$$\int_{-\infty}^{\infty}[\delta^*(x)]^2\varphi(x)\,dx-\{M_{20}\varphi(0)-M_{21}\varphi'(0)\} \tag{8.5}$$

is not an infinitesimal number. The reader could invent himself many other examples of asymptotic functions which are not standard and for which the theorems presented so far fail to be true.

2. The following question arises: Why do we not restrict ourselves to the class of all standard asymptotic functions only but work in the whole class F_0? It is a pity but this turns out to be impossible because the class of all standard asymptotic functions is not closed with respect to the operation of addition (although it is closed with respect to multiplication). For example, if δ and X^{-n} are the asymptotic functions defined in example (1.8) and example (5.7) respectively, then $\delta . X^{-n}$ is standard, indeed, but $\delta+X^{-n}$ is not standard asymptotic function although both δ and X^{-n} are standard. That is why we are not able to separate the standard asymptotic functions in a separate class and must continue to work in the framework of the whole class F_0;

3. We shall draw attention once again to the focal points of this work:

(i) Every Schwartz distribution $T\in\mathscr{D}'$ possesses kernels f in the class of the asymptotic functions F_0 ([11], Theorem (4.20));

(ii) F_0 is a ring, i. e. every two asymptotic functions can be c rrectly added and multiplied. In particular, the products like $\delta_2(x)$, $\delta(x).\Theta(x)$, $\delta(x)_{(m)}^{(0)}.X^{-n}$, $\Theta(x).X^{-n}$, etc. have sense in F_0 for every choice of the kernels δ, Θ, X^{-n} of the corresponding Schwartz distributions;

(iii) When, in particular, these kernels are standard asymptotic functions, their products are studied in a greater detail and several formulae like (2.3), (4.2), (5.3), (6.2) or (7.1) are derived. These formulae answer, in particular, to the question, how the properties of these products depend on the choice of the asymptotic functions taking part in them, being at the same time kernels of the same (fixed) Schwartz distributions;

4. Although the present paper does not offer any applications to physics its motivations lie in some problems of quantum field theory. Recall that the basic objects in that theory — the so-called "quantum fields" — are operators whose matrix elements are Schwartz distributions [16]. Nearly in this way the distributions are involved in quantum field theory. On the other hand, for some reasons (which are difficult to explain here), products of the above mentioned type: $\delta^2(x)$. $\delta(x).\Theta(x)$, $\delta(x).X^{-n}$, $\Theta(x).X^{-n}$, etc. having no sense

in the conventional theory of distributions appear at a certain stage of quantum theory and cause great complications in its mathematical apparatus [15]. The author of the present paper hopes that the results offered here could turn out to be useful for simplifying the solution to some problems of quantum theory connecting its physical consequences more closely with its mathematical principles.

References

1. Christov, Chr., T. Todorov. Asymptotic Numbers — One Generalization of the Notion of Number. — In: Bulletin de L'Institut de Physique et de Recherche Atomique. T. XXIV, 1973. 221—231.
2. Christov, Chr. Ya. Eine neue Art von verallgemeinerten Funktionen. — Nova Acta Leopoldina. Neue Folge, 212, Band 39, 1974, 181—1944.
3. Christov, Chr. Ya., T. D. Todorov. Asymptotic Numbers — Algebraic Operation with Them. — Serdica, Bulgaricae Publicationes, Vol. 2, 1976, 87—102.
4. Христов, Хр. Я., Бл. П. Дамянов. Асимптотические функции — новый класс обобщенных функций:

   I. Общая постановка задачи и определение. — Болг. физ. журнал, 6, 1978.

   II. Существование функций и однозначность их разложений. — Болг. физ. журнал, 1, 1979.

   III. Общее определение действий над ними, линейные операции. — Болг. физ. журнал, 3, 1979.

   IV. Умножение, линейные функционалы. — Болг. физ. журнал, 4, 1979.

   V. Преобразование Фурье. — Болг. физ. журнал, 7, 1980.

   VI. Представители Фурье-образов в множестве обычных функций. — Болг. физ. журнал, 8, 1981.

   VII. Замена независимой переменной. — Болг. физ. журнал, 8, 1981.
5. Todorov, T. D. Asymptotic Numbers. I. Algebraic Properties. — Bulg. J. Phys., 7, 1980, 5, 450—468.
6. Todorov, T. D. Asymptotic Numbers. II. Order Relation and Interval Topology. — Bulg. J. Phys., 7, 1980, 6, 547—562.
7. Todorov, T. D. Asymptotic Numbers, Asymptotic Functions and Distributions. A report at the couference "Operatoren — Distributionen und Verwandte Non-Standard Methoden", Oberwolfach, Fep. Rep. of Germany, 2-8 July, 1978 (Trieste Int. Report, IC/79/75).
8. Todorov, T. D. Asymptotic Functions and Multiplication of Distributions. — Bulg. J. Phys., 8, 1981, 2, 109—120.
9. Todorov, T. D. Extended Asymptotic Functions. Some Examples. — Bulg. J. Phys., 8, 1981, 3, 216—226.
10. Todorov, T. D. Quasi-Extended Asymptotic Functions. — Bulg. J. Phys., 8, 1981, 4, 314—326.
11. Todorov, T. D. Asymptotic Functions as Kernels of the Schwartz Distributions.— Bulg. J. Phys. (to be published).
12. Li Bang-He. Non Standard Analysis and Multiplication of Distributions. — Scientia Sinica, vol. XXI, No 5, 1978, 561—585.
13. Fisher, B. Products of Generalized Functions. — Studia Math., 33, 1969, 227—230.
14. Antosik, P., J. Mikusiski, R. Sikorski. Theory of Distributions. The Sequential Approach. Amsterdam, 1973.
15. Bogolubov, N. N., O. S. Parasiuk. Über Multiplikationen der Kausalfunktionen in de Quantentheorie der Felder. — Acta Math., 97, 1957, 227—266.
16. Боголюбов, Н. Н., А. А. Логунов, И. Т. Тодоров. Основы аксиоматического подхода в квантовой теории поля. М., Наука, 1969.