Monads and realcompactness

Sergo Salbany

Todor Todorov

Abstract

We give a quantifier free characterization of realcompactness and ordered realcompactness in terms of monads. We also present simple proofs of some topological facts concerning realcompact spaces.

Key words: Ordered topological space; Ordered realcompactification; Product space; Nonstandard extension; Nonstandard ordered hull; Monads

0. Introduction

Robinson has given an elegant quantifier free description of compactness in terms of nonstandard extensions and monads [6]. The purpose of this paper is to prove an analogous characterization of realcompactness and ordered realcompactness, which should be contrasted, for simplicity, with the classical ones for completely regular topological spaces (X, T) (Gillman and Jerison [3]):

(i) Every real maximal ideal of $C(X, \mathbb{R})$ is fixed.

(ii) (X, T) is isomorphic to a closed subspace in the canonical product $(\mathbb{R}, \tau)^J$, where $J = C(X, \mathbb{R})$.

(iii) Every Z-ultrafilter with the countable intersection property is fixed.

By $C(X, \mathbb{R})$ we denote the class of continuous functions $f:(X, T) \to (\mathbb{R}, \tau)$ where τ is the usual topology in $\mathbb{R}$.

Let $C^{\uparrow}(X, \mathbb{R})$ consist of all monotone nondecreasing functions in $C(X, \mathbb{R})$ and let $A(X, \mathbb{R}) = C^{\uparrow}(X, \mathbb{R}) - C^{\uparrow}(X, \mathbb{R})$ be the algebra of all differences in $C^{\uparrow}(X, \mathbb{R})$. It was shown in [7] that the following are equivalent:

(i′) Every real maximal ideal in $A(X, T)$ is fixed.

(ii′) $(X, T, \leqslant)$ is order isomorphic to a closed subspace in the canonical product $(\mathbb{R}, \tau, \leqslant)^J$, where $J = C^{\uparrow}(X, \mathbb{R})$ and $\leqslant$ is the usual order on $\mathbb{R}$.

The immediate analog of (iii) is not equivalent to (i′), as shown in Choe and Hong [1].

In what follows, we shall work in a nonstandard model with a set of individuals S that contains both X and $\mathbb{R}$ and degree of saturation k larger than $2^{2^{\aleph_0}}$ and $2^{2^{\operatorname{card} X}}$. In particular, any polysaturated model of S will do. For the basic results and notations in nonstandard analysis we refer to Hurd and Loeb [4].

1. Notations and terminology

(1) $(X, T, \leqslant)$ will represent a completely regular ordered topological space (Nachbin [5] or Fletcher and Lindgren [2]). The nonstandard extension ${}^*X$ will always be endowed with the standard topology ${}^{\mathrm{s}}T$ with basic open sets of the form ${}^*G$ where $G \in T$ and with a binary relation $\lesssim$ on ${}^*X$ given by: $\alpha \lesssim \beta$ if $({}^*f(\alpha) \leqslant {}^*f(\beta)$ or ${}^*f(\alpha) \approx {}^*f(\beta))$ for all f in $C^{\uparrow}(X, \mathbb{R})$ [7], where $\approx$ is the infinitesimal relation in ${}^*\mathbb{R}$.

(2) We denote by $\tilde{X}$ the subset of ${}^*X$ consisting of all prenearstandard points α, i.e., for which ${}^*f(\alpha)$ is a finite number in ${}^*\mathbb{R}$ for all f in $C^{\uparrow}(X, \mathbb{R})$. As usual, a point in ${}^*X$ is nearstandard if it is in the monad of some standard point in X. The nearstandard points are, obviously, prenearstandard.

(3) Let $\sim$ denote the equivalent relation in $\tilde{X}$ defined by: $\alpha \sim \beta$, for $\alpha, \beta \in \tilde{X}$, if ${}^*f(\alpha) \approx {}^*f(\beta)$, for all f in $C^{\uparrow}(X, \mathbb{R})$. The (ordered) nonstandard hull $(\hat{X}, \hat{T}, \hat{\lessgtr})$ of $(X, T, \leqslant)$ is then the factor space $(\tilde{X}, {}^{\mathrm{s}}T, \lesssim)/\sim$, where $\hat{T}$ is the quotient topology induced by the quotient mapping $q : \tilde{X} \to \hat{X}$, and $\hat{\lessgtr}$ is the order relation on $\hat{X}$ given by $q(\alpha) \hat{\lessgtr} q(\beta)$ if and only if $\alpha \lesssim \beta$. The extension $\hat{f} : \hat{X} \to \mathbb{R}$ of $f : X \to \mathbb{R}$ is defined by $\hat{f}(q(\alpha)) = \mathrm{st}({}^*f(\alpha))$ where st is the standard part mapping in ${}^*\mathbb{R}$. The ordered topological space $(\hat{X}, \hat{T}, \hat{\lessgtr})$ will be called the real compactification of $(X, T, \leqslant)$ [7].

(4) The nonordered case, i.e., the case of a topological space (X, T), will be considered as a particular case of $(X, T, \leqslant)$ when the order $\leqslant$ is discrete. In this case $C^{\uparrow}(X, \mathbb{R}) = A(X, \mathbb{R}) = C(X, \mathbb{R})$.

2. Monad characterization of ordered realcompact spaces

Definition 2.1 (order realcompactness). A completely regular ordered topological space $(X, T, \leqslant)$ is *ordered realcompact* if it is order isomorphic to a closed subspace of a product of copies of $(\mathbb{R}, \tau, \leqslant)$.

We can now formulate a purely topological, nonstandard characterization of order realcompactness: "every prenearstandard point is nearstandard".

In what follows we denote by $\{\mu(x)\colon x \in X\}$ the system of monads of $(X, T, \leqslant)$ (Hurd and Loeb [4]).

Lemma 2.2. *Let* $(X, T, \leqslant)$ *be a completely regular ordered topological space* (Nachbin [5]). *Then the monads* $\mu(x)$ *and the equivalence classes* $q^{\leftarrow}(x)$ *under the equivalence relation* $\sim$ *coincide for all standard points* $x \in X$.

This result is proved in [7, Lemma 1.6] for $\Phi^{\uparrow} = C^{\uparrow}(X, \mathbb{R})$.

Theorem 2.3. *Let* $(X, T, \leqslant)$ *be a completely regular ordered topological space. Then* $(X, T, \leqslant)$ *is an ordered realcompact topological space if and only if*

$$\tilde{X} = \bigcup_{x \in X} \mu(x). \tag{1}$$

Proof. In [7, Section 6] it is shown that $(\hat{X}, \hat{T}, \hat{\leqslant})$ is the ordered realcompactification of $(X, T, \leqslant)$. The result then follows if we prove that $(\hat{X}, \hat{T}, \hat{\leqslant}) = (X, T, \leqslant)$. Assume that (1) holds. By the lemma, $\mu(x)$ is equal to the equivalence class of x, so that $\hat{X} = q[\tilde{X}] = X$. Hence $(\hat{X}, \hat{T}, \hat{\leqslant}) = (X, T, \leqslant)$. Conversely, if $\hat{X} = X$, then

$$\tilde{X} = q^{\leftarrow}[\hat{X}] = q^{\leftarrow}[X] = \bigcup_{x \in X} q^{\leftarrow}(x) = \bigcup_{x \in X} \mu(x)$$

again by the lemma above. The proof is complete. $\square$

If the above property (1) is taken as the definition of an ordered realcompact space, then an elegant nonstandard proof can be given to show that products of ordered realcompact spaces are ordered realcompact spaces and so are closed subspaces. These results were proved in [7] by standard arguments. The much simpler nonstandard proofs follow.

Theorem 2.4. *The product of ordered realcompact topological spaces is an ordered realcompact topological space.*

Proof. Let $\{(X_i, T_i, \leqslant_i)\colon i \in I\}$ be a family of ordered realcompact topological spaces and

$$(X, T, \leqslant) = \prod_{i \in I} (X_i, T_i, \leqslant_i).$$

To prove that

$$\tilde{X} = \bigcup_{x \in X} \mu(x),$$

suppose that $\alpha \in \tilde{X}$. Observe that for each standard i in $^*I$, we have $\alpha_i \equiv {}^*\pi_i(\alpha) \in {}^*X_i$; in fact, $\alpha_i \in \tilde{X}_i$, since for any $f\colon (X_i, T_i, \leqslant_i) \to (\mathbb{R}, \tau, \leqslant)$, ${}^*f(\alpha_i) = {}^*f({}^*\pi_i(\alpha)) = {}^*(f \circ \pi_i)(\alpha)$ is a finite number in $^*\mathbb{R}$, since $f \circ \pi_i\colon (X, T, \leqslant) \to$

$(\mathbb{R}, \tau, \leqslant)$ is in $C^{\uparrow}(X, \mathbb{R})$. Hence, α_i should be a nearstandard point in ${}^*X_i$, since $(X_i, T_i, \leqslant_i)$ is ordered realcompact by assumption, i.e., there is x_i in X_i such that $\alpha_i \in \mu(x_i)$ for all standard indices i in ${}^*I$ where $\mu_i(x_i)$ is the monad of x_i in $(X_i, T_i, \leqslant_i)$. That implies $\alpha \in \mu(x)$, where x is determined by $\pi_i(x) = x_i$, $i \in I$ (Hurd and Loeb [4, (III, 1. 18) and (III, 1. 19)]). The proof is complete. $\square$

Theorem 2.5. *Closed subspaces of ordered realcompact topological spaces are ordered realcompact topological spaces.*

Proof. Let $(X, T, \leqslant)$ be an ordered realcompact space and let $(A, T_A, \leqslant_A)$ be a closed subspace. To show that $(A, T_A, \leqslant_A)$ is an ordered realcompact space, let $\alpha \in \tilde{A}$. Now $\tilde{A} \subseteq \tilde{X}$, so that $\alpha \in \mu(x)$ for some $x \in X$, since $(X, T, \leqslant)$ is ordered realcompact. Hence $\alpha \in {}^*A \cap \mu(x)$, so that $x \in A$ since A is closed. Thus,

$$\tilde{A} \subseteq \bigcup_{x \in A} \mu(x),$$

i.e., A is an ordered realcompact space. The proof is complete. $\square$

Let $(X, T, \leqslant)$ be a completely regular ordered topological space. Then $(X, T, \leqslant)$ is isomorphic to an ordered subspace of the canonical product $(\mathbb{R}, \tau, \leqslant)^J$, for $J = C^{\uparrow}(X, \mathbb{R})$ (Nachbin [5] or Fletcher and Lindgren [2]). We shall identify X with this subspace of $\mathbb{R}^J$.

The following is a characterization of the set $\tilde{X}$ of the prenearstandard points of the space X in terms of the canonical product $\mathbb{R}^J$.

Theorem 2.6. *Let $(X, T, \leqslant)$ be a completely regular ordered topological space and $\alpha \in {}^*X$. The α is a prenearstandard point of $(X, T, \leqslant)$ if and only if it is a nearstandard point of the space $(\mathbb{R}, \tau, \leqslant)^J$, where $J = C^{\uparrow}(X, \mathbb{R})$.*

Proof. Suppose $\alpha \in \tilde{X}$, i.e., ${}^*f(\alpha)$ is finite in ${}^*\mathbb{R}$ for all f in J. Define $\xi \in \mathbb{R}^J$ by $\xi_f \equiv \pi_f(\xi) = \mathrm{st}({}^*f(\alpha))$, $f \in J$, so we have (by definition of ξ) ${}^*f(\alpha) \approx \xi_f$, $f \in J$, i.e., $\alpha \in m(\xi)$ where

$$m(\xi) = \left\{\alpha \in {}^*(\mathbb{R}^J) \colon \alpha_f \approx \xi_f, \text{ for all standard } f \text{ in } {}^*J\right\}, \quad \xi \in \mathbb{R}^J,$$

are the monads of the space $\mathbb{R}^J$ (Hurd and Loeb [4, pp. 117]). Conversely, suppose $\alpha \in m(\xi)$ for some $\xi \in \mathbb{R}^J$. Then we have ${}^*f(\alpha) \approx \xi_f$, $f \in J$, which implies that ${}^*f(\alpha)$ is finite in ${}^*\mathbb{R}$ for all f in J, i.e., $\alpha \in \tilde{X}$. The proof is complete. $\square$

Theorem 2.7. *Suppose that $(X, T, \leqslant)$ is an ordered realcompact topological space. Then $(X, T, \leqslant)$ is closed in the canonical product $(\mathbb{R}, \tau, \leqslant)^J$, where $J = C^{\uparrow}(X, \mathbb{R})$.*

Proof. Let $\xi \in \mathbb{R}^J$ and suppose ${}^*X \cap m(\xi) \neq \emptyset$. To show that $\xi \in X$, i.e., $\xi_f = f(x)$, $f \in J$, for some $x \in X$, let $\alpha \in {}^*X \cap m(\xi)$. By Theorem 2.6, $\alpha \in \tilde{X}$ and by the realcompactness of X, there exists $x \in X$ such that $\alpha \in \mu(x)$. Hence $\xi_f \approx {}^*f(\alpha) \approx$

$f(x)$, $f \in J$, which implies $\xi_f = f(x)$ for all f in J, since ξ_f and $f(x)$ are both in $\mathbb{R}$. The proof is complete. $\square$

3. Monad characterization of completely regular topological spaces

For the sake of clarity and completeness we state the above results when the order is discrete, i.e., the topological case. Let (X, T) be a completely regular topological space and let $\tilde{X}$ consist of all prenearstandard points of (X, T).

Theorem 3.1. *Let (X, T) be a completely regular topological space. Then (X, T) is a realcompact topological space if and only if*

$$\tilde{X} = \bigcup_{x \in X} \mu(x). \tag{2}$$

Theorem 3.2. *The product of realcompact spaces is realcompact.*

Theorem 3.3. *Closed subspaces of realcompact spaces are realcompact.*

Theorem 3.4. *Let (X, T) be a completely regular topological space and $\alpha \in {}^*X$. Then α is a prenearstandard point of (X, T) if and only if it is a nearstandard point of (X, T) as a canonical subspace of $(\mathbb{R}, \tau)^J$ where $J = C(X, \mathbb{R})$.*

Theorem 3.5. *Suppose (X, T) is a realcompact space. Then (X, T) is closed in the canonical product $(\mathbb{R}, \tau)^J$, where $J = C(X, \mathbb{R})$.*

4. Examples

Example 4.1. $(\mathbb{R}, \tau)$ is realcompact and $(\mathbb{R}, \tau, \leqslant)$ is ordered realcompact since

$$\tilde{X} = \bigcup_{x \in \mathbb{R}} \mu(x)$$

coincides with all finite numbers in ${}^*\mathbb{R}$.

Example 4.2. Let $I = (a, b)$ for $a, b \in \mathbb{R}$, $a < b$, be an open interval in $\mathbb{R}$. Then (I, τ) is realcompact and $(I, \tau, \leqslant)$ is ordered realcompact, since in both cases we have:

$$\tilde{I} = \{\alpha \in {}^*\mathbb{R} \colon a < \alpha < b, \text{ st } \alpha \neq a, \text{ st } \alpha \neq b\}$$

so that

$$\tilde{I} = \bigcup_{a < x < b} \mu(x).$$

Example 4.3. Let $X=(-1, 1)-\{0\}$. Then, the topological space (X, τ) is realcompact while the ordered topological space $(X, \tau, \leqslant)$ is not ordered realcompact. We have (in both cases):

$$\bigcup_{x\in X} \mu(x)=\{\alpha\in {}^*\mathbb{R}\colon\ -1<\alpha<1,\ \mathrm{st}\ \alpha\neq -1,\ \mathrm{st}\ \alpha\neq 0,\ \mathrm{st}\ \alpha\neq 1\}.$$

In the first (nonordered) case, for the prenearstandard points we have the same set as above which means that (X, τ) is realcompact. For the prenearstandard points of the space $(X, \tau, \leqslant)$ we have:

$$\tilde{X}=\{\alpha\in {}^*\mathbb{R}\colon\ -1<\alpha<1,\ \mathrm{st}\ \alpha\neq -1,\ \alpha\neq 0,\ \mathrm{st}\ \alpha\neq 1\}$$

so that $(X, \tau, \leqslant)$ is not realcompact.

Remark (real maximal ideals). Let (X, τ) and $(X, \tau, \leqslant)$ be as in Example 4.3 above. According to (i) in the Introduction, all real maximal ideals of $C(X, \mathbb{R})$ are fixed in (X, τ), i.e., they are of the type

$$M_x=\{f\in C(X, \mathbb{R})\colon f(x)=0\}$$

for some x in X. On the other hand, by (i′) in the Introduction, there should exist real maximal ideals in $A(X, \mathbb{R})$ which are not fixed in $(X, \tau, \leqslant)$. For example, for $\alpha\in {}^*\mathbb{R}$, $\alpha\neq 0$, $\alpha\approx 0$, the real maximal ideal

$$M_\alpha=\{f\in A(X, \mathbb{R})\colon {}^*f(\alpha)\approx 0\}$$

is not fixed.

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6. References

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