A stronger conclusion to the classical ham sandwich theorem

John H. Elton, Theodore P. Hill

Abstract

The conclusion of the classical ham sandwich theorem for bounded Borel sets may be strengthened, without additional hypotheses — there always exists a common bisecting hyperplane that touches each of the sets, that is, that intersects the closure of each set. In the discrete setting, where the sets are finite (and the measures are counting measures), there always exists a bisecting hyperplane that contains at least one point in each of the sets. Both these results follow from the main theorem of this note, which says that for $n$ compactly supported positive finite Borel measures in $\mathbb{R}^n$, there is always an $(n - 1)$-dimensional hyperplane that bisects each of the measures and intersects the support of each measure. Thus, for example, at any given instant of time, there is one planet, one moon and one asteroid in our solar system and a single plane touching all three that exactly bisects the total planetary mass, the total lunar mass, and the total asteroidal mass of the solar system. In contrast to the bisection conclusion of the classical ham sandwich theorem, this bisection-and-intersection conclusion does not carry over to unbounded sets of finite measure.

1. Introduction

The classical ham sandwich theorem says that every collection of $n$ bounded Borel sets in $\mathbb{R}^n$ can be simultaneously bisected in Lebesgue measure by a single hyperplane. The purpose of this short note is to show that this conclusion may be strengthened, without additional hypotheses — there always exists a common bisecting hyperplane that touches each of the sets, that is, that intersects the closure of each set. This is a corollary of the main result, Theorem 4 below, which says that for compactly supported positive finite Borel measures, there is always a hyperplane that bisects each of
the measures and intersects the support of each measure. In contrast to the bisection conclusion of
the classical ham sandwich theorem, this bisection-and-intersection conclusion does not carry over
to unbounded sets of finite measure; see Example 6 below.

Many extensions and generalizations of the classical ham sandwich theorem are well known. The
conclusion for n = 3 was apparently conjectured by Steinhaus, and first proved by Banach (cf. [5,8]).
Stone and Tukey [16] extended the conclusion to n > 3, to general continuous finite measures, and
to certain general classes of surfaces that include hyperplanes. For measures with atoms (positive
masses concentrated on points), “bisection” is interpreted as in the definition below, and, in the special
case of counting measure, if all the sets consist of finitely many points each of unit mass, then there
always exists a hyperplane that simultaneously bisects all the sets (e.g. [11,13,14]). Simultaneous
bisection of general finite measures, which may have both continuous components and atoms with
general masses, was established in [7,8]. In some settings, algorithms for constructing the bisecting
hyperplane have been found (e.g. [7,10,14]). In general, 1:1 is the only ratio that any two compact sets
in the plane can be simultaneously bisected by a line in that ratio [11, Example 3.2.4], but sufficient
conditions for other ratios have recently been found [4,14].

An interesting generalization of the classical ham sandwich theorem to k + 1 ≤ n sets (cf. [6,11,17])
says that for each of the k + 1 mass distributions in R^n, there exists a k-dimensional affine subspace A
such that any hyperplane containing A has at most (n−k+1)^−1 of each mass on each side; the classical
case is k = n − 1. In another direction, several authors (e.g. [2,9,12]) have shown that any two mass
distributions in the plane can be simultaneously equipartitioned by three rays emanating from a single
point. Similar results for equipartitioning a single mass distribution, such as partitioning a single mass
distribution in the plane into four equal parts by using two lines, have also been found (cf. [2,18,19]).
More details and references on various extensions and generalizations of the ham sandwich theorem
may be found in [11].

Notation. Fix n ∈ N, and for x, y ∈ R^n, let |x| denote the Euclidean norm of x. For subsets A and B of
R^n, let d(A, B) denote the Euclidean distance between A and B, i.e. d(A, B) = inf{|x−y| : x ∈ A, y ∈ B}.
Recall that every hyperplane H in R^n may be represented by (u, c) ∈ R^{n+1} via the relationship
x ∈ H ⇔ ⟨u, x⟩ = c, where u is a point in the unit n-sphere S^n, c ≥ 0 and ⟨·, ·⟩ denotes the standard
inner product on R^n. For the hyperplane H determined by (u, c), let H^+ denote the open half-space
defined by x ∈ H^+ ⇔ ⟨u, x⟩ > c and let H^- denote the open half-space x ∈ H^- ⇔ ⟨u, x⟩ < c.

For a bounded Borel set A ⊂ R^n, let |A| denote the cardinality of A, λ(A) the Lebesgue measure
(n-dimensional volume) of A, and A the closure of A. For a finite Borel measure μ on R^n, μ is the total mass of μ, and supp(μ) the support of μ (the smallest closed set C ⊂ R^n such that
μ(C) = μ, i.e. C = C ⊂ R^n : C is closed and μ(C) = μ). A measure μ is positive if μ > 0
and is finite if μ < ∞.

Note that to be able to treat measures where a hyperplane may have positive measure, bisection
of a positive finite measure is defined to mean that no more than half the mass of the measure lies
on either side of the hyperplane (not including the hyperplane); this is equivalent to the hyperplane
being a median for the normalized measure (see [8]).

Definition. An (n − 1)-dimensional hyperplane H in R^n bisects a positive finite Borel measure μ on
R^n if μ(H^+) ≤ μ(∥x∥/2 and μ(H^-) ≤ μ(∥x∥). In the special case where the measure is the Lebesgue
measure and that where it is the counting measure (μ(A) = #A,B), H bisects a bounded Borel set
A ⊂ R^n if λ(A∩H^+) = λ(A∩H^-) = λ(A)/2, and bisects a finite set A ⊂ R^n if both #A∩H^+ ≤ #A/2
and #A∩H^- ≤ #A/2.

2. Bisecting discrete measures

Recall that a purely atomic (or discrete) measure in R^n (such as the classical binomial, Poisson,
geometric, and Bernoulli probability distributions) is a Borel measure μ = δ_a where: A is
a non-empty finite or countable subset of R^n (the atoms of μ); \{c_a\} are positive real numbers (the
masses of the atoms); and for each a ∈ A, δ_a is the Dirac measure defined by δ_a(B) = 1 if a ∈ B, and
Theorem 1. Let \( \mu_1, \ldots, \mu_n \) be purely atomic finite measures on \( \mathbb{R}^n \), with finitely many atoms. Then there exists a hyperplane \( H \) such that for all \( 1 \leq i \leq n \), \( H \) bisects \( \mu_i \) and \( \mu_i(H) > 0 \).

Although the bisection conclusion of Theorem 1 can be proved by first principles as in [8] using the Borsuk–Ulam theorem, the next lemma, an immediate corollary of the general ham sandwich theorem, will facilitate its proof.

Lemma 2. Let \( \mu_1, \ldots, \mu_n \) be purely atomic measures on \( \mathbb{R}^n \) with finitely many atoms. Then there exists a hyperplane \( H \) that bisects \( \mu_i \), for all \( 1 \leq i \leq n \).

Proof of Theorem 1. Fix \( \epsilon > 0 \), and let \( \{ A_i \} \) be the sets of atoms of \( \{ \mu_i \} \), respectively. For each \( i \), \( 1 \leq i \leq n \), reduce the mass of one of the atoms of \( \mu_i \) by some positive amount less than \( \epsilon \) (and less than the mass of the smallest atom of \( \mu_i \)) such that for the new measure \( \mu_i \),

\[
\sum_{x \in S} \mu_i^*(x) \neq \sum_{x \in A_i \setminus S} \mu_i^*(x) \quad \text{for every } S \subset A_i.
\]

(This can clearly be done since each \( A_i \) is finite.)

Now apply Lemma 2 to \( \mu_1', \ldots, \mu_n' \). The resulting hyperplane \( H_i \) bisects each \( \mu_i' \) and must in fact contain an atom of each \( \mu_i' \) (which has the same atoms as \( \mu_i \), just of different mass), or it could not bisect it, because of (1). Since the collection of all such hyperplanes is represented by the compact set \( \{(u, c) \in \mathbb{R}^{n+1} : u \in \mathbb{R}^n, |c| \leq D\} \), where \( D = \max\{|x| : x \in \bigcup_{i=1}^n A_i\} \), every sequence of such hyperplanes contains a subsequence that converges to a hyperplane in that collection. Let \( \epsilon \) approach zero along some sequence \( \delta_k \) such that the corresponding hyperplanes \( H_k \) converge to say \( H \). Clearly \( H \) bisects each \( \mu_i \), and since each \( H_i \) contains an element of \( A_i \) for each \( i \) and the \( A_i \) are finite, by passing to a subsequence it may be assumed that there is an atom of \( \mu_i \) for each \( i = 1, \ldots, n \) which belongs to all the \( H_k \), and hence to \( H \). Thus \( \mu_i(H) > 0 \), for each \( 1 \leq i \leq n \).

An easy discrete analog of Example 6 below shows that the conclusion of Theorem 1 may fail if the measures have infinitely many atoms.

Corollary 3. For every collection \( A_1, \ldots, A_n \) of non-empty finite subsets of \( \mathbb{R}^n \), there is a hyperplane \( H \) such that for each \( 1 \leq i \leq n \), \( H \) bisects \( A_i \) and \( H \cap A_i \neq \emptyset \).

Proof. Let \( \{ \mu_i \} \) be the measures with atoms \( \{ A_i \} \), respectively, and masses of each atom equal to 1 (i.e., the counting measures of the \( \{ A_i \} \)). Apply Theorem 1.

As a simple real-life two-dimensional example of Theorem 1, sprinkle some salt and pepper on a table, any amount of each. Then there is always a grain of salt and a grain of pepper and a line through both grains that has at most half of the grains of salt on each side, and also at most half of the grains of pepper on each side.

If the points in the sets in the hypothesis of Corollary 3 are also in general position (i.e., no more than \( n \) points of \( A_1 \cup \cdots \cup A_n \) are contained in any hyperplane), then there exists a hyperplane that bisects each of the sets and passes through exactly one point in each set (cf. [11, Corollary 3.1.3]).
3. Bisecting general measures

The proof of the next theorem for general finite measures only assumes the existence of a bisecting hyperplane for the case of purely atomic measures with finitely many atoms (Lemma 2 above), so it also gives a proof of the existence of bisecting hyperplanes in the general case with the Borsuk–Ulam theorem having only been used for the purely atomic case of finitely many atoms.

**Theorem 4.** For every collection $\mu_1, \ldots, \mu_n$ of compactly supported positive finite Borel measures on $\mathbb{R}^n$ there exists a hyperplane $H$ such that for each $1 \leq i \leq n$, $H$ bisects $\mu_i$ and $H \cap \text{supp}(\mu_i) \neq \emptyset$.

**Proof.** Let $C$ be a finite closed cube containing $\text{supp}(\mu_i)$ for all $i$, and fix $\epsilon > 0$. Let $P$ be a partition of $C$ into cubes (not necessarily closed or open) of diameter less than $\epsilon$, and for each small cube $c \in P$ let $x_c$ be the centroid of $c$. For each $i$, let $v_i$ be the purely atomic measure such that for $c \in P$, $v_i(x_c) = \mu_i(c)$, and the only atoms are the $(x_c)$. That is, approximate the measures with purely atomic ones by concentrating all the mass at the centroids of the cubes, for those cubes in the partition which have non-zero mass. By Theorem 1, there is a hyperplane $H = H_c$ such that for all $i$, $H$ bisects $v_i$ and $x_c \in H$ for some $x_c$ with $v_i(x_c) > 0$, so some point of support of $\mu_i$ lies within distance $\epsilon$ of $H$; that is, $d(H, \text{supp}(\mu_i)) < \epsilon$. Let $A^+ = \cup\{c \in P : c \subset H^+\}$, so $A^+$ is the union of the cubes of the partition that are entirely contained in $H^+$. Note that $\mu_i(A^+) = \mu_i(A^+), \|\mu_i\| = \|v_i\|$, and since $H$ bisects $v_i$, $\mu_i(A^+) \leq v_i(H^+) \leq \|v_i\|/2 = \|\mu_i\|/2$. Note that any point in $H^+ \cap C$ whose distance from $H$ is greater than or equal to $\epsilon$ belongs to $A^-$. $A^-$ is defined similarly; and similarly, $\mu_i(A^-) \leq \|\mu_i\|/2$.

Now let $\epsilon = 1/k, k = 1, 2, 3, \ldots$, and let $H_k, A^+_k$, and $A^-_k$ correspond to $H, A^+, A^-$ above. Since $C$ is compact, by passing to a subsequence if necessary, it may be assumed that the hyperplanes $H_k$ converge to a hyperplane $H$ in such a way that $d(H_k \cap C, H) < 1/k$, and $(u_k, c_k) \to (u, c)$ where $(u_k, c_k) \in \mathbb{R}^{n+1}$ and $(u, c) \in \mathbb{R}^{n+1}$ represent $H_k$ and $H$, respectively, as in the earlier definition of hyperplanes. Also, $d(H_k, \text{supp}(\mu_i)) < 1/k, \mu_i(A^+_k) \leq \|\mu_i\|/2$, and $\mu_i(A^-_k) \leq \|\mu_i\|/2$ for all $i$.

Note that $H^+ \subset (H^+ \setminus A^+_k) \cup A^+_k$, so $\mu_i(H^+) \leq \|\mu_i\|/2 + \mu_i(H^+ \setminus A^+_k)$ for all $k$. It will be shown below that the sets $(H^+ \cap C) \setminus A^+_k \to \emptyset$, so from the continuity theorem for measures, $\mu_i((H^+ \cap C) \setminus A^+_k) = \mu_i((H^+ \cap C) \setminus A^+_k) \to 0$, and therefore $\mu_i(H^+) \leq \|\mu_i\|/2$; and from $d(H_k, \text{supp}(\mu_i)) < 1/k$ it follows that $H \cap \text{supp}(\mu_i) \neq \emptyset$ since $\text{supp}(\mu_i)$ is closed. Similarly, $\mu_i(H^-) \leq \|\mu_i\|/2$. This will finish the proof, once it is shown that $(H^+ \cap C) \setminus A^+_k \to \emptyset$.

Now $H^+ \setminus H^+_k \to \emptyset$, because $(u, x) > \epsilon \Rightarrow (u, x) > c_k$ for sufficiently large $k$, so $x \in H^+ \Rightarrow x \in H^+_k$ for sufficiently large $k$. Suppose $x \in (H^+ \cap C \setminus H^+) \setminus A^+_k$. Then since $x \notin A^+_k, \ d(x, H_k) < 1/k$ from the definition of $A^+_k$. Since $d(H_k \cap C, H) < 1/k$, it follows that $d(x, H) < 2/k$. So if $x \in (H^+ \cap C \setminus H^+) \setminus A^+_k$, for infinitely many $k$, it would follow that $x \in H$, which is impossible since $H$ is disjoint from $H^+$. So $(H^+ \cap C \setminus H^+) \setminus A^+_k \to \emptyset$ also.

Thus, since $(H^+ \cap C) \setminus A^+_k \subset (H^+ \setminus H^+_k) \cup (H^+_k \cap C \setminus H^+) \setminus A^+_k$, $(H^+ \cap C) \setminus A^+_k \to \emptyset$ as claimed.

Without the assumption of compact support of the measures, the conclusion may fail, as is shown in Example 6 below.

Let $\mu_1, \mu_2, \mu_3$ denote the planetary, lunar, and asteroidal mass measures in our solar system, e.g., $\mu_1(B)$ is the planetary mass of the three-dimensional ball $B$, etc., Theorem 4 implies that at any given instant of time, there is one planet, one moon and one asteroid in our solar system and a single plane touching all three that exactly bisects the total planetary mass, the total lunar mass, and the total asteroidal mass of the solar system. (Note that different objects may have different mass densities, and even non-uniform mass densities, so this conclusion does not follow from the next corollary, which is a direct strengthening of the classical ham sandwich theorem.)

**Corollary 5.** For every collection $A_1, \ldots, A_n$ of $n$ bounded Borel subsets of $\mathbb{R}^n$ of positive Lebesgue measure, there exists a hyperplane $H$ such that for each $1 \leq i \leq n$, $H$ bisects $A_i$ and $H \cap A_i \neq \emptyset$.

**Proof.** The conclusion follows immediately from Theorem 4 by letting $\mu_1, \ldots, \mu_n$ be the finite Borel measures on $\mathbb{R}^n$ defined by $\mu_i(B) = \lambda(B \cap A_i)$ for all Borel sets $B \subset \mathbb{R}^n$, and observing that $\text{supp}(\mu_i) \subset A_i$. 

There may not exist bisecting hyperplanes that intersect the sets themselves, and not every bisecting hyperplane may intersect the closures of each of the sets, even in the one-dimensional setting, as can easily be seen by looking at the sets $(0, 1) \cup (1, 2)$ and $(0, 1) \cup (2, 3)$, respectively. If the sets are not bounded, the Borsuk–Ulam theorem can still be used to guarantee the existence of a bisecting hyperplane (cf. [3, Theorem 1]), but the bisection-and-intersection conclusion may fail in dimensions higher than 1 if the sets are not bounded.

**Example 6.** Let $n = 2$, and let $A_1$ and $A_2$ be the closed unbounded sets, each of area 1, given by $A_1 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, 1 \leq y \leq 1 + e^{-x}/2 \} \cup \{(x, y) \in \mathbb{R}^2 : x \geq 0, -1 \geq y \geq -1 - e^{-x}/2 \}$, and $A_2 = \{(x, y) \in \mathbb{R}^2 : x \leq 0, 2 \leq y \leq 2 + e^x/2 \} \cup \{(x, y) \in \mathbb{R}^2 : x \leq 0, -2 \geq y \geq -2 - e^x/2 \}$. It is easy to check that the only lines that bisect both sets simultaneously are horizontal lines with height in $[-1, 1]$. But none of these intersects $A_2$.

**Acknowledgements**

The authors are grateful to Professor Robert Burckel and a referee for several helpful suggestions.

**References**


