A stronger conclusion to the classical ham sandwich theorem

John H. Elton, Theodore P. Hill

A B S T R A C T

The conclusion of the classical ham sandwich theorem for bounded Borel sets may be strengthened, without additional hypotheses – there always exists a common bisecting hyperplane that touches each of the sets, that is, that intersects the closure of each set. In the discrete setting, where the sets are finite (and the measures are counting measures), there always exists a bisecting hyperplane that contains at least one point in each of the sets. Both these results follow from the main theorem of this note, which says that for n compactly supported positive finite Borel measures in $\mathbb{R}^n$, there is always an $(n-1)$-dimensional hyperplane that bisects each of the measures and intersects the support of each measure. Thus, for example, at any given instant of time, there is one planet, one moon and one asteroid in our solar system and a single plane touching all three that exactly bisects the total planetary mass, the total lunar mass, and the total asteroidal mass of the solar system. In contrast to the bisection conclusion of the classical ham sandwich theorem, this bisection-and-intersection conclusion does not carry over to unbounded sets of finite measure.

1. Introduction

The classical ham sandwich theorem says that every collection of n bounded Borel sets in $\mathbb{R}^n$ can be simultaneously bisected in Lebesgue measure by a single hyperplane. The purpose of this short note is to show that this conclusion may be strengthened, without additional hypotheses — there always exists a common bisecting hyperplane that touches each of the sets, that is, that intersects the closure of each set. This is a corollary of the main result, Theorem 4 below, which says that for compactly supported positive finite Borel measures, there is always a hyperplane that bisects each of the measures and intersects the support of each measure. In contrast to the bisection conclusion of the classical ham sandwich theorem, this bisection-and-intersection conclusion does not carry over to unbounded sets of finite measure; see Example 6 below.

Many extensions and generalizations of the classical ham sandwich theorem are well known. The conclusion for $n = 3$ was apparently conjectured by Steinhaus, and first proved by Banach (cf. [15,3]). Stone and Tukey [16] extended the conclusion to $n > 3$, to general continuous finite measures, and to certain general classes of surfaces that include hyperplanes. For measures with atoms (positive masses concentrated on points), "bisection" is interpreted as in the definition below, and, in the special case of counting measure, if all the sets consist of finitely many points each of unit mass, then there always exists a hyperplane that simultaneously bisects all the sets (e.g. [11,13,14]). Simultaneous bisection of general finite measures, which may have both continuous components and atoms with general masses, was established in [5,8]. In some settings, algorithms for constructing the bisecting hyperplane have been found (e.g. [7,10,14]). In general, 1:1 is the only ratio that any two compact sets in the plane can be simultaneously bisected by a line in that ratio [11, Example 3.2.4], but sufficient conditions for other ratios have recently been found [1,4].

An interesting generalization of the classical ham sandwich theorem to $k+1 \leq n$ sets (cf. [6,11,17]) says that for each of the $k+1$ mass distributions in $\mathbb{R}^n$, there exists a k-dimensional affine subspace A such that any hyperplane containing A has at most $(n-k+1)^{-1}$ of each mass on each side; the classical case is $k = n - 1$. In another direction, several authors (e.g. [2,9,12]) have shown that any two mass distributions in the plane can be simultaneously equipartitioned by three rays emanating from a single point. Similar results for equipartitioning a single mass distribution, such as partitioning a single mass distribution in the plane into four equal parts by using two lines, have also been found (cf. [2,18,19]). More details and references on various extensions and generalizations of the ham sandwich theorem may be found in [11].

Notation. Fix $n \in \mathbb{N}$, and for $x, y \in \mathbb{R}^n$, let $|x|$ denote the Euclidean norm of x. For subsets A and B of $\mathbb{R}^n$, let $d(A, B)$ denote the Euclidean distance between A and B, i.e. $d(A, B) = \inf\{|x-y| : x \in A, y \in B\}$. Recall that every hyperplane H in $\mathbb{R}^n$ may be represented by $(u, c) \in \mathbb{R}^{n+1}$ via the relationship $x \in H \Leftrightarrow \langle u, x\rangle = c$, where u is a point in the unit n-sphere $\mathbb{S}^n$, $c \geq 0$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$. For the hyperplane H determined by (u, c), let H^+ denote the open half-space defined by $x \in H^+ \Leftrightarrow \langle u, x\rangle > c$ and let H^- denote the open half-space $x \in H^- \Leftrightarrow \langle u, x\rangle < c$.

For a bounded Borel set $A \subset \mathbb{R}^n$, let $\#\{A\}$ denote the cardinality of A, $\lambda(A)$ the Lebesgue measure (n-dimensional volume) of A, and $\bar{A}$ the closure of A. For a finite Borel measure μ on $\mathbb{R}^n$, $\|\mu\| = \mu(\mathbb{R}^n)$ denotes the total mass of μ, and $\text{supp}(\mu)$ the support of μ (the smallest closed set $C \subset \mathbb{R}^n$ such that $\mu(C) = \|\mu\|$, i.e. $C = \cap\{\tilde{C} \subset \mathbb{R}^n : \tilde{C}$ is closed and $\mu(\tilde{C}) = \|\mu\|\}$). A measure μ is *positive* if $\|\mu\| > 0$ and is *finite* if $\|\mu\| < \infty$.

Note that to be able to treat measures where a hyperplane may have positive measure, bisection of a positive finite measure is defined to mean that no more than half the mass of the measure lies on either side of the hyperplane (not including the hyperplane); this is equivalent to the hyperplane being a median for the normalized measure (see [8]).

Definition. An $(n-1)$-dimensional *hyperplane* H in $\mathbb{R}^n$ *bisects a positive finite Borel measure* μ on $\mathbb{R}^n$ if $\mu(H^+) \leq \|\mu\|/2$ and $\mu(H^-) \leq \|\mu\|/2$. In the special case where the measure is the Lebesgue measure and that where it is the counting measure ($\mu_A(B) = \#\{A \cap B\}$), H *bisects a bounded Borel set* $A \subset \mathbb{R}^n$ if $\lambda(A \cap H^+) = \lambda(A \cap H^-) = \lambda(A)/2$, and *bisects a finite set* $A \subset \mathbb{R}^n$ if both $\#\{A \cap H^+\} \leq \#\{A\}/2$ and $\#\{A \cap H^-\} \leq \#\{A\}/2$.

2. Bisecting discrete measures

Recall that a *purely atomic* (or discrete) measure in $\mathbb{R}^n$ (such as the classical binomial, Poisson, geometric, and Bernoulli probability distributions) is a Borel measure $\mu = \sum_{a \in A} c_a \delta_a$, where: A is a non-empty finite or countable subset of $\mathbb{R}^n$ (the atoms of μ); $\{c_a\}$ are positive real numbers (the masses of the atoms); and for each $a \in A$, δ_a is the Dirac measure defined by $\delta_a(B) = 1$ if $a \in B$, and

$\delta_a(B) = 0$ if $a \notin B$. Note that $\|\mu\| = \sum_{a\in A} c_a$, so a purely atomic measure μ is always positive since $\sum_{a\in A} c_a > 0$, and is finite if $\sum_{a\in A} c_a < \infty$.

The conclusion of the next theorem is known implicitly in some cases; if the measures are counting measures and there are an odd number of points in each of the finite sets, then every bisecting hyperplane must pass through at least one point in each set. Then note that adding a single point to any of the sets does not destroy the bisection of that set by the same hyperplane. For general discrete (purely atomic) finite measures, the proof will use this idea with a simple perturbation argument.

Theorem 1. *Let $\mu_1, \ldots, \mu_n$ be purely atomic finite measures on $\mathbb{R}^n$, with finitely many atoms. Then there exists a hyperplane H such that for all $1 \le i \le n$, H bisects μ_i and $\mu_i(H) > 0$.*

Although the bisection conclusion of Theorem 1 can be proved by first principles as in [8] using the Borsuk–Ulam theorem, the next lemma, an immediate corollary of the general ham sandwich theorem, will facilitate its proof.

Lemma 2. *Let $\mu_1, \ldots, \mu_n$ be purely atomic measures on $\mathbb{R}^n$ with finitely many atoms. Then there exists a hyperplane H that bisects μ_i for all $1 \le i \le n$.*

Proof of Theorem 1. Fix $\varepsilon > 0$, and let $\{A_i\}$ be the sets of atoms of $\{\mu_i\}$, respectively. For each i, $1 \le i \le n$, reduce the mass of one of the atoms of μ_i by some positive amount less than ε (and less than the mass of the smallest atom of μ_i) such that for the new measure μ_i',

$$\sum_{x\in S} \mu_i'(x) \neq \sum_{x\in A_i\setminus S} \mu_i'(x) \quad \text{for every } S \subset A_i. \tag{1}$$

(This can clearly be done since each A_i is finite.)

Now apply Lemma 2 to $\mu_1', \ldots, \mu_n'$. The resulting hyperplane H_ε bisects each μ_i' and must in fact contain an atom of each μ_i' (which has the same atoms as μ_i, just of different mass), or it could not bisect it, because of (1). Since the collection of all such hyperplanes is represented by the compact set $\{(u, c) \in \mathbb{R}^{n+1} : u \in \mathbb{S}^n, |c| \le D\}$, where $D = \max\{|x| : x \in \bigcup_{i=1}^n A_i\}$, every sequence of such hyperplanes contains a subsequence that converges to a hyperplane in that collection. Let ε approach zero along some sequence ε_k such that the corresponding hyperplanes H_k converge to say H. Clearly H bisects each μ_i, and since each H_k contains an element of A_i for each i and the A_i are finite, by passing to a subsequence it may be assumed that there is an atom of μ_i for each $i = 1, \ldots, n$ which belongs to all the H_k, and hence to H. Thus $\mu_i(H) > 0$, for each $1 \le i \le n$.

An easy discrete analog of Example 6 below shows that the conclusion of Theorem 1 may fail if the measures have infinitely many atoms.

Corollary 3. *For every collection $A_1, \ldots, A_n$ of non-empty finite subsets of $\mathbb{R}^n$, there is a hyperplane H such that for each $1 \le i \le n$, H bisects A_i and $H \cap A_i \neq \emptyset$.*

Proof. Let $\{\mu_i\}$ be the measures with atoms $\{A_i\}$, respectively, and masses of each atom equal to 1 (i.e., the counting measures of the $\{A_i\}$). Apply Theorem 1.

As a simple real-life two-dimensional example of Theorem 1, sprinkle some salt and pepper on a table, any amount of each. Then there is always a grain of salt and a grain of pepper and a line through both grains that has at most half of the grains of salt on each side, and also at most half of the grains of pepper on each side.

If the points in the sets in the hypothesis of Corollary 3 are also in general position (i.e., no more than n points of $A_1 \cup \cdots \cup A_n$ are contained in any hyperplane), then there exists a hyperplane that bisects each of the sets and passes through *exactly* one point in each set (cf. [11, Corollary 3.1.3]).

3. Bisecting general measures

The proof of the next theorem for general finite measures only assumes the existence of a bisecting hyperplane for the case of purely atomic measures with finitely many atoms (Lemma 2 above), so it also gives a proof of the existence of bisecting hyperplanes in the general case with the Borsuk–Ulam theorem having only been used for the purely atomic case of finitely many atoms.

Theorem 4. *For every collection $\mu_1, \ldots, \mu_n$ of compactly supported positive finite Borel measures on $\mathbb{R}^n$ there exists a hyperplane H such that for each $1 \le i \le n$, H bisects μ_i and $H \cap \operatorname{supp}(\mu_i) \neq \emptyset$.*

Proof. Let C be a finite closed cube containing $\operatorname{supp}(\mu_i)$ for all i, and fix $\varepsilon > 0$. Let P be a partition of C into cubes (not necessarily closed or open) of diameter less than ε, and for each small cube $c \in P$ let x_c be the centroid of c. For each i, let υ_i be the purely atomic measure such that for $c \in P$, $\upsilon_i(x_c) = \mu_i(c)$, and the only atoms are the $\{x_c\}$. That is, approximate the measures with purely atomic ones by concentrating all the mass at the centroids of the cubes, for those cubes in the partition which have non-zero mass. By Theorem 1, there is a hyperplane $H = H_\varepsilon$ such that for all i, H bisects υ_i, and $x_c \in H$ for some x_c with $\upsilon_i(x_c) > 0$, so some point of support of μ_i lies within distance ε of H; that is, $d(H, \operatorname{supp}(\mu_i)) < \varepsilon$. Let $A^+ = \cup\{c \in P : c \subset H^+\}$, so A^+ is the union of the cubes of the partition that are entirely contained in H^+. Note that $\mu_i(A^+) = \upsilon_i(A^+)$, $\|\mu_i\| = \|\upsilon_i\|$, and since H bisects υ_i, $\mu_i(A^+) \le \upsilon_i(H^+) \le \|\upsilon_i\|/2 = \|\mu_i\|/2$. Note that any point in $H^+ \cap C$ whose distance from H is greater than or equal to ε belongs to A^+. A^- is defined similarly; and similarly, $\mu_i(A^-) \le \|\mu_i\|/2$.

Now let $\varepsilon = 1/k$, $k = 1, 2, 3 \ldots$, and let H_k, A_k^+, and A_k^- correspond to H, A^+, and A^- above. Since C is compact, by passing to a subsequence if necessary, it may be assumed that the hyperplanes H_k converge to a hyperplane H in such a way that $d(H_k \cap C, H) < 1/k$, and $(u_k, c_k) \to (u, c)$ where $(u_k, c_k) \in \mathbb{R}^{n+1}$ and $(u, c) \in \mathbb{R}^{n+1}$ represent H_k and H, respectively, as in the earlier definition of hyperplanes. Also, $d(H_k, \operatorname{supp}(\mu_i)) < 1/k$, $\mu_i(A_k^+) \le \|\mu_i\|/2$, and $\mu_i(A_k^-) \le \|\mu_i\|/2$ for all i.

Note that $H^+ \subset (H^+ \setminus A_k^+) \cup A_k^+$, so $\mu_i(H^+) \le \|\mu_i\|/2 + \mu_i(H^+ \setminus A_k^+)$ for all k. It will be shown below that the sets $(H^+ \cap C) \setminus A_k^+ \to \emptyset$, so from the continuity theorem for measures, $\mu_i(H^+ \setminus A_k^+) = \mu_i((H^+ \cap C) \setminus A_k^+) \to 0$, and therefore $\mu_i(H^+) \le \|\mu_i\|/2$; and from $d(H_k, \operatorname{supp}(\mu_i)) < 1/k$ it follows that $H \cap \operatorname{supp}(\mu_i) \neq \emptyset$ since $\operatorname{supp}(\mu_i)$ is closed. Similarly $\mu_i(H^-) \le \|\mu_i\|/2$. This will finish the proof, once it is shown that $(H^+ \cap C) \setminus A_k^+ \to \emptyset$.

Now $H^+ \setminus H_k^+ \to \emptyset$, because $\langle u, x\rangle > c \Rightarrow \langle u_k, x\rangle > c_k$ for sufficiently large k, so $x \in H^+ \Rightarrow x \in H_k^+$ for sufficiently large k. Suppose $x \in (H_k^+ \cap C \cap H^+) \setminus A_k^+$. Then since $x \notin A_k^+$, $d(x, H_k) < 1/k$ from the definition of A_k^+. Since $d(H_k \cap C, H) < 1/k$, it follows that $d(x, H) < 2/k$. So if $x \in (H_k^+ \cap C \cap H^+) \setminus A_k^+$, for infinitely many k, it would follow that $x \in H$, which is impossible since H is disjoint from H^+. So $(H_k^+ \cap C \cap H^+) \setminus A_k^+ \to \emptyset$ also.

Thus, since $(H^+ \cap C) \setminus A_k^+ \subset (H^+ \setminus H_k^+) \cup (H_k^+ \cap C \cap H^+) \setminus A_k^+$, $(H^+ \cap C) \setminus A_k^+ \to \emptyset$ as claimed.

Without the assumption of compact support of the measures, the conclusion may fail, as is shown in Example 6 below.

Letting μ_1, μ_2, μ_3 denote the planetary, lunar, and asteroidal mass measures in our solar system, e.g., $\mu_1(B)$ is the planetary mass of the three-dimensional ball B, etc., Theorem 4 implies that at any given instant of time, there is one planet, one moon and one asteroid in our solar system and a single plane touching all three that exactly bisects the total planetary mass, the total lunar mass, and the total asteroidal mass of the solar system. (Note that different objects may have different mass densities, and even non-uniform mass densities, so this conclusion does not follow from the next corollary, which is a direct strengthening of the classical ham sandwich theorem.)

Corollary 5. *For every collection $A_1, \ldots, A_n$ of n bounded Borel subsets of $\mathbb{R}^n$ of positive Lebesgue measure, there exists a hyperplane H such that for each $1 \le i \le n$, H bisects A_i and $H \cap \bar{A}_i \neq \emptyset$.*

Proof. The conclusion follows immediately from Theorem 4 by letting $\mu_1, \ldots, \mu_n$ be the finite Borel measures on $\mathbb{R}^n$ defined by $\mu_i(B) = \lambda(B \cap A_i)$ for all Borel sets $B \subset \mathbb{R}^n$, and observing that $\operatorname{supp}(\mu_i) \subset \bar{A}_i$.

There may not exist bisecting hyperplanes that intersect the sets themselves, and not every bisecting hyperplane may intersect the closures of each of the sets, even in the one-dimensional setting, as can easily be seen by looking at the sets $(0, 1) \cup (1, 2)$ and $(0, 1) \cup (2, 3)$, respectively. If the sets are not bounded, the Borsuk–Ulam theorem can still be used to guarantee the existence of a bisecting hyperplane (cf. [3, Theorem 1]), but the bisection-and-intersection conclusion may fail in dimensions higher than 1 if the sets are not bounded.

Example 6. Let $n = 2$, and let A_1 and A_2 be the closed unbounded sets, each of area 1, given by $A_1 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, 1 \leq y \leq 1 + e^{-x}/2\} \cup \{(x, y) \in \mathbb{R}^2 : x \geq 0, -1 \geq y \geq -1 - e^{-x}/2\}$, and $A_2 = \{(x, y) \in \mathbb{R}^2 : x \leq 0, 2 \leq y \leq 2 + e^{x}/2\} \cup \{(x, y) \in \mathbb{R}^2 : x \leq 0, -2 \geq y \geq -2 - e^{x}/2\}$. It is easy to check that the only lines that bisect both sets simultaneously are horizontal lines with height in $[-1, 1]$. But none of these intersects A_2.

Acknowledgements

The authors are grateful to Professor Robert Burckel and a referee for several helpful suggestions.

References

[1] I. Bárány, A. Hubard, J. Jerónimo, Slicing convex sets and measures by a hyperplane, Discrete Comput. Geom. 39 (1–3) (2008) 67–75.
[2] S. Bespamyatnikh, D. Kirkpatrick, J. Snoeyink, Generalizing ham sandwich cuts to equitable subdivisions, Discrete Comput. Geom. 24 (2000) 605–622.
[3] W. Beyer, A Zardecki, The early history of the ham sandwich theorem, Amer. Math. Monthly 111 (2004) 58–61.
[4] F. Breuer, Uneven splitting of ham sandwiches, Discrete Comput. Geom. 43 (4) (2010) 876–892.
[5] G. Cox, R. McKelvey, A ham sandwich theorem for general measures, Soc. Choice Welf. 1 (1) (1984) 75–83.
[6] V. Dol'nikov, A generalization of the ham sandwich theorem, Math. Notes 52 (2) (1992) 771–779.
[7] H. Edelsbrunner, Algorithms in Combinatorial Geometry, Springer, New York, 1987.
[8] T.P. Hill, Common hyperplane medians for random vectors, Amer. Math. Monthly 95 (1988) 437–441.
[9] H. Ito, H. Uehara, M. Yokoyama, 2-dimension ham sandwich theorem for partitioning into three convex pieces, in: Proc. Japan. Conf. Discrete Comput. Geom., JCDCG'98, in: Lecture Notes in Computer Science, vol. 1763, Springer-Verlag, Berlin, 2000, pp. 129–157.
[10] C.-Y. Lo, J. Matoušek, W. Steiger, Algorithms for ham-sandwich cuts, Discrete Comput. Geom. 11 (1994) 433–452.
[11] J. Matoušek, Using the Borsuk–Ulam Theorem, Springer, Berlin, 2003.
[12] T. Sakai, Balanced convex partitions of measures in R^2, Graphs Combin. 18 (1) (2002) 169–192.
[13] J. Spencer, Discrete ham sandwich theorems, European J. Combin. 2 (1981) 291–298.
[14] W. Steiger, J. Zhao, Generalized ham-sandwich cuts, Discrete Comput. Geom. 44 (3) (2010) 535–545.
[15] H. Steinhaus, A note on the ham sandwich theorem, Mathesis Polska 9 (1938) 26–28.
[16] A.H. Stone, J.W. Tukey, Generalized "sandwich" theorems, Duke Math. J. 9 (1942) 356–359.
[17] H. Tverberg, S. Vrecica, On generalizations of Radon's theorem and the ham sandwich theorem, European J. Combin. 14 (1993) 259–264.
[18] S. Vrecica, R. Zivaljevic, Conical equipartitions of mass distributions, Discrete Comput. Geom. 25 (3) (2001) 335–350.
[19] S. Vrecica, R. Zivaljevic, The ham sandwich theorem revisited, Israel J. Math. 78 (1992) 21–32.