EXPECTATION INEQUALITIES ASSOCIATED
WITH PROPHET PROBLEMS

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ABSTRACT

Applications of the original prophet inequalities of Krengel and Sucheston are made to problems of order selection, non-measurable stop rules, look-ahead stop rules, and iterated maps of random variables. Also, proofs are given of two results of Hill and Hordijk concerning optimal orderings of uniform and exponential distributions.

§1. INTRODUCTION

Universal inequalities comparing the two functionals

$$M = M(X_1, X_2, \dots) = E(\sup_n X_n)$$

and

$$V = V(X_1, X_2, \dots) = \sup\{EX_t : t \text{ is a stop rule for } X_1, X_2, \dots\}$$

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of sequences of random variables are called "prophet inequalities" because of the natural interpretation of M as the value to a prophet, or player with complete foresight, in an optimal stopping problem involving random variables $X_1, X_2, \dots$. First discovered by Krengel and Sucheston [22, 23], these inequalities have been the subject of a number of recent investigations (e.g., [1, 2, 4-15, 17-21, 24-27]).

In §2, the applications of prophet inequalities to inequalities involving functionals other than M or V are given, with attention focused on the fundamental prophet inequality [23]

- (1) If $X_1, X_2, \dots$ are independent and nonnegative, then $M \leq 2V$, and this bound is sharp.

(Analogous applications of other prophet inequalities to similar problems are left to the reader.)

Section 3 contains proofs of two optimal-ordering results of Hill and Hordijk [11].

§2. APPLICATIONS OF PROPHET INEQUALITIES

The initial discovery and application of prophet inequalities such as (1) were made by Krengel and Sucheston in conjunction with investigations of semi-amarts and processes with finite value [22, 23]. In this section, other applications of the basic inequality (1) are given to several optimal-stopping problems and an iterated map problem.

For the main application theorem, which follows immediately from (1), let

$$U = U(X_1, X_2, \dots)$$

be any (real-valued) functional of $X_1, X_2, \dots$. (More formally, U is a function from C , the set of infinite

sequences of probability distributions, to the real numbers. In practice, U is usually Borel measurable, with C endowed with the product topology induced by the total-variation norm topology on the space of probability distributions.)

Theorem 2.1. Let $X_1, X_2, \dots$ be independent nonnegative random variables. Then

- (i) $V \leq U$ implies $U \leq 2V$; and
- (ii) $U \leq M$ implies $M \leq 2U$.

Proof. Immediate from (1). □

Application to Order Selection

Let U_S be the value of the sequence $X_1, X_2, \dots$ to a player free to choose the order of observation of the random variables, as well as the time of stopping, that is,

$$U_S = U_S(X_1, X_2, \dots) = \sup\{V(X_{\pi(1)}, X_{\pi(2)}, \dots) : \pi \text{ is a permutation of } 1, 2, \dots\}.$$

(For a formal definition, including stochastic permutations π , see [9].)

Corollary 2.2. Let $X_1, X_2, \dots$ be independent nonnegative random variables. Then

- (i) (Hill [9]) $U_S \leq 2V$; and
- (ii) $M \leq 2U_S$.

Moreover, the bound in (i) is sharp.

(Whether or not the constant "2" in (ii) is a sharp bound is not known to the author.) Inequality (i) says that a player may never do better than double his expected value by rearranging the order of a given sequence of random variables. Inequality (ii) is immediate from (1) and the fact that $U_S \geq V$; only the question of its sharpness is of interest.

Application to Use of Non-Measurable Stop Rules

Let U_N be the value of the sequence $X_1, X_2, \dots$ to a player free to use non-measurable stop rules, i.e., integer-valued functions s for which $\{s = j\}$ can be any (not necessarily measurable) function of $X_1, \dots, X_j$. That is, U_N is the functional

$$U_N = U_N(X_1, X_2, \dots)$$

$$= \sup_s \{EX_s : s \text{ is a "non-measurable" stop rule}\}.$$

(For a formal definition, see [16].)

Corollary 2.3. Let $X_1, X_2, \dots$ be independent nonnegative random variables. Then

(i) (Hill and Pestien [16]) $U_N \leq 2V$; and

(ii) $M \leq 2U_N$.

Moreover both bounds are sharp.

Proof. The inequalities follow immediately from (1); the sharpness of (i) is in [16]. To see that the bound in (ii) is sharp, let X_1 be constant +1, and let X_2 be a "long shot" [12] given by $P(X_2 = \epsilon^{-1}) = \epsilon = 1 - P(X_2 = 0)$. Then $M = 2 - \epsilon$, and $U_N = U_S = 1$. $\square$

Application to "Look-Ahead" Stop Rules

Let $U_{A,k}$ be the value of the sequence $X_1, X_2, \dots$ to a player free to use stop rules s which allow looking ahead k steps (i.e., integer-valued measurable functions satisfying $\{s = j\} \in \sigma(X_1, \dots, X_{j+k})$), so

$$U_{A,k} = U_{A,k}(X_1, X_2, \dots)$$

$$= \sup_s \{EX_s : s \text{ is a } k\text{-step "look-ahead" stop rule}\}.$$

Corollary 2.4. Let $X_1, X_2, \dots$ be independent nonnegative random variables, and let k be a positive integer.

Then

(i) $U_{A,k} \leq 2V$; and

(ii) $M \leq 2U_{A,k}$.

Moreover, both bounds are sharp.

Proof. The inequalities follow immediately from (1).

To see that (i) is sharp, let $X_1 = \text{constant } +1$, $X_2 = \dots = X_{k+1} = \text{constant } 0$, and let X_{k+2} be a "long shot" with $P(X_{k+2} = \epsilon^{-1}) = \epsilon = 1 - P(X_{k+2} = 0)$; then $U_{A,k} = 2-\epsilon$ and $V = 1$. To see that (i) is sharp, let

$X_1 \equiv +1$, $X_2 \equiv \dots \equiv X_{k+2} = 0$, and let X_{k+3} be the "long shot" random variable just described; then $M = 2-\epsilon$ and $U_{A,k} = 1$. □

Thus (i) says that a player able to look k steps into the future never has optimal expected return more than twice that of a player who cannot look ahead, and (ii) says that a prophet's optimal expected return is never more than twice that of a player who may look a fixed number of steps into the future. On the other hand, for a fixed sequence of random variables, it is clear that

$$\lim_{k \rightarrow \infty} U_{A,k}(X_1, X_2, \dots) = M(X_1, X_2, \dots).$$

Application to Iterated Maps

Let $\phi(X, Y)$ and $\psi(X, Y)$ be the random variables $\phi(X, Y) = \max\{X, Y\}$ and $\psi(X, Y) = \max\{X, EY\}$, and define the random variables $\phi_n(X_n, \dots, X_1)$ and $\psi_n(X_n, \dots, X_1)$ inductively by

$$\begin{aligned} \phi_2(X_2, X_1) &= \phi(X_2, X_1), \quad \text{and} \\ \phi_k(X_k, \dots, X_1) &= \phi(X_k, \phi_{k-1}(X_{k-1}, \dots, X_1)) \end{aligned}$$

and

$$\begin{aligned} \psi_2(X_2, X_1) &= \psi(X_2, X_1), \quad \text{and} \\ \psi_k(X_k, \dots, X_1) &= \psi(X_k, \psi_{k-1}(X_{k-1}, \dots, X_1)). \end{aligned}$$

Application to Use of Non-Measurable Stop Rules

Let U_N be the value of the sequence $X_1, X_2, \dots$ to a player free to use non-measurable stop rules, i.e., integer-valued functions s for which $\{s = j\}$ can be any (not necessarily measurable) function of $X_1, \dots, X_j$. That is, U_N is the functional

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Application to "Look-Ahead" Stop Rules

Let $U_{A,k}$ be the value of the sequence $X_1, X_2, \dots$ to a player free to use stop rules s which allow looking ahead k steps (i.e., integer-valued measurable functions satisfying $\{s = j\} \in \sigma(X_1, \dots, X_{j+k})$), so

$$U_{A,k} = U_{A,k}(X_1, X_2, \dots)$$

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Thus (i) says that a player able to look k steps into the future never has optimal expected return more than twice that of a player who cannot look ahead, and (ii) says that a prophet's optimal expected return is never more than twice that of a player who may look a *fixed* number of steps into the future. On the other hand, for a fixed sequence of random variables, it is clear that

$$\lim_{k \rightarrow \infty} U_{A,k}(X_1, X_2, \dots) = M(X_1, X_2, \dots).$$

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Let $\phi(X, Y)$ and $\psi(X, Y)$ be the random variables $\phi(X, Y) = \max\{X, Y\}$ and $\psi(X, Y) = \max\{X, EY\}$, and define the random variables $\phi_n(X_n, \dots, X_1)$ and $\psi_n(X_n, \dots, X_1)$ inductively by

$$\begin{aligned} \phi_2(X_2, X_1) &= \phi(X_2, X_1), \quad \text{and} \\ \phi_k(X_k, \dots, X_1) &= \phi(X_k, \phi_{k-1}(X_{k-1}, \dots, X_1)) \end{aligned}$$

and

$$\begin{aligned} \psi_2(X_2, X_1) &= \psi(X_2, X_1), \quad \text{and} \\ \psi_k(X_k, \dots, X_1) &= \psi(X_k, \psi_{k-1}(X_{k-1}, \dots, X_1)). \end{aligned}$$

Then $E(\max\{X_1, \dots, X_n\}) = E(\phi_n(X_n, \dots, X_1))$, and $V(X_1, \dots, X_n) = E(\psi_n(X_n, \dots, X_1))$, so the finite version of (1) may be restated as

$$(2) \quad E[\phi_n(X_n, \dots, X_1)] \leq 2E[\psi_n(X_n, \dots, X_1)].$$

Corollary 2.5. Let $X_1, X_2, \dots$ be independent nonnegative random variables, let $g(X, Y)$ be such that $g \geq \psi$, and $g(X, Y) \geq g(X, \hat{Y})$ if $Y \geq \hat{Y}$ a.e. Define $g_n(X_n, \dots, X_1)$ inductively by $g_2(X_2, X_1) = g(X_2, X_1)$ and

$$g_k(X_k, \dots, X_1) = g(X_k, g_{k-1}(X_{k-1}, \dots, X_1)).$$

$$(3) \quad E(\max\{X_1, \dots, X_n\}) \leq 2E[g_n(X_n, \dots, X_1)].$$

Proof. Follows easily by (2) and induction. □

The iterated maps g_n need not closely resemble ordinary stopping theory functions, for example consider $g(X, Y) = \max\{X, \|Y\|_p\}$ for $p > 1$, or $g(X, Y) = (\max\{X, Y\} + \max\{X, EY\})/2$. Inequality (3) corresponds to the inequality $M \leq 2U$ in Theorem 2.1; the analog of (3) corresponding to $U \leq 2V$ is also possible under similar hypotheses.

§3. PROOFS OF TWO RESULTS IN ORDER SELECTION

The purpose of this section is to give proofs of two results, both concerning optimal stopping with order selection, which appear in [11] without proof.

Theorem 3.1 (4.6(ii) of [11]). Let $\alpha_1, \alpha_2, \dots$ be a sequence of non-increasing positive numbers, and let $X_1, X_2, \dots$ be independent random variables with distributions uniform on $[0, \alpha_1], [0, \alpha_2], \dots$ respectively. Then

$$V(X_1, X_2, \dots) = \sup\{V(X_{\pi(1)}, X_{\pi(2)}, \dots) :$$

π is a permutation of $\mathbb{N}\}$.

Proof. (due to Hordijk and Hill). The proof will be an application of Proposition 4.5 of [11]. By renormalizing, it suffices to show

$$(4) \quad V(X_1, X_\alpha, c) \geq V(X_\alpha, X_1, c)$$

for all $\alpha \in (0,1)$ and all $c \in \mathbb{R}$.

For a random variable T with values in $\{1,2,3\}$, let $R_T(X,Y,c) = X$ if $T = 1$; $= Y$ if $T = 2$; and $= c$ if $T = 3$. Also, let $T(X,Y,c) = 1$ if $X > E(\max\{Y,c\})$; $= 2$ if $X \leq E(\max\{Y,c\})$ and $Y > c$; and $= 3$ otherwise. Letting X and Y be i.i.d. $U[0,1]$, by Lemma 2.1 of [3] it follows that (4) is equivalent to

$$(5) \quad E[R_{T(X,\alpha Y,c)}(X,\alpha Y,c) - R_{T(\alpha Y,X,c)}(\alpha Y,X,c)] \geq 0.$$

To see (5), first observe that

$$(6) \quad E[R_{T(X,\alpha Y,c)}(X,\alpha Y,c) | X \in [0,\alpha]] \\ = E[R_{T(\alpha X,\alpha Y,c)}(\alpha X,\alpha Y,c)],$$

since the distribution of X given $X \in [0,\alpha]$ is uniform on $[0,\alpha]$, that is, has the same distribution as αX .

Next calculate

$$(7) \quad E[R_{T(\alpha Y,X,c)}(\alpha Y,X,c) | X \in [0,\alpha]] \\ = E[R_{T(\alpha Y,X,c)}(\alpha Y,\alpha X,c)] \\ \leq E[R_{T(\alpha Y,\alpha X,c)}(\alpha Y,\alpha X,c)],$$

where the first equality follows as in (6), and the inequality since $T(\alpha Y,\alpha X,c)$ is the optimal stop rule (by Lemma 2.1 of [3]) for $(\alpha Y,\alpha X,c)$. Together (6) and (7) imply

$$(8) \quad E[R_{T(X,\alpha Y,c)}(X,\alpha Y,c) - R_{T(\alpha Y,X,c)}(\alpha Y,X,c) | X \in [0,\alpha]] \geq 0 \text{ a.s.}$$

Similarly, conditioning on $X \in (\alpha,1]$ and using the fact that given $X \in (\alpha,1]$, the conditional distribution

of X is uniform on $(\alpha, 1]$, one has the following two relations:

$$(9) \quad E[R_{T(X, \alpha Y, c)}(\alpha Y, c) | X_1 \in (\alpha, 1]] = E[\max\{Z, c\}],$$

and

$$\begin{aligned} (10) \quad E[R_{T(\alpha Y, X, c)}(\alpha Y, X, c) | X \in (\alpha, 1]] \\ = E[R_{T(\alpha Y, X, c)}(\alpha Y, Z, c)] \\ \leq E[R_{T(\alpha Y, Z, c)}(\alpha Y, Z, c)] \\ = E[\max\{Z, c\}], \end{aligned}$$

where Z is uniform $(\alpha, 1]$ (and independent of Y, X).

From (9) and (10) follows the inequality corresponding to (8) given that $X \in (\alpha, 1]$, which together with (8) yields (5) and completes the proof. $\square$

Theorem 3.2 (4.6(iii) of [11]). Let $\alpha_1, \alpha_2, \dots$ be a sequence of non-increasing positive numbers, and let $X_1, X_2, \dots$ be independent exponentially distributed random variables with means $\alpha_1, \alpha_2, \dots$ respectively. Then

$$V(X_1, X_2, \dots) = \sup\{V(X_{\pi(1)}, X_{\pi(2)}, \dots) : \pi \text{ is a permutation of } \mathbb{N}\}.$$

Proof (due to Chris Klaassen). By Proposition 4.5 of [11] and renormalizing, it suffices to show

$$(11) \quad xe^{-\frac{c}{x}} - \frac{e^{-c}}{x} + e^{-c} \geq e^{-xe^{-\frac{c}{x}} - c} + xe^{-\frac{c}{x}}$$

for all $x \geq 1$ and all $c \geq 0$.

Substituting $y = e^{-c}$ and $\alpha = 1/x$, it suffices to show

$$(12) \quad \psi_\alpha(y) = -(1 - e^{-\alpha y}) + \alpha y^{1-\alpha} (1 - e^{-\frac{y^\alpha}{\alpha}}) \geq 0$$

for all $\alpha \in [0, 1]$ and all $y \in [0, 1]$.

Since $\psi_0(y) \equiv \psi_1(y) \equiv 0$, fix $\alpha \in (0,1)$. Let $F(y) = 1 - e^{-\alpha y}$ and $G(y) = 1 - e^{-(y^\alpha/\alpha)}$. Then

$$(13) \quad \psi_\alpha(y) = -F(y) + \alpha y^{1-\alpha} G(y); \text{ and}$$

$$(14) \quad \psi'_\alpha(y) = \alpha \{F(y) + [(1-\alpha)y^{-\alpha} - 1]G(y)\}.$$

Since F and G are non-negative on $[0,1]$, and $(1-\alpha)y^{-\alpha} - 1 \geq 0$ for $y \in I_1 = (0, (1-\alpha)^{1/\alpha}]$, from (14) it follows that $\psi'_\alpha(y) \geq 0$ for $y \in I_1$. Since $\psi_\alpha(0) = 0$ this implies $\psi_\alpha(y) \geq 0$ for all $y \in I_1$.

If $\psi_\alpha(\hat{y}) \leq 0$ for some $\hat{y} \in I_2 = ((1-\alpha)^{1/\alpha}, 1]$, then (13) would imply

$$(15) \quad G(\hat{y}) \leq \frac{1}{\alpha} \hat{y}^{\alpha-1} F(\hat{y}).$$

Since $(1-\alpha)\hat{y}^{-\alpha} - 1 < 0$ on I_2 , it follows from (14) and (15) with $\chi(\hat{y}) = \alpha + (1-\alpha)\hat{y}^{-1} - \hat{y}^{\alpha-1}$ that

$$(16) \quad \psi'_\alpha(\hat{y}) \geq F(\hat{y})\chi(\hat{y}).$$

Since $\chi(1) = 0$ and $\chi'(\hat{y}) = (1-\alpha)\hat{y}^{-2}(\hat{y}^\alpha - 1) \leq 0$, $\psi_\alpha(\hat{y}) \leq 0$ implies $\psi'_\alpha(\hat{y}) \geq 0$ for $\hat{y} \in I_2$. But the continuity of ψ_α then implies $\psi_\alpha \geq 0$ for all $y \in I_2$ also, which establishes (12), completing the proof. $\square$

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