Aspects of General Relativity in 1+1 Dimensions

Richard D. Mellinger Jr

Advisors: Scott Fraser and Thomas D. Gutierrez

A Senior Thesis
presented to
The Department of Physics
California Polytechnic State University, San Luis Obispo

December 12, 2012

Abstract

What would be the properties of a universe with only one spatial dimension and one time dimension? General relativity in 1+1 dimensions is unique since the two curvature terms in the Einstein field equations cancel. This makes the Einstein field equations algebraic rather than differential equations. This special feature can make 1+1 dimensionality attractive as an instructional tool to simplify the mathematics that many beginners find opaque. We explore the implications and features of the Einstein field equations in 1+1 dimensions and find they provide a surprisingly rich and interesting model. We then study an alternate theory and its implications for a potentially stable wormhole solution.

Contents

Part I
Introduction

In 1915 Albert Einstein published his work on what would later become known as the Einstein field equations [1]. The Einstein field equations (EFE) appear as

$$R_{\alpha\beta} - \frac{R}{2}g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G T_{\alpha\beta} \tag{1}$$

in tensor form. We are viewing them here in units such that the speed of light, c, is equal to 1, as is common in general relativity. Qualitatively this relationship was best summed up in a quote by John Wheeler: "Matter tells spacetime how to curve, and spacetime tells matter how to move." [2] This covers the notion that gravity is just an effect of the geometry of the spacetime, which is curved by matter or other forms of energy. Mathematically, The EFE are a set of nonlinear partial differential equations which relate the geometry of spacetime, $g_{\alpha\beta}$, to the energy of the system, $T_{\alpha\beta}$. Before this can be understood, we must briefly visit some notational staples of general relativity.

In Part I we introduce the major themes and notation necessary to understand Parts II and III. In Part II we explore the form of the EFE in 1+1 dimensions and analyze the implications of this pure 1+1 general relativity. In Part III, we introduce an accepted analog to the EFE for 1+1 and investigate the possibility of a stable wormhole solution. In Part IV we discuss overall conclusions for our exploration of general relativity in 1+1 dimensions and suggest some other directions that might be taken to further analyze this interesting little corner of relativity.

1 Overview of notation

1.1 Tensors

The EFE utilize the Einstein summation convention common in field equations where each subscript notates an index for the tensor. For example, if we have some made up tensor $M_{\alpha\beta}$ which exists in three spatial dimensions and one time dimension, or 3+1 dimensionality, (t, x, y, z), $M_{\alpha\beta}$ would appear as

$$M_{\alpha\beta} = \begin{bmatrix} M_{tt} & M_{tx} & M_{ty} & M_{tz} \\ M_{xt} & M_{xx} & M_{xy} & M_{xz} \\ M_{yt} & M_{yx} & M_{yy} & M_{yz} \\ M_{zt} & M_{zx} & M_{zy} & M_{zz} \end{bmatrix} \tag{2}$$

where M_{tt} is the time-time component, M_{tx} the time-x components and so on. In this way, it might be understood why there is an "s" on the end of "Einstein field equations" even though they appear as only one statement. Each object with two indices is actually a second rank tensor. In the full 3+1 dimensional spacetime, each of these contains 16 entries. All the tensors that appear in the EFE are symmetric, meaning that there are 10 potentially unique entries. Thus, in the 3+1 dimensional case, the EFE can be dissected into 10 different equations.

Repeated upper and lower indices implies a sum over those indices. For example

$$F^{\alpha\beta}M_{\alpha\beta} = \sum_{\alpha=(t,x,y,z)} \sum_{\beta=(t,x,y,z)} F^{\alpha\beta}M_{\alpha\beta} = F^{tt}M_{tt} + F^{xt}M_{xt} + F^{yt}M_{yt} + ... + F^{zz}M_{zz}. \tag{3}$$

Notice that the result will be a scalar, with no free indices.

1.2 The metric tensor

The metric tensor, $g_{\alpha\beta}$, describes the geometry of the spacetime. A diagonal metric looks, in general like

$$g_{\alpha\beta} = \begin{bmatrix} g_{tt} & 0 & 0 & 0 \\ 0 & g_{xx} & 0 & 0 \\ 0 & 0 & g_{yy} & 0 \\ 0 & 0 & 0 & g_{zz} \end{bmatrix}. \tag{4}$$

A special case of Eq. (4) that is often of interest, and the entire arena for special relativity, is that of flat spacetime. The metric for flat space in a 3+1 dimensional system is

$$\eta_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

where it is interesting to note that η_{tt} has the opposite sign of the spatial components. Historically, the sign convention was to use this in reverse, where the time component is positive and the spatial components are negative. The convention seen in Eq. (5) is more in fashion as of the writing of this paper, however the actual signs are not very important, as long as the space and time components differ by a minus sign.

Another way to display the information in the metric is in the line element. The line element, dS^2 is related to the spacetime interval between adjacent points in the spacetime and, for the diagonal case seen in Eq. (4), looks like

$$dS^2 = g_{tt}dt^2 + g_{xx}dx^2 + g_{yy}dy^2 + g_{zz}dz^2. \tag{6}$$

This touches something familiar in the case of flat space, as seen in Eq.(4), which would look like

$$dS^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{7}$$

which, resembles the spatial Pythagorean theorem in Euclidean space, but with the addition of a time dimension. Thus Eq. (6) can be understood to be the generalization of that Pythagorean relation to spacetimes that are not necessarily flat.

1.3 The terms of the EFE

In this section we will introduce the symbols seen in the EFE, term by term. The first term, as seen in Eq. (1) is $R_{\alpha\beta}$ and is called the Ricci tensor. It is defined as

$$R_{\alpha\beta} = \frac{\partial \Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\gamma}_{\alpha\gamma}}{\partial x^{\beta}} + \Gamma^{\gamma}_{\alpha\beta}\Gamma^{\delta}_{\gamma\delta} - \Gamma^{\gamma}_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} \tag{8}$$

where $\Gamma^{\gamma}_{\alpha\beta}$ is called a Christoffel symbol and is defined as

$$\Gamma^{\delta}_{\alpha\beta} = \frac{1}{2} g^{\delta\gamma} \left(\frac{\partial g_{\gamma\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\gamma\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} \right). \tag{9}$$

Notice that the Christoffel symbols are dependent on first derivatives of the metric tensor and the Ricci tensor is dependent on first derivatives of the Christoffel symbols. With respect to the Ricci tensor it is clear that the EFE will end up as a set of second order differential equations for the metric tensor, $g_{\alpha\beta}$.

The next term is dependent on the Ricci scalar, R. This is essentially just a scalar version of the Ricci tensor. The Ricci scalar is obtained by

$$R = g^{\alpha\beta} R_{\alpha\beta}. \tag{10}$$

where the upper indices on $g^{\alpha\beta}$ notate that it is the inverse of the metric with lower indices. This attribute is specific to the metric tensor. Upper indices on any other tensor means that they were raised by multiplication with the inverse metric. For example

$$R^{\mu\eta} = g^{\alpha\eta} R_{\alpha\beta} g^{\mu\beta}. \tag{11}$$

Another feature of the metric tensor that sets it apart from other tensors is that

$$g^{\alpha\beta} g_{\alpha\gamma} = \delta^{\beta}_{\gamma} \tag{12}$$

where δ^{β}_{γ} is known as the Kronecker delta and has the property that

$$\delta^{\beta}_{\gamma} = \begin{cases} 1 & \beta = \gamma \\ 0 & \beta \neq \gamma \end{cases} \tag{13}$$

meaning that

$$g^{\alpha\beta} g_{\alpha\beta} = n \tag{14}$$

where n is the number of dimensions in the system.

The stress-energy tensor, $T_{\alpha\beta}$, contains information about all the energy of the system. It is sometimes referred to as the stress-energy-momentum tensor. Though this name is more precise, it is a cumbersome title. For this reason, it is usually referred to as the stress-energy tensor and the momentum is just known to be included.

The highlights of $T_{\alpha\beta}$ are the physical interpretations of its terms. T_{tt} is related to the energy density. The time-space cross terms, such as T_{tx} or T_{yt}, are either energy flux or momentum density (which are the same thing, but different names given to different regions of the tensor). Of the purely spatial terms, the diagonals are related to the pressure in their respective directions, and the off diagonals are called shear stress. Most of these terms were included here, merely for completeness of introduction to the topic. As will be seen, in 1+1 we deal predominantly with T_{tt}, the energy density, and T_{xx}, the pressure in the x (and only) direction.

The final symbols in the EFE that need definitions are G, which is the familiar gravitational constant from Newtonian gravity and Λ which is the cosmological constant. The cosmological constant introduces an amount of energy density and pressure to otherwise empty space and is thought to be directly related to the accelerated expansion of the universe. In many toy model universes similar to the one we explore below, this is set to zero. However, as will be seen, this would be disastrous in 1+1 general relativity.

2 The covariant derivative

Another piece of notation that will be useful to know is that of the covariant derivative of a tensor, which is not unlike the general relativistic analog to the gradient. For some made up tensor, $t^{\alpha\beta}$, the covariant derivative is

$$\nabla_\gamma t^{\alpha\beta} = \frac{\partial t^{\alpha\beta}}{\partial x^\gamma} + t^{\delta\beta}\Gamma^\alpha_{\gamma\delta} + t^{\alpha\delta}\Gamma^\beta_{\gamma\delta} \tag{15}$$

and is defined such that, for any inverse metric $g^{\alpha\beta}$,

$$\nabla_\gamma g^{\alpha\beta} = 0 \tag{16}$$

because of the covariant derivative's dependence on the Christoffel symbols and the Christoffel symbols' dependence on the metric. Notice that in Eq. (16), the right hand side is not simply a scalar zero, but a zero matrix with 3 indices, because the covariant derivative of a tensor is a tensor one rank higher.

The covariant derivative is useful in many ways and provides a tidy means of expressing conservation of energy and momentum. With the covariant derivative, local conservation of energy and momentum look like

$$\nabla_\beta T^{\alpha\beta} = 0. \tag{17}$$

Notice that in Eq. (17), the repeated index β means that a sum is to be taken after the computation of the covariant derivative, in accordance with the Einstein summation convention, making the zero on the right hand side a vector of length n, where n is, again, the number of dimensions in the system.

The covariant derivative is also defined for vectors, where it takes a similar form. For a vector u_β, this would look like

$$\nabla_\alpha u_\beta = \frac{\partial u_\beta}{\partial x^\alpha} - \Gamma^\gamma_{\alpha\beta} u_\gamma. \tag{18}$$

3 Motivation

Now that we know our way around the EFE, it becomes immediately clear that the mathematics get complex enough to obscure physical meaning from the beginner. For this reason, looking at situations in which the mathematics are simplified is attractive. One of the tricks for doing this which is commonly used, not just in general relativity but in nearly all branches of physics, is to start with a system constrained to fewer dimensions. The lowest dimensional system that means anything is the case of 1 spatial and 1 time dimension (called 1+1 dimensional). As seen below, this will simplify the mathematics greatly, but reveal strange implications.

In recent years some evidence has also surfaced to support that a successful theory of quantum gravity may act two dimensional rather than four [3]. In this paper, we don't explore this idea, however it is important to note that the model that we do explore within as a toy model may have implications farther reaching then those of pure novelty.

Part II
Pure general relativity

It is generally understood that the EFE take on a unique form in 1+1 dimensions [4, 5]. In this part we will look at the effect of 1+1 dimensionality on the EFE and the implications of the form they assume.

4 The derivation

To derive the EFE for 1+1 dimensions, we look at the general case and choose the generic metric tensor

$$g_{\alpha\beta} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{19}$$

where A and B potentially have dependence on both t and x. We can make this diagonal without loss of generality because locally any system can be changed into some set of coordinates where the metric tensor is diagonal [6]. If we define F as

$$F \equiv 2AB\left[\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 B}{\partial t^2}\right] - A\left[\left(\frac{\partial B}{\partial t}\right)^2 + \frac{\partial A \partial B}{\partial x \partial x}\right] - B\left[\frac{\partial A \partial B}{\partial t \partial t} + \left(\frac{\partial A}{\partial x}\right)^2\right] \tag{20}$$

then it can be shown that calculation of the Ricci tensor and Ricci scalar for this metric yield

$$R_{\alpha\beta} = \frac{-F}{4A^2B^2}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{21}$$

and

$$R = \frac{-F}{2A^2B^2} \tag{22}$$

respectively. Putting these results into the Einstein field equations, Eq. (1), has a startling result. The Ricci tensor and Ricci scalar terms cancel! No matter what the curvature is, as long as the system is constrained to only 1+1 dimensions,

$$R_{\alpha\beta} - \frac{R}{2}g_{\alpha\beta} = 0. \tag{23}$$

This leaves Eq. (1) as

$$\Lambda g_{\alpha\beta} = 8\pi G T_{\alpha\beta} \tag{24}$$

which are very much simplified! What was originally a set of second order nonlinear partial differential equations is now a set of linear algebraic equations. Mathematically, then, this can be viewed as a success, as our initial goal was to simplify the mathematics. However, as promised, the implications of this system are very strange, and will take the majority of the remaining pages of this part to explore.

5 Initial implications

The following are observations that can be made as a direct consequence of Eq. (24).

5.1 Fulfilling qualitative expectations

The first thing one might notice when looking at our EFE, is that the system still does exactly what it should, albeit differently. Since Λ, π, and G are all constants, the linear relationship highlights the fact that changes in energy, $T_{\alpha\beta}$, cause changes in the metric, $g_{\alpha\beta}$, which defines the curvature of spacetime and vice versa. As we expect, energy defines the geometry of spacetime. As will be seen, the lack of any sort of differential equation will have far reaching consequences.

5.2 Non-zero cosmological constant

Equation (24) shows that for a system to contain any energy or mass at all in 1+1 dimensions, there has to be a cosmological constant present. To take this a step further, if there exists a single point where the energy density, T_{tt}, and a pressure, T_{xx}, are both zero then, since an all zero metric is not physical, Λ is forced to be zero. As noted before, Λ is constant, meaning that it would be zero everywhere in the spacetime. This implies that, for Eq. (24) to be satisfied, there is no energy anywhere. Thus for energy to be anywhere, there must be some energy everywhere.

5.3 Asymptotically flat spacetime

If we demand an asymptotically flat spacetime,

$$\lim_{x\to\infty} g_{\alpha\beta} = \lim_{x\to\infty} \begin{bmatrix} A(x) & 0 \\ 0 & B(x) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{25}$$

which is commonly done in general relativity, then we see that a positive Λ yields a negative T_{tt} (energy density), and a negative Λ yields a negative T_{xx} (pressure). A negative pressure has physical meaning as a tension, but a negative energy density is strange. This means that either Λ is negative or that in 1+1 dimensional systems, we can't have asymptotically flat metrics. Though there is no reason for one to expect Λ to be the same in both a 1+1 system and a 3+1 system, it is of interest to note that Λ of our own universe has been shown experimentally to be positive [7].

5.4 Lack of differential equations

Due to the loss of the Ricci terms, Eq. (24) lacks any sort of differential equation. The differential equations in higher dimensional systems are what force an interaction between the different equations wrapped up in the EFE. In this case, our diagonal metric Eq. (24) can be dissected into

$$\Lambda g_{tt} = 8\pi G T_{tt} \tag{26}$$

$$\Lambda g_{xx} = 8\pi G T_{xx} \tag{27}$$

which are completely independent of one another. In higher dimensional systems, picking a T_{tt}, for example, places restrictions on what T_{xx} must be. However, in this case, their independence does away with those restrictions. This leaves one able to put almost anything they want into Eq. (24) for either $g_{\alpha\beta}$ or $T_{\alpha\beta}$, no matter how unrelated the time and x terms are. This freedom comes at a cost, though. It does away with much of the utility of the equations as much less information can be pulled from them.

5.5 Local conservation of energy-momentum

Due to the loss of information along with the Ricci terms, the search for a way to impose some sort of constraint onto the system becomes attractive. We look into conservation of energy-momentum for a potential source of this constraint as the covariant derivative looks like it should force an interaction. As was mentioned earlier, local conservation of energy-momentum can be summed up by Eq. (17). Since Eq. (24) says that

$$T^{\alpha\beta} = \frac{\Lambda}{8\pi G} g^{\alpha\beta} \tag{28}$$

where the coefficient of $g^{\alpha\beta}$ is a constant, in 1+1 general relativity, it immediately follows that

$$\nabla_\gamma T^{\alpha\beta} = \frac{\Lambda}{8\pi G} \nabla_\gamma g^{\alpha\beta} = 0 \tag{29}$$

because of Eq. (16). Recall that local conservation and momentum called for the sums of specific derivatives of $T^{\alpha\beta}$ to be zero. What we have here is much stronger: every derivative is zero. This has the effect of trivially satisfying local conservation of energy and momentum in 1+1 which gives us absolutely no help in our quest for constraints. However, the fact that we ended up with a more general statement than needed by the conservation law warrants further exploration.

Notice that Eq. (29) satisfies

$$\nabla_\mu T_{\alpha\beta} + \nabla_\alpha T_{\mu\beta} + \nabla_\beta T_{\alpha\mu} + \nabla_\mu T_{\beta\alpha} + \nabla_\alpha T_{\beta\mu} + \nabla_\beta T_{\mu\alpha} = 0 \tag{30}$$

because each term vanishes by Eq. (29). This is interesting because Eq. (30) is the definition of a Killing tensor [8]. Killing tensors result in constants of motion for test particles. As the stress energy tensor is not generally a Killing tensor in higher dimensional systems, the resulting constant of motion will also be specific to 1+1.

By the same logic as was used to determine that the stress energy tensor was a Killing tensor in 1+1, it can be seen that the metric tensor is always a Killing tensor. A constant of motion due to the metric tensor is the square of the mass m of the test particle. The relation looks like

$$g_{\alpha\beta}P^{\alpha}P^{\beta} = -m^2 \tag{31}$$

where P^{α} is the momentum of the particle. This gives us a relationship between the geometry of the system and the particle's momentum. From Eq. (24), however, we know that $g_{\alpha\beta}$ and $T_{\alpha\beta}$ are interchangeable with the moving around of some constants. Doing so, we get

$$T_{\alpha\beta}P^{\alpha}P^{\beta} = -m^2 \frac{\Lambda}{8\pi G}. \tag{32}$$

Thus in 1+1, the stress energy tensor provides a constant of motion which couples the particle's mass to the cosmological constant, which is not a general property of stress tensors.

6 Pure general relativity conclusions

We have seen several ways in which the Einstein field equations in 1+1 are unique with respect to other dimensional systems. There are probably many more, but over the last few sections it may have become clear that this uniqueness, though interesting in its own right, limits usefulness of pure general relativity in 1+1 as an educational tool, as it doesn't act very much like higher dimensional systems. For this reason analogues to Eq. (24) have been proposed that maintain some resemblance to higher dimensional systems. In the next part we will introduce one commonly used analogue and explore the possibility of a stable wormhole solution.

Part III

Trace-dependent general relativity

As mentioned in the previous part, alternate field equations have been proposed to replace Eq. (1) in the case of 1+1 dimensionality [5, 9]. One such equation is

$$R - \Lambda = 8\pi GT \tag{33}$$

where T is the trace of $T_{\alpha\beta}$. As mentioned earlier, this would be accomplished by computing

$$T = T_{\alpha\beta}g^{\alpha\beta}. \tag{34}$$

This trace-dependent field equation is attractive because the trace and the surviving Ricci scalar cause a relationship between the time and spatial terms. Also, as discussed in [5], spacetimes governed by Eq. (33) have many things in common with higher dimensional systems governed by Eq. (1). One of the most interesting things that it shares is the prediction of black holes. To see where Eq. (33) comes from, we start with the n dimensional Eq. (1). We can take the trace by multiplying through by $g^{\alpha\beta}$ and we get

$$R - \frac{n}{2}R + n\Lambda_n = 8\pi G_n T. \tag{35}$$

It follows that

$$(1-\frac{n}{2})R+n\Lambda_n=8\pi G_nT. \tag{36}$$

It is clear that we have the same issue of Ricci terms disappearing as pure general relativity in the case that $n=2$. However, the clever authors of [9] defined

$$G_2\equiv\lim_{n\to 2}\frac{G_n}{(1-\frac{n}{2})} \tag{37}$$

for a version of Eq. (33) that depends on $\Lambda=0$. Similarly, we here define

$$\Lambda_2\equiv\lim_{n\to 2}\frac{-n\Lambda_n}{(1-\frac{n}{2})} \tag{38}$$

which gives Eq. (33), give or take some subscripts, left off for notational ease. It is important to note that due to Eq. (37) and Eq. (38), the G and Λ found in Eq. (33) are not the same G and Λ found in Eq. (1), though they serve very similar purposes mathematically. This derivation may shed light onto why this part has been entitled "trace-dependent general relativity" as our field equations here are dependent only on the traces.

7 Black holes

As previously noted, one of the benefits of Eq. (33) is that it allows for black hole solutions [5]. Here we have a quick, conceptual overview of black holes in 1+1.

It is not possible to have closed orbits in 1+1, a property unique to this dimensionality. However, we allow, in principle, for oscillations, which would be the natural 1+1 analogue to an orbit. But, unless the test particles can travel through other matter, this does not help in setting up gravitational interactions. By definition, a black hole is "black" meaning not even light can escape once past the event horizon. Thus anything on one side of the black hole can either remain on that side or end up in the black hole, it can not end up on the other side, or communicate with the other side in any way. This fact illustrates that in the case of a space with a black hole in 1+1 (a conceptual illustration for which might look like Fig. 1) the space is cleaved in two. As we move into a discussion of wormholes in the next section, basic concepts, such as if it is possible to traverse the distance between points in region A and points in region B will need to be revisited.

8 Wormhole solution

Wormholes are solutions to the EFE that include two entrances connected by way of a "short tube", despite the fact that the entrances may be vastly separated in regular space [10]. The idea that wormholes could provide shortened routes between distant points has made them invaluable in science fiction. The fact is, though that there are valid solutions to the EFE that allow for wormholes (although, for the most part they require exotic types of matter, or types of energy densities we have never found in nature). Here we take advantage of the simplified model afforded by Eq. (33) to explore a possible wormhole in 1+1.

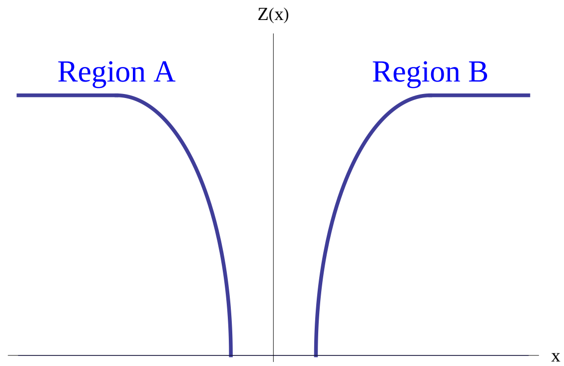

Figure 1: A sketch of what a black hole embedding in 1+1 might look like, shown to help illustrate the fact that in 1+1 a black hole cleaves the space into two separate regions, neither of which can be reached by the other.

One method used to generate a certain class of wormhole solution is to take the geometry of a spacetime with two black holes in it and "stitch" the throats (horizons) of the wormholes together [11, 12]. This creates wormhole solutions with event horizons. It is important that, in "stitching" the sections together, we choose our points wisely as to make the result smooth and continuous. This creates the wormhole classically pictured as a handle. In most cases this is not a particularly useful idea as two black holes generally move towards each other and eventually collide. However, as is pointed out in the paper by Mann, Shiekh and Tarasov [5], there is a solution to Eq. (33) which allows for N static black holes. This potentially allows for the method detailed above to be used to construct a static wormhole.

8.1 Two black holes

If one defines

$$\alpha = -\frac{\Lambda x^2}{2} - \xi + 2\sum_{i=1}^{N} M_i(|x - x_i|) \tag{39}$$

where M_i is the mass parameter of the ith black hole, x_i is its location, and ξ is an arbitrary constant, then the static N black hole metric is

$$g_{\alpha\beta} = \begin{bmatrix} -\alpha & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix}. \tag{40}$$

For our purposes, we set $N = 2$. Placing two black holes of equal mass parameters, m, symmetrically at a distance a from the origin, Eq. (39) becomes

$$\alpha = -\frac{\Lambda x^2}{2} - \xi + 2m(|x - a| + |x + a|). \tag{41}$$

As was pointed out in the previous section, this will break the entire space into three regions, each distinct from the others. α becomes

$$\alpha = -\frac{\Lambda x^2}{2} - \xi + 4m \begin{cases} -x, & x < -a \\ a, & -a < x < a \\ x, & x > a \end{cases} \tag{42}$$

As a defines the location where we put the black holes, not the entirety of the regions noted in Eq. (42) are available, due to the lack of accessibility of regions inside the event horizons. The event horizons appear where $g_{tt} = 0$, i.e. when $\alpha = 0$. With this in mind, we calculate the horizons of which there are three sets. We also calculate the locations where α flattens, by locating where the first derivative with respect to x is zero. These values can be found in Table 1.

Region	Event Horizons	Flattening
$x < -a$	$\frac{4m \pm \sqrt{16m^2 - 2\Lambda\xi}}{-\Lambda}$	$\frac{-4m}{\Lambda}$
$-a < x < a$	$\pm\sqrt{\frac{2(4ma-\xi)}{\Lambda}}$	0
$x > a$	$\frac{4m \pm \sqrt{16m^2 - 2\Lambda\xi}}{\Lambda}$	$\frac{4m}{\Lambda}$

Table 1: The important x values in the static, two black hole geometry.

If we focus on the $x > 0$ side, and rely on the fact that the function is symmetric around zero, we can simplify our discussion. Figure 2 shows an example α where $x > 0$ with some of the values noted in Table 1 labeled.

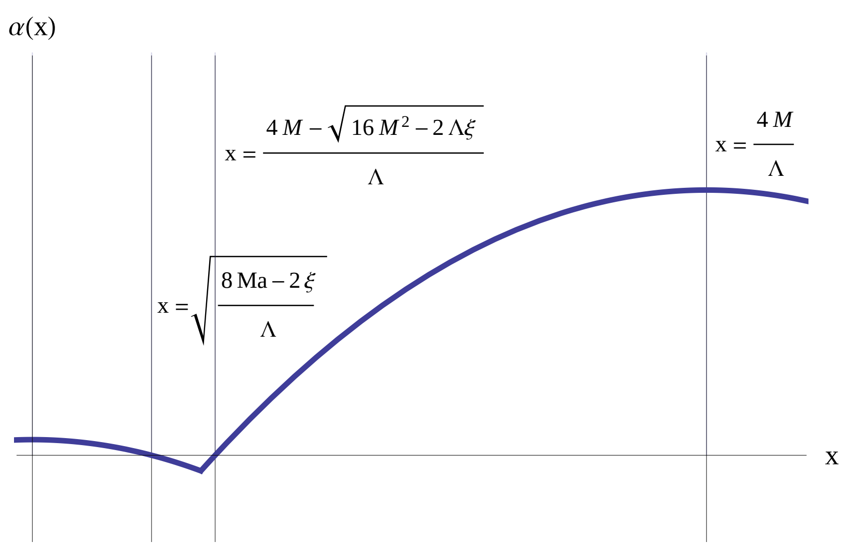

Figure 2: A plot of an example α with most of the important x values labled.

Notice that in Fig. 2, to the right of $x = \frac{4m}{\Lambda}$, the graph starts to head back down and according to the values in Table 1 there is another event horizon to the right, but only one, meaning that α is not positive again anywhere to the right of that upper event horizon. Thus there is a finite amount of space that still exists.

A few more restrictions must be noted about this setup. For us to stitch the two interior sets of event horizons together, we want said horizons to exist at real locations. From the square roots in Table 1 it must be the case that

$$8m^2 > \Lambda\xi \tag{43}$$

and either

$$4ma > \xi \quad \text{and} \quad \Lambda > 0 \tag{44}$$

or

$$4ma < \xi \quad \text{and} \quad \Lambda < 0. \tag{45}$$

Armed with the necessary assumptions and values regarding the two black hole geometry, we are prepared to discuss the wormhole.

8.2 Making the wormhole

If we stitch together the event horizons, as planned, we get something that qualitatively looks like Fig. 3 for α, where the dashed lines are just tying the points on either end together without having a real physical meaning otherwise. One might notice that Fig. 3 is actually a plot of $\sqrt{\alpha}$ rather than just α, we do this here to emphasize some of the more subtle curves in the lines. The topology and important values in the sketch remain unaffected.

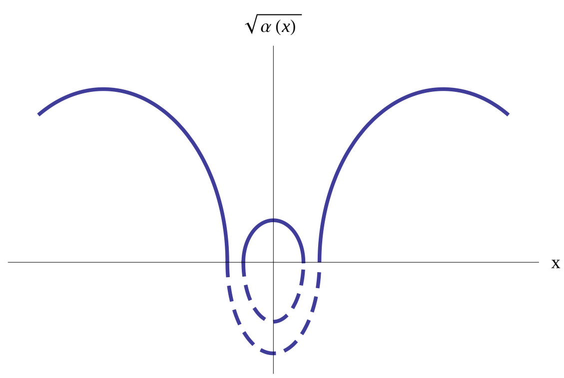

Figure 3: A conceptual representation of $\sqrt{\alpha}$ for a wormhole in 1+1 after stitching the throats of two black holes together.

A few things to notice about Fig. 3 are that the little island between the event horizons is still completely secluded with respect to the region made up by the right and left. The middle region is now closed in on itself, while travel is now possible between the left and right. As was noted in the section on black holes in 1+1, the notion of travel between two points is not as straightforward as one might think. Conceptually, one can see that a traveler using a wormhole in higher dimensions would have two routes to get from place to place: through the wormhole, or over the “regular” space between [11]. Here, there is no such choice, as the region “between” them is now self contained.

To remove the problem caused by the existence of the outer-most event horizons, and afford our traveler a second route, we make the outer route closed as well. This isn't all that outlandish as said outer event horizons force the size of the space to be finite anyway, as discussed above. We do this by “stitching” a point from the right to a point from the left. To ensure a smooth transition from one to the other, we connect them where the right- hand and left- hand sides flatten, as calculated in Table 1. The validity of this connection is discussed in more detain in the next section and is shown schematically in Fig. 4, again, the dashed lines just show the connections between the points on either end.

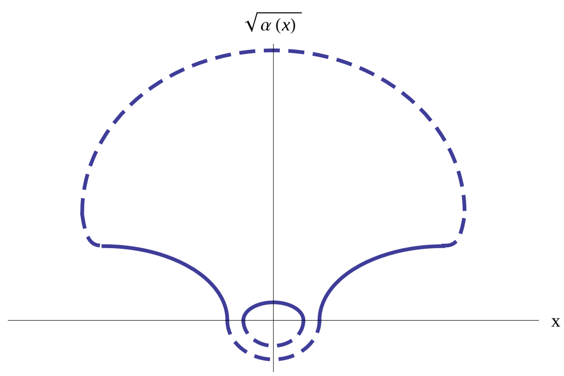

Figure 4: A conceptual representation of α for a wormhole in 1+1 after stitching the throats of two black holes together and the space closed by stitching together the points where the left and right flatten.

To better illustrate the topology of the proposed wormhole, see Fig. 5 which has the same topology as Fig. 4. This was done to better illustrate that there is an inner and an outer path which make no contact anywhere. The inner path is the one that is made by connecting the inner horizons, while the outer path is the one made by connecting the outer horizons and the wrap around. As will be seen in section 8.4, the fact that the inside path is smaller in Fig. 5 does not necessarily mean that the path is actually shorter. In the sketch, the inside path is simply smaller, so that it fits inside the sketch of the outside path.

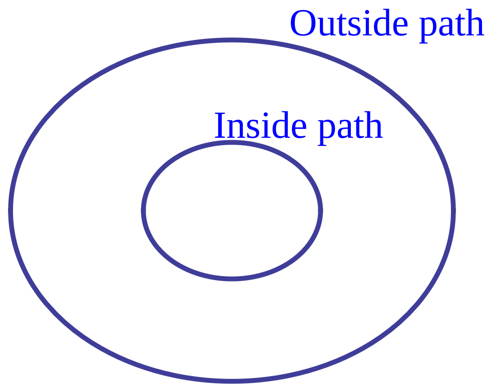

Figure 5: A topological representation of the proposed wormhole geometry.

8.3 Extrinsic curvature

In the previous section, we commented that the "stitching" from right to left will be smooth due to the fact that $\frac{d\alpha}{dx} = 0$ at the chosen locations. This is true, but the actual reasoning that this works is slightly more subtle than may have been obvious. The real measure of how smooth a transition this should be is actually continuity of extrinsic curvature. It is important that the two points we are connecting have the same extrinsic curvature, so that, when connected, there is continuity. Since we are exploring trace-dependent general relativity, we rely on a scalar analog to the extrinsic curvature [13]. This analog is

$$K \equiv u^\alpha u^\beta \nabla_\alpha n_\beta \tag{46}$$

where u^α is the tangent, n^α is the normal, and ∇_α is the covariant derivative for a vector outside a horizon. Noting that n^α will be in the x direction, and that $n \cdot u = 0$, it becomes clear that u will have only a time component. Substituting in Eq. (18), Eq. (46) becomes

$$K = u^t u^t \left(\frac{\partial n_t}{\partial t} - \Gamma^x_{tt} n_x \right) = -u^t u^t \Gamma^x_{tt} n_x. \tag{47}$$

To evaluate this, we determine some of the pieces by noting

$$g_{xx}(n^x)^2 = 1 \implies n^x = \frac{1}{\sqrt{g_{xx}}} \implies n_x = g_{xx} n^x = \frac{\sqrt{\alpha}}{\alpha} = \frac{1}{\sqrt{\alpha}} \tag{48}$$

$$g_{tt}(u^t)^2 = -1 \implies u^t = \frac{1}{\sqrt{-g_{tt}}} = \frac{1}{\sqrt{\alpha}} \tag{49}$$

and retrieving the appropriate Christoffel symbol from reference [5]

$$\Gamma^x_{tt} = \frac{1}{2} \alpha \frac{d\alpha}{dx}. \tag{50}$$

From these pieces we can further simplify the expression for the extrinsic curvature to

$$K = -\frac{1}{\alpha}\left(\frac{1}{2}\alpha\frac{d\alpha}{dx}\right)\frac{1}{\sqrt{\alpha}} = -\frac{1}{2\sqrt{\alpha}}\frac{d\alpha}{dx} \tag{51}$$

which, is always going to be zero when $\frac{d\alpha}{dx} = 0$ as long as one is not evaluating at an event horizon ($\alpha = 0$). Thus, by choosing the flat portions on the right and left to connect, we chose two locations where the extrinsic curvature is zero. More important than the actual value is the fact that they have the same extrinsic curvature. This shows that the connection was nice and smooth between the left and right.

The other "stitches" occur at event horizons, where the surface is null. It turns out that the method detailed above is not as straight forward when evaluating the extrinsic curvature of null surfaces due in part to the fact that, for a null surface, the normal and tangent vectors are actually the same direction. We don't explore this idea any more deeply here, as the ability to "stitch" together geometries at an event horizon to create a wormhole is far from a new idea [11, 12].

8.4 Proper distances

As noted in the previous section, there is now an outside and an inside path, the comparative lengths of which we still don't know. We determine this by computing the proper distances of each. To do this we will use the fact that

$$dS = \sqrt{g_{xx}}dx \tag{52}$$

and integrate. We will call the inner path ΔS_1, and the outer path ΔS_2. These would be calculated as

$$\Delta S_1 = 2\int_0^{\sqrt{\frac{2(4ma-\xi)}{\Lambda}}} \frac{dx}{\sqrt{-\frac{\Lambda x^2}{2} - \xi + 4ma}} \tag{53}$$

$$\Delta S_2 = 2\int_{\frac{4m-\sqrt{16m^2-2\Lambda\xi}}{\Lambda}}^{\frac{4m}{\Lambda}} \frac{dx}{\sqrt{-\frac{\Lambda x^2}{2} - \xi + 4mx}}. \tag{54}$$

Actually carrying out the calculations, keeping in mind the assumptions made in Eq.(43), yields

$$\Delta S_1 = \Delta S_2 = \frac{\sqrt{2}\pi}{\sqrt{\Lambda}}. \tag{55}$$

This is startling for several reasons. Firstly, the only physical parameter that it depends on is Λ. Being purely spatial, one might expect that the path lengths depended at the very least on the positioning of the black holes, a, but this is apparently not so. Secondly, the two path lengths are the same. So physically, by forcing the spacetime to fit into this wormhole geometry, the space becomes riven into two spaces that are the same length, closed loops, and not in contact anywhere. If we wanted to update Fig. 5 to reflect path lengths, they would sit right on top of each other without touching, despite existing only in one spatial dimension. Another implication of Eq. (55) is that it only yields real solutions in the case that $\Lambda > 0$, which makes Eq. (45) no longer a valid assumption.

9 Conclusions for trace-dependent wormholes

We showed that a static wormhole solution with an internal and external route is possible for the trace-dependent field equations. We determined that the two path lengths are equal and only dependent on Λ, which was unexpected. The discussion above also revealed that real solutions exist only when $8m^2 > \Lambda\xi$, $4Ma > \xi$ and $\Lambda > 0$. These inequalities may be useful for any further investigation of 1+1 geometries that contain two black holes in trace-dependent general relativity.

Part IV
General conclusions

In this treatment of general relativity in 1+1 dimensions, we did some preliminary exploration of two different types of general relativity. The first was what we called "pure general relativity". We called it this due to the fact that it was the actual form of the Einstein field equations simplified to 1+1. The second type was an established analog to the Einstein field equations specifically defined to preserve some of the characteristics of higher dimensional systems when working in 1+1. We called this "trace-dependent general relativity" due to the fact that the field equation is derived using the trace of the Einstein field equations.

For the most part our explorations of these two subjects didn't overlap in any way that allows for comparison, aside from the fact that in pure general relativity we showed that a negative Λ might be more desirable, while in trace-dependent general relativity we employ a positive Λ to construct a specific wormhole geometry. This isn't necessarily a discrepancy between the two theories because, as seen in the definition of the Λ for the trace-dependent case, it was distinct from the one in pure general relativity. In the pure general relativity section we were, for the most part, limited to commenting in abstract, while the trace-dependent equation allowed for us to explore a very specific case and find some physical relationships associated with a wormhole made out of two black holes.

General relativity in 1+1, regardless of whether one is looking at pure or trace-dependent equations, provides an interesting playground for exploration. Even in the specific nooks discussed in these pages, there is plenty more depth to be probed. Some future work that might be interesting would be to determine the dynamics of test particles in the presence of the wormhole, exploring what, if anything, pure general relativity has to say about the wormhole metric, and investigating the extrinsic curvature of the "stitching" in the trace-dependent wormhole at the event horizons.

Acknowledgments

A special thanks to my advisers, Dr. Thomas Gutierrez and Dr. Scott Fraser of Cal Poly San Luis Obispo. Their patience, know-how, intuition, and encouragement were an integral part of the project process without which the project may not have gotten past the "I wonder..." stage.
Thank you.
If I ever build a wormhole, you two will get the invites to join me on the first trip.

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