

The (q, r) -Simon Newcomb Problem

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A new statistic, the r -major index, is defined for sequences. A linear recurrence is then derived that enumerates sequences by r -descent number and r -major index.

1. INTRODUCTION

The classic Simon Newcomb problem may be described as follows. Let $Js = (j_1, j_2, \dots, j_s)$ be a sequence of non-negative integers with $n = j_1 + j_2 + \dots + j_s$. Denote by $R(Js)$ the set of sequences $f = (f(1), f(2), \dots, f(n))$ in which the integer m occurs j_m times. For an integer $r \geq 1$, the set of r -descents of f is defined as

$$\{i: f(i) \geq f(i+1) + r, 1 \leq i \leq n-1\}. \quad (1.1)$$

The cardinality of set (1.1) is denoted by $r\text{-des } f$. Then, the problem is to determine the number of sequences $f \in R(Js)$ with $1\text{-des } f = k$.

The present paper has three objectives. First, the r -major index as introduced in [12] for permutations is extended to sequences. Denoted by $r\text{-maj } f$, the r -major index of f is defined to be the sum of the elements in set (1.1) plus the cardinality of the set

$$\{(i, j): 1 \leq i < j \leq n, f(i) > f(j) > f(i) - r\}. \quad (1.2)$$

In the two special cases $r = 1$ and $r \geq s$, the r -major index respectively reduces to what are commonly known as the major index and the inversion number. It will be shown that if the q -analog, q -factorial, and q -multinomial

coefficient of n are respectively defined as

$$\begin{aligned} \text{(a)} \quad [n] &= 1 + q + \cdots + q^{n-1} \\ \text{(b)} \quad [n]! &= [1][2]\cdots[n] \\ \text{(c)} \quad \left[\begin{matrix} n \\ j_1 j_2 \cdots j_s \end{matrix} \right] &= \frac{[n]!}{[j_1]![j_2]!\cdots[j_s]!} \end{aligned} \quad (1.3)$$

then the generating function for the r -major index is

$$\sum_f q^{r\text{-maj } f} = \left[\begin{matrix} n \\ j_1 j_2 \cdots j_s \end{matrix} \right] \quad (1.4)$$

summed over $R(Js)$, which generalizes the observation of MacMahon [11] that the major index and inversion number have the same generating function.

Second, as identity (1.4) suggests, for integers $b, c \geq 1$ there is a bijection $\Phi_{b,c}: R(Js) \rightarrow R(Js)$ with the property that

$$b\text{-maj } f = c\text{-maj } \Phi_{b,c}(f) \quad (1.5)$$

for all $f \in R(Js)$. The construction of $\Phi_{b,c}$ in section 6 will even provide a proof of (1.4). Foata [6] has given another such bijection in the case $b = 1$ and $c \geq s$.

The final and main objective is to solve the (q, r) -Simon Newcomb problem of enumerating sequences by r -descents and r -major index. It will be shown that for $Js = (j_1, j_2, \dots, j_s)$ with

$$\begin{aligned} \text{(a)} \quad j_s &\geq 1 \\ \text{(b)} \quad j(m, r) &= \begin{cases} j_m + j_{m-1} + \cdots + j_{m-r+1} & \text{if } m \geq r \\ j_m + j_{m-1} + \cdots + j_1 & \text{otherwise} \end{cases} \\ \text{(c)} \quad Js - 1 &= (j_1, j_2, \dots, j_{s-1}, j_s - 1) \end{aligned} \quad (1.6)$$

the polynomial

$$M[Js, k, r] = \sum_f q^{r\text{-maj } f} \quad (1.7)$$

summed over sequences $f \in R(Js)$ with $r\text{-des } f = k$ satisfies the recurrence

$$\begin{aligned} [j_s]M[Js, k, r] &= [k + j(s, r)]M[Js - 1, k, r] \\ &\quad + q^{k + j(s, r) - 1} [n + 1 - k - j(s, r)]M[Js - 1, k - 1, r] \end{aligned} \quad (1.8)$$

where $M[Js, 0, r] = \left[\begin{matrix} n \\ j_1 j_2 \cdots j_r \end{matrix} \right]$. The more explicit formula

$$M[Js, k, r] = \sum_{l=0}^k (-1)^l \begin{bmatrix} n+1 \\ l \end{bmatrix} q^{\binom{l}{2}} \prod_{m=1}^s \begin{bmatrix} k-l+j(m, r) \\ j_m \end{bmatrix} \quad (1.9)$$

will then be derived from (1.8).

Numerous special cases of (1.8) and (1.9) may be found in the list references. When $r = 1$ identity (1.9) gives the solution to the q -Simon Newcomb problem as obtained by MacMahon [10, v. 2 p. 211]. Also see Gessel [9, p. 98]. The case $r = 1 = q$ is discussed in [1, 2, 5, 9, 10]. However, even in these cases, recurrence (1.8) seems to be new.

In the permutation case, $j_m = 1$ for $1 \leq m \leq s$, (1.7) becomes

$$A[n, k, r] = \sum_{\sigma} q^{r\text{-maj } \sigma} \quad (1.10)$$

summed over permutations σ of $\{1, 2, \dots, n\}$ with k r -descents and (1.8) reduces to

$$A[n, k, r] = [k+r]A[n-1, k, r] + q^{k+r-1}[n+1-k-r]A[n-1, k-1, r] \quad (1.11)$$

where $A[r, 0, r] = [r]!$. Recurrence (1.11) defines the (q, r) -Eulerian numbers as developed in [12]. The $(1, r)$ -Eulerian numbers have been studied in [7, 8, 13] and Carlitz [4] obtained recurrence (1.11) for the $(q, 1)$ -Eulerian numbers.

2. THE INSERTION LEMMA

The construction of $\Phi_{b,c}$ and the proof of (1.8) are based on insertion. It will be convenient to regard a sequence $f \in R(Js)$ as the word $f(1)f(2)\dots f(n)$ obtained by juxtaposing the letters $f(1), f(2), \dots, f(n)$.

A colored word F is obtained by inserting an underlined s into an $f \in R(Js-1)$. The color of F , denoted by $C(F)$, is defined to be the number of times that s appears to the right of the underlined s in F . For example, with $s = 5$ and $f = 215243152 \in R(2, 3, 1, 1, 2)$ one may obtain

$$F = 2\ 1\ 5\ 2\ 4\ 3\ 1\ \underline{5}\ 5\ 2 \quad (2.1)$$

where $C(F) = 1$. The color is just a way of indicating which s was inserted.

To observe the effect that the insertion has on the r -major index and the r -descent number, the n possible insertion positions between the letters of $f = f(1)f(2)\dots f(n-1) \in R(Js-1)$ are labeled as follows. Using the labels $\{0, 1, \dots, n-1\}$ in order, first read from right to left and label the positions that will *not* result in the creation of a new r -descent. Then, reading back from left to right the remaining positions that *will* create an r -descent are labeled. For instance, if $r = 3$, $s = 5$, and $f = 215243152 \in R(2, 3, 1, 1, 2)$, then the labels in the top and bottom rows of

$$\begin{array}{ccccccccccccccc} & \bullet & \bullet & 6 & 5 & 4 & 3 & & \bullet & 2 & 1 & 0 & & \\ & & & \downarrow & \downarrow & \downarrow & \downarrow & & & \downarrow & \downarrow & \downarrow & & \\ f = & 2 & 1 & 5 & 2 & 4 & 3 & 1 & & 5 & 2 & & & \\ & \uparrow & \uparrow & & & & & \uparrow & & & & & & \\ & 7 & 8 & \bullet & \bullet & \bullet & \bullet & 9 & \bullet & \bullet & \bullet & \bullet & & \end{array} \quad (2.2)$$

respectively indicate the positions that will not and that will result in a new 3-descent.

Let $(f; l)$ denote the colored word obtained by inserting the underlined s into position l . For example (2.2), the colored word $(f; 2)$ is given in (2.1). Note that $3\text{-des}(f; 2) = 3\text{-des } f$ and that $3\text{-maj}(f; 2) + C(f; 2) = 2 + 3\text{-maj } f$. This demonstrates the

INSERTION LEMMA. For $f \in R(Js-1)$ and $0 \leq l \leq n-1$

$$(a) \ r\text{-des}(f; l) = \begin{cases} r\text{-des } f & \text{if } 0 \leq l \leq r\text{-des } f + j(s, r) - 1 \\ 1 + r\text{-des } f & \text{otherwise} \end{cases}$$

$$(b) \ r\text{-maj}(f; l) + C(f; l) = l + r\text{-maj } f.$$

Proof For (a), note that there are $r\text{-des } f + j(s, r)$ insertion positions that will not result in a new r -descent: preceding any of the $j(s, r) - 1$ integers greater than $s - r$, in any of the r -descents, or at the extreme right end of f . As these positions are labeled first, (a) is immediate.

For (b), let m be the number of r -descents and integers in $\{s - r + 1, s - r + 2, \dots, s - 1\}$ that are to the right of position l . Let c be the number of times s appears to the right of position l . In the case $0 \leq l \leq r\text{-des } f + j(s, r) - 1$, it follows from (1.1) and (1.2) that inserting s into position l will increase the r -major index by m . The color of the resulting word is c . By the labeling, $l = m + c$ and (b) follows in this case. For $r\text{-des } f + j(s, r) \leq l \leq n - 1$, note that there are $l - m - c - 1$ integers in f to the left of position l . As a new r -descent is created in this case, the color of the resulting word plus the increase in the r -major index is $c + (l - m - c) + m = l$.

3. PROOF OF RECURRENCE (1.8)

Let $R(Js, k, r) = \{f \in R(Js) : r\text{-des } f = k\}$. Part (a) of the insertion lemma implies that the set of ordered pairs

$$\{(F, C(F)) : F \in R(Js, k, r), C(F) \in \{0, 1, \dots, j_s - 1\}\} \quad (3.1)$$

regarded as colored words is the disjoint union of the sets

$$\begin{aligned} (a) \ & \{(f; l) : f \in R(Js-1, k, r), 0 \leq l \leq k + j(s, r) - 1\} \\ (b) \ & \{(g; m) : g \in R(Js-1, k-1, r), k + j(s, r) - 1 \leq m \leq n-1\}. \end{aligned} \quad (3.2)$$

The following calculation based on (b) of the insertion lemma

$$\begin{aligned}
[j_s]M[Js, k, r] &= \sum_{(F, C(F))} q^{r-\text{maj } F + C(F)} \\
&= \sum_{(f; l)} q^{r-\text{maj}(f; l) + C(f; l)} + \sum_{(g; m)} q^{r-\text{maj}(g; m) + C(g; m)} \\
&= \sum_l q^l \sum_f q^{r-\text{maj } f} + \sum_m q^m \sum_g q^{r-\text{maj } g}
\end{aligned}$$

summed respectively over the sets of (3.1) and (3.2) establishes (1.8).

To show that $M[Js, o, r]$ is equal to the q -multinomial coefficient of n , first note that $M[J1, o, r] = 1$ and for all $s \leq r$ identity (1.8) reduces to

$$[j_s]M[Js, o, r] = [n]M[Js-1, o, r]. \quad (3.3)$$

Iteration of (3.3) yields

$$M[Js, o, r] = \frac{[n]}{[j_r]} M[Js-1, o, r] = \left[\begin{matrix} n \\ j_1 j_2 \dots j_r \end{matrix} \right].$$

4. THE EXPLICIT SOLUTION

Let $(t; q)_{n+1} = (1-t)(1-tq) \dots (1-tq^n)$ and $(q)_n = (q; q)_n = (1-q)(1-q^2) \dots (1-q^n)$. For a reference of the identities

$$\begin{aligned}
\text{(a)} \quad (t; q)_{n+1} &= \sum_{k \geq 0} (-1)^k \begin{bmatrix} n+1 \\ k \end{bmatrix} q^{\binom{k}{2}} t^k \\
\text{(b)} \quad (t; q)_{n+1}^{-1} &= \sum_{k \geq 0} \begin{bmatrix} n+k \\ k \end{bmatrix} t^k
\end{aligned} \quad (4.1)$$

see [2, p. 36]. Let $B[Js, k, r]$ denote the polynomial on the right-hand side of (1.9). To prove (1.9) it suffices to show that

$$B[Js, k, r] = \begin{cases} \begin{bmatrix} n \\ j_1 j_2 \dots j_r \end{bmatrix} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.2)$$

and that $B[Js, k, r]$ satisfies recurrence (1.8).

First, note that (a) of (4.1) implies

$$\sum_{k \geq 0} B[Js, k, r] t^k = (t; q)_{n+1} \sum_{k \geq 0} t^k \prod_{m=1}^s \begin{bmatrix} k + j(m, r) \\ j_m \end{bmatrix}. \quad (4.3)$$

In the case $r = s$, (4.3) and (b) of (4.1) show that

$$\sum_{k \geq 0} B[Js, k, r] t^k = \left[\begin{matrix} n \\ j_1 j_2 \dots j_r \end{matrix} \right] (t; q)_{n+1} \sum_{k \geq 0} \left[\begin{matrix} n+k \\ k \end{matrix} \right] t^k = \left[\begin{matrix} n \\ j_1 j_2 \dots j_r \end{matrix} \right]$$

which checks (4.2). Then using the identities

$$\begin{aligned} \text{(a)} \quad [j_s] \left[\begin{matrix} k-l+j(s, r) \\ j_s \end{matrix} \right] &= [k-l+j(s, r)] \left[\begin{matrix} k-l+j(s, r)-1 \\ j_s-1 \end{matrix} \right] \\ \text{(b)} \quad [k-l+j(s, r)] &= [k+j(s, r)] - q^{k-l+j(s, r)} [l] \\ \text{(c)} \quad \left[\begin{matrix} n+1 \\ l \end{matrix} \right] &= \left[\begin{matrix} n \\ l \end{matrix} \right] + q^{n+1-l} \left[\begin{matrix} n \\ l-1 \end{matrix} \right] \end{aligned} \quad (4.4)$$

it may be tediously verified that $B[Js, k, r]$ does indeed satisfy recurrence (1.8).

It is known that $r!$ divides the $(1, r)$ -Eulerian numbers. Identity (1.9) may be rewritten as

$$\begin{aligned} M[Js, k, r] &= \left[\begin{matrix} j(r, r) \\ j_1 j_2 \dots j_r \end{matrix} \right] \sum_{l=0}^k (-1)^l \left[\begin{matrix} n+1 \\ l \end{matrix} \right] \left[\begin{matrix} k-l+j(r, r) \\ k-l \end{matrix} \right] q^{\binom{l}{2}} \\ &\quad \times \prod_{m=r+1}^s \left[\begin{matrix} k-l+j(m, r) \\ j_m \end{matrix} \right] \end{aligned}$$

which shows that the q -multinomial coefficient of $j(r, r)$ divides the polynomial $M[Js, k, r]$.

5. THE GENERATING FUNCTION FOR THE R -MAJOR INDEX

Note that

$$\sum_{k \geq 0} M[Js, k, r] = \sum_f q^{r\text{-maj } f} \quad (5.1)$$

summed over $R(Js)$. Identity (4.3) with $t = 1$ and $|q| < 1$, and the fact that $[n]! = (q)_n(1-q)^{-n}$ imply

$$\begin{aligned} \sum_{k \geq 0} M[Js, k, r] &= (q)_n \prod_{m=1}^s \lim_{k \rightarrow \infty} \left[\begin{matrix} k+j(m, r) \\ j_m \end{matrix} \right] \\ &= (q)_n \prod_{m=1}^s (q)_{j_m}^{-1} = \left[\begin{matrix} n \\ j_1 j_2 \dots j_s \end{matrix} \right] \end{aligned} \quad (5.2)$$

which verifies (1.4).

6. THE CONSTRUCTION OF $\Phi_{c, b}$

Denote by $U(j, l)$ the set of words $u = u(1)u(2) \dots u(j)$ with letters $u(m) \in \{0, 1, \dots, l-1\}$ for $1 \leq m \leq j$ and $u(1) \leq u(2) \leq \dots \leq u(j)$. Let $W(Js)$ be the set of

words $w = u_1 u_2 \dots u_s$ obtained by juxtaposing words $u_m \in U(j_m, 1 + j(m-1, m-1))$ for $1 \leq m \leq s$. Define $\Sigma(z)$ to be the sum of the letters in a word z . First, a bijection $\Gamma_r: R(Js) \rightarrow W(Js)$ is constructed such that

$$\text{if } \Gamma_r(f) = w \text{ then } r\text{-maj } f = \Sigma(w). \quad (6.1)$$

It is then clear that the composition $\Phi_{b,c} = \Gamma_c^{-1} \Gamma_b$ satisfies (1.5).

The construction of Γ_r is inductive. When $s = 1$, $R(J1)$ and $W(J1)$ consist respectively of the words $f = 1 \ 1 \dots 1$ and $w = 0 \ 0 \dots 0$ of length j_1 . Clearly, $r\text{-maj } f = \Sigma(w) = 0$.

In general, from $f \in R(Js)$ remove the s that has the minimum effect on the r -major index. If there is more than one such s , take the one furthest to the right. Let $u_s(1)$ denote the resulting decrease in the r -major index. Remove the next s with minimum effect and set $u_s(2)$ equal to the decrease in the r -major index. Repeat until all j_s letters equal to s have been removed from f . Let g denote the resulting word and let $u_s = u_s(1)u_s(2) \dots u_s(j_s)$. It follows from the order of removal and the definition of u_s that

$$(a) \ r\text{-maj } f = r\text{-maj } g + \Sigma(u_s) \quad (6.2)$$

$$(b) \ u_s(1) \leq u_s(2) \leq \dots \leq u_s(j_s).$$

Part (b) of the insertion lemma implies that $u_s(j_s) \leq j(s-1, s-1)$ showing that $u_s \in U(j_s, 1 + j(s-1, s-1))$. Induction then completes the construction.

As an example of the process, for $r = 3$ the subscripts of the letters $s = 5$ in the word

$$f = 2 \ 1 \ 5_2 \ 2 \ 4 \ 3 \ 1 \ 5_3 \ 5_1 \ 2 \in R(2, 3, 1, 1, 3) \quad (6.3)$$

indicate the order of removal of the first $j_5 = 3$ letters. Removal gives $g = 2 \ 1 \ 2 \ 4 \ 3 \ 1 \ 2$ and $u_5 = 1 \ 6 \ 7$. Iteration then leads to $\Gamma_3(f) = u_1 u_2 \dots u_5 = 0 \ 0 \ 0 \ 1 \ 2 \ 2 \ 2 \ 1 \ 6 \ 7 \in W(2, 3, 1, 1, 3)$ with $3\text{-maj } f = 21 = \Sigma(u_1 u_2 \dots u_5)$.

The correspondence Γ_r provides another proof of (1.4). Using Γ_r and the identity

$$\sum q^{\Sigma(u_m)} = \begin{bmatrix} j(m, m) \\ j_m \end{bmatrix} \quad (6.4)$$

summed over $u_m \in U(j_m, 1 + j(m-1, m-1))$, which may be found in Carlitz [3], the calculation

$$\sum_f q^{r\text{-maj } f} = \prod_{m=1}^s \sum_{u_m} q^{\Sigma(u_m)} = \prod_{m=1}^s \begin{bmatrix} j(m, m) \\ j_m \end{bmatrix} = \begin{bmatrix} n \\ j_1 j_2 \dots j_s \end{bmatrix} \quad (6.5)$$

where the first sum is over $f \in R(Js)$, proves again that the q -multinomial coefficient of n is the generating function for the r -major index. Carlitz [3] used calculation (6.5) to prove (1.4) in the case $r \geq s$.

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