

THE COMBINATORICS OF CERTAIN PRODUCTS

DON RAWLINGS

ABSTRACT. A combinatorial interpretation for the coefficients in the expansion of $\prod (1 + ux^jy^k)(1 - ux^jy^k)^{-1}$ is given.

1. Introduction. The coefficients $A(n; x, y)$, $B(n; x, y)$ and $C(n; x, y)$ defined by

$$(1.1) \quad \sum_{n \geq 0} \frac{A(n; x, y)u^n}{(x)_n(y)_n} = \frac{1}{(u; x, y)},$$

$$(1.2) \quad \sum_{n \geq 0} \frac{B(n; x, y)u^n}{(x)_n(y)_n} = (-u; x, y),$$

$$(1.3) \quad \sum_{n \geq 0} \frac{C(n; x, y)u^n}{(x)_n(y)_n} = \frac{(-u; x, y)}{(u; x, y)},$$

where

$$(1.4) \quad (x)_n = (1 - x)(1 - x^2) \cdots (1 - x^n), \quad (x)_0 = 1,$$

$$(1.5) \quad (u; x, y) = \prod_{j, k \geq 0} (1 - ux^jy^k),$$

have been considered by a number of mathematicians. Carlitz [2] was the first to demonstrate that these coefficients are polynomials in x and y with positive integral coefficients and provided closed formulas for calculating them. In a subsequent paper Roselle [6] gave combinatorial interpretations for $A(n; x, y)$ and $B(n; x, y)$. More recently, generalizations of (1.1) and (1.2) have appeared in [3, 4, 5] and [4], respectively.

A natural question, which was suggested to me by Dominique Foata, arises from Roselle's work: Is there a combinatorial interpretation for $C(n; x, y)$? This note provides one in terms of the already known interpretations of $A(n; x, y)$ and $B(n; x, y)$ and in terms of certain statistics defined on bicolored permutations.

2. Roselle's work. The polynomials $A(n; x, y)$ and $B(n; x, y)$ may be interpreted as the generating functions for various statistics defined on the group $G(n)$ consisting of permutations of $\{1, 2, \dots, n\}$. To be precise, if the major index of a permutation $\sigma \in G(n)$, denoted $m(\sigma)$, is defined to be the sum of the elements in the set

$$(2.1) \quad \{j: \sigma(j) > \sigma(j+1), 1 \leq j \leq n-1\}$$

Received by the editors September 30, 1982.

1980 *Mathematics Subject Classification*. Primary 05A15.

©1983 American Mathematical Society
 0002-9939/83 \$1.00 + \$.25 per page

and the i -major and co- i -major indices are defined, respectively, to be

$$(2.2) \quad i(\sigma) = m(\sigma^{-1}),$$

$$(2.3) \quad c(\sigma) = \binom{n}{2} - i(\sigma),$$

then, as Roselle [6] and Foata [4] have demonstrated,

$$(2.4) \quad A(n; x, y) = \sum_{\sigma \in G(n)} x^{m(\sigma)} y^{i(\sigma)},$$

$$(2.5) \quad B(n; x, y) = \sum_{\sigma \in G(n)} x^{m(\sigma)} y^{c(\sigma)}.$$

3. Bicolored permutations. Since Carlitz [2] showed that $C(n; 1, 1) = 2^n \cdot n!$, it is clear that $G(n)$ will not provide a combinatorial structure for interpreting $C(n; x, y)$. For this reason, one is led to consider the set of bicolored permutations $BG(n)$, consisting of words $b = b(1)b(2) \cdots b(n)$ obtained by coloring each letter of some permutation $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n) \in G(n)$ either red (indicated by underlining the letter) or blue (letter not underlined). For instance,

$$(3.1) \quad b = 3 \underline{2} \underline{7} \underline{4} \underline{1} \underline{6} 5 \in BG(7).$$

Note that there are 2^n different colorings of each $\sigma \in G(n)$.

Now, possessing a combinatorial set with the correct cardinality, all that remains is to define some appropriate statistics on $BG(n)$ for which $C(n; x, y)$ is the generating function. To this end, let $W(n)$ be the set of words $w = w(1)w(2) \cdots w(n)$ of length n with letters $w(i) \in \{0, 1\}$, and let $|w|$ denote the number of letters equal to 1 in w . The reduction of a permutation γ of $\{a_1 < a_2 < \cdots < a_n\}$ is obtained by replacing a_i in γ by i for $1 \leq i \leq n$. For instance, the reduction of the permutation $\gamma = 2416$ of $\{1 < 2 < 4 < 6\}$ is 2314 .

Each $b \in BG(n)$ is now assigned to a 4-tuple (w, v, θ, α) where $w, v \in W(n)$ with $|w| = |v|$, $\theta \in G(|w|)$ and $\alpha \in G(n - |w|)$ according to the following rules:

$$(3.2) \quad w(i) = \begin{cases} 1 & \text{if } b(i) \text{ is red,} \\ 0 & \text{otherwise,} \end{cases}$$

$$(3.3) \quad v(i) = \begin{cases} 1 & \text{if } i \text{ is red in } b, \\ 0 & \text{otherwise,} \end{cases}$$

$$(3.4) \quad \theta = \text{reduction of the red subpermutation of } b,$$

$$(3.5) \quad \alpha = \text{reduction of the blue subpermutation of } b.$$

Roughly speaking, w indicates which positions of b are red, v which elements of $\{1, 2, \dots, n\}$ are placed in the red positions, θ how the red letters are arranged in the red positions, and α how the blue letters are placed in the remaining positions. For instance, the bicolored permutation b of (3.1) corresponds to the 4-tuple (w, v, θ, α) where $w = 0101110$, $v = 1101010$, $\theta = 2314$ and $\alpha = 132$. Note that the map $b \rightarrow (w, v, \theta, \alpha)$ is a bijection between $BG(n)$ and the set of such 4-tuples.

Finally, for the bicolored permutation $b \rightarrow (w, v, \theta, \alpha)$ the appropriate statistics are defined by

$$(3.6) \quad M(b) = m(\theta) + m(\alpha) + m(w),$$

$$(3.7) \quad I(b) = i(\theta) + c(\alpha) + m(v),$$

where $m(w)$ and $m(v)$ are defined in exactly the same way as the major index of a permutation.

4. The interpretation of $C(n; x, y)$. It is now possible to show that

$$(4.1) \quad C(n; x, y) = \sum_{b \in BG(n)} x^{M(b)} y^{I(b)}.$$

First, as Carlitz [2] demonstrated, identities (1.1)–(1.3) imply

$$(4.2) \quad C(n; x, y) = \sum_{k \geq 0} \binom{n}{k}_x \binom{n}{k}_y A(k; x, y) B(n-k; x, y),$$

where the x -binomial coefficient is defined by

$$(4.3) \quad \binom{n}{k}_x = \frac{(x)_n}{(x)_k (x)_{n-k}}.$$

Second, the fact that (see [1, p.40])

$$(4.4) \quad \sum x^{m(w)} = \binom{n}{k}_x,$$

where the summation is over $w \in W(n)$ with $|w| = k$, along with (2.4), (2.5), (3.6) and (3.7) allow the calculation

$$\begin{aligned} (4.5) \quad \sum_{b \in BG(n)} x^{M(b)} y^{I(b)} &= \sum_{(w, v, \theta, \alpha)} x^{m(\theta) + m(\alpha) + m(w)} y^{i(\theta) + c(\alpha) + m(v)} \\ &= \sum_{k \geq 0} \sum_{|w|=k} x^{m(w)} \sum_{|v|=k} y^{m(v)} \sum_{\theta \in G(k)} x^{m(\theta)} y^{i(\theta)} \sum_{\alpha \in G(n-k)} x^{m(\alpha)} y^{c(\alpha)} \\ &= \sum_{k \geq 0} \binom{n}{k}_x \binom{n}{k}_y A(k; x, y) B(n-k; x, y). \end{aligned}$$

Identities (4.2) and (4.5) imply (4.1).

REFERENCES

1. G. E. Andrews, *The theory of partitions*, Addison-Wesley, Reading, Mass., 1976.
2. L. Carlitz, *The expansion of certain products*, Proc. Amer. Math. Soc. **7** (1956), 558–564.
3. A. M. Garsia and I. Gessel, *Permutation statistics and partitions*, Adv. in Math. **31** (1979), 288–305.
4. D. Foata, *Aspects combinatoires du calcul des q -séries* (Séminaire d'Informatique Théorique, 1980), Institut de Programmation, Centre National Recherche Scientifique No. 248, Paris, 1980.
5. D. Rawlings, *Generalized Worpitzky identities with applications to permutation enumeration*, European J. Combin. **2** (1981), 67–78.
6. D. P. Roselle, *Coefficients associated with the expansion of certain products*, Proc. Amer. Math. Soc. **45** (1974), 144–150.

DEPARTMENT OF MATHEMATICS, CALIFORNIA POLYTECHNIC STATE UNIVERSITY, SAN LUIS OBISPO, CALIFORNIA 93407