Lifted p-Adic Homology with Compact Supports of the Weierstrass Family and Its Zeta Endomorphism

GORO KATO

School of Mathematics, The Institute for Advanced Study, Princeton, New Jersey 08540, U.S.A.

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The relations among the generators for the lifted p-adic homology with compact supports of the various subfamilies of the Weierstrass family in characteristic $p > 0$ ($p \neq 2, 3$) are explicitly given in Section 2. Then, the universal coefficient spectral sequence and the zeta endomorphism in Section 3 enable one to compute explicitly the lifted p-adic homology with compact supports of all fibres, including all the elliptic curves and all their singular degenerations in the family.

1. INTRODUCTION

Let p be a positive rational prime, $p \neq 2, 3$. Then the Weierstrass family corresponding to the field $\mathbb{Z}/p\mathbb{Z}$ is the algebraic family over $\operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})[g_2, g_3])$ defined by

$$\operatorname{Proj}\left(\frac{(\mathbb{Z}/p\mathbb{Z})[g_2, g_3, X, Y, Z]}{(\text{homogeneous ideal generated by } Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3)}\right),$$

where $(\mathbb{Z}/p\mathbb{Z})[g_2, g_3, X, Y, Z]$ is the graded $(\mathbb{Z}/p\mathbb{Z})[g_2, g_3]$-algebra such that each X, Y, Z has degree $+1$ and all of the elements in $(\mathbb{Z}/p\mathbb{Z})[g_2, g_3]$ have degree zero. We denote the Weierstrass family by $\mathbb{W}_{\mathbb{Z}/p\mathbb{Z}}$. Let U be the open subset of $\mathbb{W}_{\mathbb{Z}/p\mathbb{Z}}$ consisting of finite points, i.e.,

$$U = \mathbb{W}_{\mathbb{Z}/p\mathbb{Z}} \cap \mathbb{A}^2(\operatorname{Spec}((\mathbb{Z}/p\mathbb{Z})[g_2, g_3])),$$

the closed subscheme of $\mathbb{A}^2(\operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})[g_2, g_3]))$ whose equation is given by

$$Y^2 = 4X^3 - g_2X - g_3.$$

Note that the set of points at ∞, $\mathbb{W}_{\mathbb{Z}/p\mathbb{Z}} - U$, is a closed subscheme of

$\mathbb{W}_{\mathbb{Z}/p\mathbb{Z}}$ which is isomorphic to Spec $((\mathbb{Z}/p\mathbb{Z})[g_2, g_3])$. Therefore, each fibre of $\mathbb{W}_{\mathbb{Z}/p\mathbb{Z}}$ has exactly one point at ∞, which is a rational point in the fibre. Let $\underline{A} = \hat{\mathbb{Z}}_p[g_2, g_3]$ and let $A = (\mathbb{Z}/p\mathbb{Z})[g_2, g_3]$. Then, from the long exact sequence corresponding to the triple ((points at ∞), $\mathbb{W}_{\mathbb{Z}/p\mathbb{Z}}$, U), $\dashrightarrow H^c_{i-2}$ ("points at ∞," $\underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \to H^c_i(\mathbb{W}_{\mathbb{Z}/p\mathbb{Z}}, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \to H^c_i(U, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q})$ $\to$ --- and the first group being zero for $i \neq 2$, we have $H^c_i(\mathbb{W}_{\mathbb{Z}/p\mathbb{Z}}, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q})$ $\approx H^c_i(U, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q})$ for $i = 1$. By the definition in [5], we have

$$H^c_1(U, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) = H^3(\mathbb{A}^2(\mathrm{Spec}((\mathbb{Z}/p\mathbb{Z})[g_2, g_3])), \mathbb{A}^2(\mathrm{Spec}((\mathbb{Z}/p\mathbb{Z}) \times [g_2, g_3])) - U, \Gamma^*_{\underline{A}}(\mathbb{A}^2(\mathrm{Spec}(\hat{\mathbb{Z}}_p[g_2, g_3])))^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}). \tag{1}$$

Note also that a unique singular point of each fibre over μ on the closed subscheme "$\Delta = (g_2^3 - 27g_3^2 = 0)$" lies in the affine open U.

If one knows

(i) the lifted p-adic homology with compact supports of U and

(ii) the zeta endomorphism of the homology group,

then one can determine the lifted p-adic homology with compact supports of all the fibres in the family. This is because the zeta function of a fibre is given by

$$Z_\mu(T) = \frac{\prod_{p+q=\mathrm{odd}} P_{p,q}(T)}{\prod_{p+q=\mathrm{even}} P_{p,q}(T)},$$

where $P_{p,q}$ is the reverse characteristic polynomial of the endomorphism of the $E^2_{p,q}$-term of the universal coefficient spectral sequence $\mathrm{Tor}_p^{\underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}}(H^c_q(U, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}), W(\mathbb{k}(\mu)) \otimes_{\mathbb{Z}} \mathbb{Q})$, where $W(\mathbb{k}(\mu))$ is the complete discrete valuation ring, e.g., for a perfect field $\mathbb{k}(\mu)$. Furthermore, the above universal coefficient spectral sequence abuts upon the finitely generated lifted p-adic homology with compact supports of the fibre, which gives the zeta function of that fibre (see [5, 6] Chaps. 5 and 6). See [4] also.

The topics of this paper are (i) and (ii) above for the Weierstrass family. The preimage of $\mathrm{Spec}((\mathbb{Z}/p\mathbb{Z})[g_2, g_3, \Delta^{-1}])$ of U is the open subfamily considered in [1].

2. Module Structure

Let $A = (\mathbb{Z}/p\mathbb{Z})[g_2, g_3] = \underline{A} \otimes_{\hat{\mathbb{Z}}_p} (\mathbb{Z}/p\mathbb{Z})$. One can use the covering $\{\mathbb{A}^2(\mathrm{Spec}(A)), \mathbb{A}^2(\mathrm{Spec}(A)) - (Y^2 - 4X^3 + g_2 X + g_3 = 0)\}$ to compute the cohomology group (1) in the Introduction. Then the long sequence $\dashrightarrow^{\partial^{n-1}} H^n(\mathbb{A}^2(\mathrm{Spec}(A)), \mathbb{A}^2(\mathrm{Spec}(A)) - U, \Gamma^*_{\underline{A}}(\mathbb{A}^2(\mathrm{Spec}(\underline{A})))^\dagger \otimes_{\mathbb{Z}} \mathbb{Q})$

$\to H^n(\mathbb{A}^2(\mathrm{Spec}(A)), \Gamma^*_{\underline{A}}(\mathbb{A}^2(\mathrm{Spec}(A)))^\dagger \otimes_{\mathbf{Z}} \mathbb{Q}) \to H^n(\mathbb{A}^2(\mathrm{Spec}(A)) - U, \Gamma^*_{\underline{A}}(\mathbb{A}^2(\mathrm{Spec}(\underline{A})))^\dagger \otimes_{\mathbf{Z}} \mathbb{Q}) \xrightarrow{\partial^n} ---$ is induced. The second and third groups are cohomologies of the global sections.

THEOREM 2.1. *The $\underline{A}^\dagger \otimes_{\mathbf{Z}} \mathbb{Q}$-module $H^c_1(U, \underline{A}^\dagger \otimes_{\mathbf{Z}} \mathbb{Q})$ has the recursive cohomologous relations among the generators*

$$\begin{aligned} 2(i-1)\,\Delta C^{-i}\,dX \wedge dY &\sim (6i-13)\,6g_2 XC^{-(i-1)}\,dX \wedge dY \\ &\quad - (6i-11)\,9g_3 C^{-(i-1)}\,dX \wedge dY \\ 4(i-1)\,\Delta XC^{-i}\,dX \wedge dY &\sim (6i-11)\,g_2^2 C^{-(i-1)}\,dX \wedge dY \\ &\quad - (6i-13)\,18g_3 XC^{-(i-1)}\,dX \wedge dY, \end{aligned} \tag{CR}$$

$i \geqslant 2$, where $C = Y^2 - 4X^3 + g_2X + g_3$ and Δ is the discriminant $g_2^3 - 27g_3^2$. In particular, $H^c_1(U, \underline{A}^\dagger \otimes_{\mathbf{Z}} \mathbb{Q})$ is generated by $\{C^{-i}\,dX \wedge dY, XC^{-i}\,dX \wedge dY\}_{i \geqslant 1}$ over $\underline{A}^\dagger \otimes_{\mathbf{Z}} \mathbb{Q}$.

Proof. The cohomology group (1) in the Introduction is the abutment of the spectral sequence $H^q(\mathbb{A}^2(A), \mathbb{A}^2(A) - U, \Gamma^p_{\underline{A}}(\mathbb{A}^2(\underline{A}))^\dagger \otimes_{\mathbf{Z}} \mathbb{Q})$. Then we have the isomorphisms by Lemma 1 in [2]:

$$\begin{aligned} H^c_1(U, \underline{A}^\dagger \otimes_{\mathbf{Z}} \mathbb{Q}) &\approxeq H^2(\mathbb{A}^2(A) - U, \Gamma^*_{\underline{A}}(\mathbb{A}^2(\underline{A}))^\dagger \otimes_{\mathbf{Z}} \mathbb{Q}) \\ &\approxeq \mathrm{Coker}(\Gamma^2_{\underline{A}}(\underline{A}[X, Y, C^{-1}]^\dagger) \\ &\xleftarrow{d_1^{1,0}} \Gamma^1_{\underline{A}}(\underline{A}[X, Y, C^{-1}]^\dagger)). \end{aligned} \tag{2}$$

The cohomologous relations, induced by the map $d_1^{1,0}$ in (2), among the elements of $\Gamma^2_{\underline{A}}(\underline{A}[X, Y, C^{-1}])$ are given by

$$\begin{aligned} &2iX^kY^{j+1}C^{-i-1}\,dX \wedge dY \sim jX^kY^{j-1}C^{-i}\,dX \wedge dY \\ &12iX^{k+2}Y^jC^{-i-1}\,dX \wedge dY + kX^{k-1}Y^jC^{-i}\,dX \wedge dY \\ &\quad \sim g_2 iX^kY^jC^{-i-1}\,dX \wedge dY. \end{aligned} \tag{3}$$

Then we have

$$\begin{aligned} \frac{(6i-11)}{6(i-1)} C^{-(i-1)}\,dX \wedge dY &\sim \frac{2g_2}{3} XC^{-i}\,dX \wedge dY + g_3 C^{-i}\,dX \wedge dY \\ \frac{6i-13}{6(i-1)} XC^{-(i-1)}\,dX \wedge dY &\sim \frac{g_2^2}{18} C^{-i}\,dX \wedge dY + g_3 XC^{-i}\,dX \wedge dY. \end{aligned} \tag{4}$$

The equations (4) plainly imply Eq. (CR) in Theorem 2.1. The generation of $H^c_1(U, \underline{A}^\dagger \otimes_{\mathbf{Z}} \mathbb{Q})$ by the elements $\{C^{-i}\,dX \wedge dY, XC^{-i}\,dX \wedge dY\}_{i \geqslant 1}$

can be shown in the same way as in the case of characteristic zero (see [2]). The universal coefficient spectral sequence implies, if U_Δ is the open subfamily of non-singular fibres,

$$H_1^c(U, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \underset{\underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}}{\otimes} ((\Delta^{-1}\underline{A})^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \overset{\sim}{\rightarrow} H_1^c(U_\Delta, (\Delta^{-1}\underline{A})^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}), \qquad (5)$$

where $\Delta^{-1}\underline{A}$ denotes the localization of $\underline{A}$ at the discriminant Δ. (The latter was computed in [1] to be free of rank two over $(\Delta^{-1}\underline{A})^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}$).) Applying the long exact sequence for $k = 2, 1, 0$ in the following Note 1, we have the exact sequence

$$\begin{aligned}
0 &\to \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\text{“}\Delta\text{”}} \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q} \to \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}/\Delta \cdot \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q} \\
&\to H_1^c(U, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{\text{“}\Delta\text{”}} H_1^c(U, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \\
&\to H_1^c(U', \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}/\Delta \cdot \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \\
&\to \underline{A} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\text{“}\Delta\text{”}} \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q} \\
&\to \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}/\Delta \cdot \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q} \to 0.
\end{aligned}$$

From this, we extract the short exact sequence

$$\begin{aligned}
0 &\to H_1^c(U, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{\text{“}\Delta\text{”}} H_1^c(U, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \\
&\to H_1^c(U', \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}/\Delta \cdot \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \to 0.
\end{aligned}$$

That is, $H_1^c(U, \underline{A} \otimes_{\mathbb{Z}} \mathbb{Q})$ has no non-zero Δ-torsion; i.e., $H_1^c(U, \underline{A} \otimes_{\mathbb{Z}} \mathbb{Q})$ is torsion free. Therefore the equations (CR) tell us that there is no cohomologous relation among the set of generators in spite of the "$\dagger$" completion of the base ring A; hence, the inclusion (6) in Section 3 follows. Otherwise, the homology groups on the left in (5) would become free of rank one over $(\Delta^{-1}\underline{A})^\dagger \oplus_{\mathbb{Z}} \mathbb{Q}$.

Remark 1. The isomorphism in (5) can be given by $C^{-1}\, dX \wedge dY \to Y\, dX$ and $XC^{-1}\, dX \wedge dY \to XY\, dX$ as $(\Delta^{-1}\underline{A})^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}$-modules (see [1]).

COROLLARY 2.2. *Let U' be the closed Weierstrass subfamily over $A/(\Delta \cdot A)$, where $A = (\mathbb{Z}/p\mathbb{Z})[g_2, g_3]$ and $\Delta = g_2^3 - 27g_3^2$; i.e., U' is the closed subscheme over $A/\Delta \cdot A$ consisting of all the singular fibres of U. Then, the lifted p-adic homology with compact supports of this Weierstrass subscheme is generated by $\{C^{-i}\, dX \wedge dY,\ XC^{-i}\, dX \wedge dY\}_{i \geq 1}$ over $\underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}/\Delta \cdot \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Note 1. Since generally we have, for a non-zero-divisor $\Delta \in \underline{A}$,

$$\operatorname{Tor}^i_{\underline{A}}(H^c_1(U, \underline{A} \otimes_{\mathbb{Z}} \mathbb{Q}), \underline{A}/\Delta\underline{A})$$
$$= \begin{cases} H^c_1(U, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q})/\Delta \cdot H^c_1(U, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}), & i=0, \\ \operatorname{Ker}(\text{mult. by } \Delta), & i=1, \\ 0, & i \geqq 2, \end{cases}$$

respectively, we have the long exact sequence

$$--- \to H^c_k(U, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{\text{“}\Delta\text{”}} H^c_k(U, \underline{A} \otimes_{\mathbb{Z}} \mathbb{Q})$$
$$\to H^c_k(U', \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}/\Delta \cdot \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \to ---$$

from the corresponding universal coefficient spectral sequence (see [6, Chap. 5]).

Proof. Since $H^c_1(U', \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}/\Delta \cdot \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q})$ is obtained by taking the cohomology of the cochain complex, which is obtained by tensoring the cochain complex

$$C^*(\mathbb{A}^2(A), \mathbb{A}^2(A) - U, \Gamma^*_{\underline{A}}(\mathbb{A}^2(\underline{A}))^\dagger \otimes_{\mathbb{Z}} \mathbb{Q})$$

with $\underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}/\Delta \cdot \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\underline{A} \otimes_{\mathbb{Z}} \mathbb{Q}$ (see [5]), the assertion of Corollary 2.2 is obtained from (CR) in Theorem 2.1 by substituting $\Delta = 0$; i.e.,

$$\begin{aligned} (6i-13)\, 2g_2 X C^{-(i-1)}\, dX \wedge dY &\sim (6i-11)\, 3g_3 C^{-(i-1)}\, dX \wedge dY \\ (6i-11)\, g_2^2 C^{-(i-1)} dX \wedge dY &\sim (6i-13)\, 18 g_3 X C^{-(i-1)}\, dX \wedge dY. \end{aligned} \qquad \text{(CR}'\text{)}$$

Remark 2. Let $\bar{\Delta} = \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}/\Delta \cdot \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}$. One can observe that $H^c_1(U', \bar{\Delta}) \otimes_{\bar{\Delta}} \bar{\Delta}_{g_2}$ and $H^c_1(U', \bar{\Delta}) \otimes_{\bar{\Delta}} \bar{\Delta}_{g_3}$, where $\bar{\Delta}_{g_2}$ and $\bar{\Delta}_{g_3}$ are localizations at g_2 and g_3, respectively, are generated by $\{C^{-i}\, dX \wedge dY\}_{i \geqslant 1}$ or, since $g_2 \neq 0$ implies $g_3 \neq 0$, $\{XC^{-i}\, dX \wedge dY\}_{i \geqslant 1}$ over $\bar{\Delta}_{g_i}$, $i = 2, 3$. Note also that if $g_2 = 0$ (then $g_3 = 0$), (4) computes the homology of the singular fibre over $\mu = (g_2 = g_3 = 0)$; i.e., the corresponding homology group is trivial.

Note 2. We have the short exact sequence

$$0 \to H^c_1(U, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{\text{“}\Delta\text{”}} H^c_1(U, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \to H^c_1(U', \bar{\Delta}) \to 0$$

from the universal coefficient spectral sequence (see the proof of Theorem 2.1 and [6, Chap. 3]) induced by the short exact sequence

$$0 \to \underline{A}^{\dagger} \otimes_{\mathbf{Z}} \mathbb{Q} \xrightarrow{\text{mult. by } \Delta} \underline{A}^{\dagger} \otimes_{\mathbf{Z}} \mathbb{Q} \to \bar{\Delta} \to 0;$$

see Note 1.

COROLLARY 2.3. *Let U^0 and U^3 be the corresponding closed subfamilies of the Weierstrass obtained by pulling back the Weierstrass family to the closed subsets $(g_2 = 0)$ and $(g_2 = 3)$ of the base scheme, respectively. Then, the module structures of $H_1^c(U^0, \hat{\mathbf{Z}}_p[g_3]^{\dagger} \otimes \mathbb{Q})$ and $H_1^c(U^3, \hat{\mathbf{Z}}_p[g_3]^{\dagger} \otimes \mathbb{Q})$ over $\hat{\mathbf{Z}}_p[g_3]^{\dagger} \otimes_{\mathbf{Z}} \mathbb{Q}$ are given from* (CR) *or equations* (4) *by substituting $g_2 = 0$ and $g_2 = 3$, respectively. That is, the equations* (4) *become* (4^3) *and* (4^0) *for $g_2 = 3$ and $g_2 = 0$, respectively,*

$$\begin{aligned} \frac{6i-11}{6(i-1)} C^{-(i-1)}\, dX \wedge dY &\sim 2XC^{-i}\, dX \wedge dY + g_3 C^{-i}\, dX \wedge dY \\ \frac{2(6i-13)}{6(i-1)} XC^{-(i-1)}\, dX \wedge dY &\sim 2g_3 XC^{-i}\, dX \wedge dY + C^{-i}\, dX \wedge dY \end{aligned} \tag{4^3}$$

$$\begin{aligned} \frac{6i-11}{6(i-1)} C^{-(i-1)}\, dX \wedge dY &\sim g_3 C^{-i}\, dX \wedge dY \\ \frac{6i-13}{6(i-1)} XC^{-(i-1)}\, dX \wedge dY &\sim g_3 XC^{-i}\, dX \wedge dY. \end{aligned} \tag{4^0}$$

If $g_3 = +1$, i.e., the singular fibre over $\mu = (g_2 = 3, g_3 = 1)$, then the corresponding homology of this projective line with an ordinary double point is free of rank one; one can choose, for example, $C^{-1}\, dX \wedge dY$ as a basis element. For $g_3 = -1$ use (CR) *to have the corresponding statement. If $g_3 \neq 0$ in* (4^0), *then the open subfamily of U^0 defined by "$g_3 \neq 0$" has the homology generated by two elements $C^{-1}\, dX \wedge dY$ and $XC^{-1}\, dX \wedge dY$.*

Proof. The above statements can be observed plainly from (CR) and (4).

3. ZETA ENDOMORPHISM

Define a ring endomorphism $F^{\dagger}\colon \hat{\mathbf{Z}}_p[g_2, g_3]^{\dagger} \to \hat{\mathbf{Z}}_p[g_2\ g_3]^{\dagger}$ over $\hat{\mathbf{Z}}_p$ as $F(g_2) = g_2^p$ and $F(g_3) = g_3^p$ and let f be the pth power map of the Weierstrass scheme in characteristic p. Then the first zeta endomorphism $H_1^c(F, f)$ is induced on the lifted p-adic homology with compact supports of the Weierstrass family $M = H_1^c(U, \hat{\mathbf{Z}}_p[g_2, g_3]^{\dagger} \otimes_{\mathbf{Z}} \mathbb{Q})$. The homology

group $H_1^c(U, \hat{\mathbb{Z}}_p[g_2, g_3]^\dagger \otimes_{\mathbb{Z}} \mathbb{Q})$ becomes a vector space of dimension two after being tensored with the quotient field of the ring $\hat{\mathbb{Z}}_p[g_2, g_3]$. Since M is torsion free (see, the proof of Theorem 2.1, Notes 1 and 2 in Section 2) we have the inclusion

$$M \hookrightarrow M \otimes \hat{\mathbb{Q}}_p(g_2, g_3). \tag{6}$$

Let $\mathbb{K}^\dagger$ be the quotient field of $\hat{\mathbb{Z}}_p[g_2, g_3]^\dagger$. Then $M \otimes \hat{\mathbb{Q}}_p(g_2, g_3) \otimes_{\mathbb{Q}_p(g_2, g_3)} \mathbb{K}^\dagger$ is $H_1^c(U_g, \mathbb{K}^\dagger)$, where U_g is the generic fibre of U, i.e., the p-adic homology with compact supports of an elliptic curve. Therefore, the zeta matrix, like the one computed in [1], induces a semi-linear endomorphism of the free module

$$M \underset{\hat{\mathbb{Z}}_p[g_2, g_3]^\dagger}{\otimes} (\Delta^{-1}\hat{\mathbb{Z}}_p[g_2, g_3])^\dagger \tag{7}$$

of rank two over $(\Delta^{-1}\hat{\mathbb{Z}}_p[g_2, g_3])^\dagger$. The zeta endomorphism of M is obtained by restricting the zeta matrix of (7) on M by the inclusion (6). We now compute it explicitly as follows:

$$H_1^c(F, f)(C^{-1}\, dX \wedge dY), \tag{B_1}$$

$$H_1^c(F, f)(XC^{-1}\, dX \wedge dY). \tag{B_2}$$

(B_1) equals

$$\frac{1}{Y^{2p} - 4X^{3p} + g_2^p X^p + g_3^p}\, dX^p \wedge dY^p = \frac{p^2 X^{p-1} Y^{p-1}}{C^p - pT}\, dX \wedge dY,$$

where $C = Y^2 - 4X^3 + g_2 X + g_3$ and T is a polynomial in g_2, g_3, X, and Y. Similarly (B_2) equals

$$\frac{p^2 X C^{2p-1} Y^{p-1}}{C^p - pT}\, dX \wedge dY.$$

Rewrite $C^p - pT$ as $C^p(1 - pT/C^p)$. Then (B_1) and (B_2) become

$$\sum_{l \geq 0} p^{l+2} T^l X^{p-1} Y^{p-1} C^{-p(l+1)}\, dX \wedge dY \tag{B_1'}$$

$$\sum_{l \geq 0} p^{l+2} T^l X^{2p-1} Y^{p-1} C^{-p(l+1)}\, dX \wedge dY. \tag{B_2'}$$

Let $2j = p' - 1$, where p' is uniquely determined by p and the even power

of Y in T. Then the first equation of (3) implies that the terms $X^i Y^{2j} C^{-p(l+1)}\, dX \wedge dY$ in (B_1') and (B_2') can be written as

$$\frac{(2j-1)\cdots(2j-2\beta+1)\cdots 3\cdot 1}{k(k-1)\cdots(k-j+1)} X^i C^{-p(l+1)+j}\, dX \wedge dY.$$

Now the second equation in (3) implies that the term $X^i C^{-p(l+1)+j}\, dX \wedge dY$ can be written as a linear combination of $\{XC^{-k}\, dX \wedge dY, C^{-k}\, dX \wedge dY\}_{k \geq 1}$ by Theorem 2.1 over $\hat{\mathbb{Z}}_p[g_2, g_3]^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}$ (see [2]). Consequently, over $\Delta^{-1}\hat{\mathbb{Z}}_p[g_2, g_3]^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}$, ($B_1$) and ($B_2$) can be written as linear combinations of $C^{-1}\, dX \wedge dY$ and $XC^{-1}\, dX \wedge dY$ by (CR) in Theorem 2.1 (see also Remark 1). Therefore, the zeta matrix of the $(\Delta^{-1}\hat{\mathbb{Z}}_p[g_2, g_3])^\dagger$-module (7) is obtained. As a consequence, we have the first zeta endomorphism of $M = H_1^c(U, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q})$.

Remark 3. The zeta matrix via bounded Witt cohomology for the Weierstrass open subfamily, i.e., over $(\mathbb{Z}/p\mathbb{Z})[g_2, g_3, \Delta^{-1}]$, will be published in [3] when it has been made elegant enough. See [7] also.

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