Quantile-locating functions and
the distance between the mean and quantiles

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Given a random variable $X$ with finite mean, for each $0 < p < 1$, a new sharp bound is found on the distance between a $p$-quantile of $X$ and its mean in terms of the central absolute first moment of $X$. The new bounds strengthen the fact that the mean of $X$ is within one standard deviation of any of its medians, as well as a recent quantile-generalization of this fact by O’Cinneide.

Key words & Phrases: mean, median, quantiles, absolute central first moment, convex function.

1 Introduction

Let $X$ be a real-valued random variable with finite mean $E(X) = \mu$ and standard deviation $\sigma$. O’Cinneide (1990) gives an interesting proof of the fact, which he attributes to Hotelling and Solomons (1932), that the mean of $X$ is within one standard deviation of any of its medians. As observed by Mallows and Richter (1969), even a bit more is true: the distance between the mean and any median of $X$ is bounded not only by its standard deviation (which may be infinite), but even by its (generally) smaller central first moment. Putting $m$ for any median of $X$, one obtains

$$|EX - m| \leq \sigma$$

(1)

where the crucial second inequality in (1) is valid because, as is well known (e.g. see Bickel and Doksum, 1977, p. 54), $m$ minimizes the mapping $x \rightarrow E|X - x|$. Note that equalities throughout (1) are attained if $X$ is symmetric about $\mu$, two-valued and one of its values is taken for $m$.

In this note we use, for each $0 < p < 1$, a functional $U_p$ which is uniquely minimized by any $p$-quantile of $X$ to obtain a central first moment bound, which generalizes (1), on the distance between the mean of $X$ and any of its $p$-quantiles.

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Throughout this note, if \( a \) and \( b \) are real numbers, \( a \lor b \) \((a \land b)\) stand for their maximum (minimum) and, as is customary, \( a^+ = a \lor 0 \) and \( a^- = (-a)^+ \). Recall that the real number \( m = m_p \) is a \( p \)-quantile of \( X \) \((0 < p < 1)\), if \( P\{X \leq m\} \geq p \) and \( P\{X \geq m\} \geq 1 - p \).

2 A function uniquely minimized by \( p \)-quantiles

Given a random variable \( X \) with finite mean, for \( 0 < p < 1 \), let the functions \( U_{p,X} = U_p: \mathbb{R} \to \mathbb{R} \) be defined by

\[
U_p(x) = pE(X - x)^+ + (1 - p)E(X - x)^-. \tag{2}
\]

Note that \( U_{1/2}(x) = (1/2)E|X - x| \); the next proposition records two useful properties of \( U_p \).

**Proposition 1.** For each \( 0 < p < 1 \), \( U_p \) is convex, and \( U_p \) uniquely determines the distribution of \( X \).

**Proof:** The convexity of \( U_p \) follows easily from the convexity (for each \( y \)) of the maps \( x \to (y - x)^+ \) and \( x \to (y - x)^- \). The usual technique of integration by parts yields

\[
U_p(x) = p \int_{-\infty}^{x} P\{X > t\} \, dt + (1 - p) \int_{x}^{\infty} P\{X \leq t\} \, dt. \tag{3}
\]

Thus \( U_p \) is differentiable at continuity points of \( P\{X \leq x\} \) and if \( x \) is such a point, then the derivative \( U'_p \) satisfies

\[
U'_p(x) = -pP\{X > x\} + (1 - p)P\{X \leq x\} = P(X \leq x) - p = (1 - p) - P(X > x). \tag{4}
\]

Since the set of such \( x \)'s is dense in \( \mathbb{R} \), the distribution of \( X \) can be recovered from (4). \[\square\]

The property of \( U_p \) which will be used in deriving the mean-quantile bound mentioned in the introduction is that \( U_p \) is a \( p \)-quantile-locator in the sense of the following proposition (cf. also Ferguson, 1967, Exercise 3, page 51, where uniqueness is not claimed). Note that this includes the well-known inequality (used in (1)) that the median minimizes \( E|X - x| = 2U_{1/2}(x) \).

**Proposition 2.** For every integrable random variable \( X \) and every \( p \in (0, 1) \), the function \( U_p \) is uniquely minimized by any \( p \)-quantile of \( X \). That is, \( U_p(x) = \min \{U_p(y) : y \in \mathbb{R}\} \) if and only if \( x \) is a \( p \)-quantile of \( X \).

**Proof:** Since \( U_p \) is nonnegative, convex and unbounded on both the positive and negative rays of \( \mathbb{R} \), it has a minimum. If, in addition, \( U_p \) is differentiable (which is the case when the distribution of \( X \) is continuous) then setting the derivative \( U'_p(x) = 0 \) in formula (4) completes the proof that \( U_p \) is minimized by every \( p \)-quantile. In general, when the distribution of \( X \) may have atoms, proceed as follows.
First apply integration by parts to rewrite $U_p$, defined in (2), in the forms
\[
U_p(x) = p(EX - x) + E(X - x)^-
\]
\[
= p(EX - x) + \int_{-\infty}^{x} P[X \leq t] \, dt
\]
\[
= -(1 - p)(EX - x) + E(X - x)^+
\]
\[
= -(1 - p)(EX - x) + \int_{x}^{\infty} P[X > t] \, dt.
\] (5i)

Next, distinguish between two cases. If $x \geq m_p$ use (5i) to obtain
\[
U_p(x) = p(EX - x) + \int_{-\infty}^{m_p} P[X \leq t] \, dt + \int_{m_p}^{x} P[X \leq t] \, dt
\]
\[
\geq p(EX - x) + \int_{-\infty}^{m_p} P[X \leq t] \, dt + p(x - m_p)
\]
\[
= p(EX - m_p) + E(X - m_p)^- = U_p(m_p),
\] (6)
where the inequality is valid because $m_p$ is a $p$-quantile of $X$, and the last equality follows from (5i). The proof for the case $x < m_p$ is similar using (5ii). This completes the proof that $U_p$ is minimized by any $p$-quantile.

To see the converse note that if $x$ is strictly larger than the largest $p$-quantile, then the inequality in (6) is strict. Similarly, using (5ii), a strict inequality is obtained when $x$ is strictly smaller than the smallest $p$-quantile. Thus the minimum of $U_p$ is attained only at $p$-quantiles.

\[\square\]

3 Distance between the mean and quantiles

For an integrable random variable $X$ and each $0 < p < 1$, let $V_{p,X} = V_{p} : \mathbb{R} \to \mathbb{R}$ be defined by
\[
V_p(x) = p(EX - x)^+ + (1 - p)(EX - x)^-.
\] (7)

and let
\[
\Delta_p = U_p - V_p,
\]
where $U_p$ is as in (2).

Note that $V_p$ is piecewise linear with slope $p$ to the left of $EX$ and slope $1 - p$ to its right.

Some useful properties of $\Delta_p$ are recorded in the following lemma.

**Lemma.** For each $0 < p < 1$,

(i) $\Delta_p(x) = \begin{cases} E(X - x)^-, & x \leq EX \\ E(X - x)^+, & x \geq EX \end{cases}$ independently of $p$;

(ii) $0 \leq \Delta_p(x) \to 0$ as $|x| \to \infty$; and

(iii) $\int_{-\infty}^{\infty} \Delta_p(x) \, dx = \frac{1}{2} \text{Var} X$ (whether finite or not).
**PROOF:** (i) By definition, 
\[
\Delta_p(x) = U_p(p) - V_p(x) = p[E(X - x)^+ - (EX - x)^+] + (1 - p)[E(X - x)^- - EX - x^-]
\]
\[
= p[(E(X - x)^+ - E(X - x)^-)] - [(EX - x)^+ - (EX - x)^-)] + \{E(X - x)^- - (EX - x)^-]
\]
\[
= p[(EX - x) - (EX - x)] + [E(X - x)^- - (EX - x)^-)]
\]
\[
= [E(X - x)^-], \quad x \leq EX
\]
\[
E(X - x)^+, \quad x \geq EX.
\]
(ii) The inequality follows from (i) (or from Jensen). The asymptotic statement follows from (i) using monotone convergence.

(iii) Assume, without loss of generality, that $EX = 0$. By (i)
\[
\int_{-\infty}^{\infty} \Delta_p(x) \, dx = \int_{-\infty}^{0} E(X - x^-) \, dx + \int_{0}^{\infty} E(X - x^+) \, dx.
\]
Applying integration by parts twice and using Fubini in between to change the order of integration, one obtains
\[
\int_{-\infty}^{\infty} \Delta_p(x) \, dx = \int_{-\infty}^{0} tP\{X < t\} \, dt + \int_{0}^{\infty} tP\{X > t\} \, dt
\]
\[
= (1/2)(E(X^-)^2 + E(X^+)^2) = \left(\frac{1}{2}\right)EX^2.
\]
\[
\square
\]

**REMARK.** Applying (iii) with $p = \frac{1}{2}$ it follows that
\[
\int_{-\infty}^{\infty} \{E|X - x| - |EX - x|\} \, dx = \text{Var} \, X.
\]
Finally, Proposition 2 and the above properties of $\Delta_p$ will be used to obtain bounds on the distance between any $p$-quantile of a random variable and its mean in terms of its central absolute first moment. These bounds are analogous to the standard deviation bounds of Dharmadhikari (1991)
\[
EX - \sigma \sqrt{q/p} \leq m_p \leq EX + \sigma \sqrt{p/q}
\]
which both generalize (1) and strengthen the symmetric version of O'Cinneide (1990)
\[
|EX - m_p| \leq \sigma \sqrt{\max \{p/q, q/p\}}.
\]
Theorem 1. For \(0 < p < 1\), let \(m_p\) be a \(p\)-quantile of the random variable \(X\). If \(EX\) is finite, then (letting \(q = 1 - p\))

\[
EX - (1/2p)E |X - EX| \leq m_p \leq EX + (1/2q)E |X - EX|,
\]

and these bounds are attained.

Proof: If \(m_p \leq EX\) then

\[
p(EX - m_p) = V_p(m_p) \leq U_p(m_p) \leq U_p(EX) = pE(X - EX)^+ + (1 - p)E(X - EX)^- = (1/2)E |X - EX|, \tag{8}
\]

where the first equality follows from (7), the first inequality from Lemma (ii), the second inequality from the definition of \(U_p\), and the last equality since \(E(X - EX)^+ = E(X - EX)^- = (1/2)E |X - EX|\). If \(m_p \geq EX\), use the same argument with the fact from (7) that \((1 - p)(m_p - EX) = V(m_p)\). Equality is easily seen to be attained if \(X\) takes only two values \(a < b\) and \(P(X = a) = p\). \(\square\)

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References


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