Quantile-locating functions and the distance between the mean and quantiles

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Given a random variable X with finite mean, for each $0 < p < 1$, a new sharp bound is found on the distance between a p -quantile of X and its mean in terms of the central absolute first moment of X . The new bounds strengthen the fact that the mean of X is within one standard deviation of any of its medians, as well as a recent quantile-generalization of this fact by O'Cinneide.

Key words & Phrases: mean, median, quantiles, absolute central first moment, convex function.

1 Introduction

Let X be a real-valued random variable with finite mean $E(X) = \mu$ and standard deviation σ . O'CINNEIDE (1990) gives an interesting proof of the fact, which he attributes to HOTELLING and SOLOMONS (1932), that *the mean of X is within one standard deviation of any of its medians*. As observed by MALLOWS and RICHTER (1969), even a bit more is true: the distance between the mean and any median of X is bounded not only by its standard deviation (which may be infinite), but even by its (generally) smaller central first moment. Putting m for any median of X , one obtains

$$|EX - m| \leq E|X - m| \leq E|X - EX| \leq \sigma \quad (1)$$

where the crucial second inequality in (1) is valid because, as is well known (e.g. see BICKEL and DOKSUM, 1977, p. 54), m minimizes the mapping $x \rightarrow E|X - x|$. Note that equalities throughout (1) are attained if X is symmetric about μ , two-valued and one of its values is taken for m .

In this note we use, for each $0 < p < 1$, a functional U_p which is uniquely minimized by any p -quantile of X to obtain a central first moment bound, which generalizes (1), on the distance between the mean of X and any of its p -quantiles.

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Throughout this note, if a and b are real numbers, $a \vee b$ ($a \wedge b$) stand for their maximum (minimum) and, as is customary, $a^+ = a \vee 0$ and $a^- = (-a)^+$. Recall that the real number $m = m_p$ is a p -quantile of X ($0 < p < 1$), if $P\{X \leq m\} \geq p$ and $P\{X \geq m\} \geq 1 - p$.

2 A function uniquely minimized by p -quantiles

Given a random variable X with finite mean, for $0 < p < 1$, let the functions $U_{p,X} = U_p: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$U_p(x) = pE(X - x)^+ + (1 - p)E(X - x)^-. \quad (2)$$

Note that $U_{1/2}(x) = (1/2)E|X - x|$; the next proposition records two useful properties of U_p .

PROPOSITION 1. *For each $0 < p < 1$, U_p is convex, and U_p uniquely determines the distribution of X .*

PROOF: The convexity of U_p follows easily from the convexity (for each y) of the maps $x \rightarrow (y - x)^+$ and $x \rightarrow (y - x)^-$. The usual technique of integration by parts yields

$$U_p(x) = p \int_x^\infty P\{X > t\} dt + (1 - p) \int_{-\infty}^x P\{X \leq t\} dt. \quad (3)$$

Thus U_p is differentiable at continuity points of $P\{X \leq x\}$ and if x is such a point, then the derivative U'_p satisfies

$$U'_p(x) = -pP\{X > x\} + (1 - p)P\{X \leq x\} = P\{X \leq x\} - p = (1 - p) - P\{X > x\}. \quad (4)$$

Since the set of such x 's is dense in $\mathbb{R}$, the distribution of X can be recovered from (4). $\square$

The property of U_p which will be used in deriving the mean-quantile bound mentioned in the introduction is that U_p is a p -quantile-locator in the sense of the following proposition (cf. also FERGUSON, 1967, Exercise 3, page 51, where uniqueness is not claimed). Note that this includes the well-known inequality (used in (1)) that the median minimizes $E|X - x| = 2U_{1/2}(x)$.

PROPOSITION 2. *For every integrable random variable X and every $p \in (0, 1)$, the function U_p is uniquely minimized by any p -quantile of X . That is, $U_p(x) = \min\{U_p(y): y \in \mathbb{R}\}$ if and only if x is a p -quantile of X .*

PROOF: Since U_p is nonnegative, convex and unbounded on both the positive and negative rays of $\mathbb{R}$, it has a minimum. If, in addition, U_p is differentiable (which is the case when the distribution of X is continuous) then setting the derivative $U'_p(x) = 0$ in formula (4) completes the proof that U_p is minimized by every p -quantile. In general, when the distribution of X may have atoms, proceed as follows.

First apply integration by parts to rewrite U_p , defined in (2), in the forms

$$\begin{aligned} U_p(x) &= p(EX - x) + E(X - x)^- \\ &= p(EX - x) + \int_{-\infty}^x P\{X \leq t\} dt \end{aligned} \quad (5i)$$

$$\begin{aligned} &= -(1 - p)(EX - x) + E(X - x)^+ \\ &= -(1 - p)(EX - x) + \int_x^{\infty} P\{X > t\} dt. \end{aligned} \quad (5ii)$$

Next, distinguish between two cases. If $x \geq m_p$ use (5i) to obtain

$$\begin{aligned} U_p(x) &= p(EX - x) + \int_{-\infty}^{m_p} P\{X \leq t\} dt + \int_{m_p}^x P\{X \leq t\} dt \\ &\geq p(EX - x) + \int_{-\infty}^{m_p} P\{X \leq t\} dt + p(x - m_p) \\ &= p(EX - m_p) + E(X - m_p)^- = U_p(m_p), \end{aligned} \quad (6)$$

where the inequality is valid because m_p is a p -quantile of X , and the last equality follows from (5i). The proof for the case $x < m_p$ is similar using (5ii). This completes the proof that U_p is minimized by any p -quantile.

To see the converse note that if x is strictly larger than the largest p -quantile, then the inequality in (6) is strict. Similarly, using (5ii), a strict inequality is obtained when x is strictly smaller than the smallest p -quantile. Thus the minimum of U_p is attained only at p -quantiles. $\square$

3 Distance between the mean and quantiles

For an integrable random variable X and each $0 < p < 1$, let $V_{p,X} = V_p: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$V_p(x) = p(EX - x)^+ + (1 - p)(EX - x)^-, \quad (7)$$

and let

$$\Delta_p = U_p - V_p,$$

where U_p is as in (2).

Note that V_p is piecewise linear with slope p to the left of EX and slope $1 - p$ to its right.

Some useful properties of Δ_p are recorded in the following lemma.

LEMMA. For each $0 < p < 1$,

- (i) $\Delta_p(x) = \begin{cases} E(X - x)^-, & x \leq EX \\ E(X - x)^+, & x \geq EX \end{cases}$ independently of p ;
- (ii) $0 \leq \Delta_p(x) \rightarrow 0$ as $|x| \rightarrow \infty$; and
- (iii) $\int_{-\infty}^{\infty} \Delta_p(x) dx = \frac{1}{2} \text{Var } X$ (whether finite or not).

PROOF: (i) By definition,

$$\begin{aligned}
\Delta_p(x) &= U_p(p) - V_p(x) = p\{E(X-x)^+ - (EX-x)^+\} \\
&\quad + (1-p)\{E(X-x)^- - (EX-x)^-\} \\
&= p\{[E(X-x)^+ - E(X-x)^-] - [(EX-x)^+ - (EX-x)^-]\} \\
&\quad + \{E(X-x)^- - (EX-x)^-\} \\
&= p\{(EX-x) - (EX-x)\} + \{E(X-x)^- - (EX-x)^-\} \\
&= \{E(X-x)^- - (EX-x)^-\} \\
&= \begin{cases} E(X-x)^-, & x \leq EX \\ E(X-x)^+, & x \geq EX. \end{cases}
\end{aligned}$$

(ii) The inequality follows from (i) (or from Jensen). The asymptotic statement follows from (i) using monotone convergence.

(iii) Assume, without loss of generality, that $EX = 0$. By (i)

$$\int_{-\infty}^{\infty} \Delta_p(x) dx = \int_{-\infty}^0 E(X-x)^- dx + \int_0^{\infty} E(X-x)^+ dx.$$

Applying integration by parts twice and using Fubini in between to change the order of integration, one obtains

$$\begin{aligned}
\int_{-\infty}^{\infty} \Delta_p(x) dx &= \int_{-\infty}^0 tP\{X < t\} dt + \int_0^{\infty} tP\{X > t\} dt \\
&= (1/2)\{E(X^-)^2 + E(X^+)^2\} = \left(\frac{1}{2}\right) EX^2.
\end{aligned}$$

□

REMARK. Applying (iii) with $p = \frac{1}{2}$ it follows that

$$\int_{-\infty}^{\infty} \{E|X-x| - |EX-x|\} dx = \text{Var } X.$$

Finally, Proposition 2 and the above properties of Δ_p will be used to obtain bounds on the distance between any p -quantile of a random variable and its mean in terms of its central absolute first moment. These bounds are analogous to the standard deviation bounds of DHARMADHIKARI (1991)

$$EX - \sigma\sqrt{q/p} \leq m_p \leq EX + \sigma\sqrt{p/q}$$

which both generalize (1) and strengthen the symmetric version of O'CINNEIDE (1990)

$$|EX - m_p| \leq \sigma\sqrt{\max\{p/q, q/p\}}.$$

THEOREM 1. For $0 < p < 1$, let m_p be a p -quantile of the random variable X . If EX is finite, then (letting $q = 1 - p$)

$$EX - (1/2p)E|X - EX| \leq m_p \leq EX + (1/2q)E|X - EX|,$$

and these bounds are attained.

PROOF: If $m_p \leq EX$ then

$$\begin{aligned} p(EX - m_p) &= V_p(m_p) \leq U_p(m_p) \leq U_p(EX) \\ &= pE(X - EX)^+ + (1 - p)E(X - EX)^- = (1/2)E|X - EX|, \end{aligned} \quad (8)$$

where the first equality follows from (7), the first inequality from Lemma (ii), the second inequality from Proposition 1, the second equality from the definition of U_p , and the last equality since $E(X - EX)^+ = E(X - EX)^- = (1/2)E|X - EX|$. If $m_p \geq EX$, use the same argument with the fact from (7) that $(1 - p)(m_p - EX) = V(m_p)$. Equality is easily seen to be attained if X takes only two values $a < b$ and $P(X = a) = p$. $\square$

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