

Elemental t.g. principles of relativistic t-topos^(*) (Presheafification of matter, space, and time)

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Abstract. – We would like to solve the following problem: find a mathematical model formulating I) quantum entanglement, II) particle-wave duality, III) universal objects (ur-sub-Planck objects): to be defined in terms of direct or inverse limits (defined by universal mapping properties) giving microcosm behaviors of space-time so as to give the smooth macrocosm space-time, and IV) the “curved” space-time associated with particles with mass in microcosm consistent with the notion of a *light cone* in macrocosm. Problems I) and II) are treated in KATO G., *Europhys. Lett.*, **68** (2004) 467. In this paper, we will focus on III) and IV). As a candidate for such a model, we have introduced the category of presheaves over a site called a *t-topos*. During the last several years, the methods of category and sheaf theoretic approaches have been actively employed for the foundations of quantum physics and for quantum gravity. Particles, time, and space are presheafified in the following sense: a fundamental entity is a triple (m, κ, τ) of presheaves so that for an object V in a *t-site*, a local datum $(m(V), \kappa(V), \tau(V))$ may provide a *local state* of the particle $\underline{m} = m(V)$, *i.e.*, the localization of presheaf m at V , in the neighborhood $(\kappa(V), \tau(V))$ of \underline{m} . By presheafifying matter, space, and time, *t-topos* can provide sheaf-theoretic descriptions of *ur-entanglement* and *ur-particle and ur-wave states*⁽¹⁾ formulating the *EPR-type non-locality* and the duality in a double-slit experiment. Recall that presheaves m and m' are said to be *ur-entangled* when m and m' behave as one presheaf. Also recall: a presheaf m is said to be in *particle ur-state* (or *wave ur-state*) when the presheaf m is evaluated as $m(V)$ at a specified object V in the *t-site* (or when an object in the *t-site* is not specified). For more comments and the precise definitions of *ur-entanglement* and *particle and wave ur-states*, see the above-mentioned paper. The applications to a double-slit experiment and the *EPR-type non-locality* are described in detail in the forthcoming papers KATO G. and TANAKA T., *Double slit experiment and t-topos*, submitted to *Found. Phys.* and KAFATOS M., KATO G., ROY S. and TANAKA T., *The EPR-type non-locality and t-topos*, to be submitted to *Int. J. Pure Appl. Math.*, respectively. By the notion of *decompositions* of a presheaf and of an object of the *t-site*, *ur-sub-Planck* objects are defined as *direct* and *inverse limits*, respectively, in Definitions 2.1 and 2.4 in what will follow.

Notion of a site. – As a domain category for t -topos, we consider a *site*, i.e., a category with a *Grothendieck topology*. This generalization to a site from a topological space is crucial for the t -topos theory since we need more than one morphism between two objects. Further, the notion of a *site* is appropriate for the *universal mapping characterization* for *ur-sub-Planck* objects as limits. For a further description of a *site*, see [1–3], or [4] for the logic aspect of topoi. We shall review the concept of a *site*:

Definition 1.1. A category S is said to be a *site* if for each object U of S , there is a collection of families of morphisms of S :

$$\text{Cov. } U = \{\{f_i : U_i \rightarrow U\}_{i \in I}\} \quad (1)$$

satisfying the following axioms:

(T.1) For $\{f_i : U_i \rightarrow U\}_{i \in I} \in \text{Cov. } U$ and for $V \xrightarrow{h} U$ in S , there exists $U_i \times_U V$ in S , and in the commutative diagram

$$\begin{array}{ccc} U_i & \longleftarrow & U_i \times_U V \\ \downarrow & & \downarrow \\ U & \longleftarrow & V \end{array} \quad (2)$$

we have $\{U_i \times_U V \xrightarrow{\text{proj}} V\} \in \text{Cov. } V$.

(T.2) For a covering of U , i.e., $\{f_i : U_i \rightarrow U\}_{i \in I} \in \text{Cov. } U$ and for a covering of each U_i , i.e., $\{g_j : U_{ij} \rightarrow U_i\}_{j \in I_i} \in \text{Cov. } U_i$, the composition $\{f_i \circ g_j : U_{ij} \rightarrow U\}_{i \in I, j \in I_i} \in \text{Cov. } U$.

(T.3) An isomorphism $\{V' \rightarrow V\} \in \text{Cov. } U$.

Then, $\{f_i : U_i \rightarrow U\}_{i \in I}$ is said to be a covering family of U .

A subcategory of a site is said to be a *subsite* if each object of the subcategory has covering families satisfying the above (T.1), (T.2), and (T.3).

Definition 1.2. Let \hat{S} be the category of presheaves from a site S to a product category of categories indexed by a set Γ :

$$\hat{S} = \left(\prod_{\alpha \in \Gamma} C_\alpha \right)^{S^{opp}}. \quad (3)$$

Then category \hat{S} is said to be a t -topos (or *temporal topos*), where C_1 , and C_2 , $1, 2 \in \Gamma$, are the microcosm (in the quantum-mechanical level) and macrocosm (in the classical level) discrete categories, respectively.

Hypotheses on κ and τ . i) Presheaves of space and time κ and τ are sheaves. That is, the following is exact: for a covering family $\{f_i : U_i \rightarrow U\}_{i \in I}$

$$\kappa(U) \xrightarrow{\kappa(f_i)} \prod_i \kappa(U_i) \begin{array}{c} \xrightarrow{\kappa(\rho_1)} \\ \xrightarrow{\kappa(\rho_2)} \end{array} \prod_{i,k} \kappa(U_i \times U_k) \quad (4)$$

(and the same for presheaf τ .)

ii) κ and τ are ur-entangled over a subsite.

Remark 1.3. Let m and m' be objects of \hat{S} . First assume that m and m' are not ur-entangled. Let $m(V)$ and $m'(V')$ be the corresponding particle ur-states of m and m' over generalized time periods V and V' of S , respectively. Consider also the corresponding space-time neighborhoods $(\kappa(V), \tau(V))$ and $(\kappa(V'), \tau(V'))$ of $m(V)$ and $m'(V')$, respectively. One

can take V and V' “small enough” in S by *decomposing* (the definition is given later) the generalized time periods V and V' so that $(\kappa(V), \tau(V))$ and $(\kappa(V'), \tau(V'))$ are non-intersecting local neighborhoods of $m(V)$ and $m'(V')$. Next consider the case where m and m' are *wr-entangled*. That is, we now can let $V = V'$ in the above. Then the non-intersecting local space-time neighborhoods become both $(\kappa(V), \tau(V))$. In order to avoid this contradiction, we can assume that space and time presheaves κ and τ depend upon a particle. Namely, presheaves κ and τ cannot be considered unless a particle (presheaf) is chosen first.

Definition 1.4. We will consider some t-topos theoretic consequences of presheafified particles, space, and time in this section. First, we recall the following axiom in [5] on the direction of a morphism in the site S for two observed states. When $\tau(V)$ precedes $\tau(U)$, there exists a morphism from V to U in S . Note that not every morphism from V to U indicates the corresponding $\tau(V)$ preceding $\tau(U)$. This is another reason why the general notion of a site is better suited than the induced category from a topological space where only one morphism (inclusion) exists between two objects. Let the linear physical time $\tau(V)$ precede $\tau(U)$ and let g be the corresponding morphism from V to U in the site S . We will characterize the wave ur-state between $\tau(V)$ and $\tau(U)$ by all the possible factorizations of morphism g as follows.

Recall also from [5] that under the assumption in the above paragraph, we can define

$$\{V \xrightarrow{g} U\} = \left\{ (\alpha, W, \beta) \mid g = \beta \circ \alpha, \quad \text{where } V \xrightarrow{\alpha} W \quad \text{and} \quad W \xrightarrow{\beta} U \right\}. \quad (5)$$

In the forthcoming paper [6] on a double-slit experiment, Definition 2.1 of the factorization of g plays an important role indicating the Feynman diagram and the wave ur-state between the particle ur-states determined by V and U . Now we begin: let m and P be presheaves manifested at V and V' , respectively. Next, we will give a general notion of a relativistic observation as a morphism in a category where the image of the morphism should be regarded as the information obtained by the observation. (For the notion of a non-relativistic observation, see [5].) A t-topos theoretic *relativistic observation (measurement)* of object $m \in Ob(\hat{S})$ over $V' \in Ob(S)$ by another object $P \in Ob(\hat{S})$ over $V \in Ob(S)$ in a non-discrete category is defined as follows: there exists a decomposition (or a factorization) of a morphism $V \xrightarrow{\varphi} V'$ as

$$V = V_0 \longrightarrow V_1 \longrightarrow \dots \longrightarrow V_N = V'$$

satisfying the following i) and ii):

- i) each $\tau(V_j)$ precedes $\tau(V_{j+1})$ for $j = 0, 1, \dots, N - 1$,
- ii) there is a morphism $s_{V'}^{V'} : m(V') \rightarrow P(V)$.

Definition 1.5. When such a decomposition in Definition 1.4 exists, the states $m(V')$ and $P(V)$ are said to be *mutually in a light cone*.

Note 1.6. A maximal number of objects and morphisms giving such a decomposition of $V \xrightarrow{\varphi} V'$ is the notion of a *microdecomposition*. That is, each $V_j \xrightarrow{\varphi_j} V_{j+1}$ of the above decomposition is a micromorphism. Namely, $V_j \xrightarrow{\varphi_j} V_{j+1}$ cannot be further factored as $V_j \xrightarrow{\alpha_j} W_j \xrightarrow{\beta_j} V_{j+1}$ so that the objects V_j, W_j, V_{j+1} in the t-site may give the linear time relationship among three corresponding ur-states as satisfying $\beta_j \circ \alpha_j = \varphi_j$. The concept of a micromorphism is defined in [5].

Remarks 1.7. 1) The above exactness (4) in i) means that space and time appear to be a continuum at the “macro level” when the global object $\kappa(U)$ is obtained by pasting “micro level pieces” $\kappa(U_i)$. Let \hat{S} be the full subcategory of sheaves. As for ii), for an object U of a subsite S' , $(\kappa(U), \tau(U))$ is the usual space-time in C_2 or in C_1 . Let us observe the following: a morphism from U_i to U , where U_i is an object in a covering family of U , induces

the “restriction” morphisms from $\kappa(U)$ to $\kappa(U_i)$ and from $\tau(U)$ to $\tau(U_i)$ of space and time. That is, intuitively speaking, the smaller U_i is, the smaller the space $\kappa(U_i)$ becomes, and the smaller U_i is, the shorter the time $\tau(U_i)$ becomes.

2) By the definition of a light cone, we have the following consequence for a micromorphism.

That is, for a micromorphism $V_j \xrightarrow{\varphi_j} V_{j+1}$, if a factorization $V_j \xrightarrow{\alpha_j} W_j \xrightarrow{\beta_j} V_{j+1}$ satisfying $\beta_j \circ \alpha_j = \varphi_j$ exists, then by the definition of a micromorphism, $\tau(W_j)$ cannot be in the light cone where $\tau(V_j)$ and $\tau(V_{j+1})$ belong. Note also that time sheaf τ can be replaced by a “particle” presheaf with which τ is associated.

3) In [7], a full fledged relativistic t-topos is to be developed where, for example, the notion of a light cone can be defined also in terms of the presheaf γ associated with a photon.

Limits and universal objects. – We will focus on space and time presheaves κ and τ which are observable in the classical macro level over an object V of the t-site S , over which, for the sake of simplicity, κ and τ are ur-entangled. The ur-entangled pair $(\kappa(V), \tau(V))$ could be considered for our study in what will follow; however, first we will consider κ and τ separately. Compare our approach with [8] on very similar topics. For the generalizations of inverse and direct limits of functors defined over a category, see the forthcoming treatise, see [2], or consult with [3].

Let us consider a decomposition of an object U of S in terms of a covering family as in Definition 1.1. One of the reasons why we consider a covering family $\{f_i : U_i \rightarrow U\}_{i \in I}$ of U rather than just a sequence $\dots \rightarrow W' \rightarrow W \rightarrow U$ is that by the sheaf property, $\kappa(U)$ is obtained by “pasting” local data $\{\kappa(U_i)\}$. That is, we need local data to get the global $\kappa(U)$ by (4). Then consider a covering family $\{f_{ij} : U_{ij} \rightarrow U_i\}_{j \in I_i}$ of each U_i . And similarly, consider a covering family $\{f_{ijk} : U_{ijk} \rightarrow U_{ij}\}_{k \in I_{ij}}$ of U_{ij} . By (T.2) of Definition 1.1, we have the following inverse system of the covering families of U by composing covering morphisms f_i, f_{ij}, \dots :

$$\{f_i : U_i \rightarrow U\} \longleftarrow \{f_i \circ f_{ij} : U_{ij} \rightarrow U\} \longleftarrow \{f_i \circ f_{ij} \circ f_{ijk} : U_{ijk} \rightarrow U\} \longleftarrow \dots \quad (6)$$

Definition 2.1. One can consider the inverse limit $\varprojlim \{U_{ijk} \xrightarrow{f_i \circ f_{ij} \circ f_{ijk} \dots} U\}$ of the above inverse system (6). For all $i \in I, j \in I_i, k \in I_{ij}, \dots$, consider

$$\varprojlim \{U_{ijk} \dots\} \quad (7)$$

and the corresponding direct limits

$$\{\varinjlim \kappa(U_{ijk} \dots)\} \quad \text{and} \quad \{\varinjlim \tau(U_{ijk} \dots)\}. \quad (8)$$

Then we can define the following “smallest possible objects” in the sense of universal mapping properties of limits. The inverse limit (6) is said to be an *ur-sub-Planck decomposition* of U , and the direct limits in (8) are said to be a *ur-sub-Planck decomposition* for $\kappa(U)$ and an *ur-sub-Planck decomposition* for $\tau(U)$, respectively. Those individual direct limit objects $\varinjlim \kappa(U_{ijk} \dots)$ and $\varinjlim \tau(U_{ijk} \dots)$ are said to be an *ur-sub-Planck space* and an *ur-sub-Planck time* with respect to $\kappa(U)$ and $\tau(U)$.

Remark 2.2. For a given U in S , by considering an ur-sub-Planck decomposition $\varprojlim \{U_{ijk} \dots\}$ of U , we get not only the *shortest* generalized time period but also the *virtually initial* stage of U in the following sense. The morphism, for example, from each U_i of the covering family $\{f_i : U_i \rightarrow U\}_{i \in I}$ to U need not imply that $\tau(U_i)$ precedes $\tau(U)$ as we noted in Definition 1.4. Yet, there exists a morphism from the inverse limit $\varprojlim \{U_{ijk} \dots\}$ to every $\{U_{ijk} \dots\}$ as if $\varprojlim \{U_{ijk} \dots\}$ were the earliest generalized time for U . Namely, *mini-bang-like objects*

associated with particles to space-time presheaves exist which can last only the periods of ur-sub-Planck time $\varinjlim \tau(U_{ijk\dots})$ as defined in Definition 2.1.

Remark 2.3. In order to formulate a t-topos theoretic ultra early universe, we need to pose three technical hypotheses. We will come back to this topic in a later paper.

Definition 2.4. We can consider a decomposition of a particle $\underline{m} = m(U)$ into micro objects $\{\underline{m}_\alpha\}$ in terms of presheaves as done in Definition 2.1. First, rewrite the presheaf $m \in \text{Ob}(\hat{S})$ as a direct sum $\sum_\alpha m_\alpha$ of subpresheaves and next, rewrite each m_α as $\sum_\beta m_{\alpha\beta}$, and similarly for $m_{\alpha\beta}$ as $\sum_\gamma m_{\alpha\beta\gamma}$. For these decompositions, define morphisms as

$$m \longleftarrow \sum_\alpha m_\alpha \longleftarrow \sum_\alpha \sum_\beta m_{\alpha\beta} \longleftarrow \sum_\alpha \sum_\beta \sum_\gamma m_{\alpha\beta\gamma} \longleftarrow \dots \quad (9)$$

Then consider

$$\varprojlim \sum_\alpha \sum_\beta \sum_\gamma \dots m_{\alpha\beta\gamma\dots} \quad \text{and} \quad \varprojlim m_{\alpha\beta\gamma\dots} \quad (10)$$

Then $\varprojlim \sum_\alpha \sum_\beta \sum_\gamma \dots m_{\alpha\beta\gamma\dots}$ in (10) is said to be an *ur-sub-Planck decomposition* of m , and $\varprojlim m_{\alpha\beta\gamma\dots}$ is said to be an *ur-sub-Planck subobject* of \hat{S} for m .

On the t.g. principles of relativistic t-topos. – Let m and m^* be presheaves associated with particles and let (κ, τ) and (κ^*, τ^*) be the space-time presheaves associated with m and m^* , respectively. We define the space-time presheaf $(\tilde{\kappa}, \tilde{\tau})$ determined by (κ, τ) and (κ^*, τ^*) by the product of (κ, τ) and (κ^*, τ^*) in the category \hat{S} of sheaves. Namely, we have $\tilde{\kappa} = \kappa \times \kappa^*$ and $\tilde{\tau} = \tau \times \tau^*$. Note that a measurement (an observation), in the sense of Definition 1.4 in [5], of $\tilde{\kappa}$ by P in \hat{S} over an object V in S does not give any information on κ . That is, one cannot compose a morphism from $\tilde{\kappa}$ to P with the projection from $\tilde{\kappa} = \kappa \times \kappa^*$ to κ . On the contrary, if both κ and κ^* are measured by P , then the compositions of projections and the observation morphisms give a measurement of the product $\tilde{\kappa} = \kappa \times \kappa^*$.

Next for a given space-time sheaves (κ, τ) , which is associated with a presheaf m , representing a particle \underline{m} , as described in the above paragraph, we will consider a morphism between two observers P and Q of m in \hat{S} . By using the above interference of space-time presheaves and the relativistic observation in Definition 1.4, one can give the *macro-global* relativistic definition of the morphism from P to Q . However, we will give a *micro-local* version of the morphism P to Q in this paper. Namely, such a morphism is defined over a single object of the t-site S . (See also [7] for a full treatment.) Recall first that space and time sheaves κ and τ are entangled over a t-subsite. *Fundamental relativistic commutative diagrams* (micro-local case) over a morphism from V to V' of S can be given as

$$\begin{array}{ccc} (\kappa(V), \tau(V)) \xleftarrow{=} (\kappa(V), \tau(V)) & & (\kappa(V'), \tau(V')) \xleftarrow{=} (\kappa(V'), \tau(V')) \\ \downarrow & \text{and} & \downarrow \\ P(V) \xrightarrow{E-L(V)} Q(V) & & P(V') \xrightarrow{E-L(V')} Q(V') \end{array} \quad (\text{F.D.})$$

connected with functorially induced morphisms from the right square diagram over V' to the left commutative diagram over V . Then the natural transformation $E-L$ over V and V' is said to be an *Einstein-Lorentz natural transformation* for observer presheaves P and Q . Notice that for the micro-local case, the above space-time presheaves (κ, τ) are determined by presheaves P , Q and m . Namely, (κ, τ) are the product of space-time sheaves associated with P , Q , and m in the above sense. A detailed study of (F.D.) will appear in [7].

Final remarks. – i) The principle of general covariance and the principle of equivalence are interpreted as the commutativity of diagram (F.D.). Compare with [9] for the morphisms in (F.D.), *i.e.*, natural transformations evaluated at objects of the t-site.

ii) Note also that one of the reasons for the presheafification of matter, space, and time is to avoid the undesired Cantor-Dedekind-Euclidean-type notion of point singularities. Namely, even the t-topos theoretic smallest possible objects, defined in terms of limits (*i.e.*, by the universal mapping properties of limits), are still *objects*, possibly having many set-theoretic elements in those objects. That is, singularities need not be points in the Cantor-Dedekind-Euclidean sense. For careful analysis in terms of Abstract Differential Geometry of such singularities of the classical theory, see [10–12].

iii) Summarizing, the theory of t-topos is a theory where an entanglement (the EPR-type non-locality), wave-particle duality (the principle of complementarity), the notion of a light cone and fundamental relativistic concepts (as the above commutative diagram (F.D.)) are defined in terms of presheaves (and sheaves) over a category with a Grothendieck topology. The next step in our theory might be to introduce a formulation of scaling. For example, following Butterfield and Isham [13], their *fundamental composition principle* becomes the commutative diagram in [5] by replacing the Hilbert space with our product category. (Note also that our formulation of Kochen-Specker Theorem is given in [5].) More importantly, in addition to space and time presheaves, we must introduce new presheaves to induce dynamic notions in t-topos theory.

iv) The gravitational dynamics might be interpreted as the product $\tilde{\kappa} = \kappa' \times \kappa'' \times \dots \kappa'''$ of sheaves associated with particles with mass, *i.e.*, determined by “geometry of space”. On the other hand, as in i) above, there exist objects V and V' in the t-site S which determine the states of m , P , and Q so that the vertical morphisms in (F.D.) are associated with an equivalent acceleration to $\tilde{\kappa} = \kappa' \times \kappa'' \times \dots \kappa'''$. For arbitrary V and V' in S , the lower square commutativity of (F.D.) corresponds to the differential equations of Einstein which is associated with the principle of general covariance. For further analysis, see the forthcoming [7].

v) As for singularities, as we mentioned in ii) above, the theory of t-topos is a pointless and background space-timeless theory as the theory developed in [12]. Namely, there is no infinity, *i.e.*, no singularity in our topos theoretic (categorical, cohomological algebraic) approach, where the usual general relativity based on a background manifold induces all those pathologies.

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