ON THE NOTION OF PRECOHOMOLOGY

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Dedicated to Professor SAUL LUBKIN.

ABSTRACT. For a cochain complex one can have the cohomology functor. In this paper we introduce the notion of precohomology for a cochain that is not a complex, i.e., $d^{q+1} \circ d^q$ may not be zero. Such a cochain, with objects and morphisms of an abelian category $A$, is called a cochain precomplex whose category is denoted by $\text{Pco}(A)$. If a cochain precomplex is actually a cochain complex, then the notion of precohomology coincides with that of cohomology, i.e., precohomology is a generalization of cohomology. For a left exact functor $F$ from an abelian category $A$ to an abelian category $B$, the hyperprecohomology of $F$ is defined, and some properties are given. In the last section, a generalization of an inverse limit, called a preinverse limit, is introduced. We discuss some of the links between precohomology and preinverse limit.

Introduction

Let $\mathbf{Z}$ be the ring of integers and let $A$ be an abelian category. Suppose a sequence of objects and morphisms in $A$ is given

$$\vdots \rightarrow C_{q-1} \rightarrow C_q \rightarrow C_{q+1} \rightarrow \cdots,$$

which may not satisfy $d^q \circ d^{q-1} = 0$ for certain $q \in \mathbf{Z}$. Then one may not be able to take the cohomology at $C_q$. We will introduce a functor


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for such a cochain by initially complexifying the cochain to a cochain complex, then taking the cohomology of the complex. For diagram (or element) chasing, we use an exact imbedding of $A$ into the category of abelian groups. It should be noted that precohomology is a self-dual construction and that it is not an exact connected sequence of functors. Furthermore, for each $n \in \mathbb{Z}$, $\text{Ph}^n$ is half exact. Hence, they are not derived functors, see §1.

1. Precohomology

Let $A$ be an abelian category, and let $\text{Co}(A)$ and $\text{Co}^+(A)$ be the categories of cochain complexes and positive cochain complexes of objects in $A$, respectively.

**Definition 1.1.** A sequence of objects and morphisms of $A$,

$$
\cdots \rightarrow C_{q-1} \xrightarrow{d_q} C_q \xrightarrow{d_q} C_{q+1} \xrightarrow{d_q} \cdots
$$

is said to be a cochain precomplex, whose category is denoted by $\text{Pco}(A)$, and $\text{Pco}^+(A)$ denotes the category of positive cochain precomplexes. A morphism $(f_q)_{q \in \mathbb{Z}} : (C_q, d_q)_{q \in \mathbb{Z}} \rightarrow (D_q, e_q)_{q \in \mathbb{Z}}$ in $\text{Pco}(A)$ is a sequence of morphisms $f_q : C_q \rightarrow D_q$ such that the diagram

$$
\cdots \rightarrow C_q \xrightarrow{d_q} C_{q+1} \xrightarrow{d_q} \cdots
$$

$$
\downarrow f_q \quad \quad \quad \quad \downarrow f_{q+1}
$$

$$
\cdots \rightarrow D_q \xrightarrow{e_q} D_{q+1} \xrightarrow{e_q} \cdots
$$

commutes, i.e., $f_{q+1} \circ d_q = e_q \circ f_q$ for $q \in \mathbb{Z}$.

**Note.** A cochain precomplex $(C_q, d_q)_{q \in \mathbb{Z}}$ is a cochain complex if $d_{q+1} \circ d_q = 0$ for $q \in \mathbb{Z}$.

**Lemma 1.2.** Let $(C_q, d_q)_{q \in \mathbb{Z}}$ be an object in $\text{Pco}(A)$. Then $(C_q/\text{Im} \ d_{q-1} \circ d_{q-2}, "d_q")_{q \in \mathbb{Z}}$, abbreviated as $("C_q")_{q \in \mathbb{Z}}$, is an object in $\text{Co}(A)$, where $"d_q"$ is the morphism induced by $d_q$ as will be described below in the proof.
Proof. Let

\[ \cdots \rightarrow C_q \xrightarrow{d_{q-2}} C_{q-1} \xrightarrow{d_{q-1}} C_q \xrightarrow{d_q} \cdots \]

be a cochain precomplex in \( \text{Pco} (\mathcal{A}) \). Then the morphism "\( d'' \)" is defined as the morphism

\[ \begin{array}{c}
C_q / \text{Im} \ d_{q-1} \circ d_{q-2} \\
\xrightarrow{"d''"} C_{q+1} / \text{Im} \ d_q \circ d_{q-1}
\end{array} \]

such that "\( d'' [c_q] \)" = \( [d_q c_q] \) in \( C_{q+1} / \text{Im} \ d_{q+1} \circ d_{q-2} \) for \( [c_q] \in C_q / \text{Im} \ d_{q-1} \circ d_{q-2} \). Note "\( d'' \)" is well-defined. It remains to demonstrate that "\( d'' \circ d'' \) (\( [c_q] \)) = 0. By the above definition, "\( d' \circ d'' \) (\( [c_q] \)) = \( [d_{q+1} \circ d_q (c_q)] \) = 0 holds in \( C_{q+2} / \text{Im} \ d_{q+2} \circ d_q \).

Remark. The assignment of an object \((C_q, d_q)_{q \in \mathbb{Z}}\) in \( \text{Pco} (\mathcal{A}) \) to the object \((C_q / \text{Im} \ d_{q-1} \circ d_{q-2}, "d''")_{q \in \mathbb{Z}}\) is a right exact functor.

Note. We call this process (functor) \((C_q, d_q)_{q \in \mathbb{Z}} \rightarrow ("C_q", "d''")_{q \in \mathbb{Z}}\) the complexifying functor of the precomplex \((C_q, d_q)_{q \in \mathbb{Z}}\).

Definition 1.3. For an object \((C_q, d_q)_{q \in \mathbb{Z}}\) in \( \text{Pco} (\mathcal{A}) \), define the \( q \)-th precohomology of \((C_q, d_q)_{q \in \mathbb{Z}}\), denoted as \( \text{Ph}_q (C^*) \), by

\[ \text{Ph}_q (C^*) = H^q (\cdots \rightarrow C_q / \text{Im} \ d_{q-1} \circ d_{q-2} \xrightarrow{"d''"} \cdots) \]

\[ = \text{Ker} "d'' / \text{Im} "d'' \text{-1}" \]

i.e., by the \( q \)-th cohomology of the cochain complex derived from the cochain precomplex \((C_q, d_q)_{q \in \mathbb{Z}}\).

Note. We have \( \text{Ker} "d'' = \{[c_q] \in C_q / \text{Im} \ d_{q-1} \circ d_{q-2} \mid d_q (c_q - d_{q-1} c_{q-1}) = 0 \} \) and \( \text{Im} "d'' \text{-1}" = \{[c_q] \in C_q / \text{Im} \ d_{q-1} \circ d_{q-2} \mid c_q = d_{q-1} (c_{q-1}) \} \) for some \( c_{q-1} \in C_{q-1} \).

From this note, we plainly have the following proposition.

Proposition 1.4. Precohomology is a generalization of cohomology in the sense that precohomology coincides with cohomology in the case when a cochain precomplex is a cochain complex.
**Definition 1.5.** Let $(C_q, d^q, q \in \mathbb{Z})$ be a cochain precomplex in $\text{Preco} (A)$, then the dual-complexifying functor of the precomplex $(C_q, d^q, q \in \mathbb{Z})$ is defined as $(\text{Ker } d^{q+1} \circ d^q, 'd^q, q \in \mathbb{Z})$, where '$d^q$' is the restriction of $d^q$ on the subobject $\text{Ker } d^{q+1} \circ d^q$ of $C_q$. The object which was obtained above is a cochain complex, denoted by $(C_q', 'd^q, q \in \mathbb{Z})$ or simply by $(C_q')_q \in \mathbb{Z}$. Define the $q$-th dual-precohomology $'\text{Ph}_q (C^*)$ of a precomplex $C^*$ as

$'\text{Ph}_q (C^*) = \text{Ker } 'd^q / \text{Im } 'd^{q-1}$. 

**Theorem 1.6.** (Self-Duality of Precohomology). The canonical map from $(C^*)'$ to $(C^*)'$ induces an isomorphism from $'\text{Ph}_q (C^*)$ to $\text{Ph}_q (C^*)$ for each $q \in \mathbb{Z}$.

**Proof.** We will give a proof using [4]. Let us denote the canonical map $'\text{Ph}_q (C^*) \rightarrow \text{Ph}_q (C^*)$ by $\Phi$, i.e., for the cohomologous class $\overline{x}$ of $x \in \text{Ker } 'd^q$, $\Phi (\overline{x}) = \pi_q (i_q x)$, where $i$ is the monomorphism $\text{Ker } d^{q+1} \circ d^q \rightarrow C_q$ and $\pi_q$ denotes the projection $C_q \rightarrow C_q / \text{Im } d^{q-1} \circ d^{q-2}$. Notice $\pi_q (i_q x) = [x]$, where $[x] \in (C^*)' = C_q / \text{Im } d^{q-1} \circ d^{q-2}$. This map is well-defined since $''d^q'' ([x]) = 0$ holds in $(C^{q+1})'$. This is because $x \in \text{Ker } 'd^q$, i.e., $'d^q (x) = d^q (x) = 0$ in $(C^{q+1})'$. First we will show that $\Phi$ is monomorphic. Suppose $[x] = 0$, then $[x] \in \text{Im } ''d^{q-1}''$. Hence $x = d^{q-1} (x^{q-1})$ as in the note after Def 1.3. We need to check $x^{q-1} \in \text{Ker } d^{q-1} \circ d^{q-1} = (C^{q-1})'$. $d^{q-1} (x^{q-1}) = d^q x = 0$ holds from the above. Secondly, we will prove $\Phi$ is epimorphic. Let $[x'] \in \text{Ph}_q (C^*)$. Then, since $[x] \in \text{Ker } ''d^q''$, $d^q (x - d^{q-1} x') = 0$ holds for some $x' \in C^{q-1}$. Then $\Phi (x - d^{q-1} x') = [x - d^{q-1} x'] = [x]$ holds since $-d^{q-1} x' = d^{q-1} (-x')$. Notice also $x - d^{q-1} x' \in \text{Ker } d^{q+1} \circ d^q = (C^')$.

**Proposition 1.7.** (Half-Exactness). Let $0 \rightarrow C_1^* \rightarrow C_2^* \rightarrow C_3^* \rightarrow 0$ be a short exact sequence in $\text{Preco} (A)$. Then, for each $q \in \mathbb{Z}$, the sequence

$\overline{a^q} : \text{Ph}_q (C_1^*) \rightarrow \text{Ph}_q (C_2^*) \rightarrow \text{Ph}_q (C_3^*)$ is exact at $\text{Ph}_q (C_1^*)$.

**Proof.** Suppose $\overline{a^q} ([x]) = [\overline{b^q} (x)] = 0$ holds in $\text{Ph}_q (C_3^*)$. That is, $[\overline{b^q} (x)] \in \text{Im } ''d^q_{3-1}''$ holds, which implies $\beta^q (x) = d_{3-1}^{-1} (y)$ for some $y \in C_{3-1}^q$. Since $d_{3-1}^{-1}$ is an epimorphism, there exists $x' \in C_{2-1}^q$ such
that $\beta^{q-1}(x') = y$. Let $x'' = d_{q-1}^q x'$. We obtain $\beta^q(x'' - x) = 0$ since $\beta^q(x'') - \beta^q(x) = d_{q-1}^q x' - d_{q-1}^q (x') - \beta^q(x) = d_{q-1}^q (y) - \beta^q(x) = 0$. Therefore one can find $z \in C_1$ such that $\alpha^q(z) = x'' - x$ by the exactness. We need to prove "$d_1^q [z] = 0$, i.e.,

$$d_1^q z - d_1^q d_1^{q-1} z' = 0$$

holds for some $z' \in C_1$.

We have that

$$\alpha^{q+1} d_1^q z - \alpha^{q+1} d_1^q d_1^{q-1} z' = \alpha^{q+1} d_1^q z - d_2^q d_2^{q-1} \alpha^{q-1} z' =$$

$$= d_2^q (x^1 (z) - d_2^{q-1} \alpha^{q-1} z').$$

Therefore, it is sufficient to prove $[\alpha^q(z)] \in \text{Ker } "d_2^q", i.e., to show $[x'' - x] \in \text{Ker } "d_2^q"$. Choose $x' - x^0 \in C_2$, where $x^0$ is chosen such that $d_2^q x - d_2^q d_2^{q-1} x^0 = 0$ for $[x] \in \text{Ker } "d_2^q"$ above. Then

$$d_2^q (x'' - x - d_2^{q-1} (x' - x^0)) = d_2^q x'' - d_2^q x - d_2^q d_2^{q-1} (x' - x^0)$$

$$= d_2^q (d_2^{q-1} x' - x - d_2^{q-1} (x' - x^0)) = 0$$

holds. Hence $\text{Ph}_q$ is a half-exact functor.

**REMARK 1.8.** Consider the following short exact sequence of precomplexes, denoted as $0 \rightarrow ^2\mathbb{Z} \rightarrow ^3\mathbb{Z} \rightarrow ^4\mathbb{Z} \rightarrow 0$, of $\text{Pco}^+ (\Lambda)$:
Then the complexifying functor " " applied to the above implies the diagram

\[
\begin{array}{cccc}
0 & \to & 0 & \to \\
\uparrow & & \uparrow & \\
0 & \to & 0 & \to \\
\uparrow & & \uparrow & \\
0 & \to & 0 & \to \\
\uparrow & & \uparrow & \\
0 & \to & 0 & \to \\
\end{array}
\]

From this sequence of complexes, if \( \Phi^* \) were an exact connected sequence of functors, one would obtain

\[
\begin{align*}
0 & \to \Phi_0(\mathbb{Z}) \to \Phi_0(\mathbb{Z}) \to \Phi_0(\mathbb{Z}) \to \Phi_1(\mathbb{Z}) \to \Phi_1(\mathbb{Z}) \to \ldots \\
\end{align*}
\]

Hence, \( \Phi^n, n \in \mathbb{Z} \), is not an exact connected sequence of functors.

REMARK 1.9. The right derived functors of \( \Phi^0 \) on \( \text{Pco}^+ (A) \) are given by

\[
\left\{\begin{array}{ll}
\Phi^0 = \text{Ker } (d^0), & n = 0 \\
\text{Coker } (d^0), & n = 1 \\
0, & n \geq 2.
\end{array}\right.
\]

2. Hyperprecohomology of a left exact functor

Let \( A \) and \( B \) be abelian categories and let \( F : A \to B \) be a left exact additive functor.

DEFINITION 2.1. Let \( (C_q, d^q)_{q \in \mathbb{Z}} \in \text{Pco}^+ (A) \). By the complexifying functor, denoted by " " in the previous section, one has
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\((C^q / \text{Im } dq^{-1} \circ dq^{-2}, "d^q")_{q \in \mathbb{Z}}\) as an object \(\text{Co}^+(A)\). We will abbreviate the above associated cochain complex as "\(C^*\)". Then \(F"C^*"\) is an object of \(\text{Co}^+(B)\). The \(q\)-th hyperprecohomology of \(F\) evaluated at \(C^*\), denoted as \(\text{Ph}_q F(C^*)\), is defined as the \(q\)-th hyperderived functor of \(F\) evaluated at "\(C^*\)".

**Note 1.** We have the following diagram of categories and functors

\[
\begin{array}{ccc}
\text{Co}^+(A) & \xrightarrow{\text{Co}(F)} & \text{Co}^+(B) \\
\downarrow H^0 & & \downarrow H^0 \\
A & \xrightarrow{F} & B,
\end{array}
\]

where functors \((C^q, d^q)_{q \in \mathbb{Z}} \xrightarrow{F"C^*"} \text{Co}^+(A), "C^*" \xrightarrow{F"C^*"} \text{Co}^+(B)\) and \(H^0 : \text{Co}^+(A) \xrightarrow{\text{Ker } d^0} A\) is defined by \(H^0 ("C^*") = \text{Ker } d^0\) and \(F : A \xrightarrow{\text{Ker } d^0} B\) by \(\text{Ker } d^0 \xrightarrow{\text{Ker } F d^0} F(\text{Ker } d^0)\). Notice \(F(\text{Ker } d^0) \xrightarrow{\text{Ker } F d^0}\) holds since \(F\) is left exact. Then there are induced spectral sequences

\[
\begin{align*}
E_2^{p, q} &= H^p (R^q F ("C^*")) = \\
&= H^p (- - \rightarrow R^q F ("C^p") \rightarrow R^q F ("C^p+1") \rightarrow - - -)
\end{align*}
\]

\[
(2.1.1)
\]

(2.1.2)

with their abutement the hyperprecohomology \(\text{Ph}_p F(C^*)\), where \(R^p F\) denotes the \(p\)-th derived functor of \(F\).

Furthermore, (2.1.1) can be extended to

\[
(2.1.1')
\]

see [2, pp. 118].
REMARK. We have the commutative diagram of categories and functors:

See Definition 2.1 and the above Note 1 for the description of each functor. The composition of functors leaving $Pco^+(A)$ to $B$, counter-clockwise, defines the zero-th hyperprecohomology $Ph^0F(C^*)$ of $F$ at $C^*$ in $Pco^+(A)$. The composition of functors leaving $Pco^+(B)$ to $B$, clockwise, defines the zero-th precohohomology of $F(C^*)$.

3. Preinverse Limit

Let $(C^q, d^q)_{q \in \mathbb{Z}}$ be a cochain precomplex and be regarded as an inverse system:

\[
\begin{array}{ccccccc}
\cdots & \cdots & \cdots & d^q & d^{q-1} & C^{q-1} & \longrightarrow & C^q & \longrightarrow & C^{q+1} & \longrightarrow & \cdots
\end{array}
\]

DEFINITION 3.1. Let $A$ be an abelian category such that denumerable direct products of objects exist and such that the denumerable direct product functor is exact. Let $C^0 = C^1 = \Pi_{q \in \mathbb{Z}} C^q$ and define a morphism

\[
\delta^0 : C^0 \rightarrow C^1
\]

by $\pi_{q+1} \circ \delta^0 = d^q \circ \pi_q - d^q d^{q-1} \circ \pi_{q-1}$, where $\pi_q : \Pi_{C^q \rightarrow C^q}$ is the projection. Let $C^n = 0$ for $n \neq 0, 1$ and $\delta^n = 0$ for $n \neq 0$. Then

\[
\begin{array}{ccccccc}
0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
\]
is a cochain complex, denoted by $C^*$. Define the preinverse limit, denoted as $\text{Pim}$,

$$\text{Pim} C^q = H^0 (C^*) = \text{Ker} \delta^0$$

and define the 1-st preinverse limit, denoted as $\text{Pim}^1$,

$$\text{Pim}^1 C^q = H^1 (C^*) = C^1 / \text{Im} \delta^0$$

\textbf{Note.} $\lim C^q \subseteq \text{Pim} C^q$ and $\lim^1 C^q \longrightarrow \text{Pim}^1 C^q$ hold, where $\lim$ and $\lim^1$ are the usual inverse limits.

\textbf{Theorem 3.2.} Let $(C^q, d^q)_{q \in \mathbb{Z}}$ be a cochain precomplex, regarded as an inverse system. There exists an isomorphism

$$\prod C^q / \text{Pim} C^q \cong \text{Pim}^1 C^q / \prod \text{Ph} C^q$$

where $\prod$ is the canonical epimorphism $\prod C^q \rightarrow \prod C^q$.

\textbf{Proof.} Consider the following diagram.
From the definition of "dq", one has \( \Pi \text{Ker} "dq" = \text{Ker} "d_0" \) and \( \Pi \text{Im} "dq" = \text{Im} "d_0" \). Hence, the commutative diagram

\[
\begin{array}{c}
0 \rightarrow \Pi \text{Im} "d_0-1" \rightarrow \Pi \text{Ker} "d_0" \rightarrow \Pi \text{Ker} "d_0/Im "d_0-1" \rightarrow 0 \\
0 \rightarrow \text{Im} "d_0" \rightarrow \text{Ker} "d_0" \rightarrow \Pi \text{Ph} q(C^*) \rightarrow 0 \\
0 \rightarrow \text{Im} "d_0" \rightarrow \Pi "c_0" \rightarrow \text{Pim}^1 "c_0" \rightarrow 0
\end{array}
\]

implies, by a well-known lemma applied to the second and third short exact sequences, the following exact sequence,

\[
0 \rightarrow \text{Ker} l' \rightarrow \text{Ker} l \rightarrow \text{Ker} l'' \rightarrow \text{Coker} l' \rightarrow \text{Coker} l \rightarrow \text{Coker} l'' \rightarrow 0.
\]

Hence, one obtains the isomorphism

\[
\text{Coker} l = \Pi q \in \mathbb{Z} "Cq" / \text{Pim}^1 "Cq" \cong \text{Coker} l'' = \Pi q \in \mathbb{Z} \text{Ph} q (C^*).
\]

REFERENCES