

Second Leray spectral sequence of relative hypercohomology

(generalized Mayer–Vietoris sequence)

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ABSTRACT A second Leray spectral sequence of relative hypercohomology is constructed. (This is skew in generality to an earlier one constructed by S. Lubkin [(1968) *Ann. Math.* 87, 105–255].) The Mayer–Vietoris sequence of relative hypercohomology [Lubkin, S. (1968) *Ann. Math.* 87, 105–255] is also generalized.

This note provides a generalization of the Leray spectral sequence. This material was originally presented in the seminar “Zeta Matrices of an Algebraic Family,” given by S.L. at Harvard University during the fall of 1969. Our second Leray spectral sequence of relative hypercohomology has important applications in algebraic geometry, complex analytic geometry, and the theory of hyperfunctions of several variables.

Also we generalize the Mayer–Vietoris sequence in ref. 1, I.6, Corollary 4.3. (This generalized Mayer–Vietoris sequence was also presented in the same seminar, “Zeta Matrices of an Algebraic Family,” in 1969.)

1. The second Leray spectral sequence of hypercohomology modulo an open subset

THEOREM. Let X and Y be topological spaces; let f be a continuous map from X to Y ; let U be an open subset in X ; and let V be an open subset in Y . Let us denote by $\text{Co}^+(\mathcal{S}(X))$ the category of positive cochain complexes of sheaves of abelian groups on X . For each open subset W in Y and $\mathcal{F}^* \in \text{Co}^+(\mathcal{S}(X))$, define $\mathbf{H}_Y^q(X, U, \mathcal{F}^*)$ to be the sheaf associated to the presheaf $(W \mapsto H^q(f^{-1}(W), f^{-1}(W) \cap U, \mathcal{F}^*))$. Then there is induced a first-quadrant cohomological spectral sequence with $E_2^q = H^p(Y, V, \mathbf{H}_Y^q(X, U, \mathcal{F}^*))$ abutting to $H^n(X, f^{-1}(V) \cup U, \mathcal{F}^*)$.

Note. We call this spectral sequence “the second Leray spectral sequence of relative hypercohomology” because there is another well-known Leray spectral sequence of relative hypercohomology (ref. 1, I.6; pp. 151–152), which we now call the first Leray spectral sequence of relative hypercohomology. This second Leray spectral sequence of relative hypercohomology is very general (but is skew in generality with respect to the first Leray spectral sequence of relative hypercohomology as defined in ref. 1, I.6, pp. 151–152).

As special cases of the theorem we have the following corollaries. [If $f = \text{id}_X$, we write $\mathbf{H}^q(X, U, \mathcal{F}^*)$ for $\mathbf{H}_X^q(X, U, \mathcal{F}^*)$.]

COROLLARY 1. If f is an identity map, then $E_2^q = H^p(X, V, \mathbf{H}^q(X, U, \mathcal{F}^*))$ with the abutment $H^n(X, V \cup U, \mathcal{F}^*)$.

COROLLARY 2. If f is an identity map and if $U = \phi$, then $E_2^q = H^p(X, V, \mathcal{H}^q(\mathcal{F}^*))$ with the abutment $H^n(X, V, \mathcal{F}^*)$, in which $\mathcal{H}^q(\mathcal{F}^*) = \ker(\mathcal{F}^q \rightarrow \mathcal{F}^{q+1})/\text{Im}(\mathcal{F}^{q-1} \rightarrow \mathcal{F}^q)$. Note that Corollary 2 is the second spectral sequence of relative hypercohomology as defined in ref. 1, I.6, p. 141, equation 2. Thus, that spectral sequence is a special case of the theorem.

COROLLARY 3. If f is an identity map and if $V = \phi$, then

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$E_2^q = H^p(X, \mathbf{H}^q(X, U, \mathcal{F}^*))$ and the abutment is $H^n(X, U, \mathcal{F}^*)$.

Note. Let X be an open subset D in $\mathbb{C}^n = \underbrace{\mathbb{C}X \cdots \mathbb{C}X}_{n \text{ times}}$ and let

$U = D - \mathbb{R}^n \cap D$ and let \mathcal{F}^* be the single sheaf \mathcal{O} , sheaf of germs of holomorphic functions on D , then $\mathbf{H}^n(D, D - \mathbb{R}^n \cap D, \mathcal{O})$ is called the sheaf of hyperfunctions on $\mathbb{R}^n \cap D$ (ref. 2).

COROLLARY 4. Let f be a continuous map from X to Y ; let $U = V = \phi$; let \mathcal{F}^* be a single sheaf $\mathcal{F} \in \mathcal{S}(X)$; let $(R^q f_*)_{\text{pre}}(\mathcal{F})$ be the presheaf defined by $W \mapsto H^q(f^{-1}(W), \mathcal{F})$; and let $(R^q f_*)(\mathcal{F})$ be the sheaf associated to this presheaf. Then in this special case we obtain the most familiar, ordinary Leray spectral sequence with $E_2^q = H^p(Y, (R^q f_*)(\mathcal{F}))$ and with abutment $H^n(X, \mathcal{F})$.

Note. In this case, when $U = V = \phi$ and $\mathcal{F}^* = \mathcal{F}$ in the theorem, we have that $\mathbf{H}_Y^q(X, \mathcal{F}) = (R^q f_*)(\mathcal{F})$.

COROLLARY 5. If f is a continuous map from X to Y , $\mathcal{F}^* \in \text{Co}^+(\mathcal{S}(X))$, and $U = V = \phi$, then we have a spectral sequence of ref. 3, chapter III, section 2.

Proof of the theorem. We use the theorem of the spectral sequence of a composite functor. First note that the abelian category $\text{Co}^+(\mathcal{S}(X))$ has enough injectives (ref. 1, I.1, Theorem 2) and that the relative hypercohomology is a system of derived functors on $\text{Co}^+(\mathcal{S}(X))$ (ref. 1, I.1, pp. 117–118). We must first show that $H^0(Y, V, \mathbf{H}_Y^0(X, U, \mathcal{F}^*)) = H^0(X, f^{-1}(V) \cup U, \mathcal{F}^*)$ for $\mathcal{F}^* \in \text{Co}^+(\mathcal{S}(X))$. This is true because $W \mapsto H^0(f^{-1}(W), f^{-1}(W) \cap U, \mathcal{F}^*)$ is obviously a sheaf. It remains to show that $\mathbf{H}_Y^0(X, U, \mathcal{F}^*)$ is flask for an injective object \mathcal{F}^* in $\text{Co}^+(\mathcal{S}(X))$ —i.e., a cochain complex \mathcal{F}^* such that $\mathcal{F}^0 \rightarrow \mathcal{F}^1 \rightarrow \cdots \rightarrow \mathcal{F}^{n+1} \rightarrow \cdots$ is exact, such that \mathcal{F}^j is injective, all $j \geq 0$ and such that $\ker(\mathcal{F}^0 \rightarrow \mathcal{F}^1)$ is injective (ref. 1, I.1, Theorem 2). But $\mathbf{H}_Y^0(X, U, \mathcal{F}^*) = \mathbf{H}_Y^0(X, U, \mathcal{F})$ in which $\mathcal{F} = \mathcal{H}^0(\mathcal{F}^*)$, so that \mathcal{F} is injective (ref. 1, I.1, Theorem 2). Therefore it remains to show that $\mathbf{H}_Y^0(X, U, \mathcal{F})$ is flask when \mathcal{F} is an injective sheaf. If W is an open subset in Y , then it suffices to show that $\Gamma(Y, \mathbf{H}_Y^0(X, U, \mathcal{F})) \rightarrow \Gamma(W, \mathbf{H}_Y^0(X, U, \mathcal{F}))$ is an epimorphism. Let $s_1 \in H^0(f^{-1}(W), f^{-1}(W) \cap U, \mathcal{F}) = \Gamma(W, \mathbf{H}_Y^0(X, U, \mathcal{F}))$ and let $s_2 \in \Gamma(U, \mathcal{F})$ be the zero section of U . Then there exists $s_3 \in \Gamma(U \cup f^{-1}(W), \mathcal{F})$ such that $s_3|_{f^{-1}(W)} = s_1$ and $s_3|_U = s_2$. Hence we have $s \in \Gamma(X, \mathcal{F})$ such that $s|(U \cup f^{-1}(W)) = s_3$ because \mathcal{F} is flask. Then $s \in H^0(X, U, \mathcal{F})$ maps into s_1 .

2. The generalized Mayer–Vietoris sequence

THEOREM. Let X be a topological space; let U, U', V , and V' be open subsets in X such that $U' \subset U$ and $V' \subset V$; and let $\mathcal{F}^* \in \text{Co}^+(\mathcal{S}(X))$. Then there is induced a long exact sequence

$$\begin{aligned} \cdots \xrightarrow{d^{n-1}} H^n(U \cup V, U' \cup V', \mathcal{F}^*) &\xrightarrow{\bar{p}^n} H^n(U, U', \mathcal{F}^*) \\ &\oplus \\ &H^n(V, V', \mathcal{F}^*) \\ &\xrightarrow{\bar{\alpha}^n} H^n(U \cap V, U' \cap V', \mathcal{F}^*) \\ &\xrightarrow{d^n} H^{n+1}(U \cup V, U' \cup V', \mathcal{F}^*) \rightarrow \cdots, \end{aligned}$$

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & H^0(U' \cup V', \mathcal{J}) & \xrightarrow{\rho_1} & H^0(U', \mathcal{J}) \oplus H^0(V', \mathcal{J}) & \xrightarrow{\alpha_1} & H^0(U' \cap V', \mathcal{J}) \rightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & H^0(U \cup V, \mathcal{J}) & \xrightarrow{\rho_2} & H^0(U, \mathcal{J}) \oplus H^0(V, \mathcal{J}) & \xrightarrow{\alpha_2} & H^0(U \cap V, \mathcal{J}) \rightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & H^0(U \cup V, U' \cup V', \mathcal{J}) & \xrightarrow{\bar{\rho}^0} & H^0(U, U', \mathcal{J}) \oplus H^0(V, V', \mathcal{J}) & \xrightarrow{\bar{\alpha}^0} & H^0(U \cap V, U' \cap V', \mathcal{J}) \rightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
& & 0 & & 0 & & 0
\end{array}$$

FIG. 1. Commutative diagram.

where the map $\bar{\rho}^n$ is induced by restrictions, and the map $\bar{\alpha}^n$ is induced by the restriction $H^n(U, U', \mathcal{J}^*) \rightarrow H^n(U \cap V, U' \cap V', \mathcal{J}^*)$ and the negative of the restriction $H^n(V, V', \mathcal{J}^*) \rightarrow H^n(U \cap V, U' \cap V', \mathcal{J}^*)$.

Proof. In general if we are given an exact sequence of functors and natural transformations $0 \rightarrow F \rightarrow G \rightarrow H$ (from one abelian category, having enough injectives, into another abelian category) such that $G(I) \rightarrow H(I)$ is an epimorphism whenever I is injective, and if H is left exact and G is half exact, then there is induced a long exact sequence $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow R^1F \rightarrow R^1G \rightarrow R^1H \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} R^nF \rightarrow R^nG \rightarrow R^nH \xrightarrow{d^n} R^{n+1}F \rightarrow \dots$ (ref. 4, III, Exercise 5, p. 52). Therefore it suffices to show that, for $\mathcal{J}^* \in \text{Co}^+(\mathcal{S}(X))$, the sequence

$$\begin{aligned}
0 \rightarrow H^0(U \cup V, U' \cup V', \mathcal{J}^*) & \xrightarrow{\bar{\rho}^0} H^0(U, U', \mathcal{J}^*) \oplus H^0(V, V', \mathcal{J}^*) \\
& \xrightarrow{\bar{\alpha}^0} H^0(U \cap V, U' \cap V', \mathcal{J}^*) \quad [1]
\end{aligned}$$

is exact, in which $\bar{\rho}^0$ and $\bar{\alpha}^0$ are defined as in the statement of the theorem, and also to show that for an injective object \mathcal{J}^* in $\text{Co}^+(\mathcal{S}(X))$ the sequence

$$\begin{aligned}
0 \rightarrow H^0(U \cup V, U' \cup V', \mathcal{J}^*) & \xrightarrow{\bar{\rho}^0} H^0(U, U', \mathcal{J}^*) \oplus H^0(V, V', \mathcal{J}^*) \\
& \xrightarrow{\bar{\alpha}^0} H^0(U \cap V, U' \cap V', \mathcal{J}^*) \rightarrow 0 \quad [2]
\end{aligned}$$

is exact. Equation 1 is plainly true from the definitions of $\bar{\rho}^0$ and $\bar{\alpha}^0$. To prove Eq. 2, we can replace \mathcal{J}^* by $\mathcal{J} = \mathcal{H}^0(\mathcal{J}^*)$, which is injective (ref. 1, I.1, Theorem 2). Consider the commutative diagram with exact columns shown in Fig. 1. We must show that the bottom row is exact. But by the nine lemma, it suffices to show that the top two rows are exact. Therefore in order to

prove that the map $\bar{\alpha}^0$ in Eq. 2 is an epimorphism, we are reduced to the case in which $U' = V' = \phi$, $\mathcal{J}^* = \mathcal{J}$ a single injective sheaf. But then the restriction map $H^0(U, \mathcal{J}) \rightarrow H^0(U \cap V, \mathcal{J})$ is an epimorphism because \mathcal{J} is flack.

q.e.d.

Remark 1. In ref. 1, I.6, Corollary 4.3, Mayer-Vietoris sequences and the first Leray spectral sequence of relative hypercohomology are established for punctual cohomology and punctual sheaves. Of course those results [and also the first and second spectral sequences of relative hypercohomology (ref. 1, I.6, pp. 140-141) in that paper] work equally well for ordinary sheaf cohomology by the methods analogous to this note.

Conversely, the second Leray spectral sequence of relative hypercohomology and the generalized Mayer-Vietoris sequence established in this paper hold equally well for punctual cohomology and punctual sheaves by the analogue of the technique in ref. 1, Chapter 1.

Remark 2. The first and second Leray spectral sequence of relative hypercohomology; the generalized Mayer-Vietoris sequence, and the two spectral sequences of relative hypercohomology go through equally well to combinatorial punctual hypercohomology of positive cochain complexes of étale sheaves on a proscheme, as defined in ref. 5, Chapter 1.

Note Added in Proof. In the theorem (resp.: Corollary 1; in Corollary 3) in Section I, the sheaf $\mathcal{H}^q(X, U, \mathcal{J}^*)$ is concentrated on the closed subset $D = \bar{f}(X - U)$ (resp.: $D = X - U$; $D = X - U$) of Y (resp.: X ; X). Therefore, $E_2^{p,q} = H^p(D, D \cap V, \mathcal{H}^q(X, U, \mathcal{J}^*))$.

1. Lubkin, S. (1968) *Ann. Math.* 87, 105-255.
2. Harvey, R. (1969) *Am. J. Math.* 91, 853-873.
3. Grothendieck, A. & Dieudonné, J. (1961) *Éléments de Géométrie Algébrique* (Publ. Math. I.H.E.S. 11).
4. Cartan, H. & Eilenberg, S. (1956) *Homological Algebra* (Princeton Univ. Press, Princeton, NJ).
5. Lubkin, S. (1967) *Am. J. Math.* 89, 443-548.