Mapping Galileo's Trajectory

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The NASA Galileo mission was mapped out using a patched conics approximation. Galileo launched from Earth, underwent a gravity assist from Venus back to Earth for another gravity assist. Galileo then came back to Earth for one more gravity assist to propel it to Jupiter. A simulation ignoring all perturbations (i.e. third body effects, zonal harmonics, solar wind, drag) was carried out. The simulation was able to make it to the final Earth flyby before diverging from the actual trajectory. The simulated and actual flyby dates all differed by less than 5 hours, and the simulated and actual flyby altitudes differed by less than 2,000 km.

Nomenclature

A	Parameter used to determine Lambert trajectory
$C(\cdot)$, $S(\cdot)$	Stumpff functions
e	Eccentricity
$F(\cdot)$	Function used for Newton's method
f, $\dot{f}$, g, $\dot{g}$	Lagrange coefficients
h	Specific angular momentum, km^2/s
i	Inclination, rad
M	Amount to move position vector for course correction, km
$\mathbf{Q}$	Transformation matrix
$\mathbf{r}$	Position vector, km
r	Magnitude of position vector, km
$\hat{\mathbf{u}}$	Unit vector giving direction of course correction
$\mathbf{u}^*$	Rotated vector giving magnitude and direction of position change, km
$\hat{\mathbf{u}}^*$	Rotated unit vector giving direction of course correction
$\mathbf{v}$	Velocity vector, km/s
$\hat{\mathbf{v}}$	Unit vector giving direction of velocity vector
v	Magnitude of velocity vector, km/s
$\hat{\mathbf{w}}$	Unit vector orthogonal to orbital plane
$y(\cdot)$	Parameter function used to determine Lambert trajectory
z	Parameter used to determine Lambert trajectory and universal variable
α	Reciprocal of semimajor axis, km^{-1}
β	Angle to rotate $\hat{\mathbf{u}}$ counterclockwise, rad
Δt	Change in time, s
$\Delta\theta$	Change in true anomaly, rad
θ	True anomaly, rad
μ	Gravitational parameter, km^3/s^2
χ	Universal variable, km$^{1/2}$
Ω	Right ascension of the ascending node, rad
ω	Argument of periapse, rad
$\boldsymbol{\varpi}$	Absolute angular velocity vector, rad/s

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Subscripts

0	Initial guess for Newton's method
1	Starting index for Lambert trajectory
2	Ending index for Lambert trajectory
G	Galileo
G/P	Galileo relative to a planet
i	inclination
P	Planet
p	Perifocal frame
r	Radial
S	Sun
Ω	Right ascension of the ascending node
ω	Argument of periapse

I. Introduction

Galileo was a NASA spacecraft sent to Jupiter. It's objectives were to investigate the chemical composition and physical nature of Jupiter's atmosphere and its satellites as well as study the structure and dynamics of Jupiter's magnetosphere.[1] The purpose of this project was to accurately map out the trajectory from Earth to Jupiter.

Originally, Galileo was scheduled for a 2.5 year direct inject transfer, but this plan was cancelled due to the Space Shuttle Challenger accident.[2] As a result, the Centaur upper stage rocket was replaced with the less powerful Inertial Upper Stage (IUS) rocket. Trajectory engineers at NASA's Jet Propulsion Laboratory (JPL) concluded that the best way to reach Jupiter was a Venus-Earth-Earth Gravity Assist (VEEGA) trajectory.

A VEEGA trajectory is one in which a spacecraft uses a gravity assist from Venus, followed by two Earth gravity assists before arriving at the destination. This trajectory was chosen for a number of reasons. One reason is that Galileo's VEEGA trajectory only requires a small amount of fuel for course corrections ($\Delta v \approx 125$ m/s). In addition, the VEEGA trajectory allowed Galileo to pass two asteroids, 951 Gaspra and 243 Ida. These were the first close encounters of an asteroid by a spacecraft.[3] Another reason for the VEEGA trajectory was that it allowed a launch window of 41 days.[1]

In this report, the methodology for mapping the orbits is discussed followed by the results of doing so. After this, a discussion of the results is presented as well as a conclusion and plan for future work. In the appendix, a list of relevant algorithms are given.

II. Methodology

The goal of this senior project was to map the Galileo trajectory to Jupiter as accurately as possible by treating every orbit as a two-body problem. This is called the method of patched conics.[4] Only the strongest gravitational force is considered at any point in the trajectory. That is, within the Earth's sphere of influence, only the gravity of the Earth is taken into account. The same is true for Venus. Outside the sphere of influence of Earth and Venus, only the gravity of the Sun is taken into account. It is called the method of patched conics because different conic sections are being patched together. Within the Earth's and Venus' sphere of influence, the conic section is a hyperbola. Otherwise the conic section is an ellipse. The goal is to produce a picture that looks like Fig. 1.

A. Initializing Galileo's Trajectory

Galileo launched on October 18, 1989 on the Space Shuttle Atlantis at 16:54 universal time coordinated (UTC). It arrived at Venus at a closest approach altitude of 16,123 km on February 10, 1990 at 5:59 UTC.[1] To model this, Earth's position on the launch date was determined as well as Venus' position on the arrival date using the available ephemeris data. From this, Lambert's method was used to find the appropriate starting velocity vector for Galileo. Lambert's method finds the starting and ending velocity vectors given two position vectors and the time separating them. The algorithm for Lambert's method is given in appendix

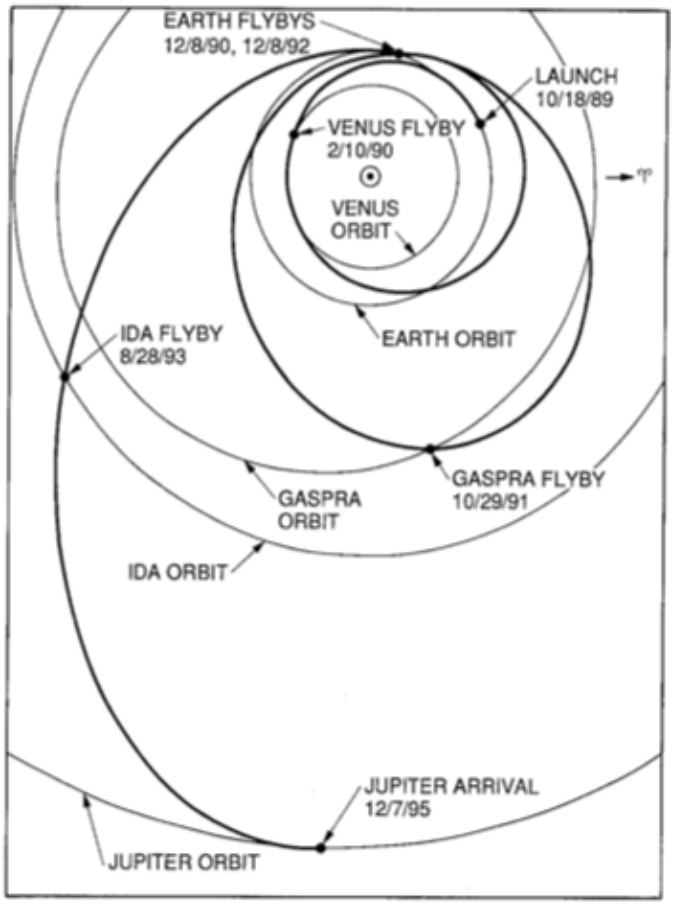

Figure 1. This is Galileo's trajectory as mapped out by NASA.[1]

A. This trajectory ensured that Galileo would hit Venus' sphere of influence.

Although the Lambert's trajectory was used between Earth and Venus, if Galileo's orbit is propagated, it will not hit Venus. This is due to the patched conics approximation. The Lambert's trajectory assumes that for an interplanetary orbit, the only gravitational force is from the Sun. Once Galileo enters the sphere of influence, the orbit changes to a hyperbola, and the only gravitational force is from Venus. It turns out that if uncorrected, Galileo will have a closest approach altitude of about 6,000 km, 10,000 km too close. Thus, something must be done to correct the trajectory.

B. Correcting Galileo's Position

Correcting a trajectory in a three dimensional space is non-trivial. Galileo's trajectory is defined by the gravitational force it is in, it's position vector at a specific time, and it's velocity vector at a specific time. The gravitational force cannot be changed, so there are seven parameters for changing Galileo's trajectory, the three components of the position, the three components of the velocity, and the time at which to change them.

In an effort to simplify this, the time at which Galileo enters Venus' sphere of influence was arbitrarily chosen to make corrections. For subsequent flybys, this will be when Galileo enters Earth's sphere of influence. The velocity of Galileo with respect to Venus when entering the sphere of influence is known (at least to one decimal place).[1] It turns out that the Lambert trajectory matches this relative velocity. Thus, it was decided to keep the velocity vector the same. This leaves three parameters to vary.

This can be reduced to two parameters. Freeze the simulation the moment Galileo enters a sphere of influence and imagine looking in the direction of Galileo's velocity vector. Now attach a polar coordinate

frame to the tail end of the velocity vector with the vertical axis orthogonal to the orbital plane. Galileo's position vector (not velocity vector) can be moved to a point on this plane (the plane orthogonal to Galileo's velocity vector). This is shown in Fig. 2. The point that the position vector was moved to is characterized by only two parameters: an angle and a magnitude.

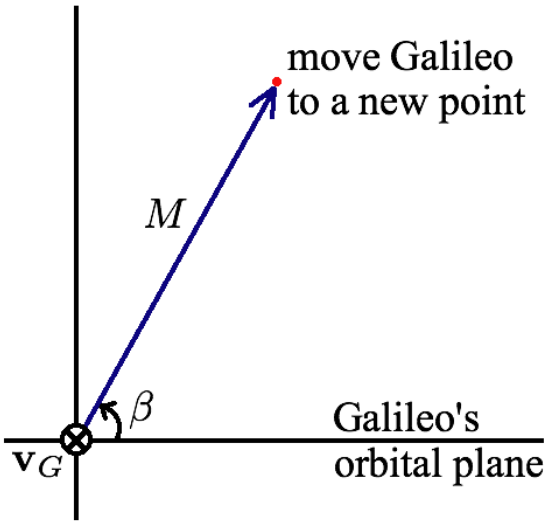

Figure 2. Attach a polar coordinate system to the tail of Galileo's velocity vector $\mathbf{v}_G$, and move Galileo to a new point specified by an angle β and a magnitude M.

This can be done through the use of a quaternion.[4,5] An algorithm for the above method is given in appendix B. To determine the optimal angle and amount to move the position vector, a brute force method was used. The two parameters were varied, and the pair that put Galileo closest to the subsequent flyby altitude was used.

C. Determining Galileo's Trajectory

The position correction described above must be carried out for each flyby. Otherwise, determining the rest of the trajectory is done by propagating Galileo's orbit. A straight forward way of orbit propagation is the universal variable approach. This algorithm is given in appendix C. It is a good idea to not only propagate Galileo's orbit, but Earth's, Venus', and Jupiter's orbits as well. This way, all bodies are kept at the same moment in time, and the ephemeris data only needs to be used to initialize the orbits. This is important because the simulated flybys (and planetary arrival in Jupiter's case) will not occur at the time recorded by NASA. They will vary by a few hours. By propagating Earth's, Venus', and Jupiter's orbits, the Julian date does not need to be tracked. This saves some headache in converting the Julian date to the Gregorian date.

Since this is a patched conics problem, Galileo's orbit is an ellipse when outside a planet's sphere of influence and a hyperbola when inside a planet's sphere of influence. In terms of propagating Galileo's orbit, this requires that its state vector be converted to a relative frame while inside a planet's sphere of influence. An algorithm for doing this is in appendix D.

III. Results

In this section, the results of the simulation of Galileo's trajectory is presented. The simulation followed the procedure described in the methodology section above.

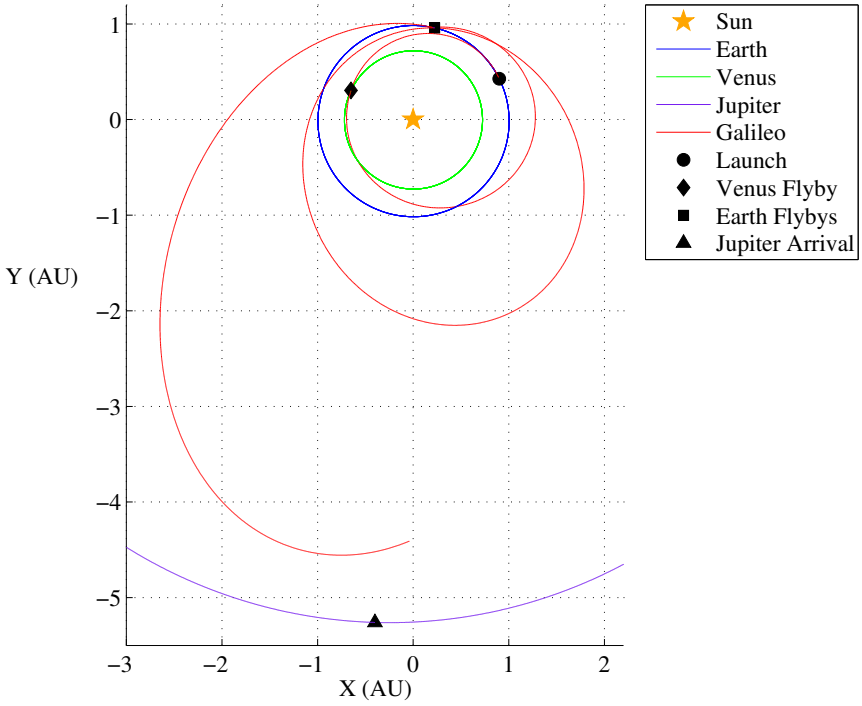

Figure 3. This is the simulated trajectory of Galileo using the procedure outlined in the methodology.

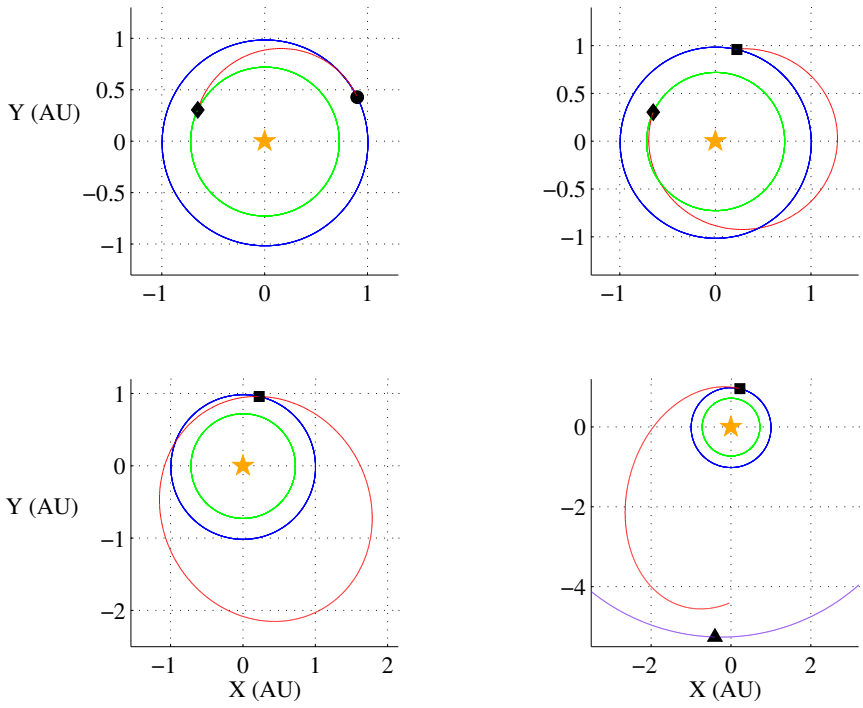

Figure 4. The upper left plot shows the transfer from Earth at launch to Venus. The upper right plot shows the transfer from the Venus flyby to Earth. The bottom left plot shows the transfer from the first Earth flyby back to Earth. The bottom right plot shows the transfer from the second Earth flyby to the end. The legend from Fig. 3 applies to these plots as well.

The simulated trajectory is shown in Fig. 3. Although Fig. 3 is useful as a comparison to Fig. 1, it is difficult to see each individual transfer. For this reason, Fig. 4 was included to show each leg of the mission. Begin with the top left plot of Fig. 4. The trajectory starts at the black circle. Galileo travels to the black diamond for the Venus flyby. Immediately before entry into the sphere of influence, Galileo's trajectory was corrected. The position vector was moved 21,500 km at an angle of 249 degrees (all angles are measured counterclockwise for this project – see Fig. 2). This correction ensured that Galileo would enter Earth's sphere of influence for the subsequent flyby. In the simulation, closest approach occurred on 2/10/90 at 4:38 UTC at an altitude of 14,591 km. The actual date was 2/10/90 at 5:59 UTC. This is a difference of about 1.35 hours. The actual altitude at closest approach was 16,123 km, for a difference of 1,532 km.

After the Venus flyby, Galileo continued toward Earth. Referencing the top right of Fig. 4, the trajectory moves to the black square. Before entering the sphere of influence, the trajectory was corrected again to ensure the next flyby. This correction moved the position vector 866,200 km at an angle of 170.5 degrees. For the simulation, closest approach was on 12/9/90 at 1:10 UTC at an altitude of 2,619 km. The actual flyby was on 12/8/90 at 20:35 UTC. The simulated date is off by about 4.58 hours. The actual flyby altitude was 960 km, so the simulation differed by 1,659 km.

After the first Earth flyby, Galileo continues on its trajectory. After about two years, it returned to Earth for the second and final Earth flyby. This is shown in the bottom left plot of Fig. 4. Before entry into the sphere of influence, the trajectory was corrected for the final time. The position vector was moved 42,840 km at an angle of 359 degrees. The simulated second Earth flyby occurred on 12/8/92 at 14:06 UTC. The actual date was 12/8/1992 at 15:35, about a 1.48 hour difference. The simulated flyby altitude was −1,440 km (i.e. below the surface of the Earth), and the actual was 300 km. This is a difference of 1,740 km.

As seen in Fig. 3 and the bottom right of Fig. 4, Galileo did not make it to Jupiter for the simulation. This is because the course correction for the second Earth flyby was not sufficient. This is discussed in the next section. The results of the simulation are summarized in Table 1.

Table 1. These are the Galileo trajectory simulation results.

	Venus Flyby	First Earth Flyby	Second Earth Flyby
Simulated Date	2/10/90 at 4:38 UTC	12/9/90 at 1:10 UTC	12/8/92 at 14:06 UTC
Actual Date	2/10/90 at 5:59 UTC	12/8/90 at 20:35 UTC	12/8/92 at 15:35 UTC
Date Difference	1.35 hours	4.58 hours	1.48 hours
Simulated Altitude	14,591 km	2,619 km	−1,440 km
Actual Altitude	16,123 km	960 km	300 km
Altitude Difference	1,532 km	1,659 km	1,740 km

IV. Discussion

As stated above, the simulation never made it to Jupiter. Before entry into the Earth's sphere of influence for the second flyby, a course correction was attempted again, but there was no correction that would send Galileo to Jupiter. This is clear from the flyby altitude. The best course correction gives a trajectory below the surface of the Earth. This means that the magnitude of the velocity vector was not large enough. The velocity could only be increased more by making the altitude more negative, but this also does not get Galileo to Jupiter. One possible explanation for this is the accumulated error over the trajectory. The error in velocity starts small, but increases as time continues. Also, the influence of the moon could affect Galileo during the Earth flybys. Galileo passes the moon at an altitude of about 111,000 km for the second flyby.[1]

Aside from that, though, the simulation was fairly close. The time differences are within a few hours and the altitudes are within 2,000 km. In terms of astronomical distances, these are very small. The course corrections themselves may seem large, but those are also small in terms of astronomical distances. One astronomical unit (AU) is about 149,598,000 km. The largest course correction is roughly 0.6% of 1 AU, and from Fig. 3, the course corrections are not visible to the eye.

Although the course corrections are relatively small, they still need to be present. The course corrections show that error accumulates during the simulation. Third body effects are the main source of error. In

particular, the Sun's gravity no doubt is a factor during flybys. For the patched conics approximation, it was ignored, which caused error. Also, a planet's gravity would have an appreciable effect when Galileo is close to, but outside its sphere of influence. Obviously, these perturbations contribute to trajectory error over time. Other perturbations include solar wind, drag, and zonal harmonics, particularly J_2.

Another source of error are the burns that Galileo made during the mission. Information on how large the burns were is readily available on the internet, but that is not the full story. The moments the burns were initiated is only given approximately, and the burn direction is not given. For the simulation, these burns were taken into account, but not at the actual time and direction. For the simulation, the burns were done after Galileo exits a planet's sphere of influence. They were modeled as instantaneous burns tangential to the velocity vector. This uncertainty also contributed to trajectory error.

V. Conclusion and Future Work

Using strictly Keplerian orbits for mapping Galileo's trajectory was mostly successful. An approximate mapping was obtained from launch to the second Earth flyby. The only piece that is missing is the transfer to Jupiter. The perturbations in Galileo's trajectory made this not possible for the simulation. The main perturbations were third body effects and uncertainty in course correction maneuvers. Smaller perturbations such as solar wind, drag, and zonal harmonics also introduced error.

Future work for this is to include third body effects. If the above claim is correct, then including the third body effects should make the trajectory much more accurate. This should give a better indication of its effect on an interplanetary trajectory.

Appendix

Here, the algorithms necessary to determine Galileo's trajectory are presented. These are not derivations. First, the solution to Lambert's problem is shown followed by a method for correcting Galileo's trajectory. Subsequently, an algorithm for propagating an orbit using the universal variable is given. Finally, an algorithm for converting Galileo's state vector from a heliocentric frame to a planetocentric frame is given. The MATLAB code that was used for the Galileo simulation is available upon request from Dr. Abercromby.

Before beginning, some notes on notation are in order. Scalars are always in non-bolded italics. They can be either upper or lower case. Vectors are always bolded lower case. Whenever a vector norm is shown, it is referring to the Euclidean norm (L_2 norm). All vectors are column vectors. These algorithms will not work if they are row vectors. Lastly, matrices are always bolded upper case.

A. Lambert Trajectory

The inputs to obtaining Lambert's trajectory are the starting position vector $\mathbf{r}_1$, the ending position vector $\mathbf{r}_2$, and the the time to get from the beginning to the end Δt. The outputs are the starting velocity vector $\mathbf{v}_1$ and the ending velocity vector $\mathbf{v}_2$.

Begin by finding the magnitudes of the position vectors.

$$r_1 = ||\mathbf{r}_1|| \tag{1}$$

$$r_2 = ||\mathbf{r}_2|| \tag{2}$$

If the Z-component of $\mathbf{r}_1 \times \mathbf{r}_2 \geq 0$, then the change in true anomaly is

$$\Delta\theta = \arccos\left(\frac{\mathbf{r}_1^T \mathbf{r}_2}{r_1 r_2}\right). \tag{3}$$

Otherwise, the change in true anomaly is

$$\Delta\theta = 2\pi - \arccos\left(\frac{\mathbf{r}_1^T \mathbf{r}_2}{r_1 r_2}\right). \tag{4}$$

Define a parameter

$$A = \sin(\Delta\theta)\sqrt{\frac{r_1 r_2}{1 - \cos(\Delta\theta)}} \tag{5}$$

to be used throughout the algorithm.

The Stumpff functions will be used periodically. They are given as

$$S(z) = \begin{cases} \frac{\sqrt{z}-\sin\sqrt{z}}{(\sqrt{z})^3} & z > 0 \\ \frac{\sinh\sqrt{-z}-\sqrt{-z}}{(\sqrt{-z})^3} & z < 0 \\ \frac{1}{6} & z = 0 \end{cases} \tag{6}$$

$$C(z) = \begin{cases} \frac{1-\cos\sqrt{z}}{z} & z > 0 \\ \frac{\cosh\sqrt{-z}-1}{-z} & z < 0 \\ \frac{1}{2} & z = 0 \end{cases} \tag{7}$$

Now, Newton's method is used to minimize a function. That function is given as

$$F(z) = \left(\frac{y(z)}{C(z)}\right)^{\frac{3}{2}} S(z) + A\sqrt{y(z)} - \sqrt{\mu}\Delta t, \tag{8}$$

where

$$y(z) = r_1 + r_2 + A\frac{zS(z)-1}{\sqrt{C(z)}} \tag{9}$$

and μ is the gravitational parameter. Newton's method requires the derivative of $F(z)$. This is given as

$$F'(z) = \begin{cases} \left(\frac{y(z)}{C(z)}\right)^{\frac{3}{2}} \left[\frac{1}{2z}\left(C(z) - \frac{3}{2}\frac{S(z)}{C(z)}\right) + \frac{3}{4}\frac{S^2(z)}{C(z)}\right] + \frac{A}{8}\left(3\frac{S(z)}{C(z)}\sqrt{y(z)} + A\sqrt{\frac{C(z)}{y(z)}}\right) & z \neq 0 \\ \frac{\sqrt{2}}{40}y^{\frac{3}{2}}(0) + \frac{A}{8}\left(\sqrt{y(0)} + A\sqrt{\frac{1}{2y(0)}}\right) & z = 0 \end{cases} \tag{10}$$

Initialize the parameter $z = 0$ and update using the Newton step

$$z \leftarrow z - \frac{F(z)}{F'(z)}, \tag{11}$$

where the left arrow means replace. Iterate Eqns. (6) – (11) until the ratio $\frac{F(z)}{F'(z)}$ is less than or equal to some tolerance. The smaller the tolerance, the more accurate the solution. However, a smaller tolerance requires more time to converge.

It is a good idea to set a maximum number of iterations. That way, if convergence is slow, iteration will not continue for too long. It is also good practice to print a warning to the screen if the maximum number of iterations is reached because this means convergence to the specified tolerance was not met.

Next, calculate the Lagrange coefficients

$$f = 1 - \frac{y(z)}{r_1}, \tag{12}$$

$$g = A\sqrt{\frac{y(z)}{\mu}}, \tag{13}$$

and

$$\dot{g} = 1 - \frac{y(z)}{r_2}. \tag{14}$$

Finally, calculate the starting and ending velocity vectors.

$$\mathbf{v}_1 = \frac{1}{g}(\mathbf{r}_2 - f\mathbf{r}_1) \tag{15}$$

$$\mathbf{v}_2 = \frac{1}{g}(\dot{g}\mathbf{r}_2 - \mathbf{r}_2) \tag{16}$$

B. Trajectory Correction

The inputs for changing changing Galileo's trajectory via the polar coordinate method are Galileo's classical orbital elements in the heliocentric frame, the distance the position vector is being moved M, and the angle specifying the direction in which to move it β. The output is Galileo's new position vector $\mathbf{r}_G$.

To begin, Galileo's velocity vector needs to be converted to the perifocal frame. This can be done with

$$\mathbf{v}_{G,p} = \frac{\mu_S}{h} \begin{bmatrix} -\sin\theta \\ e + \cos\theta \\ 0 \end{bmatrix} \tag{17}$$

where θ is Galileo's true anomaly, e is Galileo's eccentricity, h is Galileo's specific angular momentum, μ_S is the gravitational parameter of the Sun, and the subscript p denotes the perifocal frame. To get the direction of this vector, divide by the norm.

$$\hat{\mathbf{v}} = \frac{\mathbf{v}_{G,p}}{||\mathbf{v}_{G,p}||} \tag{18}$$

The hat on the left hand side of Eqn. (18) signifies that it is a unit vector.

In the perifocal frame, the unit vector orthogonal to the orbital plane is $\hat{\mathbf{w}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. In order to get the direction to move the position vector, take a cross product

$$\hat{\mathbf{u}} = \hat{\mathbf{v}} \times \hat{\mathbf{w}}. \tag{19}$$

Using the right hand rule, $\hat{\mathbf{u}}$ is the unit vector pointing at an angle of 0 degrees (see Fig. 2). To rotate $\hat{\mathbf{u}}$ counterclockwise, the quaternion vector rotation

$$\hat{\mathbf{u}}^* = \hat{\mathbf{u}}\cos\beta + \hat{\mathbf{v}}(\hat{\mathbf{u}}^T\hat{\mathbf{v}})(1 - \cos\beta) + (\hat{\mathbf{u}} \times \hat{\mathbf{v}})\sin\beta \tag{20}$$

can be used where β is the angle through which to rotate $\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}^*$ is the rotated unit vector. For example, if $\beta = \frac{\pi}{2}$, then $\hat{\mathbf{u}}^*$ would be pointing at an angle of 90 degrees (see Fig. 2). To get the desired magnitude by which to move the position vector, simply multiply by a scalar

$$\mathbf{u}^* = M\hat{\mathbf{u}}^*. \tag{21}$$

The resulting vector is still in the perifocal frame. To convert back to the heliocentric frame, define three transformation matrices. They are given as

$$\mathbf{Q}_\omega = \begin{bmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{22}$$

$$\mathbf{Q}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix}, \tag{23}$$

and

$$\mathbf{Q}_\Omega = \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{24}$$

where ω is Galileo's argument of perigee, i is Galileo's inclination, and Ω is Galileo's right ascension. Multiplying these three matrices

$$\mathbf{Q} = \mathbf{Q}_\omega\mathbf{Q}_i\mathbf{Q}_\Omega \tag{25}$$

gives a transformation matrix from the heliocentric frame to the perifocal frame. In order to convert from the perifocal frame to the heliocentric frame, the inverse of Eqn. (25) is needed. However, since $\mathbf{Q}$ is a transformation matrix, it is also an orthogonal matrix, so $\mathbf{Q}^{-1} = \mathbf{Q}^T$.

Finally, to correct the trajectory of Galileo, move the position vector

$$\mathbf{r}_G \leftarrow \mathbf{r}_G + \mathbf{Q}^T\mathbf{u}^* \tag{26}$$

where the left arrow means replace.

C. Propagating an Orbit

The required inputs for propagating an orbit are the initial position and velocity vectors $\mathbf{r}_0$ and $\mathbf{v}_0$ and the time through which to propagate Δt. The outputs are the final position and velocity vectors $\mathbf{r}$ and $\mathbf{v}$.

To start, find the magnitude of the initial state vector.

$$r_0 = ||\mathbf{r}_0|| \tag{27}$$

$$v_0 = ||\mathbf{v}_0|| \tag{28}$$

Now find the radial component of the velocity

$$v_{r0} = \frac{\mathbf{r}_0^T \mathbf{v}_0}{r_0} \tag{29}$$

and the reciprocal of the semimajor axis

$$\alpha = \frac{2}{r_0} - \frac{v_0^2}{\mu} \tag{30}$$

where μ is the gravitation parameter of the body being orbited around. For Galileo, $\mu = \mu_S$ when the Sun is the main body and $\mu = \mu_P$ when a planet is the main body. For a planet, $\mu = \mu_S$.

The next step is to use Newton's method to find the universal variable χ. To do this, start by initializing the universal variable to

$$\chi_0 = \sqrt{\mu}|\alpha|\Delta t \tag{31}$$

where Δt is the time through which to propagate. Evaluate the function

$$F(\chi) = \frac{r_0 v_{r0}}{\sqrt{\mu}}\chi^2 C(z) + (1 - \alpha r_0)\chi^3 S(z) + r_0\chi - \sqrt{\mu}\Delta t \tag{32}$$

and its derivative

$$F'(\chi) = \frac{r_0 v_{r0}}{\sqrt{\mu}}\chi\left[1 - \alpha\chi^2 S(z)\right] + (1 - \alpha r_0)\chi^2 C(z) + r_0 \tag{33}$$

where $z = \alpha\chi^2$ and $S(z)$ and $C(z)$ are given in Eqns. (6) and (7), respectively. Update the universal variable with

$$\chi \leftarrow \chi - \frac{F(\chi)}{F'(\chi)} \tag{34}$$

where the left arrow means replace. Iterate Eqns. (32) – (34) until the ratio $\frac{F(\chi)}{F'(\chi)}$ is less than or equal to some tolerance. The smaller the tolerance, the more accurate the solution. However, a smaller tolerance requires more time to converge.

It is a good idea to set a maximum number of iterations. That way, if convergence is slow, iteration will not continue for too long. It is also good practice to print a warning to the screen if the maximum number of iterations is reached because this means convergence to the specified tolerance was not met.

Next, find the Lagrange coefficients for the position vector.

$$f = 1 - \frac{\chi^2}{r_0}C(z) \tag{35}$$

$$g = \Delta t - \frac{1}{\sqrt{\mu}}\chi^3 S(z) \tag{36}$$

These yield the propagated position vector

$$\mathbf{r} = f\mathbf{r}_0 + g\mathbf{v}_0. \tag{37}$$

Now, normalize the position vector.

$$r = \frac{\mathbf{r}}{||\mathbf{r}||} \tag{38}$$

This can be used to find the Lagrange coefficients for the velocity vector.

$$\dot{f} = \frac{\sqrt{\mu}}{r r_0}\left[\alpha\chi^3 S(z) - \chi\right] \tag{39}$$

$$\dot{g} = 1 - \frac{\chi^2}{r} C(z) \tag{40}$$

Finally, the propagated velocity vector is found with

$$\mathbf{v} = \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0. \tag{41}$$

D. Heliocentric to Planetocentric Frame

The inputs for converting from a heliocentric frame to a planetocentric frame are the position and velocity vectors of both Galileo ($\mathbf{r}_G$ and $\mathbf{v}_G$) and the planet ($\mathbf{r}_P$ and $\mathbf{v}_P$) and the classical orbital elements of the planet. The outputs are the position and velocity vectors of Galileo relative to the planet $\mathbf{r}_{G/P}$ and $\mathbf{v}_{G/P}$.

To begin, find the absolute angular velocity of the planet in the perifocal frame. This is done with

$$\boldsymbol{\varpi}_{P,p} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{\frac{\mu_S}{||\mathbf{r}_P||^3}} \end{bmatrix} \tag{42}$$

where $\mathbf{r}_P$ is the position vector of the planet, μ_S is the gravitational parameter of the Sun, and the subscript p denotes the perifocal frame. This equation assumes that the planet is in a circular orbit. Although this is not true, both Venus and Earth have a very small eccentricity, so it is a reasonable approximation.

Next, calculate a transformation matrix. This is done via Eqns. (22) – (25) where the orbital elements used are the planet's orbital elements. The matrix $\mathbf{Q}$ is the transformation matrix from the heliocentric frame to the perifocal frame. In order to convert from the perifocal frame to the heliocentric frame, the inverse of Eqn. (25) is needed. However, since $\mathbf{Q}$ is a transformation matrix, it is also an orthogonal matrix, so $\mathbf{Q}^{-1} = \mathbf{Q}^T$.

Now convert the planet's absolute angular velocity to the heliocentric frame. This can be done with

$$\boldsymbol{\varpi}_P = \mathbf{Q}^T \boldsymbol{\varpi}_{P,p}. \tag{43}$$

All that is left is to find the relative state vector between Galileo and planet. The relative position vector is

$$\mathbf{r}_{G/P} = \mathbf{r}_G - \mathbf{r}_P \tag{44}$$

where the subscript G/P denotes Galileo relative to the planet. Finally, the relative velocity vector is found with

$$\mathbf{v}_{G/P} = \mathbf{v}_G - \mathbf{v}_P - \boldsymbol{\varpi}_P \times \mathbf{r}_{G/P}. \tag{45}$$

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