ON THE CONSTRUCTION OF GENERALIZED MEASURE PRESERVING TRANSFORMATIONS WITH GIVEN MARGINALS

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Abstract

Measure preserving transformations generate stationary processes and vice versa. Which processes (X_i) correspond to the class of *generalized* measure preserving transformations? We give necessary conditions and show that they are sufficient for 2-valued processes as far as the marginals of (X_0, X_1, X_2, X_3) are concerned. The general problem remains open. Our main tool is a construction of a class of generalized measure preserving transformations which may be of independent interest.

§1. Introduction

The notion of generalized measure preserving transformation (gmp-transformation) was introduced in [K] as a mathematical model for the movement of sets of incompressible objects subject to interaction. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. A gmp-transformation is a map $\phi : \mathcal{A} \rightarrow \mathcal{A}$ which is order preserving and which preserves μ . In other words, $A \subset B$ implies $\phi(A) \subset \phi(B)$, and $\mu(\phi(A)) = \mu(A)$ holds for all $A \in \mathcal{A}$.

A measure preserving transformation $\tau : \Omega \rightarrow \Omega$ induces a gmp-transformation ϕ_τ by setting $\phi_\tau(A) = \tau^{-1}A$. In general, however, a gmp-transformation need not commute with the formation of unions or intersections. We only have

$$\phi(A \cap B) \subset \phi(A) \cap \phi(B) \quad (A, B \in \mathcal{A}) \quad (1.1)$$

and

$$\phi(A \cup B) \supset \phi(A) \cup \phi(B) \quad (A, B \in \mathcal{A}). \quad (1.2)$$

Let $(X_n)_{n \geq 0}$ be a real-valued stochastic process defined on a probability space $(\Omega', \mathcal{A}', P)$. It is well known that there exists a measure preserving τ on a suitable probability space $(\Omega, \mathcal{A}, \mu)$ and a measurable f on Ω such that (X_n) and $(f \circ \tau^n)$ have the same joint distributions if and only if the distribution of (X_n) is stationary. We propose to study the corresponding problem for gmp-transformations.

As is shown in [K] and [LW], the map $f \rightarrow f \circ \tau$ can be extended to gmp-transformations by putting, for real-valued measurable f ,

$$T_\phi f(\omega) = \sup\{t \in \mathbb{R} : \omega \in \phi(\{f > t\})\}.$$

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T_ϕ is nonlinear, in general, and satisfies $\{T_\phi f \geq t\} = \phi(\{f \geq t\})$. If $\phi = \phi_\tau$, then $T_\phi f = f \circ \tau$.

We say that a process $(X_n)_{n \geq 0}$ can be represented by a gmp-transformation if there exists a gmp-transformation ϕ on a suitable probability space $(\Omega, \mathcal{A}, \mu)$, and a measurable f such that the sequence $(T_\phi^n f)$ has the same joint distribution as (X_n) . Our question now is: Which processes can be represented by a gmp-transformation?

We shall consider the following conditions:

(M $_\cap$) (Monotonicity condition for intersections): For all $n \in \mathbb{N}$ and all $t_0, \dots, t_{n-1} \in \mathbb{R}$

$$P \left(\bigcap_{i=0}^{n-1} \{X_i \geq t_i\} \right) \leq P \left(\bigcap_{i=0}^{n-1} \{X_{i+1} \geq t_i\} \right).$$

(M $_\cup$) (Monotonicity condition for unions): For all $n \in \mathbb{N}$ and all $t_0, \dots, t_{n-1} \in \mathbb{R}$

$$P \left(\bigcup_{i=0}^{n-1} \{X_i \geq t_i\} \right) \geq P \left(\bigcup_{i=0}^{n-1} \{X_{i+1} \geq t_i\} \right).$$

We shall see below that these monotonicity conditions are necessary for a process to permit the representation by a gmp-transformation. This is a fairly simple consequence of (1.1) and (1.2). We do not know if all processes satisfying these two conditions can be represented by a gmp-transformation. In this direction we obtain only a very special result: If (X_0, X_1, X_2, X_3) assumes only two values and the monotonicity conditions above hold, a representation for these 4-dimensional marginals is obtained. Even this special case requires a considerable argument. Our proof relies on a general method for constructing gmp-transformations which satisfy certain priority rules. This is presented in Section 2, and it seems of independent interest. Progress on the main problem seems to require new methods for the construction of gmp-transformations. It is clear that the class of gmp-transformations is very rich, but at present only few methods of construction are available.

Any gmp-transformation ϕ induces a transformation, also denoted by ϕ , in the measure algebra $\bar{\mathcal{A}}$ obtained from $\mathcal{A}$ by identifying sets that differ only by null sets. It will be convenient to look mainly at the measure algebra, and we shall be satisfied with constructing ϕ on it.

§2. Construction of gmp-Transformations by Priority Rules

In [K], examples of gmp-transformations on a *finite* set Ω with counting measure were obtained by prescribing certain priorities for the points of the space. We now introduce an extension of this idea to general measure spaces.

Theorem 2.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space supporting an ergodic invertible measure preserving transformation τ . Let $\{\Omega_1, \Omega_2, \dots\}$ be a partition of Ω into finitely many or countably many (disjoint) measurable sets. For each $i = 1, 2, \dots$, let $\{\Omega_{i1}, \Omega_{i2}, \dots\}$ be a measurable partition of Ω . Put $\Omega^0 = \emptyset$, $\Omega^i := \bigcup_{k=1}^i \Omega_k$,*

$\Omega_i^j := \bigcup_{k=1}^j \Omega_{ik}$, and for any $A \in \mathcal{A}$, $A^i = A \cap \Omega^i$. There exists a gmp-transformation ϕ with the following property: For any $i \geq 1$ and $j \geq 2$ and any $A \in \mathcal{A}$ if

$$\mu((\Omega_{ij} \cap \phi(A^i)) \setminus \phi(A^{i-1})) > 0$$

then

$$\mu(\Omega_i^{j-1} \setminus \phi(A^i)) = 0.$$

Remark. The heuristic meaning of the property above is the following: We imagine that the image $\phi(A)$ is constructed by first mapping $A \cap \Omega_1$, then mapping $A \cap \Omega_2$, then $A \cap \Omega_3$, etc. So, after i steps one has constructed $\phi(A^i)$. If the part of the image constructed in the i^{th} step contains a non-null subset of Ω_{ij} , then Ω_i^{j-1} must already be filled up, i.e., Ω_i^{j-1} must be (mod null sets) a subset of $\phi(A^i)$. In other words, when a subset of Ω_i is mapped, one maps as much as possible into Ω_{i1} , then as much as possible into Ω_{i2} etc. The sets Ω_{ij} are called priority sets for Ω_i .

Proof: Step 1. (Construction of $\phi(A \cap \Omega_1)$). This step consists of a countable sequence of substeps. In the first substep, one tries to map as much as possible from $A \cap \Omega_1$ into Ω_{11} , in the second substep one maps as much as possible from the remaining part of $A \cap \Omega_1$ into Ω_{12} , etc. Formally, the first substep of Step 1 is defined by an inductive procedure. Set

$$B_0^1 = A^1 \cap \Omega_{11} = R_0^1.$$

We shall have $\phi(R_0^1) = B_0^1$. (On $\Omega_1 \cap \Omega_{11}$, ϕ is the identity map.) We imagine that we have two copies of Ω , the original space and a second copy for the images. We paint that part of A which has already been mapped red and the image blue in the second copy. So at this point, R_0^1 is painted red and B_0^1 blue. Now, we take the part of A^1 which is not yet painted and try to map it into the unpainted part of Ω_{11} by τ . Formally

$$\begin{aligned} B_1^1 &= \tau(A^1 \setminus R_0^1) \cap (\Omega_{11} \setminus B_0^1) \\ R_1^1 &= \tau^{-1} B_1^1. \end{aligned}$$

Then we continue with τ^2 in the same way:

$$\begin{aligned} B_2^1 &= \tau^2(A^1 \setminus (R_0^1 \cup R_1^1)) \cap (\Omega_{11} \setminus (B_0^1 \cup B_1^1)) \\ R_2^1 &= \tau^{-2} B_2^1. \end{aligned}$$

It is clear how to continue.

Suppose that a subset F of positive measure of Ω_{11} is not painted blue in this procedure. As τ is ergodic,

$$F^* = \bigcup_{k=0}^{\infty} \tau^{-k} F$$

is almost all of Ω . Any point in $F^* \cap A^1$ must be painted red. In this case almost all of A^1 has already been mapped into Ω_{11} and therefore will not be mapped into $\Omega_{12} \cup \Omega_{13} \cup \dots$ in the future steps.

If no such F exists, Ω_{11} is painted blue and we perform the second substep of Step 1, replacing Ω_{11} by Ω_{12} and A^1 by $A^1 \setminus \bigcup_{m=0}^{\infty} R_m^1$, the unpainted part of A^1 . If we write

B_{m1}^1 and R_{m1}^1 for the sets B_m^1 and R_m^1 constructed in the first substep above, the next sequence of steps could start with

$$A_2^1 = A^1 \setminus \bigcup_{m=0}^{\infty} R_{m1}^1$$

and

$$B_{02}^1 = A_2^1 \cap \Omega_{12} = R_{02}^1.$$

Next put

$$\begin{aligned} B_{12}^1 &= \tau(A_2^1 \setminus R_{02}^1) \cap (\Omega_{12} \setminus B_{02}^1) \\ R_{12}^1 &= \tau^{-1} B_{12}^1, \end{aligned}$$

etc. One continues as above. In this substep either almost all of Ω_{12} is painted blue or almost all of A_2^1 is painted red. In the latter case, $\phi(A^1)$ shall be the union of all blue sets constructed so far; in the former case one must now repeat the construction with Ω_{13} and the unpainted part of A^1 .

As $(\Omega_{11}, \Omega_{12}, \dots)$ is a partition of Ω , almost all of A^1 will be painted red after finitely many or denumerably many substeps. $\phi(A^1)$ shall be the union of all sets painted blue in this procedure.

Step 2. Step 2 is just the same as Step 1 except that $A \cap \Omega_1$ is now replaced by $A \cap \Omega_2$ and the sets $\Omega_{11}, \Omega_{12}, \dots$ are replaced by

$$\Omega_{21} \setminus \phi(A^1), \Omega_{22} \setminus \phi(A^1), \dots$$

(The set $\phi(A^1)$ has already been painted in Step 1 and is no longer available as an image.) If $B_{01}^2, B_{11}^2, B_{21}^2, \dots$ is the family of all blue sets constructed in the first substep of Step 2, $B_{02}^2, B_{12}^2, \dots$ the family of all blue sets constructed in the second substep of Step 2, etc. let $\phi(A \cap \Omega^2)$ be the union of $\phi(A \cap \Omega_1)$ and all these blue sets found in Step 2.

In Step 3 repeat the construction with $A \cap \Omega_3$ and with the sets $\Omega_{31} \setminus \phi(A \cap \Omega^2), \Omega_{32} \setminus \phi(A \cap \Omega^2)$, etc. Finally $\phi(A)$ is the union of all blue sets constructed in Steps 1, 2, 3, ... It is clear that $\mu(\phi(A)) = \mu(A)$ since A is mod μ the disjoint union of red sets R_{mi}^j and $\phi(A)$ the corresponding disjoint union of blue sets $B_{mi}^j = \tau^m R_{mi}^j$. If $\tilde{A} \supset A$, then, at each stage of the construction, the total blue set for $\tilde{A}$ is at least as large as the corresponding blue set for A . The unpainted part of $\tilde{A}$ is, at each stage, at least as large as the unpainted part of A in the corresponding step of the construction of $\phi(A)$. Hence $\phi(\tilde{A}) \supset \phi(A)$. By the construction, if $\mu((\Omega_{ij} \cap \phi(A^i)) \setminus \phi(A^{i-1})) > 0$, then this means that a subset of Ω_{ij} of positive measure was painted blue in Step i . This happens only if almost all of Ω_i^{j-1} was painted blue before. Hence the gmp-transformation ϕ has the desired property. $\square$

Remark. If $\mu(A \cap \Omega_i) \leq \mu(\Omega_i^j \setminus \phi(A^{i-1}))$, then $\phi(A^i) \setminus \phi(A^{i-1})$ is, mod μ , a subset of Ω_i^j . (If $\mu(\Omega_{ik} \cap (\phi(A^i) \setminus \phi(A^{i-1})))$ is positive for some $k > j$, then $\Omega_i^j \setminus \phi(A^{i-1})$ must have been painted blue.)

§3. Representation of Processes by gmp-Transformations

Theorem 3.1. *The conditions $(M_\cap)$ and $(M_\cup)$ are necessary for a process (X_m) to admit a representation by a gmp-transformation.*

Proof. Assume (X_m) admits a representation on a probability space $(\Omega, \mathcal{A}, \mu)$. Then there exists a measurable f and a gmp-transformation ϕ such that $(T_\phi^m f)$ has the same joint distribution as (X_m) . Using $\{T_\phi^k f \geq t\} = \phi^k(\{f \geq t\})$ ($k \geq 0$) and (1.1) we obtain

$$\begin{aligned} P(X_0 \geq t_0, \dots, X_{n-1} \geq t_{n-1}) &= \mu \left(\bigcap_{i=0}^{n-1} \phi^i(\{f \geq t_i\}) \right) \\ &= \mu \left(\phi \bigcap_{i=0}^{n-1} \phi^i(\{f \geq t_i\}) \right) \\ &\leq \mu \left(\bigcap_{i=0}^{n-1} \phi^{i+1}(\{f \geq t_i\}) \right) \\ &= P(X_1 \geq t_0, \dots, X_n \geq t_{n-1}). \end{aligned}$$

The symmetric argument with (1.2) shows that also $(M_\cup)$ is necessary. $\square$

We do not know if the combined conditions $(M_\cap)$ and $(M_\cup)$ are sufficient. It even seems hard to answer this problem when the process takes only two values, say 0 and 1, and we ask only that for any fixed n there exists a ϕ and f such that $(X_i)_{i=0}^n$ and $(T_\phi^i f)_{i=0}^n$ have the same joint distribution. In this case, f is an indicator function $f = 1_{A_0}$, and we have $T_\phi^k f = 1_{\phi^k(A_0)}$.

Most of the remainder of this paper will be devoted to showing that the combination (M) of $(M_\cup)$ and $(M_\cap)$ is sufficient for this subproblem when $n \leq 3$.

Let $(\Omega, \mathcal{A}, \mu)$ be a nonatomic probability space supporting an ergodic invertible measure preserving transformation τ in Ω . E.g., $(\Omega, \mathcal{A}, \mu)$ is the unit interval with Lebesgue measure. Clearly, we can assume $(\Omega', \mathcal{A}', P) = (\Omega, \mathcal{A}, \mu)$ replacing the original process $(X_i)_{i=0}^n$ by a process with the same distribution defined on $(\Omega, \mathcal{A}, \mu)$.

Let us reformulate the conditions $(M_\cap)$ and $(M_\cup)$ for 0-1-valued processes $X_i = 1_{A_i}$ ($i = 0, \dots, n$) on $(\Omega, \mathcal{A}, \mu)$. For any nonempty subset $I = \{i_1, i_2, \dots, i_k\}$ of $\{0, 1, \dots, n-1\}$ put

$$A_{\cap I} := \bigcap_{\nu=1}^k A_{i_\nu}, \quad A_{\cup I} := \bigcup_{\nu=1}^k A_{i_\nu}$$

and

$$A_{\cap I+1} := \bigcap_{\nu=1}^k A_{i_\nu+1}, \quad A_{\cup I+1} := \bigcup_{\nu=1}^k A_{i_\nu+1}.$$

It is an exercise to show that (M) is equivalent to the condition (M') that for all $n \in \mathbb{N}$, both

$$\mu(A_{\cap I}) \leq \mu(A_{\cap I+1}) \quad \text{for all } I \subset \{0, \dots, n-1\},$$

and

$$\mu(A_{\cup I}) \geq \mu(A_{\cup I+1}) \quad \text{for all } I \subset \{0, \dots, n-1\}.$$

Note that (M') implies $\mu(A_i) = \mu(A_{i+1})$ ($0 \leq i \leq n-1$) by taking $I = \{i\}$.

By Theorem 3.1, (M') is necessary for the existence of a gmp-transformation ϕ with $\phi^i(A_0) = A_i$ ($1 \leq i \leq n$). It seems natural to conjecture that (M') is also sufficient, but we can prove this only for $n \leq 3$. For larger n , the present approach gets extremely involved, and a new, possibly more canonical construction seems desirable. For $n \leq 3$, we actually prove a stronger result.

Let $B_0, B_1, \dots$ be measurable sets. Define $B_{\cap I}$ and $B_{\cup I}$ just like $A_{\cap I}$ and $A_{\cup I}$.

Theorem 3.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a nonatomic probability space supporting an invertible ergodic measure preserving transformation τ in Ω . Let $A_0, \dots, A_{n-1}$ and $B_0, \dots, B_{n-1}$ be measurable sets, $n \leq 3$. The condition (M'') that*

$$\mu(A_{\cap I}) \leq \mu(B_{\cap I})$$

and

$$\mu(A_{\cup I}) \geq \mu(B_{\cup I})$$

hold for all $I \subset \{0, \dots, n-1\}$ is necessary and sufficient for the existence of a gmp-transformation ϕ with $B_i = \phi(A_i)$ ($i = 0, \dots, n-1$).

Proof. The necessity holds for all n and is easily established using (1.1) and (1.2).

With the help of Theorem 2.1, the proof of the sufficiency is easy for $n = 2$: Map $\Omega_1 := A_0 \cap A_1$ into $\Omega_{11} := B_0 \cap B_1$. Then map $\Omega_2 := A_0 \setminus A_1$ into $\Omega_{21} := B_0$, then $\Omega_3 = A_1 \setminus A_0$ into $\Omega_{31} := B_1$, and finally $\Omega_4 := (A_0 \cup A_1)^c$ into $\Omega_{41} = \Omega$. We leave the details as an exercise. Note that (M'') implies $\mu(A_0) = \mu(B_0)$, $\mu(A_1) = \mu(B_1)$, and $\mu(A_0 \cap A_1) \leq \mu(B_0 \cap B_1)$.

$n = 3$: We consider the partition of Ω induced by the sets A_0, A_1, A_2 . It consists of the sets

$$\begin{aligned} \Sigma_1 &:= A_0 \cap A_1 \cap A_2 & \Sigma_2 &:= A_0 \cap A_1 \cap A_2^c \\ \Sigma_3 &:= A_0 \cap A_1^c \cap A_2 & \Sigma_4 &:= A_0 \cap A_1^c \cap A_2^c \\ \Sigma_5 &:= A_0^c \cap A_1 \cap A_2 & \Sigma_6 &:= A_0^c \cap A_1 \cap A_2^c \\ \Sigma_7 &:= A_0^c \cap A_1^c \cap A_2 & \Sigma_8 &:= A_0^c \cap A_1^c \cap A_2^c. \end{aligned}$$

The partition $\{\Sigma_1^*, \Sigma_2^*, \dots, \Sigma_8^*\}$ induced by the sets B_0, B_1, B_2 is defined in exactly the same way: $\Sigma_1^* := B_0 \cap B_1 \cap B_2$, etc. Set

$$\delta_i = \mu(\Sigma_i^*) - \mu(\Sigma_i) \quad (i = 1, \dots, 8).$$

(The actual sizes of the sets are less important than the differences δ_i .) Assume (M'') holds. In terms of the δ_i 's this means that the following identities and inequalities are satisfied:

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 = 0 \quad (3.1)$$

$$\delta_1 + \delta_2 + \delta_5 + \delta_6 = 0$$

$$\delta_1 + \delta_3 + \delta_5 + \delta_7 = 0,$$

$$\delta_1 \geq 0 \quad (3.2)$$

$$\delta_1 + \delta_2 \geq 0$$

$$\delta_1 + \delta_3 \geq 0$$

$$\delta_1 + \delta_5 \geq 0$$

$$\delta_2 + \delta_3 + \delta_5 + 2\delta_1 \geq 0 \quad (3.3)$$

(The last inequality follows from $\sum_{i=1}^7 \delta_i \leq 0$ by subtracting the three identities in (3.1).)

The construction of ϕ shall be based on Theorem 2.1. The priorities will depend on the δ 's. In all cases, we shall start by mapping Σ_1 into Σ_1^* because Σ_1 is a subset of all A_i , so that $\phi(\Sigma_1)$ must be a subset of all B_i .

If we specify priority sets $\Omega_{k1}, \Omega_{k2}, \dots, \Omega_{km}$ which do not cover all of Ω , this shall mean that there is one additional set $\Omega_{k,m+1}$ which is the complement of the union of $\Omega_{k1}, \dots, \Omega_{km}$.

Case 1. $\delta_3 \geq 0$. In this case, we start by mapping those Σ_i which are contained in A_1 . Σ_3^* will be big enough to receive all of Σ_3 , and no part of Σ_1^* will be needed as image of Σ_3 . Therefore we can map Σ_2 into $\Sigma_1^* \cup \Sigma_2^*$. Possibly, Σ_5 is too big to be mapped into Σ_5^* , but we shall see that no harm is done mapping the surplus into Σ_6^* .

Formally, set

$$\Omega_1 = \Sigma_1, \quad \Omega_{11} = \Sigma_1^*$$

$$\Omega_2 = \Sigma_2, \quad \Omega_{21} = \Sigma_2^*, \quad \Omega_{22} = \Sigma_1^*$$

(so, tacitly, $\Omega_{23} = \Omega \setminus (\Omega_{21} \cup \Omega_{22})$).

$$\Omega_3 = \Sigma_5, \quad \Omega_{31} = \Sigma_5^*, \quad \Omega_{32} = \Sigma_1^*, \quad \Omega_{33} = \Sigma_6^*$$

$$\Omega_4 = \Sigma_6, \quad \Omega_{41} = B_1, \quad \Omega_{42} = B_1^c$$

(By now we know how to map subsets of A_1 .) Next, set

$$\Omega_5 = \Sigma_3, \quad \Omega_{51} = \Sigma_3^*$$

$$\Omega_6 = \Sigma_4, \quad \Omega_{61} = B_0$$

$$\Omega_7 = \Sigma_7, \quad \Omega_{71} = B_2$$

$$\Omega_8 = \Sigma_8, \quad \Omega_{81} = \Omega.$$

This determines ϕ . We have to check $\phi(A_i) = B_i$ for $i = 0, 1, 2$:

($i = 0$): $A_0 = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$. $\Sigma_1 \cup \Sigma_2$ is mapped into $\Sigma_1^* \cup \Sigma_2^*$ since $\delta_1 + \delta_2 \geq 0$. Σ_3 goes into Σ_3^* since $\delta_3 \geq 0$. Σ_4 goes into B_0 since $\mu(A_0) = \mu(B_0)$. Hence $\phi(A_0) = B_0$.

($i = 1$): $\phi(A_1) = B_1$ is clear.

($i = 2$): Constructing $\phi(A_2)$, first Σ_1 is mapped into Σ_1^* . $A_2 \cap \Sigma_2$ is empty. Therefore, when A_2 is mapped, the priorities for Σ_2 do not matter. Σ_5 is mapped into $\Sigma_1^* \cup \Sigma_5^*$ since no part of this set was consumed by $\phi(A_2 \cap \Sigma_2)$. Thus, Σ_5 goes into B_2 . Σ_3 goes into $\Sigma_3^* \subset B_2$; finally Σ_7 goes into B_2 since $\mu(B_2) = \mu(A_2)$. Thus, $\phi(A_2) \subset B_2$. As $\mu(\phi(A_2)) = \mu(A_2) = \mu(B_2)$, we have $\phi(A_2) = \phi(B_2)$.

The cases $\delta_2 \geq 0$ and $\delta_5 \geq 0$ are symmetric. Thus, we can assume $\delta_3 < 0$, $\delta_2 < 0$, $\delta_5 < 0$ in the sequel.

Case 2. $\delta_1 + \delta_2 + \delta_3 \geq 0$. This time, let the sets Ω_i and Ω_{ij} be defined by

$$\begin{aligned}\Omega_1 &= \Sigma_1, & \Omega_{11} &= \Sigma_1^* \\ \Omega_2 &= \Sigma_2, & \Omega_{21} &= \Sigma_2^*, & \Omega_{22} &= \Sigma_1^* \\ \Omega_3 &= \Sigma_5, & \Omega_{31} &= \Sigma_5^*, & \Omega_{32} &= \Sigma_1^*, & \Omega_{33} &= \Sigma_6^* \cup \Sigma_2^* \\ \Omega_4 &= \Sigma_3, & \Omega_{41} &= \Sigma_3^*, & \Omega_{42} &= \Sigma_1^*, & \Omega_{43} &= \Sigma_5^* \cup \Sigma_7^* \\ \Omega_5 &= \Sigma_4, & \Omega_{51} &= B_0 \\ \Omega_6 &= \Sigma_6, & \Omega_{61} &= B_1 \\ \Omega_7 &= \Sigma_7, & \Omega_{71} &= B_2 \\ \Omega_8 &= \Sigma_8, & \Omega_{81} &= \Omega.\end{aligned}$$

We must, again, check $\phi(A_i) = B_i$ ($i = 0, 1, 2$):

($i = 0$): $\Sigma_1 \cup \Sigma_2$ is mapped into $\Sigma_1^* \cup \Sigma_2^*$ again. Recall that we can assume $\delta_2 < 0$. Thus Σ_2^* is covered by $\phi(\Sigma_2)$. Next, Σ_3 is mapped into $\Sigma_1^* \cup \Sigma_3^*$ since $\delta_1 + \delta_2 + \delta_3 \geq 0$ means that there is enough space left over in $\Sigma_1^* \cup \Sigma_1^* \cup \Sigma_3^*$; all this space must actually be in $\Sigma_1^* \cup \Sigma_3^*$ because Σ_2^* is already covered. B_0 is large enough to receive the remaining part Σ_4 of A_0 . Hence $\phi(A_0) \subset B_0$; and then $\mu(A_0) = \mu(\phi(A_0)) = \mu(B_0)$ yields $\phi(A_0) = B_0$.

($i = 1$): This is even simpler and therefore deleted.

($i = 2$): $\phi(\Sigma_1 \cup \Sigma_5)$ fits into $\Sigma_1^* \cup \Sigma_5^*$. Then any part of Σ_3 which is not mapped into $\Sigma_1^* \cup \Sigma_3^*$ is mapped into $\Sigma_5^* \cup \Sigma_7^* \subset B_0$. Finally, Σ_7 is mapped into B_2 , too.

The cases $\delta_1 + \delta_2 + \delta_5 \geq 0$ and $\delta_1 + \delta_3 + \delta_5 \geq 0$ are symmetric. Thus it remains to study

Case 3: $\delta_1 + \delta_2 + \delta_3 < 0$, $\delta_1 + \delta_2 + \delta_5 < 0$, $\delta_1 + \delta_3 + \delta_5 < 0$. Recall that we can also assume $\delta_2 < 0$, $\delta_3 < 0$, $\delta_5 < 0$, and have $\delta_1 + \delta_2 \geq 0$, $\delta_1 + \delta_3 \geq 0$, $\delta_1 + \delta_5 \geq 0$, $\delta_1 > 0$ from (3.2). Let E be a subset of Σ_1^* having measure $\mu(E) = \delta_1$, and let E_2, E_3, E_5 be three subsets of E with

$$\mu(E_i) = |\delta_i| \quad (i = 2, 3, 5)$$

such that no point of E belongs to all three sets E_2, E_3, E_5 . It is possible to find such subsets since (3.2) holds and

$$2\delta_1 + \delta_2 + \delta_3 + \delta_5 \geq 0.$$

Using $\delta_1 + \delta_2 + \delta_3 < 0$, we can assume that $E = E_2 \cup E_3$. (The sets E_2, E_3 need not be disjoint.)

As $\mu(\Sigma_5) - \mu(\Sigma_5^*) = -\delta_5 = \mu(E_5)$, and as $E_2 \cap E_5$ and $E_3 \cap E_5$ are disjoint, we can find three disjoint subsets $\Sigma_{51}, \Sigma_{52}, \Sigma_{53}$ of Σ_5 such that

$$\begin{aligned}\mu(\Sigma_{51}) &= \mu(\Sigma_5^*) \\ \mu(\Sigma_{52}) &= \mu(E_2 \cap E_5) \\ \mu(\Sigma_{53}) &= \mu(E_3 \cap E_5).\end{aligned}$$

Let ϕ be the gmp-transformation obtained by applying Theorem 2.1 with

$$\begin{aligned}\Omega_1 &= \Sigma_1, & \Omega_{11} &= \Sigma_1^* \setminus E \\ \Omega_2 &= \Sigma_2, & \Omega_{21} &= \Sigma_2^*, & \Omega_{22} &= E_2 \\ \Omega_3 &= \Sigma_3, & \Omega_{31} &= \Sigma_3^*, & \Omega_{32} &= E_3, & \Omega_{33} &= \Sigma_4^* \\ \Omega_4 &= \Sigma_{51}, & \Omega_{41} &= \Sigma_5^*, \\ \Omega_5 &= \Sigma_{52}, & \Omega_{51} &= E_2 \cap E_5, & \Omega_{52} &= \Sigma_6^*, & \Omega_{53} &= B_1 \setminus (E_3 \cap E_5) \\ \Omega_6 &= \Sigma_{53}, & \Omega_{61} &= E_3 \cap E_5, & \Omega_{62} &= \Sigma_7^*, & \Omega_{63} &= B_2 \setminus \Sigma_7^* \\ \Omega_7 &= \Sigma_4, & \Omega_{71} &= B_0 \\ \Omega_8 &= \Sigma_6, & \Omega_{81} &= B_1 \\ \Omega_9 &= \Sigma_7, & \Omega_{91} &= B_2 \\ \Omega_{10} &= \Sigma_8, & \Omega_{10,1} &= \Omega\end{aligned}$$

Again, we check that $\phi(A_i) = B_i$ for $i = 0, 1, 2$.

($i = 0$): When $\phi(A_0)$ is constructed, first $\Omega_1 \cap A_0 = \Sigma_1$ goes into $\Sigma_1^* \setminus E$ and fills it up. Then $\Omega_2 \cap A_0$ is mapped. Part of it goes into Σ_2^* and fills it up, and the surplus goes into $E_2 \subset \Sigma_1^*$. Now, $\Sigma_2^* \cup E_2$ is filled up.

Next, $\Omega_3 \cap A_0 = \Sigma_3$ is mapped. First Σ_3^* is filled, then $E_3 \setminus E_2$ is filled (since E_2 was full already). The surplus goes into Σ_4^* . It fits into this set because $B_0 \setminus \Sigma_4^*$ is already filled up at this time.

$\Omega_4 \cap A_0, \Omega_5 \cap A_0$ and $\Omega_6 \cap A_0$ are empty. $\Omega_7 \cap A_0 = \Sigma_4$. None of this set is mapped to $B_0 \setminus \Sigma_4^*$ since that set is full by now. In view of $\mu(A_0) = \mu(B_0)$ there must be just enough space in Σ_4^* left over to receive the image of $\Omega_7 \cap A_0$. Hence $\phi(A_0) = B_0$.

($i = 1$): $\Omega_1 \cap A_1 = \Sigma_1$ goes into $\Sigma_1^* \setminus E$. Next, $\Omega_2 \cap A_1 = \Sigma_2$ goes into $\Sigma_2^* \cup E_2$ and fills it up. $\Omega_3 \cap A_1$ is empty. $\Omega_4 \cap A_1 = \Sigma_{51}$ goes into Σ_5^* and fills it. $\Omega_5 \cap A_1 = \Sigma_{52}$. The first priority for this set would be $E_2 \cap E_5$, but all of E_2 is already occupied. So, this set is mapped into Σ_6^* and into $B_1 \setminus (E_3 \cap E_5)$. There is enough space in these sets since they form the remainder of B_1 except for $E_3 \cap E_5$ and the subset Σ_{53} of A_1 remains to be mapped. (Recall that $\mu(\Sigma_{53}) = \mu(E_3 \cap E_5)$.) Thus, Σ_{52} is mapped into $B_1 \setminus (E_3 \cap E_5)$. Next, $\Omega_6 \cap A_1 = \Sigma_{53}$ is mapped, and it gets its first priority since $E_3 \cap E_5$ was kept in reserve. The remaining part of A_1 is Σ_6 . It fits into B_1 . Hence, $\phi(A_1) = B_1$.

($i = 2$): $\Omega_1 \cap A_2 = \Sigma_1$ is mapped onto $\Sigma_1^* \setminus E$. $\Omega_2 \cap A_2$ is empty. $\Omega_3 \cap A_2 = \Sigma_3$ is mapped to $\Sigma_3^* \cup E_3$ and fills this set. $\Omega_4 \cap A_2 = \Sigma_{51}$ fits into $\Sigma_5^* \subset B_2$. $\Omega_5 \cap A_2 = \Sigma_{52}$ fits into $E_2 \cap E_5$ which is still unoccupied since Σ_3 was mapped to $\Sigma_3^* \cup E_3$ and this set is disjoint from $E_2 \cap E_5$. $\Omega_6 \cap A_2 = \Sigma_{53}$ has $E_3 \cap E_5$ as first priority. This set is occupied, but the next priorities are in B_2 . $\Omega_7 \cap A_2$ and $\Omega_8 \cap A_2$ are empty. $\Omega_9 \cap A_2 = \Sigma_7$ goes into B_2 . Thus $\phi(A_2) = B_2$. $\square$

Remarks: (1) Clearly, only the measure algebra corresponding to $(\Omega, \mathcal{A}, \mu)$ matters. Thus, we can delete the assumption of existence of τ if $(\Omega, \mathcal{A}, \mu)$ is nonatomic and $\mathcal{A}$ countably generated.

(2) For $n = 2$, the present conditions are equivalent to the requirement that $\mu(B_i) = \mu(A_i)$ ($i = 0, \dots, n - 1$), and

$$\mu(A_{\cap I}) \leq \mu(B_{\cap I})$$

for all $I \subset \{0, \dots, n - 1\}$. For $n = 3$ however, the condition that $\mu(A_{\cup I}) \geq \mu(B_{\cup I})$ for all I cannot be replaced by the condition that $\mu(B_i) = \mu(A_i)$ for all i , even when $B_i = A_{i+1}$. The following sets A_i ($i = 0, \dots, 3$) in $\Omega = [0, 1]$ with Lebesgue measure μ can serve as an example: $A_0 := [0, .4]$, $A_1 := [.2, .6]$, $A_2 := [0, .2] \cup [.4, .6]$, $A_3 := [.1, .3] \cup [.4, .5] \cup [.6, .7]$. We have $\mu(A_i) = .4$ for all i , $\mu(A_i \cap A_j) = .2$ for all $i \neq j$ and $\mu(A_0 \cap A_1 \cap A_2) = 0 < \mu(A_1 \cap A_2 \cap A_3) = .1$. However, $\mu(A_0 \cup A_1 \cup A_2) = .6 < \mu(A_1 \cup A_2 \cup A_3) = .7$.

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