Prophet Inequalities for Parallel Processes

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Communicated by the Editors

Generalizations of prophet inequalities for single sequences are obtained for optimal stopping of several parallel sequences of independent random variables. For example, if \( \{X_{i,j}, 1 \leq i \leq n, 1 \leq j < \infty \} \) are independent non-negative random variables, then

\[
E(\sup X_{i,j}) \leq (n+1) \max_{t} \sup \{ E(X_{i,t}) : t \text{ is a stop rule for } X_{1,1}, X_{1,2}, \ldots \},
\]

and this bound is best possible. Applications are made to comparisons of the optimal expected returns of various alternative methods of stopping of parallel processes.

1. INTRODUCTION

For a sequence of non-negative or uniformly bounded random variables \( X = (X_1, X_2, \ldots) \), let

\[
V(X) = \sup \{ EX_t : t \text{ is a stop rule for } X \}
\]

denote the optimal expected return of the process \( X \). Typical "prophet inequalities" associated with the process \( X \) are the following three results, each of which is sharp.
THEOREM A [1]. If $X_1, X_2, \ldots$ are independent with values in $[0, 1]$, then

$$E(\sup X_i) \leq 2V(X) - (V(X))^2.$$ 

COROLLARY B [3]. If $X_1, X_2, \ldots$ are independent non-negative random variables, then

$$E(\sup X_i) \leq 2V(X).$$

COROLLARY C [2]. If $X_1, X_2, \ldots$ are independent non-negative random variables with values in $[0, 1]$, then

$$E(\sup X_i) - V(X) \leq \frac{1}{4}.$$

The main purpose of this paper is to prove generalizations of A, B, C to prophet inequalities for several parallel sequences and to apply these inequalities to obtain comparisons of optimal stopping of parallel processes under various observation alternatives.

Throughout this paper, $X_1, \ldots, X_n$ are sequences of independent integrable random variables ($X_i = (X_{i,1}, X_{i,2}, \ldots)$), and

$$\hat{V} = \hat{V}(X_1, \ldots, X_n) = \max \{ V(X_1), \ldots, V(X_n) \}$$

is the optimal expected return of a player who may choose any of the processes $X_1, \ldots, X_n$ he desires, and then proceed as in ordinary optimal stopping of the chosen process.

Intuitively, one may think of a player who is free to choose one of several parallel processes (say different stocks) to observe, and then stop at any time he wants, based only on the information accumulated up to that time, and receiving as reward the value of the process at the time of stopping. It is assumed that the player knows only the distributions of the random variables in each process and is not allowed to switch processes once the observation has begun; under an optimal strategy (choice of process and stop rule), the player's expected reward is then $\hat{V}$. How well can an ordinary player do compared to someone with insider information or predictive powers, say, who has knowledge not only of the distribution of the random variables, but also of the actual values of the variables themselves? Such a player may not only select the best process to observe, but also even stop that process at its highest value, thereby receiving an expected value of $E(\sup X_i \cdot j)$. The following inequalities (1)–(5) quantify the maximum advantage that a player with full information (foresight or
insider information) has over an ordinary player in such a situation, with several parallel processes.

**Theorem 1.1.** If \(0 \leq X_{i,j} \leq 1\) for all \(i = 1, ..., n\) and \(j = 1, 2, ..., \), then

\[
E(\sup_{i,j} X_{i,j}) \leq \min\{1, (n + 1) V - n \hat{V}^2\};
\]

(1)

if, in addition, \(X_1, ..., X_n\) are independent, then

\[
E(\sup_{i,j} X_{i,j}) \leq 1 - (1 - \hat{V})^{n+1},
\]

(2)

and both bounds are best possible.

**Corollary 1.2.** If \(X_{i,j} \geq 0\) for all \(i = 1, ..., n\) and \(j = 1, 2, ..., \), then

\[
E(\sup_{i,j} X_{i,j}) \leq (n + 1) \hat{V};
\]

(3)

furthermore, this bound is best possible, even if \(X_1, ..., X_n\) are independent.

**Corollary 1.3.** If \(0 \leq X_{i,j} \leq 1\) for all \(i = 1, ..., n\) and \(j = 1, 2, ..., \), then

\[
E(\sup_{i,j} X_{i,j}) - \hat{V} \leq 1 - n^{-1} \quad \text{for all} \quad n > 1;
\]

(4)

if, in addition, \(X_1, ..., X_n\) are independent, then

\[
E(\sup_{i,j} X_{i,j}) - \hat{V} \leq n(n + 1)^{-(n + 1)/n} \quad \text{for all} \quad n \geq 1,
\]

(5)

and these bounds are best possible.

For \(n = 1\), (1) and (2) coincide and give Theorem A, (3) gives Corollary B, and (5) gives Corollary C; the sharp bound in (4) for \(n = 1\) is of course that given by (5) for \(n = 1\), namely, \(\frac{1}{4}\). Theorem 1.1 and Corollary 1.3 are stated for random variables taking values in \([0, 1]\), but they may be reformulated immediately for the general case of uniformly bounded random variables taking values in \([a, b]\) by rescaling; e.g., in this case the bound in (4) becomes \((b - a)(1 - n^{-1})\). The same remark applies to Theorem 3.2 below.

**Remark.** Prophet inequalities in a somewhat different multi-dimensional setting have been considered by Krengel and Sucheston [4].
2. PROOFS OF (1)–(5)

First, several examples will be given to show that (1)–(5) are sharp.

**Example 2.1.** Fix $v \in [0, 1]$, let $X_{1,1} = X_{2,1} = \cdots = X_{n,1} = v$, let $X_{i,j} = 0$ for all $j > 2$ and all $i = 1, \ldots, n$, and let $X_{1,2}, X_{2,2}, \ldots, X_{n,2}$ be jointly distributed random variables with $P(X_{1,2} = 1$ and $X_{k,2} = 0$ for all $k \neq i) = \min\{n^{-1}, v\}$ for all $i = 1, \ldots, n$, and $P(X_{i,2} = 0$ for all $i = 1, \ldots, n) = 1 - \min\{1, nv\}$.

In Example 2.1, $\hat{V} = V(X_1) = \cdots = V(X_n) = v$, and if $v \leq n^{-1}$, then $E(\sup X_{i,j}) = (n + 1)v - nv^2$. On the other hand, if $v \geq n^{-1}$, then $\sup X_{i,j} = 1$, so in either case $E(\sup X_{i,j}) \geq \min\{1, (n + 1)v - nv^2\}$, which shows that the best possible bound in (1) is at least the given one. By maximizing $\min\{1 - v, nv(1 - v)\}$ over $v \in [0, 1]$, this example shows that the best possible bound in (4) is at least the given one.

**Example 2.2.** Fix $v \in [0, 1]$, let $X_{1,1} = X_{2,1} = \cdots = X_{n,1} = v$, and $X_{i,j} = 0$ for all $j > 2$ and all $i = 1, \ldots, n$, and let $X_{1,2}, X_{2,2}, \ldots, X_{n,2}$ be i.i.d. random variables with $P(X_{1,2} = 1) = v = 1 - P(X_{1,2} = 0)$.

In Example 2.2, $\hat{V} = V(X_1) = \cdots = V(X_n) = v$, and $E(\sup X_{i,j}) = 1 - (1 - v)^{n+1}$, which shows that the best possible bound in (2) is at least the given one. Maximizing $1 - (1 - v)^{n+1} - v$ for $v \in [0, 1]$ does the same thing for (5).

**Example 2.3.** Fix $\varepsilon \in (0, 1)$. Let $X_{1,1} = X_{2,1} = \cdots = X_{n,1} = 1$, let $X_{i,j} = 0$ for all $j > 2$ and all $i = 1, \ldots, n$, and let $X_{1,2}, X_{2,2}, \ldots, X_{n,2}$ be i.i.d. with $P(X_{1,2} = \varepsilon^{-1}) = \varepsilon = 1 - P(X_{1,2} = 0)$.

In Example 2.3, $\hat{V} = V(X_1) = \cdots = V(X_n) = 1$ and $E(\sup X_{i,j}) = \varepsilon^{-1}(1 - (1 - \varepsilon)^n) + (1 - \varepsilon)^n \to n + 1$ as $\varepsilon \to 0$; this shows that the best possible bound in (3) is at least the given one, even if $X_1, \ldots, X_n$ are independent.

The following lemma records some elementary inequalities for real numbers which will play key roles in the proofs of (1)–(5); $a \vee b$ denotes $\max\{a, b\}$.

**Lemma 2.4.** Let $c_1, c_2, \ldots$ be constants in $[0, 1]$. Then

(i) $\max\{c_1, \ldots, c_n\} \leq 1 - \prod_{i=1}^{n} (1 - c_i)$;
(ii) $(c_1 + c_2 - c_1c_2) \vee c_3 \leq c_1 + (1 - c_1)(c_2 \vee c_3)$;
(iii) $(1 - c_1)(1 - c_2 \vee c_3) \leq (1 - c_1 \vee c_2)(1 - c_3)$ if $c_3 \leq c_1$; and
(iv) $c_1 \vee c_2 + c_3(1 - c_2 \vee c_4) \leq c_1 + (c_2 \vee c_3)(1 - c_4)$ if $c_3 \vee c_4 \leq c_1$. 

By passing to limits, it suffices to establish (1)–(5) for finite sequences, so assume $X_{i,j} \equiv 0$ for all $j > N$ and for all $i = 1, \ldots, n$. Let

$$v_{i,k} = \sup \{ E(X_{i,t}) : t \text{ is a stop rule for } X_{i,1}, X_{i,2}, \ldots, t \geq k \},$$

$$\hat{v}_k = \max_{1 \leq i \leq n} v_{i,k},$$

and

$$M_k = \max_{1 \leq i \leq n} X_{i,j}; \quad \text{so } v_{i,1} = V(X_i) \text{ and } \hat{v}_1 = \hat{V}.$$

**Proof of (1).** Calculate

$$E(\sup_{i,j} X_{i,j})$$

\[
\begin{align*}
& \overset{(a)}{=} E(M_N) \\
& \overset{(b)}{\leq} E(\hat{v}_1 \lor M_N) \\
& \overset{(c)}{\leq} E[(\hat{v}_1 \lor M_{N-1}) + (1 - \hat{v}_1 \lor M_{N-1})(X_{1,N} \lor \cdots \lor X_{n,N})] \\
& \overset{(d)}{\leq} E \left[ (\hat{v}_1 \lor M_{N-1}) + \sum_{i=1}^{n} [(1 - \hat{v}_1 \lor X_{i,1} \lor \cdots \lor X_{i,N-1}) X_{i,N}] \right] \\
& \overset{(e)}{=} E(\hat{v}_1 \lor M_{N-1}) + \sum_{i=1}^{n} [E(X_{i,N})(1 - E(\hat{v}_1 \lor X_{i,1} \lor \cdots \lor X_{i,N-1}))] \\
& \overset{(f)}{\leq} E(\hat{v}_1 \lor M_{N-2}) \\
& \quad + \sum_{i=1}^{n} E(X_{i,N-1} \lor EX_{i,N})(1 - E(\hat{v}_1 \lor X_{i,1} \lor \cdots \lor X_{i,N-2})) \\
& \overset{(g)}{\leq} E(\hat{v}_1 \lor M_{N-2}) + \sum_{i=1}^{n} v_{i,N-1}(1 - E(\hat{v}_1 \lor X_{i,1} \lor \cdots \lor X_{i,N-2})) \\
& \quad \vdots \\
& \overset{(h)}{\leq} \hat{v}_1 + \sum_{i=1}^{n} v_{i,1}(1 - \hat{v}_1) \overset{(i)}{\leq} (n + 1) \hat{v}_1 - n\hat{v}_1^2,
\end{align*}
\]

where: (a) follows since $X_{i,j} \equiv 0$ for $j > N$; (b) is trivial; (c) by Lemma 2.4(i) with $c_1 = \hat{v}_1 \lor M_{N-1}$, $c_2 = X_{i,1} \lor \cdots \lor X_{n,N}$, and $c_3 = \cdots = c_n = 0$; (d) since (by non-negativity) $X_{1,N} \lor \cdots \lor X_{n,N} \leq \sum_{i=1}^{n} X_{i,N}$, and since $\hat{v}_1 \lor X_{i,1} \lor \cdots \lor X_{i,N-1} \leq \hat{v}_1 \lor M_{N-1}$ for all $i = 1, \ldots, n$; (e) by independence of $X_{i,1}, X_{i,2}, \ldots$ for each $i = 1, \ldots, n$; (f) by repeated use of Lemma 2.4(iv), first with $c_1 = \hat{v}_1 \lor M_{N-2} \lor X_{1,N-1} \lor \cdots \lor X_{n-1,N-1}$,
\( c_2 = X_{n,N-1}, \ c_3 = E(X_{n,N}), \ \text{and} \ c_4 = \delta_1 \lor X_{n,1} \lor \cdots \lor X_{n,N-2}, \) then with 
\( c_1 = \delta_1 \lor M_{n-2} \lor X_{1,N-1} \lor \cdots \lor X_{n-2,N-1}, \ c_2 = X_{n-1,N-1}, \ c_3 = \text{E}(X_{n-1,N}), \) and \( c_4 = \delta_1 \lor X_{n-1,1} \lor \cdots \lor X_{n-1,N-2}, \) etc; (g) since (by backward induction) \( v_{i,N-1} = E(X_{i,N-1} \lor EX_{i,N}) \); (h) by repeated applications of the arguments in (e)-(g); and (i) follows from the definition of \( \delta_1. \quad \blacksquare \)

**Proof of (2).** Calculate

\[
E(\sup X_{i,j}) = E(M_N)
\]

\[
\leq E[M_{N-1} + (1 - M_{N-1}) (X_{i,N} \lor \cdots \lor X_{n,N})]
\]

\[
\leq E \left[ M_{N-1} + (1 - M_{N-1}) \left( 1 - \prod_{i=1}^{n} (1 - X_{i,N}) \right) \right]
\]

\[
\leq E \left[ M_{N-1} + (1 - M_{N-1}) \left( 1 - \prod_{i=1}^{n} (1 - v_{i,N}) \right) \right]
\]

\[
= 1 - E \left[ (1 - M_{N-1}) \left( \prod_{i=1}^{n} (1 - v_{i,N}) \right) \right]
\]

\[
\leq 1 - E \left[ (1 - \delta_1 \lor M_{N-2}) \prod_{i=1}^{n} (1 - v_{i,N}) \right]
\]

\[
\leq 1 - E \left[ (1 - \delta_1 \lor M_{N-2}) \prod_{i=1}^{n} (1 - (X_{i,N-1} \lor v_{i,N})) \right]
\]

\[
\leq 1 - E \left[ (1 - \delta_1 \lor M_{n-2}) \prod_{i=1}^{n} (1 - v_{i,N-1}) \right]
\]

\[
\vdots \]

\[
\leq 1 - (1 - \hat{\delta}_1)^n \prod_{i=1}^{n} (1 - v_{i,1})
\]

\[
\leq 1 - (1 - \hat{\delta}_1)^{n+1}
\]

where: (a) follows since \( X_{i,j} \equiv 0 \) for \( j > N \); (b) by Lemma 2.4(i) with \( c_i = M_{N-1}, \ c_2 = X_{1,N} \lor \cdots \lor X_{n,N} \) and \( c_3 = \cdots = c_n = 0 \); (c) by Lemma 2.4(i) with \( c_i = X_{i,N} \); (d) by the independence of \( M_{N-1} \) and \( X_{1,N}, X_{2,N}, \ldots, X_{n,N} \); (e) by rearrangement; (f) since \( \delta_1 \lor M_{N-1} \geq M_{N-1} \); (g) by repeated use of Lemma 2.1(iii), first with \( c_3 = \delta_1 \lor M_{N-2} \lor X_{1,N-1} \lor \cdots \lor X_{n-1,N-1}, \ c_2 = X_{n,N-1}, \) and \( c_1 = v_{n,N} \) etc.; (h) since (by backward induction) \( v_{i,N-1} = E(X_{i,N-1} \lor v_{i,N}) \) and since \( M_{N-2} \) is independent of \( X_{1,N-1}, \ldots, X_{n,N-1} \); (i) by repeated applications of the
arguments in (f)–(h); (j) since $\theta_i \geq v_{i,1}$ for all $i = 1, \ldots, n$; and (k) since $\theta_1 = \hat{V}$.

Proofs of (3)–(5). Inequality (3) follows easily from (1) by truncation and rescaling, (4) follows from (1) by maximizing $\min \{1, (n + 1) \hat{V} - n\hat{V}^2\} - \hat{V}$ on $0 \leq \hat{V} \leq 1$, and (5) follows from (2) by maximizing $1 - (1 - \hat{V})^{n+1} - \hat{V}$ on $0 \leq \hat{V} \leq 1$.

3. APPLICATIONS TO NON-PROPHET STOPPING PROBLEMS

The purpose of this section is to compare $\hat{V}$, the optimal expected return of a player who is free to choose one of several parallel processes to play, with the optimal expected return of a player with various other alternatives for the same parallel processes, namely: cyclic observation of all processes where one first observes the first random variable in the first sequence, then the first in the second sequence, etc.; multiple observation where one first observes simultaneously all the first random variables in each sequence, then all the second random variables in each sequence, etc.; sequential observation where one may observe any particular sequence as long as desired, then switch to any other process, and so on, always observing the current process from the point at which it was left off before; and order-selection (or non-sequential) observation where one may observe the random variables in the array $X_{i,j}$ in any desired order, as a function only of past outcomes.

Definition 3.1. For $\alpha = C, M, R, S$, $V_{\alpha}(X_1, \ldots, X_n)$ is the optimal expected return under cyclic, multiple, order-selection, and sequential observation, respectively.

(For technical details concerning the definition of order-selection, the reader is referred to [1]; the other definitions are obvious. It follows easily from Theorem 3.11 in [1] that deterministic, or non-randomized, order-selectors yield as high a return as randomized order-selectors, so the definitions of $V_R$ and $V_S$ can be made even simpler.)

Since $V_{\alpha} \leq E(\sup X_{i,j})$, the inequalities (1)–(5) immediately yield upper bounds for comparisons between $V_{\alpha}$ and $\hat{V}$; examples based on the various sampling schemes can be constructed to show that these resulting inequalities are even sharp, except for the analogs of (2), (3), and (5) for $\alpha = C, M, S$ in the case when $X_1, \ldots, X_n$ are independent, which may be further improved by replacing $n+1$ by $n$ in the expressions on the right-hand sides. The next theorem gives the improved analog of (2); those for (3) and (5) follow similarly.
Theorem 3.2. If $X_1, ..., X_n$ are independent and $0 \leq X_{i,j} \leq 1$ for all $i$ and $j$, then

$$V_\alpha \leq 1 - (1 - \hat{V})^n$$

for $\alpha = C, M, S$, \hspace{1cm} (6)

and this bound is best possible.

Sketch of Proof. First check, using backward induction (within steps) and forward induction and Lemma 2.4(i) and (ii), that

$$V_M \leq 1 - \prod_{i=1}^{n} (1 - V(X_i)),$$

which easily yields (6) for $\alpha = M$.

Similarly it can be checked using backward induction, forward induction, and Lemma 2.4(ii), that for any independent random variables $Y_1, Y_2, ...$ taking values in $[0, 1]$, and any partition $(I_j)_{j=1}^{n}$ of $\mathbb{N}$ ($I_i = \{m_{i,1} < m_{i,2} < \cdots \}$),

$$V(Y_1, Y_2, ...) \leq 1 - \prod_{i=1}^{n} (1 - V(Y_{m_{i,1}}, Y_{m_{i,2}}, ...)).$$ \hspace{1cm} (7)

Inequality (6) for $\alpha = C$ then follows easily from (7) taking $(Y_1, Y_2, ...) = (X_{1,1}, X_{2,1}, ..., X_{n,1}, X_{1,2}, X_{2,2}, ...)$ and $I_1 = \{1, n+1, 2n+1, \ldots \}$, $I_2 = \{2, n+2, 2n+2, \ldots \}$ etc. The case $\alpha = S$ follows similarly.

Acknowledgments

The authors are grateful to Professors Lester Dubins and Ulrich Krengel for invitations to a conference at Oberwolfach where these ideas originated, to the University of Cambridge and the Georgia Institute of Technology for reciprocal invitations to pursue this research and to the referees and associate editor for several suggestions.

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