Modeling Glacial Termination

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Abstract

Forcing factors possibly driving the glacial termination cycle are examined with a differential equation model. The overlying question is whether or not the Milankovitch cycle alone is driving the glacial cycle, or whether there are other factors causing the observed pattern of ice ages followed by a rapid warming. The question is examined from a graphical and mathematical point of view by first examining the non-linear phase locking with Tziperman *et al.* model and analyzing several runs of the model with different parameters in place. The conclusions of Denton *et al.*, that the mechanism is actually hemispheric teleconnections causing carbon dioxide release and warming, are examined. A carbon dioxide term is then introduced to the Tziperman model, and the results are analyzed and discussed. The main conclusion reached is that the model shows a good correlation with the observed cycle of glaciation and termination, but that this fit with the model does not show that the insolation variation is acting alone to drive the cycle. Instead, it appears to be driven by internal physical mechanisms, timed by the Milankovitch cycles.

Acknowledgements

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APPROVAL PAGE

This report is being submitted as my senior project in conjunction with the Departments of Mathematics and NRES.

TITLE: Modeling Glacial Termination

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1 Introduction

Around twenty thousand years ago, most of the Northern Hemisphere was buried under sheets of ice and snow. Now, the glaciers have mostly retreated, and the planet is rapidly warming overall. On a much larger time scale, throughout Earth's history there has been a cycle of slow cooling over approximately one hundred thousand years, known as ice ages, followed by a rapid warming over a much shorter time of a few thousand years, known as terminations. The mechanisms that trigger and cause this pattern of slow ice accumulation and cooling, followed by a very rapid warming on a time scale measured in thousands of years, or kiloyears, remain undetermined. The cycle has followed the same sawtooth pattern of slow cooling followed by rapid warming for most of the Pleistocene, or the last 5 million years.

Two papers are analyzed for the project. The first, 'The Last Glacial Termination' [2], investigates the cycle from an empirical perspective, using climate data. The second paper, 'Consequences of pacing the Pleistocene 100 kyr ice ages by nonlinear phase locking to Milankovitch forcing' [1], provides a simple, nonlinear differential equation model relating change in ice volume to changes in solar energy reaching the Earth (referred to as insolation), ice area, and precipitation.

My main goal is to recreate the model runs as closely as possible using the same data, and to investigate further whether the Milankovitch cycle is acting as the main forcing on the ice age cycle, or is merely a correlation. A secondary goal is to create a simple differential equation model relating carbon dioxide release to ice volume.

Section 2 introduces the ice age cycle, and shows the proxy record of dissolved δ^{18}O related to ice volume and the temperature record. The insolation record is also introduced, showing July insolation at 65°N over the last 900,000 years. Section 3 introduces Tziperman's model, with a brief explanation of the model and the concept of nonlinear forcing, followed by the results of the model runs. A thought experiment considering different levels of atmospheric forcing is considered. Model runs are shown and discussed, as well as conclusions reached by Tziperman *et al.* concerning the mechanisms causing the ice age cycle. Section 4 compares and contrasts Tziperman *et al.*'s conclusion with Denton *et al.*'s, and discusses their conclusions concerning atmospheric teleconnections and the correlation of carbon dioxide release with modeled ice volume and proxy ice volume. The model of Tziperman *et al.* is run with the introduction of a CO_2 forcing term, and the results are discussed. Section 5 discusses conclusions and possibilities for further research and future projects.

2 The glacial or ice age cycle — patterns in Earth's climate

2.1 Ice ages and terminations

Throughout the Pleistocene, basically over the last 5 million years or so, there has been a cycle of slow cooling over a period of approximately 100,000-120,000 years, followed by a much faster warming over a period of approximately 10,000-20,000 years. It's worth noting at this point that around 1.2 million years ago (mya) there is a sudden spike in ice volume change and mean temperature, and a lengthening of the period from around 41 kiloyears (kyr) to the recent period of around 100 kyr [8, 9]. The reason for this is unknown. The time scale of this project is 900 kyr to near present.

The ice ages, or accumulation stages, show a gradual increase in ice volume in the Northern Hemisphere (NH). Eventually huge ice sheets form, with a total volume of around 45 milion cubic kilometers. At this maximum volume, something triggers a termination, or a rapid melting of the NH ice sheets and subsequent warming over a much shorter period of time, less than 20 kyr. The minimum volume of ice is around 3 million cubic kilometers. At this volume, ice once again begins to accumulate in the NH, the planet cools overall, and the cycle begins once again. But what exactly causes this? First we need to examine the pattern more closely, and the data that give us the records of the ice ages.

2.2 δ^{18}O

The amount of dissolved oxygen 18 isotope, δ^{18}O, in the sea water serves as a proxy record for ice volume and secondarily mean global temperature. Because the most common isotope of oxygen, δ^{16}O is less massive, water molecules formed with δ^{16}O evaporate at a greater rate from the

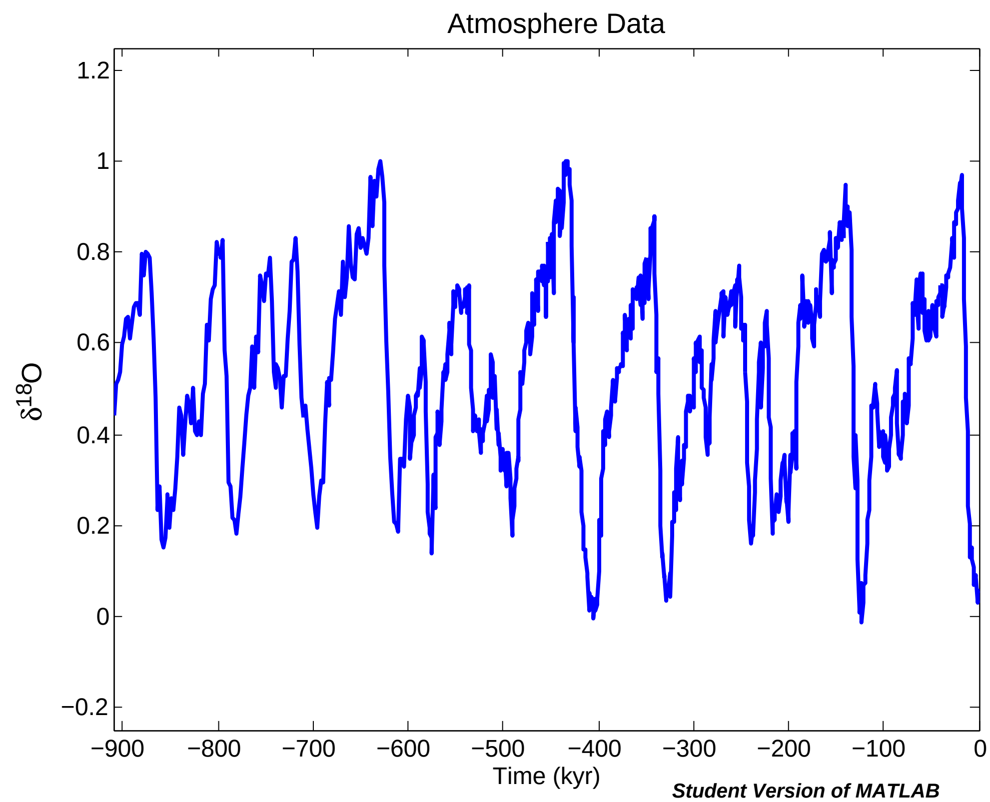

Figure 1: δ^{18}O climate record as a proxy of ice volume and mean temperature. Notice the 'sawtooth' pattern. This shows a slow build up of δ^{18}O, corresponding with an increase in ice volume and a decrease in mean global temperature, followed by an rapid decrease in δ^{18}O and a corresponding decrease in ice volume and increase in mean global temperature. Note an example of the slow ice accumulation, rapid melting pattern between approximately 500 and 400 and between 400 and 300 kyr ago. This is from the data set created by Liesecki and Raymo. [3]

sea surface, and are then precipitated over land, forming ice in the Northern Hemisphere during glaciation periods. Because water molecules formed with δ^{18}O are heavier and left behind in seawater, the ratio of $\delta^{18}\text{O}/\delta^{16}\text{O}$ increases as ice volume increases. Conversely, during interglacial periods or terminations, land ice melted and precipitated back into the oceans putting more δ^{16}O in the seawater, and the ratio decreases. This record is stored in marine sediments, and cores of the sediments can be analyzed to find the ice volume record. [6]

The record created by Liesecki and Ryamo [3], (Figure 1), is correlated with summer insolation by a differential equation incorporating insolation (solar energy reaching the Earth, explained below) at 65°N (Appendix B). This means that there is a dependence on insolation in the record. However, there is a more recent $\delta^{18}O$ record which is independent of insolation and shows a similar pattern [10].

2.3 Insolation and the Milankovitch cycles

'Insolation' is a term for the instantaneous energy from the Sun reaching the top of the atmosphere, measured in Watts/square meter. It varies with the normal 1-year period of the Earth's orbit around the sun due to the eccentricity of Earth's orbit, which along with the tilt of the Earth and atmospheric factors creates the seasons. It also varies on a much longer time scale of thousands of years. Obliquity is the Earth's tilt in relation to the plane of its orbit, and this varies around a 40,000 year scale. Eccentricity is the ellipsicity of the Earth's orbit around the Sun, and its cycle is around 100,000 and 400,000 years. Precession, or the movement of the axis of rotation, changes around 20,000 years (Figure 2). These patterns were first analyzed by Milankovitch, and are known as the Milankovitch cycle. He was also the first to postulate that these variations correlate with climate change, and suggested that they were the cause of the ice ages [7].

2.4 A thought experiment—rock world, water world, and the real world

A thought experiment is helpful to imagine the different interactions in the climate acting together to produce the ice age cycle. Let's first imagine the earth as a bleak, moon-like rock with no water or atmosphere. On this barren, rocky Earth, everything is pretty homogenous, except for random impact craters here and there. Assume nothing is going on inside the Earth either to generate heat-no leftover energy from the birth of the planet remains. Because there is no atmosphere or water to store energy and move it around, the surface will cool off and heat up as fast as the heat capacity of the rock allows. On this rock planet, it seems likely that the cycles of hot and cold surface temperature would correlate almost exactly with the insolation, or energy hitting the surface.

We can then imagine a water world, with no atmosphere or land mass, instead imagining the planet as a homogenous sphere covered with water sloshing around. As this water Earth orbits the sun, it receives different amounts of solar radiation at different latitudes, correlating with the

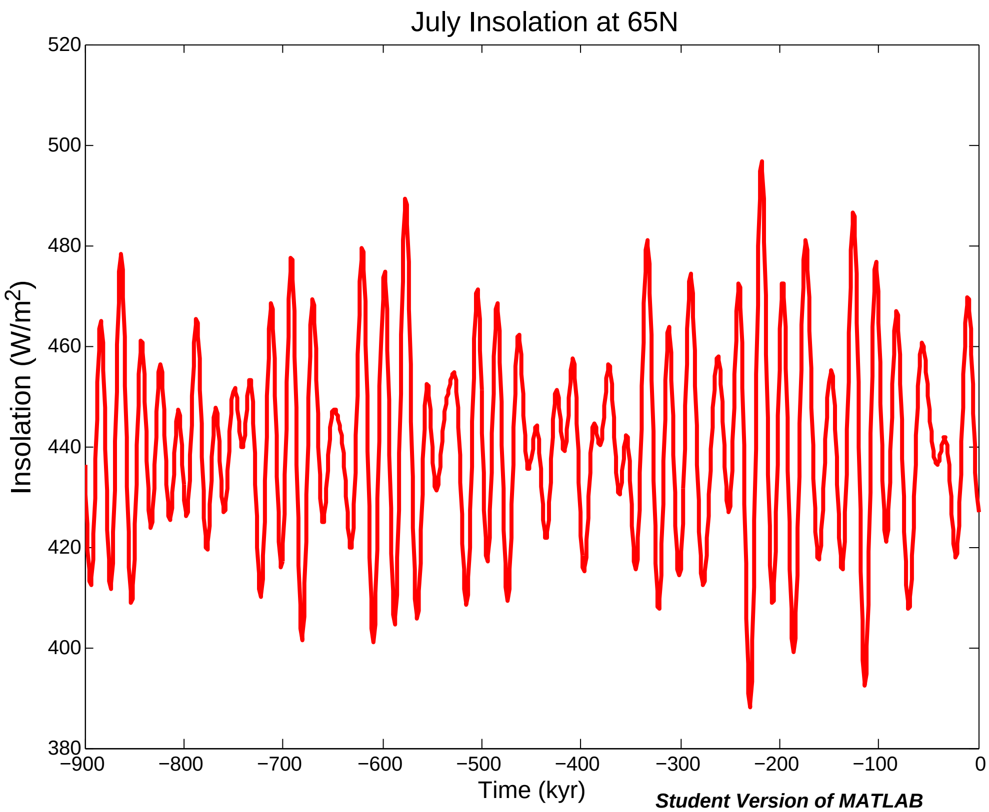

Figure 2: Insolation at 65° N. Notice that there are several cycles shown, from the large eccentricity period of approximately 400 kyr, as seen between 600 and 200 kyr ago, to the much smaller precession period of around 20, as seen between 200 and 100 kyr ago. Notice also that the pattern does not correspond with the pattern shown in Figure 1, and that the pattern is symmetrical as opposed to the sawtooth pattern seen in Figure 1. This implies that the Milankovitch cycle is not directly forcing the ice age cycle even though it may be acting as a 'clock'.

normal orbital variation and the Milankovitch cycles. Because there is no atmosphere or land, energy can only be stored in the water itself. It seems that this planet would have cycles of cooling and warming, or lots of ice/little ice that would correlate closely with the Milankovitch cycles, but because of the heat capacity, transparency, and fluidity of water, it would be much different than the rock Earth model. Now we have convection, layers, and currents generated by energy differences in the water.

Of course in the real world, this is even more complicated. Our Earth has a lot of water sloshing around, but also has land and an atmosphere. Land cools off and heats up much faster than water due to the much higher heat capacity of water and its transparency. The atmosphere is mostly nitrogen and oxygen, but also contains trace amounts of carbon dioxide, methane, and water vapor, all strong green house gasses which absorb and re-emit wavelengths in the thermal infrared, causing a net warming effect on the atmosphere even in parts per million. These gases are dissolved in water, and may be released by changes in ocean currents causing cold, dissolved gas-rich water to rise to the surface and release the gas, warming the atmosphere more and in effect creating a feedback loop, rapidly warming the planet [2].

3 Nonlinear forcing and Tziperman's model

3.1 Tziperman's model

The model of Tziperman *et al.* is a relatively simple differential equation model,

$$\frac{dV}{dt} = [(p_0 - kV)(1 - a_{si})] - [S(1 + \nu(t)) + S_M I(t)], \tag{1}$$

where V is volume of land ice in km^3 and V_{max} is the maximum ice volume, V_{min} is the minimum ice volume, t is time in kiloyears (kyr), p_0 is a precipitation term representing accumulation of ice in km^3, k is a friction term in units of $\frac{1}{40\text{kyr}}$, S and S_M are negative forcing term in units of $\frac{\text{km}^3}{\text{kyr}}$, $a_{\text{si-on}}$ and $a_{\text{si-off}}$ are sea ice area terms which are >0 when sea ice is 'on' or present, or 0 when it is off, or not present; $I(t)$ is the July Insolation at 65°N, and with noise, $\nu(t)$=$\nu(n\Delta t)$ where

$$\nu_n = R * \nu_{n-1} + \sqrt{1 - R^2 * \theta_n}. \tag{2}$$

The noise term was not used in the model runs for this project. MATLAB was used with the differential equation program 'ode45' to generate solutions to equation 1 (Figure 3, Figure 4). The insolation term $I(t)$ uses NOAA climate data [4] linearly interpolated to make it continuous. See Appendix B for MATLAB code and values of constants used.

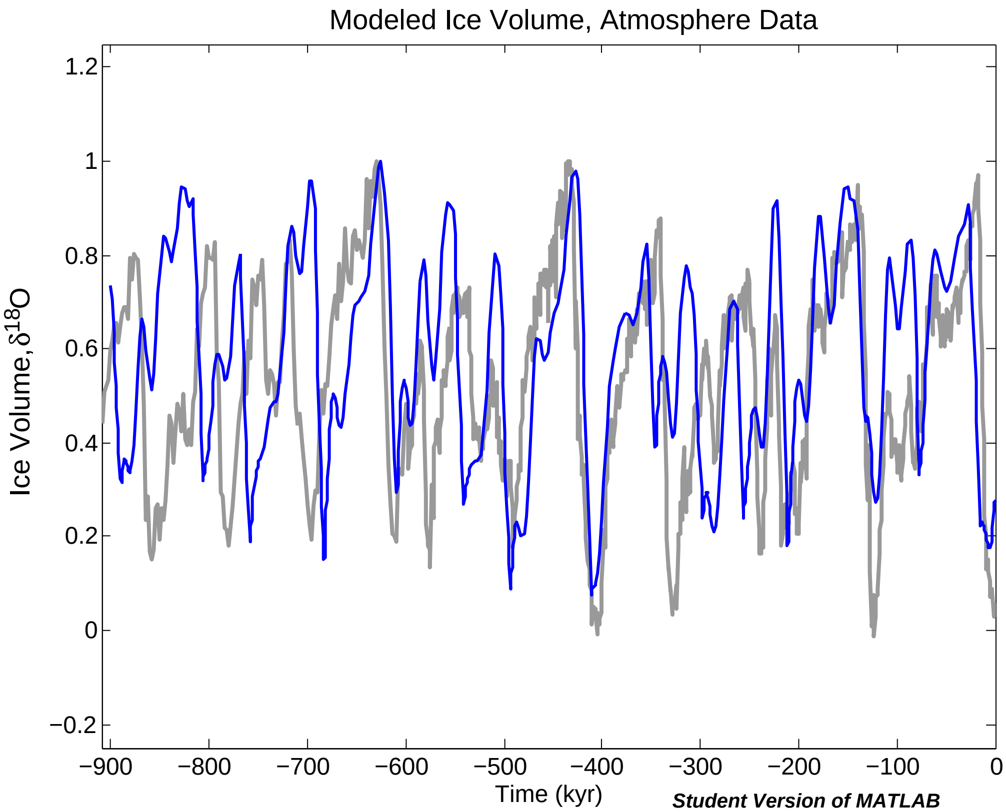

Figure 3: Tziperman model run (blue line) plotted with the δ^{18}O curve (grey line) from figure 1. Notice the correlation between the two at some points.

3.2 Nonlinearity and climate forcing

The formal definition of linearity in a differential equation is that for any two functions $y_1(t)$, $y_2(t)$, $c_1y_1(t) + c_2y_2(t)$ is a solution for arbitrary real numbers c_1, c_2. If this does not hold, then the differential equation is nonlinear. So, nonlinearity in simple terms means that the output of the differential equation is not proportional to the input. In a linear equation, multiplying the input by some constant will result in an output that's proportional to the constant. In a nonlinear system, however, a miniscule change in the input may cause a drastically disproportionate change in the output.

The model attempts to show nonlinear phase locking of the ice age cycle to the Milankovitch cycle. Nonlinear phase locking is a phenomenon whereby a nonlinear oscillation causes another

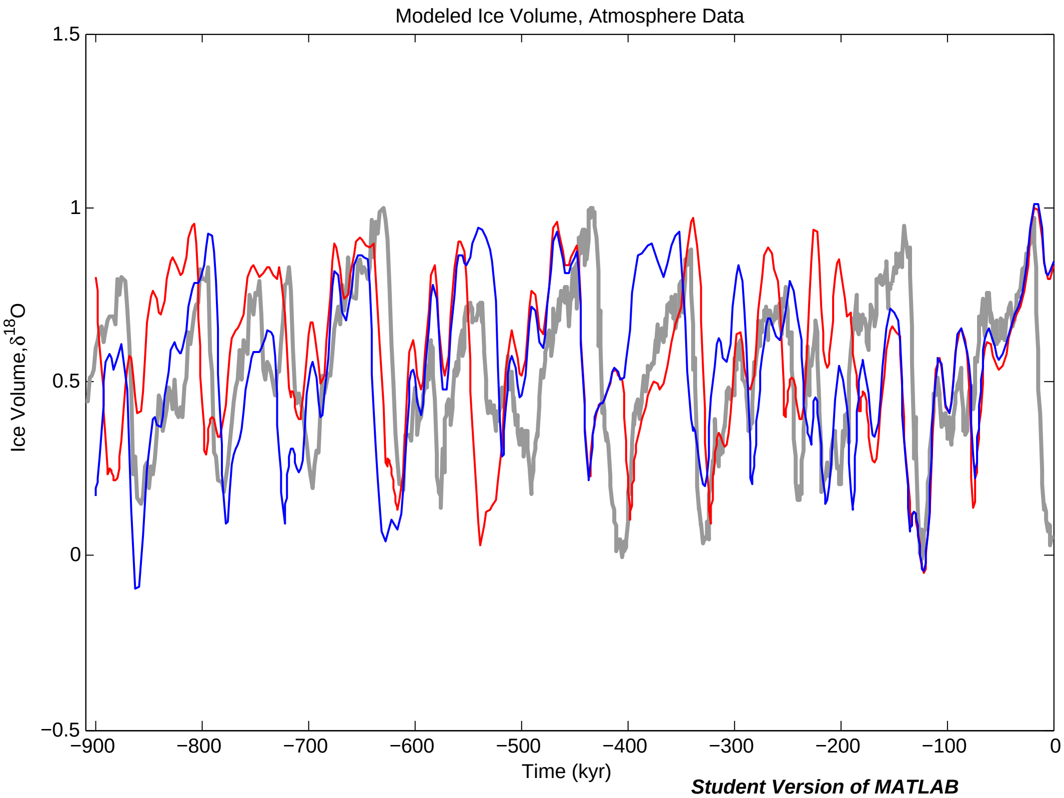

Figure 4: Model run with two initial conditions for V (red and blue lines), plotted with the δ^{18}O curve (grey line). Notice that the curves converge very closely at some points (-500 to -410 kyr and -120 to 0 kyr), and diverge at others (-400 to -350 kyr and -900 to -700 kyr).

oscillation to become locked to the same phase. This was first discovered and quantified by the Dutch physicist Huygens, when he noticed that two pendulums suspended from the same beam would fall into phase with each other, regardless of the original phases of the two. This was caused by minute vibrations passing between the pendulums on the beam. Another example is a group of fireflies in a field. If the fireflies are far apart, they will be blinking out of sync with one another, at random. If they are close though they will begin to blink in phase. In this case there is no actual friction between the two; it is instead caused by a biological response in the fireflies brains. [1]

This explains how a relatively small input, such as the forcing terms in Tziperman *et al.*'s model, can cause the phase of the glacial cycle to 'lock' with it (Figure 3). However, as Tziperman

[1] Thanks to Dr. Dave Camp for the fireflies analogy.

et al. state, this does not mean conclusively that Milankovitch is the only pacing mechanism for the ice age cycle. Any nonlinear forcing can cause a similar result.

In Tziperman *et al.*'s model, the change in energy reaching the Earth caused by the Milankovitch cycles acts to force the glacial cycle. In this way the Milankovitch cycle can be thought of like an external pendulum timing the ice age cycle. Other factors causing the forcing in the model are precipitation and accumulation of land ice, causing an increase in volume, and a decrease caused by sublimation and melting of the ice. Tziperman *et al.* included a random noise term meant to simulate variability in the climate. This was not used in this study. In Tziperman *et al.*'s model runs, a filter was also used on the insolation term, $I(t)$, which acted to smooth the data. This may explain the better fit obtained to the δ^{18}O curve of their model runs compared to the model runs shown here, in spite of having used the same values for initial conditions and constants.

When the model is run with two initial conditions, convergence is seen (Figure 4), but not as completely or quickly as in Tziperman *et al.*'s model runs [1]. Many different initial conditions were tried together, along with adjusting the frictional term k. While different degrees of convergence were seen, the results of Tziperman *et al.*, which showed almost complete convergence, were not duplicated. The reason for this may be the lack of a filter on the $I(t)$ term which Tziperman *et al.* used on their model. It should also be noted that if the model is robust, solutions of two or more initial conditions should eventually converge and not diverge again, regardless of how far apart they are initially.

4 Physical mechanisms driving the cycle: ice-albedo feedback and CO_2 release

4.1 A physical explanation of the mechanisms causing the glacial cycle

Tziperman's main conclusion is that although the model shows good correlation with the δ^{18}O proxy record of ice volume, this is probably independent of the mechanisms that actually cause the ice age cycle. It's important to note again that Liesecki and Raymo's δ^{18}O record [3] was tuned with a similar nonlinear differential equation, using a similar insolation data set from the same latitude, 65°N. So the model runs actually show a correlation between two differential equation models, and not a correlation between a model and pure data. In the conclusion of the paper, ice albedo feedback is hypothesized to be the primary physical mechanism causing the sawtooth pattern. The basic idea of ice albedo feedback is that since ice is highly reflective, having a high albedo, more incoming energy is reflected back out into space. So as ice accumulates, more energy is reflected, the planet cools, precipitation occurs and more ice forms. At a certain point there is so much ice, and the planet is so cold that precipitation no longer occurs, and accumulation ceases. Eventually this accumulated ice will begin melting, and a similar positive feedback loop will lead to more energy being absorbed and more ice melting away [1].

4.2 Denton *et al.*'s approach

Denton *et al.* look at the problem in the context of atmospheric and oceanic physics. Their main point is that while changes in ice albedo feedback definitely have an effect on the cycle, the main driving mechanism may actually be the release of carbon dioxide sequestered in cold ocean water, causing a positive feedback and rapidly melting the accumulated ice. At maximum ice volume, there is enough melt water entering the Atlantic to change the meridional circulation of the Atlantic. The injection of cold fresh water causes changes in the thermodynamic balance of the oceans and atmosphere. The overturning in the North Atlantic is reduced, moving the ITCZ south, and increasing upwelling in the Southern Hemisphere. This upwelling releases large amounts of CO_2 dissolved in the upwelling water. The subsequent greenhouse effect of the atmospheric CO_2 then rapidly increases temperatures and melts the remaining ice [2].

4.3 Carbon dioxide and Tziperman's model

The record of Earth's atmospheric composition is stored in ice, inside microscopic air bubbles. Using the Vostok CO_2 record obtained from NOAA [5], atmospheric carbon dioxide is compared to the Tziperman model and the $\delta^{18}O$ record (Figure 5). Peaks in carbon dioxide correlate with minima in ice volume throughout the time interval, around 420 kyr. This implies that peaks in CO_2 have an affect on the termination of the ice ages. The Vostok CO_2 data is then introduced into

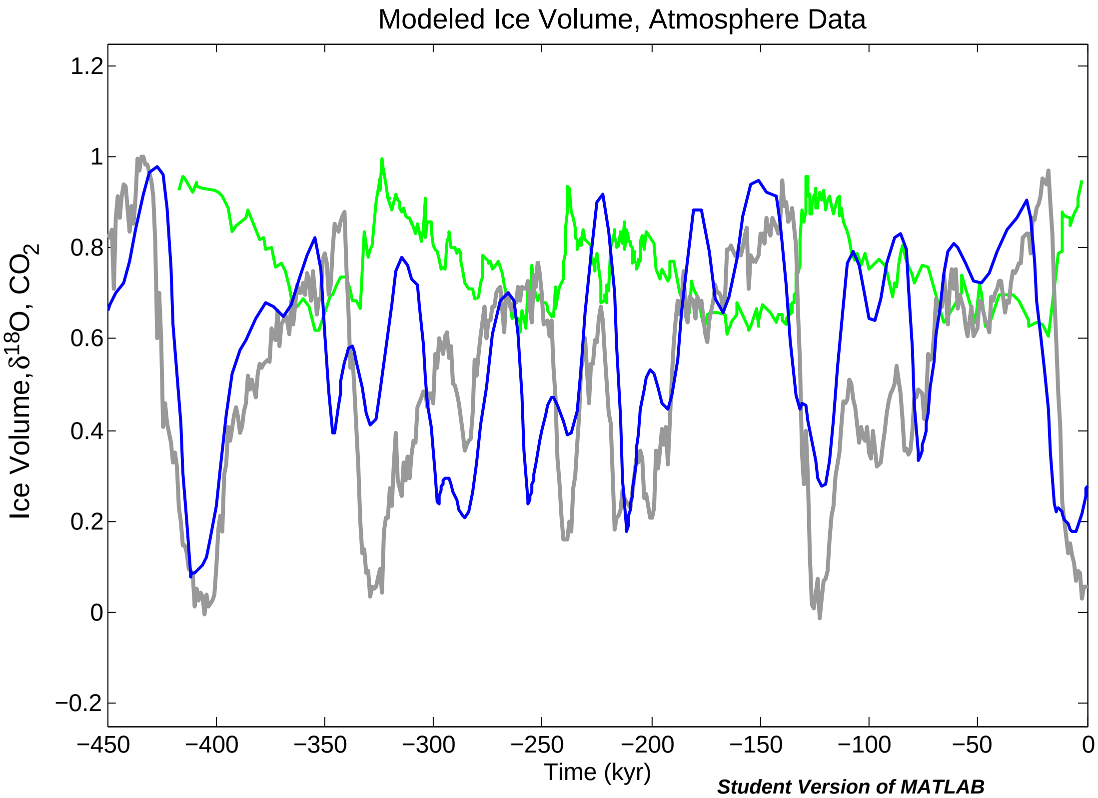

Figure 5: Model run with CO_2 and δ^{18}O. Vostok CO_2 data (green line), correlated with the same Tziperman model run as Figure 3 (blue line), and the δ^{18}O modeled ice volume (grey line). Notice that the peaks of the CO_2 plot show a close correlation with the minima in ice volume shown in the δ^{18}O curve at approximately 325, 225, and 125 kyr ago.

the Tziperman model as a negative forcing term (Figure 6). The Tziperman model was modified to include CO_2 as a negative forcing term, with the new equation being:

$$\frac{dV}{dt} = [(p_0 - kV)(1 - a_{si})] - [S(1 + \nu(t)) + S_M I(t)] - [\beta(CO_2)], \tag{3}$$

where β is an arbitrary coefficient and CO_2 is from the Vostok CO_2 data. Linear interpolation was used to convert the CO_2 data to a continuous function of t. Because the Vostok record only extends to around 2300 years before present, a simple linear extropolation was used at the last data point to match the time scale of the Tziperman and δ^{18}O models at t=0. The forcing term was scaled with an arbitrary coefficient. The value of the coefficient changed the output considerably,

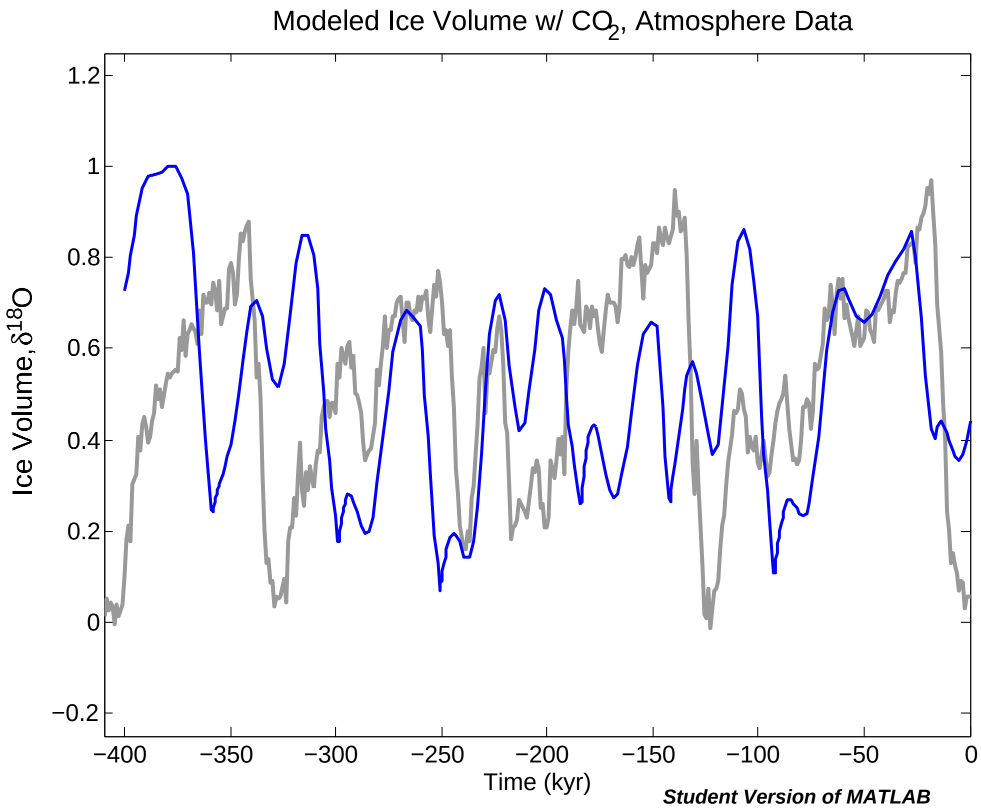

Figure 6: Tziperman model run with CO_2 introduced as an additional forcing term. The coefficient used for the CO_2 forcing term was 50 for this model run. Notice that the model solution is not as closely correlated to the δ^{18}O curve as it is in Figure 3.

especially at values around 5×10^6 (Figure 7). The result shows varying degrees of correlation with the δ^{18}O plot, and it should be noted that no conclusions can be drawn from this other than that the introduction of the CO_2 forcing term has an appreciable effect on the output of the model. The addition of CO_2 to the Tziperman model was motivated by Denton *et al.*'s conclusions that carbon dioxide has a major effect on the termination cycle. For future research, in order to model this connection in a more robust way, the CO_2 term could be incorporated into its own equation, and could be modeled independently of CO_2 data.

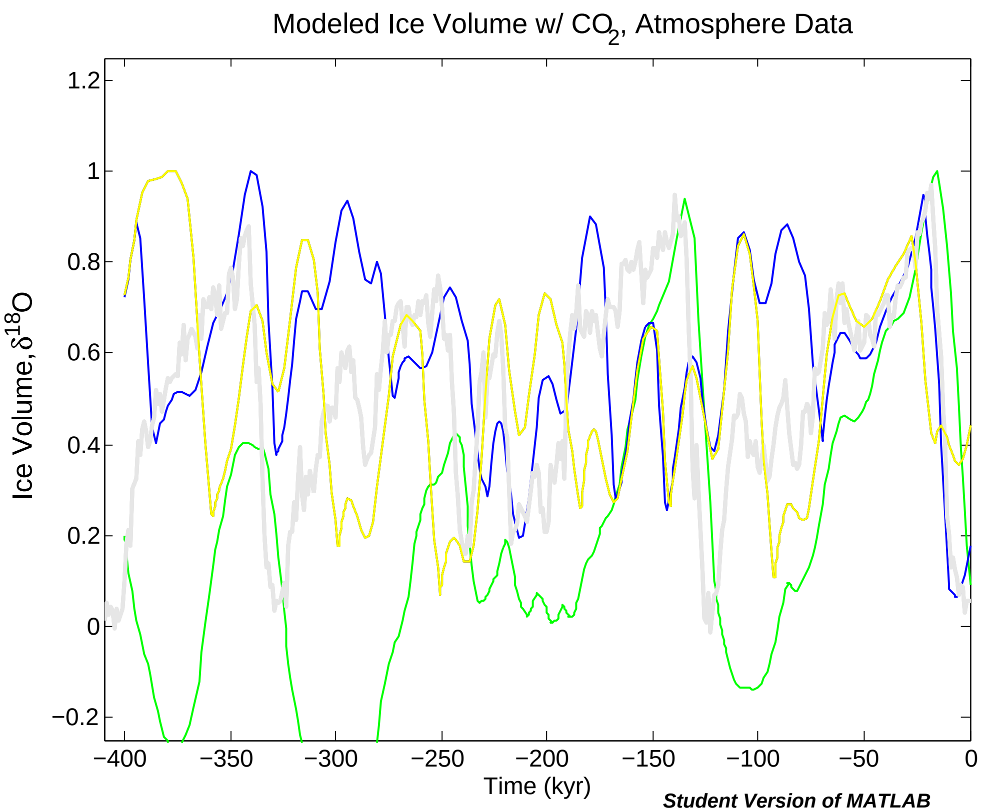

Figure 7: Tziperman + CO_2 model run with different coefficients. Different values are: 50 (yellow line), 5000 (blue line), and $5{\times}10^6$ (green line). Notice the change of the solution to something completely off scale at $5{\times}10^6$.

5 Discussion and further research

5.1 Discussion and conclusions

We have seen that the ice age cycle is much more complex than it may at first appear. The cycle of slow cooling followed by rapid warming, appears to be correlated with the cycle of changes in solar insolation, as Milankovitch postulated. The model of Tziperman *et al.* [1] shows a good fit with the dissolved δ^{18}O record. However, this correlation does not necessarily mean that the Milankovitch is directly causing the cycle, although it appears to be acting as an external clock through nonlinear forcing. According to Denton *et al.*, the actual forcing is more likely a combination of ice albedo feedback and changes in overturning ocean circulation releasing carbon dioxide. This causes a rapid warming, and melting of Northern Hemisphere ice. This is followed by a slow accumulation of ice and snow over a much longer period, and the cycle continues [2].

The introduction of CO_2 to the model is interesting in terms of experimenting with the model, but the result is not a true correlation with the δ^{18}O curve. Even if it was an exact correlation, it would not prove or disprove the hypothesis that CO_2 release causes the ice age cycle. If it was a truly robust connection it would not depend so much on the coefficient, but this means nothing in terms of the validity of Denton *et al.*'s conclusions. The Tziperman model is not based directly on physical laws, i.e., 'first principles'. In that case, it would be derivable from $F = ma$, and the units would have to be the same throughout in order for the model to work. Instead, the model combines many factors with very different units, for example, ice volume in km^3 and CO_2 in parts per million, scaled to 1 for the dependent variable and to kiloyears for the independent or time variable. The purpose of the Tziperman and similar models is to examine complex systems with many interacting factors, and often to model events in the distant past (such as this project), or events which could happen in the future. Their purpose is not to make engineering decisions or derive physical constants. The main point of this project has been to explore the methods and potential of mathematical modeling, and to use modeling to explore the problem of the ice age cycle, and what causes this long-term variation in Earth's climate

5.2 Further research

There are many possibilities for further research into this interesting and complex problem. Some possibilities include creating a new model with carbon dioxide as the main forcing term, or using a similar model to look at a larger or smaller time scale than the 900 and 450 kyr scale examined here. For a project examining a smaller time scale, a model incorporating CO_2 could be used to examine the short term and longer-reaching affects of the recent anthropogenic carbon release.

For a more complete analysis of Tziperman *et al.*'s model, the random noise term, $\nu(t)$, could be incorporated, and/or the filter used to smooth the $I(t)$ term. Additionally, algorithms could be developed to systematically determine constants in order to maximize correlation between the model and record.

A very pressing question in climate studies today is, of course, 'How is the climate changing, and where will it end up in the future?' This question could be examined with a model predicting future climate change, either on a human lifetime scale, or the 100,000 year time scale investigated in this project. We know that the ice age cycle experienced a drastic change around 1.2 million years ago, with the period between ice ages changing from approximately 41 kyr to approximately 100 kyr [8,9]. What caused this change is unknown. Could there be a similar change in the future, and could it be caused by a massive release of CO_2, either natural or anthropogenic? Potential models to be studied are those of Saltzman [11].

Another interesting project would be to develop models incorporating the atmospheric physics discussed by Denton *et al.* [2]. Carbon dioxide could be included as a term independent of an external data set. Oceanic factors, such as overturning and upwelling circulation, could also be included.

A Liesecki and Raymo's model

The benthic δ^{18}O stack data from Liesecki and Raymo [3] was tuned with the simple nonlinear differential equation model for ice volume,

$$\frac{dV}{dt} = \frac{1 + / - b}{T_m}(x - y), \tag{4}$$

where y is the target ice volume, x is the forcing term of summer insolation at 65°N, T_m is a mean time constant, and b is a nonlinearity.

B MATLAB code

```
function ode_tziperman

% Run these commands by typing
% >> ode_tziperman
% at the command line.  Cannot be run by
% copying and pasting into the matlab
% command line.


t0 = -900;                        % initial time
V0 = 32e6;                      % initial data for u(t) as a vector
%V02 = 25e6;
tfinal = 0;                     % final time
%fcnevals = 0;                   % counter for number of function evaluations

% solve ode:
%options = odeset('AbsTol',tol,'RelTol',tol);
[t1,V1] = ode45(@tziperman,[t0 tfinal],V0);
%plot two initial conditions
[t2,V2] = ode45(@tziperman,[t0 tfinal],V02);
figure(1)
clf

load('d18O.txt')
d18Omax = max(d18O(1:1000,2));
d18Oave = mean(d18O(1:1000,2));
plot(-d18O(:,1),0.5*(d18O(:,2)-d18Oave)/(d18Omax-d18Oave)+0.5,...
```

```
   'LineWidth',2,'Color',[.6 .6 .6])
hold on;
Vmax = max(V1(:)); Vave = mean(V1(:));
plot(t1,0.5*(V1(:)-Vave)/(Vmax-Vave)+0.5,'r','linewidth',1)
%plot two initial conditions
plot(t2,0.5*(V2(:)-Vave)/(Vmax-Vave)+0.5,'b','linewidth',1)

axis([-910 0 -0.25 1.25])
title('\fontsize{14}Modeled Ice Volume, Atmosphere Data')
ylabel('\fontsize{14}Ice Volume, \delta^{18}O')
xlabel('\fontsize{12}Time (kyr)')
set(gca,'fontsize',12)

function tziperman = tziperman(t,V)

persistent I_model a_si
if isempty(I_model)
    load('orbit91')
    I_model = orbit91(:,6)-mean(orbit91(:,6));
    I_model = I_model/sqrt(var(I_model));
    I_model = smooth(I_model);
    a_si = 0;
end

% to convert from Sv to km^3/kyr
conv_factor = 86400*365;

p_0 = 0.26*conv_factor;
a_si_on = 0.46;
S_M = 0.03*conv_factor;
S = 0.23*conv_factor;
k = 0.7/40;
Vmax = 45*10^6;
Vmin = 3*10^6;
I_model_flip = I_model(5001:-1:1);
I_model_cont=linterp(-5000:0,I_model_flip,t);
if (V >= Vmax ), a_si = a_si_on; end
if (V <= Vmin ), a_si = 0; end

tziperman = (p_0-k*V)*(1-a_si)-(S+S_M*I_model_cont);

%carbon dioxide term added:


%vostokCO2 goes back 417,160
```

```
%downloaded vostok data and did linear extropalation
   load('vostokCO2.txt');
   mean_CO2= mean(vostokCO2(:,4));
   Carbon= flipdim(vostokCO2(:,4),1)-mean_CO2;
   stdv= sqrt(var(Carbon));
   Carbon= Carbon/stdv;
   Carbon_time=-flipdim(vostokCO2(:,3),1)/1000;
   Carbon(364)=(306.3-mean_CO2)/stdv;
   Carbon_time(364)=0;
   Carbon_smooth= smooth(Carbon_time, Carbon);
end
Carbon_cont=linterp(Carbon_time,Carbon_smooth,t);
% to convert from Sv to km^3/kyr
conv_factor = 86400*365;
%conv_factor = conv_factor/2;

p_0 = 0.26*conv_factor;
a_si_on = 0.46;
S_M = 0.03*conv_factor;
S = 0.23*conv_factor;
k = 0.7/40;
Vmax = 45*10^6;
Vmin = 3*10^6;
CO2Coeff = 5e1;
I_model_flip = I_model(5001:-1:1);
I_model_cont=linterp(-5000:0,I_model_flip,t);
if (V >= Vmax ), a_si = a_si_on; end
if (V <= Vmin ), a_si = 0; end

tziperman = (p_0-k*V)*(1-a_si)-(S+S_M*I_model_cont)-CO2Coeff*(Carbon_cont);
```

Notes: Changing any of the terms in the model generally produces a significant change in the output. The friction term, k, shows very different results for different values, and effects the convergence of the two initial conditions. Changes of as little as +/- 0.01 in the numerator of the k term show changes, and when the numerator is above 1.3, the solutions show good convergence, but a poor fit to the δ^{18}O plot and a reduction in amplitude. When the numerator in the k term is below 0.5, the solutions show poor convergence and poor fit to the δ^{18}O plot. Small changes to any of the constants p_0, S, or S_M also show a significant change, probably due to the large conversion factor.

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