m-ISOMETRIC TRANSFORMATIONS OF HILBERT SPACE, III

Jim Agler and Mark Stankus

Department of Mathematics, Univ. Of Calif., San Diego
La Jolla, CA 92093-0112, U.S.A.

In part 1 and 2 of this series [AgSt1], [AgSt2], an operator T was defined to be an m -isometry if $\sum_{k=0}^m (-1)^k \binom{m}{k} (T^*)^{m-k} T^{m-k} = 0$. A model for m -isometric operators was given as multiplication by $e^{i\phi}$ on a Hilbert space whose inner product is defined in terms of periodic distributions. We related this model theory for the case when $m = 2$ to a disconjugacy theory for a subclass of Toeplitz operators of the type studied by Boutet de Monvel and Guillemin, classical function theoretic ideas on the Dirichlet space, and the theory of nonstationary stochastic processes.

This third paper of the series consists of Sections 8,9 and 10 and turns to the study of 2-isometries possessing a cyclic vector and gives some concluding remarks and presents a list of open questions related to all three parts of the paper.

Section 8 is devoted to the study of 2-isometries possessing a cyclic vector. It turns out that if T is a 2-isometry possessing a cyclic vector, then

$$\dim \ker (||T^*T - 1|| - (T^*T - 1)) = 1$$

and furthermore if $f_0 \neq 0$ is chosen so that

$$(T^*T - 1)f_0 = ||T^*T - 1||f_0,$$

then f_0 is cyclic for T if and only if T is pure. Here, a 2-isometry T is said to be pure if it has no isometric direct summand. If one uses the cyclic vector f_0 , then one can construct a model for T based on a probability measure on the unit circle. This model leads to a rich function theoretic analysis of 2-isometries.

Section 9 derives a number of qualitative facts about the existence and uniqueness of certain types of extensions, so called Brownian unitary and isometric extensions, of a 2-isometry possessing a cyclic vector.

Section 10 gives some remarks and open questions related to all three parts of the paper.

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Introduction

In this paper we shall continue studying the bounded linear transformations T of a complex Hilbert space $\mathcal{H}$ that satisfy an identity of the form

$$T^*mT^m - \binom{m}{1}T^{*m-1}T^{m-1} + \binom{m}{2}T^{*m-2}T^{m-2} - \dots + (-1)^m = 0$$

for a positive integer m by specializing to the case of $m = 2$. Operators T satisfying the above equation are said to be m -isometries.

Some notation and results from the part I and II of the series [AgSt1],[AgSt2]:

We now recall a few results and restate a few definitions from part 1 and 2 of this series. When expedient, we shall specialize the results and definitions to the case of 2-isometries.

If T is a 2-isometry, then Δ_T is defined to be the quantity $T^*T - 1$. It was shown that if T is a 2-isometry, then Δ_T is a positive operator and in the case that T is finitely cyclic, Δ_T is compact.

A 2-isometry T is said to be pure if it has no isometric direct summand.

DDO's of order m are the objects which were used in Section 3 ([AgSt1]) to give a distributional model for bicyclic invertible $(m+1)$ -isometries. We now restate the definition of DDO.

Let $\mathcal{D} = C^\infty(\partial\mathbb{D})$, the Frechet space of infinitely differentiable functions on the unit circle. Let $\mathcal{D}'$ denote the dual of $\mathcal{D}$, the space of distributions on the circle.

We define a linear operator $D: \mathcal{D} \rightarrow \mathcal{D}$ via the formula

$$D\varphi = \frac{1}{i} \frac{d}{d\theta} \varphi.$$

Now recall that if $\beta \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ then $\varphi\beta \in \mathcal{D}'$ can be defined by

$$\varphi\beta(\psi) = \beta(\varphi\psi), \quad \psi \in \mathcal{D}.$$

Thus, if $\beta \in \mathcal{D}'$, β can be regarded naturally as a map $\beta: \mathcal{D} \rightarrow \mathcal{D}'$ by defining

$$\beta(\varphi) = \varphi\beta.$$

Definition A *distribution differential operator* DDO of order 0 is a map $L: \mathcal{D} \rightarrow \mathcal{D}'$ that has the form $L = \beta_0$ for some $\beta_0 \in \mathcal{D}'$, $\beta_0 \neq 0$. A *distribution differential operator* DDO of order 1 is a map $L: \mathcal{D} \rightarrow \mathcal{D}'$ that has the form $L = \beta_1 D + \beta_0$ where β_0 and $\beta_1 \in \mathcal{D}'$ and $\beta_1 \neq 0$.

If L is a DDO and $\varphi, \psi \in \mathcal{D}$, let us agree to define $\psi L\varphi: \mathcal{D} \rightarrow \mathcal{D}'$ by

$$(\psi L\varphi)(\chi) = \psi L(\varphi\chi), \quad \chi \in \mathcal{D}.$$

With this definition observe that if L is a DDO of order m and $\varphi, \psi \in \mathcal{D}$, then $\psi L\varphi$ is a DDO of order $\leq m$.

DTO's of order m are the objects which were used in Section 3 to give a distributional model for cyclic $(m+1)$ -isometries. We now restate the definition of DTO.

Define $\mathcal{D}_a \subset \mathcal{D}$ by

$$\mathcal{D}_a = \{\varphi \in \mathcal{D}: \hat{\varphi}(n) = 0 \text{ if } n < 0\}.$$

We let P denote the canonical projection of $\mathcal{D}$ onto $\mathcal{D}_a$ defined by

$$(P\varphi)^\wedge(n) = \begin{cases} \hat{\varphi}(n) & n \geq 0 \\ 0 & n < 0 \end{cases}.$$

In like fashion, let $\mathcal{D}'_a$ denote the space of analytic distributions defined by

$$\mathcal{D}'_a = \{u \in \mathcal{D}': \hat{u}(n) = 0 \text{ if } n < 0\}.$$

$\mathcal{D}'_a$ can be regarded as the space of boundary values of analytic functions on $\mathbb{D}$ whose power series coefficients form a temperate sequence. We let P denote the canonical projection of $\mathcal{D}'$ onto $\mathcal{D}'_a$ defined by

$$(Pu)^\wedge(n) = \begin{cases} \hat{u}(n) & n \geq 0 \\ 0 & n < 0 \end{cases}.$$

Definition A *distribution Toeplitz operator* (DTO) is a linear mapping $A: \mathcal{D}_a \rightarrow \mathcal{D}'_a$ that has the form

$$A = PL|_{\mathcal{D}_a}$$

for some DDO L . If A is a DTO, we define the *order of A* to be the order of L where L is as above.

If $T \in \mathcal{L}(\mathcal{H})$ is a 2-isometry and $\gamma \in \mathcal{H}$, then since the unilateral array $\langle T^{k_1}\gamma, T^{k_2}\gamma \rangle$ is linear on diagonals it is natural to define the *slope μ and intercept β of (T, γ)* to be the elements of $\mathcal{D}'$ defined by the formulas

$$\hat{\mu}(k) = \begin{cases} \langle T^{k+1}\gamma, T\gamma \rangle - \langle T^k\gamma, \gamma \rangle & k \geq 0 \\ \langle T\gamma, T^{-k+1}\gamma \rangle - \langle \gamma, T^{-k}\gamma \rangle & k < 0 \end{cases}$$

and

$$\hat{\beta}(k) = \begin{cases} \langle T^k\gamma, \gamma \rangle & k \geq 0 \\ \langle \gamma, T^{-k}\gamma \rangle & k < 0 \end{cases}.$$

On the other hand there exist a pair of distributions β_0 and β_1 such that

$$A = P(\beta_1 D + \beta_0)|_{\mathcal{D}_a}.$$

Furthermore, the formulas that relate the above distributions are given by $\beta_1 = \mu$ and $\beta_0 = \beta + (1 - P)(D\mu)$.

The modelling of multicyclic (resp., multi bicyclic invertible) m -isometries requires the use of matrices whose elements are DTO (resp., DDO).

If n_1 and n_2 are positive integers, recall that $\mathcal{D}^{n_1, n_2}$ denotes the space of $n_1 \times n_2$ matrices with entries in $\mathcal{D}$. Likewise, let $\text{DDO}_m^{n_1, n_2}$ denote the space of $n_1 \times n_2$ matrices with DDO entries L_{rs} with the order of $L_{rs} \leq m$ and let $\text{DTO}_m^{n_1, n_2}$ denote the space of $n_1 \times n_2$ matrices with DTO entries A_{rs} with the order of $A_{rs} \leq m$.

We remarked above that a distributional model for invertible n -bicyclic 2-isometries was developed in Section 3 using DDO and DTO. The rest of this introduction will restate the notation which was used to obtain

- (1) an element of $\text{DDO}^{n, n}$ from an invertible n -bicyclic m -isometry,
- (2) an invertible n -bicyclic m -isometry from certain elements of $\text{DDO}^{n, n}$,
- (3) an element of $\text{DTO}^{n, n}$ from an n -cyclic m -isometry and
- (4) an n -cyclic m -isometry from certain elements of $\text{DTO}^{n, n}$.

If $\mathcal{H}$ is a Hilbert space, $T \in \mathcal{L}(\mathcal{H})$ is an invertible m -isometry and $\gamma \in \mathcal{H}$ is a nonzero vector, then γ is bicyclic for $T|_{\bigvee \{T^k\gamma: k \in \mathbb{Z}\}}$ and so there exists a uniquely associated Dirichlet operator given by Theorem 3.14 (of Section 3). We will denote this associated Dirichlet operator by $(T, \gamma)^\wedge$.

If $T \in \mathcal{L}(\mathcal{H})$ is an m -isometry and $\gamma \in \mathcal{H}$ one can set

$$\mathcal{H}_\gamma = \bigvee \{T^k\gamma: k \geq 0\}$$

and

$$T_\gamma = T|_{\mathcal{H}_\gamma}.$$

With this setup, T_γ is a cyclic m -isometry and γ is a cyclic vector for T_γ . Accordingly by Theorem 3.23, there exists a unique analytic Dirichlet operator A with the property that (T_γ, γ) is unitarily equivalent to $(M_A, 1)$. We shall let $(T, \gamma)^\wedge$ denote the unique DDO such that $A = P(T, \gamma)^\wedge|_{\mathcal{D}_a}$.

There is a certain amount of ambiguity in the $(T, \gamma)^\wedge$ notation. Observe that if $T \in \mathcal{L}(\mathcal{H})$ is an invertible m -isometry and $\gamma \in \mathcal{H}$ then according to the remark following the proof of Theorem 3.14 $(T, \gamma)^\wedge$ is the unique Dirichlet operator such that

$$\langle \varphi(T)\gamma, \psi(T)\gamma \rangle = (T, \gamma)^\wedge(\varphi)(\bar{\psi})$$

for all $\varphi, \psi \in \mathcal{D}$. On the other hand, $(T, \gamma)^\wedge$ as just defined in previous paragraph has the defining property

$$\begin{aligned} \langle \varphi(T)\gamma, \psi(T)\gamma \rangle &= A(\varphi)(\bar{\psi}) \\ &= (T, \gamma)^\wedge(\varphi)(\bar{\psi}) \end{aligned}$$

for all $\varphi, \psi \in \mathcal{D}_a$. Recalling the remark following the proof of Lemma 3.3, we find that these two definitions of $(T, \gamma)^\wedge$ agree.

If $L \geq 0$ is a DDO, we define the *form space* of L , H_L^2 , to be the completion of $\mathcal{D}$ with respect to the sesquilinear form $[\cdot, \cdot]_L$ on $\mathcal{D} \times \mathcal{D}$ given by

$$[\varphi, \psi] = L(\varphi)(\bar{\psi}).$$

Specifically, one sets $\mathcal{M}_L = \{\varphi \in \mathcal{D} : [\varphi, \varphi]_L = 0\}$, observes by the Cauchy-Schwarz inequality that $[\cdot, \cdot]_L$ induces an inner product on $\mathcal{D}/\mathcal{M}_L$, and lets H_L^2 denote the completion of $\mathcal{D}/\mathcal{M}_L$.

Likewise, if $A \geq 0$ is a DTO, we define the *form space* of A , H_A^2 , to be the completion of $\mathcal{D}_a$ with respect to the sesquilinear form $[\cdot, \cdot]_A$ on $\mathcal{D}_a \times \mathcal{D}_a$ given by

$$[\varphi, \psi]_A = A(\varphi)(\bar{\psi}).$$

If $L \geq 0$ is a DDO, then one can attempt to define a linear operator M_L on H_L^2 via the formula

$$(M_L\varphi)(e^{i\theta}) = e^{i\theta} \varphi(e^{i\theta}).$$

In general, this operator need not be well-defined on $\mathcal{D}/\mathcal{M}_L$ and, even if it is, may not extend to a bounded operator on H_L^2 .

Definition A *Dirichlet operator* is a DDO L with the property that either there exists a constant $c > 1$ such that

$$0 \leq L - c^{-2}e^{-i\theta}L e^{i\theta}$$

or $\text{ord}(L) = 0$ and $L \geq 0$.

Corresponding definitions of positivity for DTO A , a form space of A and M_A on H_A^2 are given in Section 3.

The following definition precisely identifies those DTO A with the property that $A \geq 0$ and M_A is a well-defined bounded operator.

Definition An *analytic Dirichlet operator* is a DTO A with the property that either there exists a constant $c > 1$ such that

$$0 \leq A - c^{-2} e^{-i\theta} A e^{i\theta},$$

or $\text{ord}(A) = 0$ and $A \geq 0$.

Finally, If $\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \in \mathcal{D}^{n,1}$, we define $\psi^* \in \mathcal{D}^{1,n}$ by setting

$$\psi^* = (\bar{\psi}_1, \dots, \bar{\psi}_n).$$

In Section 5 ([AgSt2]), a certain class of 2-isometries motivated by statistical considerations was introduced, the Brownian unitaries. The irreducible Brownian unitaries correspond to the time shift operator on a scaled Brownian motion process and are referred to as Brownian shifts (Definition 5.5). Each Brownian shift is determined by a nonrandom rotation angle θ and a positive covariance scalar σ . The Brownian shifts in addition to their appearance as the time shift for perhaps the most ubiquitous of nonstationary Gaussian stochastic processes have elegant characterizations on the level of operator theory (Proposition 5.6) as well as within the field of DDO (Proposition 5.2). The principal result of Section 5, the Lifting Theorem for 2-Isometries (Theorem 5.80), asserts that the general 2-isometry T with $\|\Delta_T\| \leq \sigma^2$ can be represented as the restriction to an invariant subspace of a Brownian unitary of covariance σ .

We now recall the definition of a Brownian shift and Brownian unitary.

Definition The *Brownian shift of covariance σ ($\sigma > 0$) and angle θ* is the block operator $B_{\sigma, e^{i\theta}}$ acting on $H^2 \oplus \mathbb{C}$ defined by

$$B_{\sigma, e^{i\theta}} = \begin{bmatrix} S & \sigma(1 \otimes 1) \\ 0 & e^{i\theta} \end{bmatrix}.$$

The Brownian shift of covariance 0 and angle θ is the operator multiplication by $e^{i\theta}$ acting on $\mathbb{C}$.

We now define Brownian unitary. Following this definition is a proposition describing the block form decomposition of Brownian unitaries.

Definition A *Brownian unitary of covariance 0* is a unitary operator. A *Brownian unitary of covariance $\sigma > 0$* is an operator which is unitarily equivalent to

$$U \oplus \int_{\oplus} B_{\sigma, e^{i\theta}}^{(n(\theta))} d\mu(\theta)$$

(the U summand may be absent) acting on $\mathcal{H} \oplus \int_{\oplus} (H^2 \oplus \mathbb{C})^{(n(\theta))} d\mu(\theta)$ where $U \in \mathcal{L}(\mathcal{H})$ is unitary, μ is a nonzero finite positive measure on $[0, 2\pi)$ and $n: [0, 2\pi) \rightarrow \mathbb{N} \cup \{\infty\}$ is a μ -measurable multiplicity function.

A characterization of Brownian unitaries can be given in terms of block operators.

Proposition B is a Brownian unitary of covariance 0 if and only if B is unitary. $B \in \mathcal{L}(\mathcal{K})$ is a Brownian unitary of covariance $\sigma > 0$ if and only if B has the block matrix form

$$B = \begin{bmatrix} V & \sigma E \\ 0 & U \end{bmatrix}$$

with respect to a decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ where

- (i) $V \in \mathcal{L}(\mathcal{K}_1)$ is an isometry,
- (ii) $U \in \mathcal{L}(\mathcal{K}_2)$ is unitary, and
- (iii) $E: \mathcal{K}_2 \rightarrow \mathcal{K}_1$ is an isometry which maps $\mathcal{K}_2$ onto $\ker V^*$.

A second class of operators discussed at length in Section 5 is the collection of direct integrals with multiplicity in θ of Brownian shifts with *varying* covariance which we refer to as Brownian isometries. This class not only arises naturally in Theorem 6.20 which studies the extensions of 2-isometries defined as multiplication by $e^{i\theta}$ on the form domain of Toeplitz operators but also arises naturally in the general theory of Brownian unitary extensions which is studied in Section 9. Both the classes of Brownian unitaries and Brownian isometries possess elegant C^* -algebraic characterizations (Theorems 5.20 and 5.48).

We define

$$B_{\sigma E, U} = \begin{bmatrix} M_{\lambda} & \sigma E \\ 0 & U \end{bmatrix}$$

whenever $U \in \mathcal{L}(\mathcal{K})$ is a unitary operator and $E: \mathcal{K} \rightarrow \mathcal{H}$ is a Hilbert space isomorphism.

Definition A Brownian isometry of covariance 0 is an isometry. A *Brownian isometry of covariance $\sigma > 0$* is an operator B such that $\sigma^2 = \|B^*B - I\|$ and which is unitarily equivalent to

$$V_0 \oplus \int_{\oplus} B_{tE_t, U_t} d\mu(t)$$

(the V_0 summand may be absent) acting on $\mathcal{H} \oplus \int_{\oplus} H_{\mathcal{K}_t}^2 \oplus \mathcal{K}_t d\mu(t)$ where $V_0 \in \mathcal{L}(\mathcal{H})$ is an isometry, μ is a nonzero positive measure on $(0, \sigma]$, $U_t \in \mathcal{L}(\mathcal{K}_t)$ is unitary for μ -a.e. t ,

$E_t \in \mathcal{L}(\mathcal{K}_t)$ is a Hilbert space isomorphism, $B_{tE_t, U_t} \in \mathcal{L}(H_{\mathcal{K}_t}^2 \oplus \mathcal{K}_t)$ as given above and the following two measurability conditions are satisfied:

$t \mapsto \dim \mathcal{K}_t$ is a μ -measurable function

If $X_n = \{t \in (0, \sigma] : \dim \mathcal{K}_t = n\}$, $\mu_n(E) = \mu(E \cap X_n)$ for all measurable E and for all $n \in \mathbb{N} \cup \{\infty\}$, and $\mathcal{K}_{t_1}$ and $\mathcal{K}_{t_2}$ are identified for $t_1, t_2 \in X_n$, then the maps on X_n given by $t \mapsto E_t$ and $t \mapsto U_t$ are μ_n measurable. (Recall that $t \mapsto E_t$ is measurable if and only if $t \mapsto \langle E_t x, y \rangle$ is measurable for all x and y .)

A characterization of Brownian isometries can be given in terms of block operators.

Proposition B is a Brownian isometry of covariance 0 if and only if B is an isometry. $B \in \mathcal{L}(\mathcal{K})$ is a Brownian isometry of covariance $\sigma > 0$ if and only if $\sigma^2 = \|B^*B - I\|$ and B has the block matrix form

$$(5.38) \quad B = \begin{bmatrix} V & \sigma E \\ 0 & U \end{bmatrix}$$

with respect to a decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ where

- (i) $V \in \mathcal{L}(\mathcal{K}_1)$ is an isometry,
- (ii) $U \in \mathcal{L}(\mathcal{K}_2)$ is unitary ,
- (iii) $E: \mathcal{K}_2 \rightarrow \mathcal{K}_1$ is an injective contraction which maps $\mathcal{K}_2$ into $\ker V^*$, and
- (iv) E^*E commutes with U .

Brief Description of Sections 8 through 10 Sections 8 and 9 are devoted to the study of cyclic 2-isometries. It turns out that if T is a cyclic 2-isometry, then

$$\dim \ker (||\Delta_T|| - \Delta_T) = 1$$

and furthermore if $f_0 \neq 0$ is chosen so that $\Delta_T f_0 = ||\Delta_T|| f_0$, then f_0 is cyclic for T if and only if T is pure (Theorem 8.19). Here, a 2-isometry T is said to be pure if it has no isometric direct summand. If one chooses $\gamma = f_0$, then the DDO L attached to T turns out to have a particularly nice form. This allows one to construct a model for T based on a probability measure on the unit circle (Theorem 8.32). This model leads to a rich function theoretic analysis of 2-isometries.

The class of 2-isometries has also been independently studied by Stefan Richter [R1], [R2]. His work develops a Dirichlet space type model for 2-isometries by exploiting a beautiful extension of the wandering subspace argument for isometries to the class of 2-isometries which has considerable overlap with Section 8 of this paper.

Section 9 uses Theorem 8.19 to derive a number of qualitative facts about the existence and uniqueness of Brownian unitary and isometric extensions of a cyclic 2-isometry.

Section 10 gives some remarks and open questions related to all three parts of the paper.

§8. A cyclic vector

In this section we shall establish that if T is a pure cyclic 2-isometry and if $f_0 \in \ker(\|\Delta_T\| - \Delta_T)$ is non-zero, then f_0 is a cyclic vector for T . We note that this result has already been established in the Wiener-Hopf case with bounded intercept ((6.24) with $n = 1$). Our proof in this section uses the lifting theorem for 2-isometries (Theorem 5.80). It turns out that if f_0 is chosen as above for a cyclic 2-isometry, then $A = P(T, f_0)^\wedge | \mathcal{D}_a$ has the property that its slope and intercept are colinear. This results in the space H_A^2 having a very pleasing form.

We begin with the following elementary lemma.

Lemma 8.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be a 2-isometry of covariance $\sigma > 0$. There exists a Brownian unitary extension B of covariance σ for T of the form*

$$B = \begin{bmatrix} U_0 & 0 & 0 \\ 0 & V & \sigma E \\ 0 & 0 & U \end{bmatrix}$$

acting on $\mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2$ such that

$$(8.2) \quad P_{0 \oplus \mathcal{K}_1 \oplus 0} \mathcal{H} \subseteq \bigvee \{V^k E U^j P_{0 \oplus 0 \oplus \mathcal{K}_2} h : h \in \mathcal{H}, k \geq 0, j \text{ an integer}\}.$$

PROOF. By Theorem 5.80, there exists a Brownian unitary B_1 of covariance σ for T given by

$$B_1 = \begin{bmatrix} U_1 & 0 & 0 \\ 0 & V_1 & \sigma E_1 \\ 0 & 0 & U_2 \end{bmatrix}$$

acting on $\mathcal{J}_0 \oplus \mathcal{J}_1 \oplus \mathcal{J}_2$ such that

$$\mathcal{H} \subseteq \mathcal{J}_0 \oplus \mathcal{J}_1 \oplus \mathcal{J}_2, \quad B_1 \mathcal{H} \subseteq \mathcal{H}$$

and $B_1 | \mathcal{H} = T$. Set

$$\mathcal{M}_2 = \bigvee \{U_2^j P_{\mathcal{J}_2} \mathcal{H} : j \text{ an integer}\},$$

the smallest reducing subspace for U_2 containing $P_{\mathcal{J}_2} \mathcal{H}$, and let

$$\mathcal{M}_1 = \bigvee \{V_1^k E_1 \mathcal{M}_2 : k \geq 0\},$$

the smallest invariant subspace for V_1 containing $E_1 \mathcal{M}_2$.

If $P_{\mathcal{J}_1}\mathcal{H} \subseteq \mathcal{M}_1$, then the result follows. Otherwise, note that $E_1\mathcal{M}_2 \subseteq \ker(V_1^*)$ and so $\mathcal{M}_1$ reduces V_1 . Thus B_1 is unitarily equivalent to

$$\begin{bmatrix} U_1 & 0 & 0 & 0 & 0 \\ 0 & V_{11} & 0 & \sigma E_{11} & 0 \\ 0 & 0 & V_{12} & 0 & \sigma E_{12} \\ 0 & 0 & 0 & U_{21} & 0 \\ 0 & 0 & 0 & 0 & U_{22} \end{bmatrix}$$

acting on $\mathcal{J}_0 \oplus \mathcal{M}_1 \oplus (\mathcal{J}_1 \ominus \mathcal{M}_1) \oplus \mathcal{M}_2 \oplus (\mathcal{J}_2 \ominus \mathcal{M}_2)$ and $\mathcal{J}_0 \oplus \mathcal{M}_1 \oplus (\mathcal{J}_1 \ominus \mathcal{M}_1) \oplus \mathcal{M}_2 \oplus 0$ is an invariant subspace for B_1 which contains $\mathcal{H}$. The restricted operator after a shuffle has the form

$$B_2 = \begin{bmatrix} V_{12} & 0 & 0 & 0 \\ 0 & U_1 & 0 & 0 \\ 0 & 0 & V_{11} & \sigma E_{11} \\ 0 & 0 & 0 & U_{21} \end{bmatrix}$$

where by construction

$$(8.3) \quad P_{\mathcal{M}_1}\mathcal{H} \subseteq \bigvee \{V_{11}^k E_{11} U_{21}^j P_{\mathcal{M}_2}\mathcal{H} : k \geq 0, j \text{ an integer}\}.$$

Extending the V_{12} direct summand of B_2 to a unitary operator thus gives a Brownian unitary extension B of B_2 which (8.3) guarantees is a taut extension of the desired form. This concludes the proof of Lemma 8.1.

Proposition 8.4. *Let $T \in \mathcal{L}(\mathcal{H})$ be a 2-isometry of covariance $\sigma > 0$ and assume γ is cyclic for T . There exist positive measures μ_0 and μ supported on $\partial\mathbb{D}$ and an isometric transformation*

$$L: \mathcal{H} \rightarrow L^2(\mu_0) \oplus H_{L^2(\mu)}^2 \oplus L^2(\mu)$$

such that if B is defined by

$$(8.5) \quad B = \begin{bmatrix} M_{e^{i\theta}} & 0 & 0 \\ 0 & M_\lambda & \sigma \\ 0 & 0 & M_{e^{i\theta}} \end{bmatrix}$$

acting on $L^2(\mu_0) \oplus H_{L^2(\mu)}^2 \oplus L^2(\mu)$, then

$$LT = BL,$$

and

$$L(\gamma) = 1 \oplus F \oplus 1$$

for some $F \in H_{L^2(\mu)}^2$. Furthermore, if $f_0 \in \ker(\sigma^2 - \triangle_T)$, then

$$L(f_0) = 0 \oplus 0 \oplus g$$

for some $g \in L^2(\mu)$.

PROOF. By Lemma 8.1, there exists a Brownian unitary $B_1 \in \mathcal{L}(\mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2)$ of covariance σ given by

$$B_1 = \begin{bmatrix} U_0 & 0 & 0 \\ 0 & V_1 & \sigma E_1 \\ 0 & 0 & U \end{bmatrix}$$

where $\mathcal{H} \subseteq \mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2$, $B_1 \mathcal{H} \subseteq \mathcal{H}$ and $B_1|_{\mathcal{H}} = T$ and such that (8.2) holds. Assume that γ has coordinates,

$$\gamma = \gamma_0 \oplus \gamma_1 \oplus \gamma_2.$$

Let $\mathcal{M}_0$ be the $*$ -cyclic subspace for U_0 generated by γ_0 , $\mathcal{M}_2$ be the $*$ -cyclic subspace for U generated by γ_2 and $\mathcal{M}_1$ be the smallest invariant subspace for V_1 containing $E_1 \mathcal{M}_2$.

By the spectral theorem, there exist positive measure μ_0 and μ supported on $\partial \mathbb{D}$ and Hilbert space isomorphisms $W_0: \mathcal{M}_0 \rightarrow L^2(\mu_0)$ and $W_2: \mathcal{M}_2 \rightarrow L^2(\mu)$ such that

$$\begin{aligned} W_0(U_0|_{\mathcal{M}_0}) &= M_{e^{i\theta}} W_0, \\ W_0(\gamma_0) &= 1, \\ W_2(U|_{\mathcal{M}_2}) &= M_{e^{i\theta}} W_2, \quad \text{and} \\ W_2(\gamma_2) &= 1. \end{aligned}$$

Furthermore, by the multiplicity theory of pure isometries, the fact that $E_1|_{\mathcal{M}_2}$ is a Hilbert space isomorphism from $\mathcal{M}_2$ onto the initial space of $V_1|_{\mathcal{M}_1}$ implies there exists a Hilbert space isomorphism $W_1: \mathcal{M}_1 \rightarrow H_{L^2(\mu)}^2$ such that

$$\begin{aligned} W_1(V_1|_{\mathcal{M}_1}) &= M_\lambda W_1 \quad \text{and}, \\ W_1 E_1 W_2^{-1}: L^2(\mu) &\rightarrow H_{L^2(\mu)}^2 \end{aligned}$$

is the inclusion map.

Now, observe that $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \mathcal{M}_2$ is a reducing subspace for B_1 . Furthermore, if $W = W_0 \oplus W_1 \oplus W_2$ and if B is defined by (8.5), then $W: \mathcal{M} \rightarrow L^2(\mu_0) \oplus H_{L^2(\mu)}^2 \oplus L^2(\mu)$ is a Hilbert space isomorphism such that

$$(8.6) \quad W(B_1|_{\mathcal{M}}) = BW.$$

Also, condition (8.2) implies that $\gamma_1 \in \mathcal{M}_1$. Hence $\gamma \in \mathcal{M}$ and since γ is a cyclic vector for T , so also $\mathcal{H} \subseteq \mathcal{M}$. Thus, if $L: \mathcal{H} \rightarrow L^2(\mu_0) \oplus H_{L^2(\mu)}^2 \oplus L^2(\mu)$ is defined by $L = W|_{\mathcal{H}}$, then (8.6) implies that

$$LT = BL.$$

By construction, $L(\gamma)$ has the form,

$$1 \oplus F \oplus 1$$

for some $F \in H_{L^2(\mu)}^2$. To see that $L(f_0)$ has the form,

$$L(f_0) = 0 \oplus 0 \oplus g$$

for some $g \in L^2(\mu)$, simply note that since $\text{cov}(B) = \text{cov}(T)$,

$$f_0 \in \ker(\sigma^2 - \Delta_T) \subseteq \ker(\sigma^2 - \Delta_B) = 0 \oplus 0 \oplus L^2(\mu).$$

This concludes the proof of Proposition 8.4.

Before proceeding we remark that in the proof of Proposition 8.4 the measure μ_0 could be 0. This occurs if and only if T is pure (Theorem 9.3) and corresponds to a case where the $L^2(\mu_0)$ summand is absent in the various assertions of Proposition 8.4.

Recalling Proposition 1.24, we observe that if T is a cyclic 2-isometry and $\sigma = \text{cov}(T) > 0$, then $\ker(\sigma^2 - \Delta_T) \neq \{0\}$ (compact operators attain their norm). Consequently, if γ is a cyclic vector for T and L and B are as in Proposition 8.4, then there exist $g \in L^2(\mu)$, $F \in H_{L^2(\mu)}^2$, and a sequence of polynomials such that

$$(8.7) \quad p_n(B) \begin{bmatrix} 1 \\ F \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}.$$

The above convergence when considered as a concrete approximation result on the spaces $L^2(\mu_0)$, $H_{L^2(\mu)}^2$ and $L^2(\mu)$ turns out to have several consequences. We begin with the following lemma.

Lemma 8.8. *With the setup of Proposition 8.4, if a sequence of polynomials $\{p_n\}$ is chosen so that (8.7) holds, then*

- (i) $p_n \rightarrow 0$ in $L^2(\mu_0)$
- (ii) $p_n \rightarrow g$ in $L^2(\mu)$
- (iii) $g(e^{i\theta}) \neq 0$ μ almost every $e^{i\theta}$,

and there exists a holomorphic function $\zeta(\lambda)$ defined on $\mathbb{D}$ such that

$$(iv) \quad F(\lambda, e^{i\theta}) = \sigma \frac{\zeta(\lambda) - \frac{1}{g(e^{i\theta})}}{\lambda - e^{i\theta}} g(e^{i\theta}) \text{ for every } \lambda \in \mathbb{D}.$$

In particular, $\zeta \in H^2$,

- (v) $\zeta(\lambda) \rightarrow \frac{1}{g(e^{i\theta})}$ as $\lambda \rightarrow e^{i\theta}$ non tangentially for μ almost every $e^{i\theta}$, and
- (vi) $\zeta(re^{i\theta})g(e^{i\theta}) \rightarrow 1$ in $L^2(\mu)$ as $r \rightarrow 1$.

PROOF. First note that (i) and (ii) follow immediately from (8.7) and the special form of B . Note also that

$$(8.9) \quad p_n(\lambda)F(\lambda, e^{i\theta}) + \sigma \frac{p_n(\lambda) - p_n(e^{i\theta})}{\lambda - e^{i\theta}} \rightarrow 0$$

in $H^2_{L^2(\mu)}$. Since the point evaluations from $H^2_{L^2(\mu)}$ into $L^2(\mu)$ are continuous, (8.9) actually holds as a convergence in $L^2(\mu)$ for each fixed $\lambda \in \mathbb{D}$. Hence by (ii) and by passing to a subsequence if necessary we may assume that

$$(8.10) \quad p_n(\lambda) \left(F(\lambda, e^{i\theta}) + \frac{\sigma}{\lambda - e^{i\theta}} \right) \rightarrow \sigma \frac{g(e^{i\theta})}{\lambda - e^{i\theta}}$$

in $L^2(\mu)$ for each fixed $\lambda \in \mathbb{D}$. Since for each fixed $\lambda \in \mathbb{D}$, $p_n(\lambda)$ is a sequence of scalars, and since $g \neq 0$ in $L^2(\mu)$, we deduce from (8.10) that there exists a non vanishing function $\eta: \mathbb{D} \rightarrow \mathbb{C}$ such that

$$(8.11) \quad \eta(\lambda) \left(F(\lambda, e^{i\theta}) + \frac{\sigma}{\lambda - e^{i\theta}} \right) = \sigma \frac{g(e^{i\theta})}{\lambda - e^{i\theta}}.$$

Note that (8.11) implies that $\eta(\lambda)$ is analytic, a fact which will imply that (iii) holds. We argue by contradiction. If $g(e^{i\theta}) = 0$ for every $e^{i\theta} \in E$ where $\mu(E) > 0$, then (8.11) and the fact that $\eta(\lambda)$ is nonvanishing imply that

$$F(\lambda, e^{i\theta}) + \frac{\sigma}{\lambda - e^{i\theta}} \equiv 0 \quad , \quad \lambda \in \mathbb{D}$$

for μ almost every $e^{i\theta} \in E$. Since

$$\frac{1}{\lambda - e^{i\theta}} \notin H^2$$

for any $e^{i\theta}$, this contradicts the fact that $F(\lambda, e^{i\theta}) \in H^2$ for μ almost every $e^{i\theta}$. Hence (iii) holds.

Now define ζ by setting

$$\zeta(\lambda) = \frac{1}{\eta(\lambda)} \quad , \quad \lambda \in \mathbb{D}.$$

Note that (iv) follows immediately from (8.11). Also, observe that (v) follows from (iv) and the fact

$$(\lambda - e^{i\theta})F(\lambda, e^{i\theta}) \rightarrow 0$$

as $\lambda \rightarrow e^{i\theta}$ non tangentially for μ almost every $e^{i\theta}$. Also, observe that (iv) implies that

$$(8.12) \quad \sigma \zeta(\lambda) g(e^{i\theta}) = (\lambda - e^{i\theta}) F(\lambda, e^{i\theta}) + \sigma$$

a fact which implies that $\zeta \in H^2$.

Finally, to see (vi), we estimate $\|\zeta(re^{i\theta})g(e^{i\theta}) - 1\|$ using (8.12). Indeed,

$$\begin{aligned}
\sigma^2 \int |\zeta(re^{i\theta})g(e^{i\theta}) - 1|^2 d\mu &= (1-r)^2 \int |F(re^{i\theta}, e^{i\theta})|^2 d\mu \\
&= (1-r)^2 \sup_{\substack{h \in L^2(\mu) \\ \|h\| \leq 1}} \left| \int F(re^{i\theta}, e^{i\theta}) \overline{h(e^{i\theta})} d\mu \right|^2 \\
&= (1-r)^2 \sup_{\substack{h \in L^2(\mu) \\ \|h\| \leq 1}} \left| \left\langle F, \frac{1}{1-re^{-i\theta}\lambda} h(e^{i\theta}) \right\rangle_{H^2_{L^2(\mu)}} \right|^2 \\
&\leq (1-r)^2 \|F\|^2 \sup_{\substack{h \in L^2(\mu) \\ \|h\| \leq 1}} \left\| \frac{1}{1-re^{-i\theta}\lambda} h(e^{i\theta}) \right\|^2 \\
&\rightarrow 0
\end{aligned}$$

since

$$\left\| \frac{1}{1-re^{-i\theta}\lambda} h(e^{i\theta}) \right\|^2 = \frac{1}{1-r^2} \|h\|^2.$$

This concludes the proof of Lemma 8.8.

The next lemma represents the analysis of an attempt to recapture the vector

$$\begin{bmatrix} 1 \\ F \\ 1 \end{bmatrix}$$

as an element in the cyclic subspace for B generated by

$$\begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}.$$

Lemma 8.13. *With the setups of Proposition 8.4 and Lemma 8.8, if ζ_r is defined by $\zeta_r(\lambda) = \zeta(r\lambda)$, then*

$$\zeta_r(B) \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ F \\ 1 \end{bmatrix}.$$

PROOF. First observe that

$$\zeta_r(B) \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma \frac{\zeta_r(\lambda) - \zeta_r(e^{i\theta})}{\lambda - e^{i\theta}} g(e^{i\theta}) \\ \zeta_r(e^{i\theta})g(e^{i\theta}) \end{bmatrix}.$$

By Lemma 8.8 (vi), $\zeta_r(e^{i\theta})g(e^{i\theta}) \rightarrow 1$ in $L^2(\mu)$.

To see that

$$(8.14) \quad \sigma \frac{\zeta_r(\lambda) - \zeta_r(e^{i\theta})}{\lambda - e^{i\theta}} g(e^{i\theta}) \rightarrow F(\lambda, e^{i\theta})$$

in $H_{L^2(\mu)}^2$, note first that Lemma 8.8 (iv) implies that

$$\sigma \zeta_r(\lambda)g(e^{i\theta}) = (r\lambda - e^{i\theta})F(r\lambda, e^{i\theta}) + \sigma,$$

and

$$\sigma \zeta_r(e^{i\theta})g(e^{i\theta}) = (re^{i\theta} - e^{i\theta})F(re^{i\theta}, e^{i\theta}) + \sigma.$$

Hence, if $L_r: H_{L^2(\mu)}^2 \rightarrow H_{L^2(\mu)}^2$ is defined by

$$L_r(G)(\lambda, e^{i\theta}) = (1 - r) \frac{G(r\lambda, e^{i\theta}) - G(re^{i\theta}, e^{i\theta})}{\lambda - e^{i\theta}}$$

then

$$\sigma \frac{\zeta_r(\lambda) - \zeta_r(e^{i\theta})}{\lambda - e^{i\theta}} g(e^{i\theta}) = rF(r\lambda, e^{i\theta}) - e^{i\theta} L_r(F)(\lambda, e^{i\theta}).$$

Since $rF(r\lambda, e^{i\theta}) \rightarrow F(\lambda, e^{i\theta})$ in $H_{L^2(\mu)}^2$, (8.14) will follow once it is shown that

$$(8.15) \quad L_r \rightarrow 0 \quad \text{strongly in } \mathcal{L}(H_{L^2(\mu)}^2).$$

The assertion in (8.15) is an immediate consequence of the following two facts.

$$(8.16) \quad \|L_r\| \leq \frac{1}{2}.$$

$$(8.17) \quad L_r(\lambda^n h(e^{i\theta})) \rightarrow 0 \quad \text{if } n \geq 0 \quad \text{and } h \in L^2(\mu).$$

To prove (8.16), note first from the calculation that proved Lemma 8.8 (vi) that if $H \in H_{L^2(\mu)}^2$, then

$$(8.18) \quad \int |H(re^{i\theta}, e^{i\theta})|^2 d\mu(\theta) \leq \frac{1}{1 - r^2} \|H\|^2.$$

Thus if $G \in H_{L^2(\mu)}^2$ and $L_r(G)$ is expanded into a power series in λ ,

$$\begin{aligned} \|L_r G\|^2 &= (1 - r)^2 \sum_{\ell=1}^{\infty} r^{2\ell} \int |(M_{\lambda}^{*\ell} G)(re^{i\theta}, e^{i\theta})|^2 d\mu \\ &\leq (1 - r)^2 \sum_{\ell=1}^{\infty} r^{2\ell} \frac{1}{1 - r^2} \|M_{\lambda}^{*\ell} G\|^2 \\ &\leq \frac{r^2}{(1 + r)^2} \|G\|^2 \end{aligned}$$

and we see that (8.16) holds.

To prove (8.17), we calculate directly that

$$L_r(\lambda^n h(e^{i\theta})) = (1-r)r^n h(e^{i\theta})(\lambda^{n-1} + \lambda^{n-2}e^{i\theta} + \dots + e^{i(n-1)\theta}).$$

Hence,

$$\|L_r(\lambda^n h(e^{i\theta}))\|^2 = (1-r)^2 r^{2n} \|h\|^2 n,$$

a fact which implies (8.17) and concludes the proof of Lemma 8.13.

We now are in a position to prove our promised result, that the eigenfunction for Δ_T corresponding to the eigenvalue $\|\Delta_T\|$ is cyclic for T if T is a pure cyclic 2-isometry.

Theorem 8.19. *If T is a pure cyclic 2-isometry and $f_0 \in \ker(\|\Delta_T\| - \Delta_T)$ is nonzero, then f_0 is cyclic for T .*

PROOF. Apply Proposition 8.4 and Lemmas 8.8 and 8.13 to obtain that

$$\begin{bmatrix} 0 \\ F \\ 1 \end{bmatrix} \in L(\mathcal{H}).$$

Since also

$$\begin{bmatrix} 1 \\ F \\ 1 \end{bmatrix} \in L(\mathcal{H})$$

we deduce that

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in L(\mathcal{H}).$$

Setting

$$\mathcal{H}_0 = L^{-1}(H^2(\mu_0))$$

and

$$\mathcal{H}_1 = L^{-1} \left(\bigvee \left\{ B^k \begin{bmatrix} 0 \\ F \\ 1 \end{bmatrix} : k \geq 0 \right\} \right),$$

we see that $\mathcal{H}_0$ and $\mathcal{H}_1$ are invariant for T and $\mathcal{H}_0 \perp \mathcal{H}_1$. Furthermore, since γ is cyclic for T and $\gamma \in \mathcal{H}_0 + \mathcal{H}_1$, $\mathcal{H}_0 + \mathcal{H}_1 = \mathcal{H}$. Hence $\mathcal{H}_0$ is reducing for T and since T is assumed pure, $\mathcal{H}_0 = \{0\}$. Since Lemma 8.13 implies that f_0 is a cyclic vector for $\mathcal{H}_1$ it follows that f_0 is a cyclic vector for T .

Theorem 8.19 implies that if T is a cyclic 2-isometry with $\text{cov}(T) > 0$, then

$$\dim \ker(\|\Delta_T\| - \Delta_T) = 1.$$

To see this simply observe that if $\ker(\|\Delta_T\| - \Delta_T)$ contains two linear independent vectors, then there exist a nonzero vector f_0 in

$$\ker(\|\Delta_T\| - \Delta_T) \cap \text{ran } T.$$

Such a vector could not possibly be cyclic for the pure part of T .

Theorem 8.19 has as an application the fact that a cyclic pure 2-isometry has a particular model in a nice form.

If T denotes the ordinary Dirichlet shift (i.e., $T = M_A$ with $L = \frac{d\theta}{2\pi} D + \frac{d\theta}{2\pi}$), then T is commonly represented as M_z on the space of functions $f \in H^2$ such that

$$(8.20) \quad \sum_{k=0}^{\infty} (k+1) |\hat{f}(k)|^2 < \infty.$$

We notice that (8.20) can be reformulated as the condition

$$\|f\|_{H^2(\frac{d\theta}{2\pi})}^2 + \sum_{k=0}^{\infty} \left\| f - \sum_{\ell=0}^k \hat{f}(\ell) e^{i\ell\theta} \right\|_{H^2(\frac{d\theta}{2\pi})}^2 < \infty$$

i.e., f is in $H^2(\frac{d\theta}{2\pi})$ and the sum of the squares of the errors in approximating f by its Fourier series converges.

Definition 8.21 For μ a probability measure on $\partial\mathbb{D}$, let $H_{\mu,\sigma}^2$ denote the Hilbert space of analytic functions f defined on $\mathbb{D}$ such that

- (i) $f \in H^2$,
- (ii) there exists $\tilde{f} \in L^2(\mu)$ such that $f(\lambda) \rightarrow \tilde{f}(e^{i\theta})$ as $\lambda \rightarrow e^{i\theta}$ non tangentially for μ almost every $e^{i\theta}$, and
- (iii) $\sum_{k=0}^{\infty} \left\| \tilde{f} - \sum_{\ell=0}^k \hat{f}(\ell) e^{i\ell\theta} \right\|_{L^2(\mu)}^2 < \infty$ and equipped with norm defined by,

$$\|f\|_{\mu,\sigma}^2 = \|\tilde{f}\|_{L^2(\mu)}^2 + \sigma^2 \sum_{k=0}^{\infty} \left\| \tilde{f} - \sum_{\ell=0}^k \hat{f}(\ell) e^{i\ell\theta} \right\|_{L^2(\mu)}^2.$$

The function theory that leads to the completeness of the space $H_{\mu,\sigma}^2$ and the density of polynomials is based on the following observation.

$$(8.22) \quad \left\{ \begin{array}{l} \text{The map } L: H_{\mu,\sigma}^2 \rightarrow H_{L^2(\mu)}^2 \oplus L^2(\mu) \\ \text{defined by } L(f) = \sigma \frac{f(\lambda) - \tilde{f}(e^{i\theta})}{\lambda - e^{i\theta}} \oplus \tilde{f} \\ \text{is an isometry.} \end{array} \right.$$

Both the facts that $L(f) \in H_{L^2(\mu)}^2 \oplus L^2(\mu)$ and that L is an isometry follow from an examination of the power series formula,

$$\frac{f(\lambda) - \tilde{f}(e^{i\theta})}{\lambda - e^{i\theta}} = e^{-i\theta} \sum_{k=0}^{\infty} \left(\tilde{f}(e^{i\theta}) - \sum_{\ell=0}^k \tilde{f}(\ell) e^{i\ell\theta} \right) e^{-ik\theta} \lambda^k.$$

To see that $H_{\mu,\sigma}^2$ is complete, assume that $\{f_j\}$ is a Cauchy sequence in $H_{\mu,\sigma}^2$. By (8.22) there exists $F \in H_{L^2(\mu)}^2$ such that

$$\frac{f_j(\lambda) - \tilde{f}_j(e^{i\theta})}{\lambda - e^{i\theta}} \rightarrow F(\lambda, e^{i\theta}) \quad \text{in } H_{L^2(\mu)}^2,$$

and there exists $g \in L^2(\mu)$ such that

$$\tilde{f}_j \rightarrow g \quad \text{in } L^2(\mu).$$

Following the proof of Lemma 8.8 (iv), we deduce that there exists a holomorphic function $\eta(\lambda)$ such that

$$(8.23) \quad \eta(\lambda) - g(e^{i\theta}) = (\lambda - e^{i\theta})F(\lambda, e^{i\theta}).$$

Now, (8.23) implies that $\eta \in H^2$, and as in the proof of Lemma 8.8 (v), it is easy to deduce from (8.23) that $\eta(\lambda) \rightarrow g(e^{i\theta})$ as $\lambda \rightarrow e^{i\theta}$ non tangentially for μ almost every $e^{i\theta}$. Thus, $g = \tilde{\eta}$. Consequently,

$$\frac{\eta(\lambda) - \tilde{\eta}(e^{i\theta})}{\lambda - e^{i\theta}} = F(\lambda, e^{i\theta}) \in H_{L^2(\mu)}^2,$$

and $f_j \rightarrow \eta$ in $H_{\mu,\sigma}^2$. Thus, $H_{\mu,\sigma}^2$ is complete.

The proof that the analytic polynomials are dense in $H_{\mu,\sigma}^2$ will follow from the following lemma.

Lemma 8.24. *If $f \in H_{\mu,\sigma}^2$ and f_r is defined by $f_r(\lambda) = f(r\lambda)$ for $r < 1$, then $f_r \in H_{\mu,\sigma}^2$ and $f_r \rightarrow f$ in $H_{\mu,\sigma}^2$ as $r \rightarrow 1$.*

PROOF. By (8.23), it suffices to show

$$\left[\frac{f(r\lambda) - f(re^{i\theta})}{\lambda - e^{i\theta}} \right] \rightarrow \left[\frac{f(\lambda) - \tilde{f}(e^{i\theta})}{\lambda - e^{i\theta}} \right].$$

That $f(re^{i\theta}) \rightarrow \tilde{f}(e^{i\theta})$ in $L^2(\mu)$ follows as in the proof of Lemma 8.8 (vi).

To see that

$$(8.25) \quad \frac{f(r\lambda) - f(re^{i\theta})}{\lambda - e^{i\theta}} \rightarrow \frac{f(\lambda) - \tilde{f}(e^{i\theta})}{\lambda - e^{i\theta}},$$

set

$$F(\lambda, e^{i\theta}) = \frac{f(\lambda) - \tilde{f}(e^{i\theta})}{\lambda - e^{i\theta}}$$

and note that

$$\frac{f(r\lambda) - f(re^{i\theta})}{\lambda - e^{i\theta}} = rF(r\lambda, e^{i\theta}) - e^{i\theta}L_r(F)(\lambda, e^{i\theta})$$

where L_r is defined as in the proof of Lemma 8.13. Since $L_r \rightarrow 0$ strongly and since $rF(r\lambda, e^{i\theta}) \rightarrow F(\lambda, e^{i\theta})$, (8.25) holds and the proof of Lemma 8.24 is complete.

To see how Lemma 8.24 implies that the polynomials are dense in $H_{\mu, \sigma}^2$ observe that if f is holomorphic on a neighborhood of $\mathbb{D}^-$, then $f|_{\partial\mathbb{D}} \in \mathcal{D}_a$. Also, if $f_k \in \mathcal{D}_a$ is defined by

$$f_k(e^{i\theta}) = \sum_{\ell=0}^k \hat{f}(\ell) e^{i\ell\theta},$$

then $f_k \rightarrow f$ in $\mathcal{D}_a$. Hence using Proposition 1.20

$$\begin{aligned} \|f - f_k\|_{\mu, \sigma}^2 &= \|(f - f_k)(M_{\mu, \sigma})1\|^2 \\ &\rightarrow 0. \end{aligned}$$

Since f_k is a polynomial, this proves the following lemma.

Lemma 8.26. *The polynomials are dense in $H_{\mu, \sigma}^2$.*

If μ is a probability measure on $\partial\mathbb{D}$ and $\sigma > 0$, let $M_{\mu, \sigma}$ denote the operator “multiplication by λ ” on $H_{\mu, \sigma}^2$ i.e.

$$M_{\mu, \sigma} f(\lambda) = \lambda f(\lambda).$$

Proposition 8.27. *If μ is a probability measure on $\partial\mathbb{D}$ and $\sigma > 0$, then $M_{\mu, \sigma}$ is a pure cyclic 2-isometry of covariance σ . Furthermore, 1 is a unit eigenvector for $\Delta_{M_{\mu, \sigma}}$ corresponding to the eigenvalue σ^2 and*

$$(8.28) \quad \langle \Delta_{M_{\mu, \sigma}} f, g \rangle_{\mu, \sigma} = \sigma^2 \langle \tilde{f}, \tilde{g} \rangle_{L^2(\mu)}$$

for all $f, g \in H_{\mu, \sigma}^2$.

PROOF. Note first using the identity,

$$\frac{\lambda f(\lambda) - e^{i\theta} \tilde{f}(e^{i\theta})}{\lambda - e^{i\theta}} = \lambda \frac{f(\lambda) - \tilde{f}(e^{i\theta})}{\lambda - e^{i\theta}} + \tilde{f}(e^{i\theta})$$

that

$$\left\| \frac{\lambda f(\lambda) - e^{i\theta} \tilde{f}(e^{i\theta})}{\lambda - e^{i\theta}} \right\|_{H_{L^2(\mu)}^2}^2 = \left\| \frac{f(\lambda) - \tilde{f}(e^{i\theta})}{\lambda - e^{i\theta}} \right\|_{H_{L^2(\mu)}^2}^2 + \|\tilde{f}\|_{L^2(\mu)}^2.$$

Thus, by (8.22),

$$(8.29) \quad \|\lambda f\|_{\mu,\sigma}^2 = \|f\|_{\mu,\sigma}^2 + \sigma^2 \|f^\sim\|_{L^2(\mu)}^2.$$

Since

$$\|f^\sim\|_{L^2(\mu)}^2 \leq \|f\|_{\mu,\sigma}^2,$$

(8.29) implies that $M_{\mu,\sigma}$ is bounded. Also, (8.29) implies that $M_{\mu,\sigma}$ is a 2-isometry and that (8.28) holds. $M_{\mu,\sigma}$ is cyclic by Lemma 8.26.

To see that 1 is a eigenvector for $\Delta_{M_{\mu,\sigma}}$ first note by the definition of $\|\cdot\|_{\mu,\sigma}$ and (8.28) that $\|1\| = 1$ and $\|\Delta_{M_{\mu,\sigma}} 1\| \leq \sigma^2$. Thus since (8.28) implies

$$\sigma^2 = \langle \Delta_{M_{\mu,\sigma}} 1, 1 \rangle \leq \|\Delta_{M_{\mu,\sigma}}\| = \sigma^2,$$

in fact $\|\Delta_{M_{\mu,\sigma}}\| = \sigma^2$ and 1 is an eigenvector for $\Delta_{M_{\mu,\sigma}}$ corresponding to the eigenvalue σ^2 . Finally, $M_{\mu,\sigma}$ is pure by Lemma 1.25 since $1 \in \text{ran } \Delta_{M_{\mu,\sigma}}$ and 1 is cyclic for $M_{\mu,\sigma}$. This concludes the proof of Proposition 8.27.

We write $\Delta_{\mu,\sigma}$ for $\Delta_{M_{\mu,\sigma}}$.

The following proposition will turn out to give a complete set of unitary invariants for pure cyclic 2-isometries when combined with Theorem 8.32 below.

Proposition 8.30. *If μ_1 and μ_2 are probability measures on $\partial\mathbb{D}$ and $\sigma_1, \sigma_2 > 0$, the M_{μ_1,σ_1} is unitarily equivalent to M_{μ_2,σ_2} if and only if $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$.*

PROOF. Assume that $U: H_{\mu_1,\sigma_1}^2 \rightarrow H_{\mu_2,\sigma_2}^2$ is a Hilbert space isomorphism such that

$$(8.31) \quad M_{\mu_2,\sigma_2} U = U M_{\mu_1,\sigma_1}.$$

Then necessarily,

$$\sigma_2^2 = \|\Delta_{\mu_2,\sigma_2}\| = \|\Delta_{\mu_1,\sigma_1}\| = \sigma_1^2,$$

and

$$U \left(\ker (\sigma_1^2 - \Delta_{\mu_1,\sigma_1}) \right) = \ker (\sigma_2^2 - \Delta_{\mu_2,\sigma_2}).$$

Now, Proposition 8.27 implies that both

$$1 \in \ker (\sigma_1^2 - \Delta_{\mu_1,\sigma_1}) \quad \text{and} \quad 1 \in \ker (\sigma_2^2 - \Delta_{\mu_2,\sigma_2}).$$

Also by the remark following the proof of Theorem 8.19,

$$\dim \ker (\sigma_1^2 - \Delta_{\mu_1,\sigma_1}) = \dim \ker (\sigma_2^2 - \Delta_{\mu_2,\sigma_2}) = 1.$$

Hence, there exists $\alpha \in \partial\mathbb{D}$ such that

$$U(1) = \alpha.$$

Using (8.31) we obtain that

$$U(p) = \alpha p$$

for all polynomials p , a fact which implies that

$$\begin{aligned} \sigma_2^2 \langle p, q \rangle_{L^2(\mu_2)} &= \langle \Delta_{\mu_2, \sigma_2} p, q \rangle_{L^2(\mu_2)} \\ &= \langle U^* \Delta_{\mu_1, \sigma_1} U p, q \rangle_{L^2(\mu_2)} \\ &= \langle \Delta_{\mu_1, \sigma_1} p, q \rangle_{L^2(\mu_1)} \\ &= \sigma_1^2 \langle p, q \rangle_{L^2(\mu_1)}. \end{aligned}$$

Since $\sigma_1 = \sigma_2$, this calculation shows that $\mu_1 = \mu_2$. This establishes Proposition 8.30. The following result is the converse of Proposition 8.27.

Theorem 8.32. *If T is a pure cyclic 2-isometry with $\text{cov}(T) = \sigma$, then there exists a unique probability measure μ on $\partial\mathbb{D}$ such that T is unitarily equivalent to $M_{\mu, \sigma}$.*

PROOF. The uniqueness of μ follows from Proposition 8.30. By Theorem 8.29, if f_0 is a unit vector in $\ker(\sigma^2 - \Delta_T)$, then f_0 is cyclic for T . Hence, by Proposition 8.4 with $\gamma = f_0$, there exists a positive measure ν on $\partial\mathbb{D}$, a unit vector $g \in L^2(\nu)$, and an isometry $L: \mathcal{H} \rightarrow H_{L^2(\nu)}^2 \oplus L^2(\nu)$ such that

$$(8.33) \quad LT = \begin{bmatrix} M_\lambda & \sigma \\ 0 & M_{e^{i\theta}} \end{bmatrix} L$$

and

$$L(f_0) = \begin{bmatrix} 0 \\ g \end{bmatrix}.$$

If the measure μ is defined by

$$\mu = |g|^2 \nu,$$

then since g is a unit vector in $L^2(\nu)$, μ is a probability measure.

Computing using (8.33) we find that

$$\begin{aligned} \langle p(T)f_0, q(T)f_0 \rangle &= \langle L(p(T)f_0), L(q(T)f_0) \rangle \\ &= \left\langle p \left(\begin{bmatrix} M_\lambda & \sigma \\ 0 & M_{e^{i\theta}} \end{bmatrix} \right) \begin{bmatrix} 0 \\ g \end{bmatrix}, q \begin{bmatrix} M_\lambda & \sigma \\ 0 & M_{e^{i\theta}} \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix} \right\rangle \\ &= \langle p, q \rangle_{L^2(\mu)} - \sigma^2 \left\langle \frac{p(\lambda) - p(e^{i\theta})}{\lambda - e^{i\theta}}, \frac{q(\lambda) - q(e^{i\theta})}{\lambda - e^{i\theta}} \right\rangle_{H_{L^2(\mu)}^2}^2 \\ &= \langle p, q \rangle_{H_{\mu, \sigma}^2}. \end{aligned}$$

Thus, T is unitarily equivalent to $M_{\mu, \sigma}$ via the map

$$p(T)f_0 \mapsto p.$$

This establishes Theorem 8.32.

§9. Brownian Unitaries Revisited

In this section we shall use Theorem 8.19 to derive a number of qualitative facts about the existence and uniqueness of Brownian unitary and Brownian isometric extensions of a cyclic 2-isometry T . The main results of the section are Theorems 9.3, 9.4 and 9.5.

Theorem 9.3 identifies those cyclic 2-isometries T of covariance σ that possess a pure Brownian unitary extension of covariance σ . Theorem 9.4 establishes that there exists a unique minimal Brownian unitary extension of covariance σ for a cyclic 2-isometry T of covariance σ and identifies that extension concretely in terms of a geometric criterion termed “tautness” (defined in Definition 9.1 below). Finally, Theorem 9.5 establishes the following interesting phenomenon: if $T \in \mathcal{L}(\mathcal{H})$ is a cyclic 2-isometry of covariance σ and $B \in \mathcal{L}(\mathcal{K})$ is any Brownian isometric extension for T of covariance σ , then there exists a Hilbert space $\mathcal{M}$ with $\mathcal{H} \subseteq \mathcal{M} \subseteq \mathcal{K}$ such that $\mathcal{M}$ is reducing for B and such that $B|_{\mathcal{M}}$ is the minimal unitary extension of T .

We begin with the following notion motivated by Lemma 8.1 for a general 2-isometry not assumed cyclic.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces with $\mathcal{H} \subseteq \mathcal{K}$. Let $T \in \mathcal{L}(\mathcal{H})$ be a 2-isometry with covariance σ . Let $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2$ and let $B \in \mathcal{L}(\mathcal{K})$ be a Brownian unitary of covariance σ ,

$$B = \begin{bmatrix} U_0 & 0 & 0 \\ 0 & V & \sigma E \\ 0 & 0 & U \end{bmatrix},$$

such that $B\mathcal{H} \subseteq \mathcal{H}$ and $B|_{\mathcal{H}} = T$.

Definition 9.1. B is a *taut extension* of T if

- (i) $P_{\mathcal{K}_0}\mathcal{H}$ is $*$ -cyclic for U_0
- (ii) $P_{\mathcal{K}_2}\mathcal{H}$ is $*$ -cyclic for U
- (iii) $P_{\mathcal{K}_1}\mathcal{H} \subseteq \bigvee \{V^k E \mathcal{K}_2 : k \geq 0\}$.

In the above definition and throughout this section we will abuse notation somewhat by simply denoting the orthogonal projections $P_{\mathcal{K}_0 \oplus 0 \oplus 0}$, $P_{0 \oplus \mathcal{K}_1 \oplus 0}$, and $P_{0 \oplus 0 \oplus \mathcal{K}_2}$ by $P_{\mathcal{K}_0}$, $P_{\mathcal{K}_1}$, and $P_{\mathcal{K}_2}$ respectively. In the definition also note that (i) (resp. (ii)) is equivalent to the condition that $\mathcal{K}_0$ (resp. $\mathcal{K}_2$) is the smallest reducing subspace for U_0 (resp., U) containing $P_{\mathcal{K}_0}\mathcal{H}$ (resp., $P_{\mathcal{K}_2}\mathcal{H}$). Finally, since $V^*E = 0$, (iii) is equivalent to the condition that $P_{\mathcal{K}_1}\mathcal{H}$ is in the smallest reducing subspace for V containing $E\mathcal{K}_2$.

Lemma 8.1 implies that every 2-isometry of covariance σ possesses a taut extension to a Brownian unitary of covariance σ . To see this note that if B is *any* Brownian unitary extension for T as in Lemma 8.1, then if $\mathcal{K}'_0$, $\mathcal{K}'_1$ and $\mathcal{K}'_2$ are defined by

$$\begin{aligned}\mathcal{K}'_0 &= \bigvee \left\{ U_0^j P_{\mathcal{K}_0} \mathcal{H} : j \in \mathbb{Z} \right\}, \\ \mathcal{K}'_2 &= \bigvee \left\{ U^j P_{\mathcal{K}_2} \mathcal{H} : j \in \mathbb{Z} \right\}, \quad \text{and} \\ \mathcal{K}'_1 &= \bigvee \left\{ V^k E \mathcal{K}'_2 : k \geq 0 \right\},\end{aligned}$$

then $\mathcal{K}'_0 \oplus \mathcal{K}'_1 \oplus \mathcal{K}'_2$ is reducing for B . If we set $\mathcal{K}' = \mathcal{K}'_0 \oplus \mathcal{K}'_1 \oplus \mathcal{K}'_2$, then $B|_{\mathcal{K}'}$ is a taut extension of T .

The next lemma hints at a certain structural invariance in the concept of purity relative to extensions of cyclic 2-isometries. It is the heart of the matter in analyzing when a cyclic 2-isometry has a pure Brownian extension.

Lemma 9.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be a cyclic 2-isometry of covariance $\sigma > 0$. T has the form $U_0 \oplus T_0$ acting on $\mathcal{H}_0 \oplus \mathcal{H}_1$ where U_0 is unitary and T_0 is pure. Furthermore, every Brownian unitary extension B of T of covariance σ has the form $B = U_0 \oplus B_1$ acting on $\mathcal{H}_0 \oplus \mathcal{K}_1$ where $\mathcal{K}_1 \supseteq \mathcal{H}_1$ and B_1 is a Brownian unitary extension of T_0 of covariance σ .*

PROOF. By Theorem 1.26, T has the form $V \oplus T_0$ where V is an isometry and T_0 is pure. Also, since $\sigma > 0$,

$$\text{ind}(T_0) \leq -1.$$

Consequently, since T is cyclic,

$$-1 \leq \text{ind}(T) = \text{ind}(V) + \text{ind}(T_0) \leq \text{ind}(V) - 1.$$

Hence, V is unitary.

To see the second assertion of the lemma, note that Proposition 5.90 implies that if $\mathcal{K} \supseteq \mathcal{H}_0 \oplus \mathcal{H}_1$ and B is any Brownian unitary extension for T of covariance σ then since B is also an extension for U_0 , $\mathcal{H}_0$ is a reducing subspace for B . This concludes the proof of Lemma 9.2.

Theorem 9.3. *Let T be a cyclic 2-isometry of covariance σ . T has a pure Brownian unitary extension of covariance σ if and only if T is pure.*

PROOF. If T is not pure and not an isometry, then by Lemma 9.2, T has a unitary summand. Hence every Brownian unitary extension of T of the same covariance as T has

a unitary summand, i.e., is not pure. If T is an isometry, then $\sigma = 0$ and again the result is clear.

Conversely, suppose that T is pure. Let $f_0 \in \ker(\Delta_T - \|\Delta_T\|)$ be nonzero. By Theorem 8.19, f_0 is cyclic for T . By Proposition 8.4, there exist B , L , μ_0 and μ such that B has the form given in (8.5), $LT = BL$, and $L(f_0) = 0 \oplus 0 \oplus g$ for some $g \in L^2(\mu)$. But $0 \oplus H_{L^2(\mu)}^2 \oplus L^2(\mu)$ is reducing for B and contains $\mathcal{H}$ (since $LT = BL$ and f_0 is cyclic for T) so that $B|_{0 \oplus H_{L^2(\mu)}^2 \oplus L^2(\mu)}$ is a pure Brownian unitary extension of covariance σ . This establishes Theorem 9.3.

Before proceeding we remark that Theorem 9.3 does not hold for 2-cyclic 2-isometries. Indeed, if B is a Brownian shift of covariance $\sigma > 0$, then $B \oplus S$, which is not pure, has an extension to the pure Brownian unitary $B \oplus B$.

If $T \in \mathcal{L}(\mathcal{H})$ is a 2-isometry of covariance σ , let us agree to say that a Brownian unitary extension $B \in \mathcal{L}(\mathcal{K})$ of T of covariance σ is *minimal* if $\mathcal{K}$ is the smallest reducing subspace for B containing $\mathcal{H}$. Note that Lemma 5.32 and Lemma 5.90 imply via Zorn's Lemma that minimal extensions always exist. The following theorem establishes not only the uniqueness of minimal lifts for cyclic T , but also that minimality is identical to the concept of tautness in the cyclic case.

Theorem 9.4. *Let $T \in \mathcal{L}(\mathcal{H})$ be a cyclic 2-isometry of covariance $\sigma > 0$. $B \in \mathcal{L}(\mathcal{K})$ is a minimal Brownian unitary extension of covariance σ for T if and only if B is a taut extension of T . Furthermore, any two taut extensions of T are unitarily equivalent via a Hilbert space isomorphism that fixes $\mathcal{H}$.*

PROOF. First assume that $B \in \mathcal{L}(\mathcal{K})$ is a minimal Brownian unitary extension for T of covariance σ . We need to show that B is taut. By Lemma 9.2, we may assume that T is pure. Thus, B has the form,

$$B = \begin{bmatrix} V & \sigma E \\ 0 & U \end{bmatrix}$$

with respect to a decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. Now, if $f_0 \neq 0$ is chosen in $\ker(\sigma^2 - \Delta_T)$, then by Theorem 8.19, f_0 is cyclic for T . Defining $\mathcal{M}_1 \subseteq \mathcal{K}_1$ and $\mathcal{M}_2 \subseteq \mathcal{K}_2$ by

$$\begin{aligned} \mathcal{M}_2 &= \bigvee \{U^j f_0 : j \in \mathbb{Z}\}, \quad \text{and} \\ \mathcal{M}_1 &= \bigvee \{V^k E \mathcal{M}_2 : k \geq 0\}, \end{aligned}$$

we see that $\mathcal{M}_1 \oplus \mathcal{M}_2$ is reducing for B and that $\mathcal{H} \subseteq \mathcal{M}_1 \oplus \mathcal{M}_2$ by the cyclicity of f_0 . Since B is minimal, it follows that $\mathcal{K}_1 = \mathcal{M}_1$ and $\mathcal{K}_2 = \mathcal{M}_2$. Thus, by the construction of $\mathcal{M}_1$ and $\mathcal{M}_2$, B is a taut extension for T .

The remainder of Theorem 9.4 will follow from the uniqueness of taut extensions. By Lemma 9.2, it suffices to establish this uniqueness in the case when T is pure. To establish this latter fact, let

$$B_1 = \begin{bmatrix} V_1 & \sigma E_1 \\ 0 & U_1 \end{bmatrix}$$

and

$$B_2 = \begin{bmatrix} V_2 & \sigma E_2 \\ 0 & U_2 \end{bmatrix}$$

be taut pure extensions of T acting on $\mathcal{K}_1 \oplus \mathcal{J}_1$ and $\mathcal{K}_2 \oplus \mathcal{J}_2$, respectively. Choose a nonzero vector $f_0 \in \ker(\sigma^2 - \Delta_T)$ and let $g_1 = P_{\mathcal{J}_1} f_0$ and $g_2 = P_{\mathcal{J}_2} f_0$. Densely define a linear map $W: \mathcal{K}_1 \oplus \mathcal{J}_1 \rightarrow \mathcal{K}_2 \oplus \mathcal{J}_2$ by requiring

$$W \begin{pmatrix} V_1^k E_1 U_1^j g_1 \\ U_1^\ell g_1 \end{pmatrix} = \begin{pmatrix} V_2^k E_2 U_2^j g_2 \\ U_2^\ell g_2 \end{pmatrix}$$

for $k \geq 0$ and $j, \ell \in \mathbb{Z}$. A direct calculation using the facts that $V_1^* E_1 = 0$ and $V_2^* E_2 = 0$ shows that W extends by continuity to an isometry. Using the tautness of B_1 and B_2 , the cyclicity of f_0 , and the facts that E_1 and E_2 map onto $\ker V_1^*$ and $\ker V_2^*$ allow one to conclude that W is densely defined and has dense range. Finally, note that

$$W(T^k f_0) = W \left(B_1^k \begin{bmatrix} 0 \\ g_1 \end{bmatrix} \right) = B_2^k \begin{bmatrix} 0 \\ g_2 \end{bmatrix} = T^k f_0,$$

i.e. W fixes $\mathcal{H}$, and that $WB_1 = B_2W$. This concludes the proof of Theorem 9.4.

If $T \in \mathcal{L}(\mathcal{H})$ is a 2-isometry of covariance σ and $B \in \mathcal{L}(\mathcal{K})$ is a Brownian isometric extension of covariance σ for T , let us agree to say that B is a *minimal Brownian isometric extension of covariance σ for T* if $\mathcal{K}$ is the smallest reducing subspace for B containing $\mathcal{H}$. Note that the existence of minimal Brownian isometric extensions is a tautology based on Zorn's Lemma and the fact that the collection of Brownian isometries is closed with respect to restrictions to reducing subspaces (a consequence of Theorem 5.48).

In the cyclic case we can identify not only the minimal Brownian unitary extension as in Theorem 9.4 but also the minimal Brownian isometric extension of covariance σ of a cyclic 2-isometry T of covariance σ (they are identical!).

Theorem 9.5. *Let $T \in \mathcal{L}(\mathcal{H})$ be a cyclic 2-isometry of covariance σ . If $B \in \mathcal{L}(\mathcal{K})$ is a Brownian isometric extension of T of covariance σ , then B decomposes*

$$B = B_0 \oplus B_1$$

with respect to a decomposition $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$ where B_0 is a Brownian unitary of covariance σ and $\mathcal{H} \subseteq \mathcal{K}_0$. In particular, a cyclic 2-isometry of covariance σ has a unique minimal

Brownian isometric extension of covariance σ and this extension is the unique minimal Brownian unitary extension of T of covariance σ .

PROOF. As in the proof of Theorem 9.4, Lemma 9.2 implies that we may assume that T is pure. If $B \in \mathcal{L}(\mathcal{K})$ is a Brownian isometric extension of T of covariance σ , then B has the form

$$B = \begin{bmatrix} V & \sigma E \\ 0 & U \end{bmatrix}$$

acting on $\mathcal{K} = \mathcal{M}_0 \oplus \mathcal{M}_1$ where U commutes with E^*E , $E^*E \leq 1$ and $V^*E = 0$. By Theorem 8.19, there exists a unit vector $g \in \mathcal{M}_1$ such that $0 \oplus g \in \mathcal{H}$ and $0 \oplus g \in \ker(\Delta_B - \sigma^2)$ is cyclic for T . Computation of Δ_B yields the equation

$$\langle \sigma^2 E^* E g, g \rangle = \sigma^2.$$

Since $E^*E \leq 1$, $E^*Eg = g$. Thus, since E^*E commutes with U , $E^*EU^jg = U^jg$ for all integers j .

If we define $\mathcal{M}_2$ and $\mathcal{M}_3$ by

$$\mathcal{M}_2 = \bigvee \{U^jg : j \in \mathbb{Z}\}$$

and

$$\mathcal{M}_3 = \mathcal{M}_1 \ominus \mathcal{M}_2,$$

then B decomposes with respect to $\mathcal{K} = \mathcal{M}_0 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$ as

$$B = \begin{bmatrix} V & \sigma E|_{\mathcal{M}_2} & \sigma E|_{\mathcal{M}_3} \\ 0 & U|_{\mathcal{M}_2} & 0 \\ 0 & 0 & U|_{\mathcal{M}_3} \end{bmatrix}$$

where $E|_{\mathcal{M}_2}$ is an isometry and $E(\mathcal{M}_2) \perp E(\mathcal{M}_3)$. Since $V^*E = 0$, it follows that if

$$\begin{aligned} \mathcal{M}_4 &= \bigvee \{V^k E(\mathcal{M}_2) : k \geq 0\}, \quad \text{and} \\ \mathcal{M}_5 &= \mathcal{M}_0 \ominus \mathcal{M}_4, \end{aligned}$$

then B decomposes with respect to $\mathcal{K} = \mathcal{M}_4 \oplus \mathcal{M}_5 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$ as

$$B = \begin{bmatrix} V|_{\mathcal{M}_4} & 0 & \sigma E|_{\mathcal{M}_2} & 0 \\ 0 & V|_{\mathcal{M}_5} & 0 & \sigma E|_{\mathcal{M}_3} \\ 0 & 0 & U|_{\mathcal{M}_2} & 0 \\ 0 & 0 & 0 & U|_{\mathcal{M}_3} \end{bmatrix}$$

where $E|_{\mathcal{M}_2}$ is a surjective isometry onto $\ker(V|_{\mathcal{M}_4})^*$. Since by construction, $g \in 0 \oplus 0 \oplus \mathcal{M}_2 \oplus 0$, the facts that B extends T and g is cyclic for T imply that

$$\mathcal{H} \subseteq \mathcal{M}_4 \oplus 0 \oplus \mathcal{M}_2 \oplus 0.$$

Thus, the first assertion of Theorem 9.5 follows by setting

$$\mathcal{K}_0 = \mathcal{M}_4 \oplus 0 \oplus \mathcal{M}_2 \oplus 0 \quad \text{and}$$

$$\mathcal{K}_1 = 0 \oplus \mathcal{M}_5 \oplus 0 \oplus \mathcal{M}_3.$$

To see the second assertion of the theorem note that the first assertion of the theorem implies that it is enough to show that the minimal Brownian unitary extension of covariance σ for T is a minimal Brownian isometric extension of covariance σ for T . But since by Lemma 5.32 any reducing subspace for B containing $\mathcal{H}$ gives a Brownian unitary of covariance σ , this follows immediately. This concludes the proof of Theorem 9.5.

§10. Some remarks and open questions.

In this section, which is written much less formally than the rest of the paper, we shall make a number of comments regarding the intrinsic operator theory of m -isometries as well as the interplay of the theory with other areas of mathematics. Along the way a number of interesting open questions will arise.

When is Δ_T compact?

If T is a finitely cyclic 2-isometry, then Δ_T is compact (Proposition 1.24). Furthermore, the results of Section 6 indicate that the eigenfunctions of Δ_T are destined to play a significant role in the theory of 2-isometries.

Now, if $w = (w_0, \dots, w_{m-2})$ is an $m-1$ -tuple of L^∞ functions on $\partial\mathbb{D}$, then one can define a differential operator L_w by

$$L_w = \left(\frac{1}{i} \frac{d}{d\theta} \right)^{m-1} + \sum_{\ell=0}^{m-2} w_\ell \left(\frac{1}{i} \frac{d}{d\theta} \right)^\ell.$$

Furthermore, one may compress L_w to H^2 and obtain a Toeplitz operator in the sense of [M-G]

$$A_w = P_{H^2} L_w |_{H^2}.$$

Now, in the same manner as considered in the beginning of Section 6 for the case $m=2$, in the case where there exists a positive constant ϵ such that

$$A_w - \epsilon \geq 0$$

A_w may be regarded as an analytic Dirichlet operator and thus induces via Theorem 3.49 a unique m -isometry M_w . Furthermore, just as in Section 6, the formula of (6.6) still holds,

$$(10.1) \quad A_w^{-1} = \Delta_w.$$

Thus, by the spectral results of [M-G], Δ_T is compact for the wide and naturally occurring class of cyclic m -isometries that have the form $T = M_w$.

Unfortunately, that Δ_T is not in general compact even in the cyclic case can be seen from the following simple example of a cyclic 3-isometry. Let $c > 0$, $\{w_n\}_{n=1}^\infty$ be a square summable sequence of complex numbers and $\{\alpha_n\}$ be a sequence in $\partial\mathbb{D}$ which does not contain any of its accumulation points. Set

$$(10.2) \quad T = \bigoplus_{n=1}^{\infty} \begin{bmatrix} \alpha_n & c \\ 0 & \alpha_n \end{bmatrix} \in \mathcal{L} \left(\bigoplus_{n=1}^{\infty} \mathbb{C}^2 \right)$$

and

$$(10.3) \quad \gamma = \bigoplus_{n=1}^{\infty} \begin{bmatrix} 0 \\ w_n \end{bmatrix} \in \bigoplus_{n=1}^{\infty} \mathbb{C}^2.$$

Using (10.2) observe that

$$\frac{1}{2}(yx-1)^2(T) = \bigoplus_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & c^2 \end{bmatrix}$$

so that

$$(yx-1)^3(T) = 2 \left(T^* \frac{1}{2}(yx-1)^2(T)T - \frac{1}{2}(yx-1)^2(T) \right) = 0.$$

Thus T is a 3-isometry and $\Delta_T = \frac{1}{2}(yx-1)^2(T)$ is not compact. We now show that γ is a cyclic vector for T . First observe that $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is cyclic for $\begin{bmatrix} \alpha_n & c \\ 0 & \alpha_n \end{bmatrix}$. It, therefore, suffices to show that for each $j \geq 1$, there exist a sequence of polynomials $\{p_{\ell_j}\}_{\ell=1}^\infty$ such that

$$(10.4) \quad \lim_{\ell \rightarrow \infty} p_{\ell_j}(T)\gamma = \bigoplus_{n=1}^{\infty} \begin{bmatrix} 0 \\ \delta_{nj} \end{bmatrix}$$

where δ_{nj} is the Kronecker symbol. Using (10.2), and the facts that c and w_n are nonzero one sees that (10.4) is equivalent to

$$(10.5) \quad \begin{cases} \lim_{\ell \rightarrow \infty} p'_{\ell_j}(\alpha_n) = 0 \\ \lim_{\ell \rightarrow \infty} p_{\ell_j}(\alpha_n) = \delta_{nj} \end{cases} \quad (n \geq 1, \quad j \geq 1).$$

To see (10.5), fix $j \geq 1$. Let G_j be an open ball containing α_j and H_j be a simply connected open set containing α_n for $n \neq j$ such that $H_j \cap G_j = \emptyset$. By Runge's Theorem, there exist a sequence of polynomials $p_{\ell_j}(z)$ such that $p_{\ell_j}(z)$ converges uniformly to 1 on G_j and $p_{\ell_j}(z)$ converges uniformly to 0 on H_j . By Cauchy's Integral formula, $p'_{\ell_j}(z)$ converges uniformly to 0 on $G_j \cup H_j$. With this construction, (10.5) holds. In summary, we have constructed a cyclic 3-isometry T such that Δ_T is not compact.

We now prove a generalization of Proposition 1.24 which shows that the example just considered depended on the fact that 3 is odd.

Proposition 10.6. *If $T \in \mathcal{L}(\mathcal{H})$ is a finitely cyclic m -isometry and m is even, then Δ_T is compact.*

PROOF. We first fix some notation. Let $\mathcal{K}(\mathcal{H})$ be the collection of compact operators in $\mathcal{L}(\mathcal{H})$. $\mathcal{K}(\mathcal{H})$ is a norm-closed ideal in $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is a unital C^* -algebra [Ca]. The canonical map $\pi_0 : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is a unital $*$ -representation. Since $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is a unital C^* -algebra, there exist a Hilbert space $\mathcal{M}$ and an injective unital $*$ -representation $\pi : \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{M})$ [C].

Now, since T is finitely cyclic and, by Lemma 1.21, the approximate point spectrum lies in the unit circle, T is Fredholm. Thus $\pi_0(T)$ is invertible in the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and $\pi(\pi_0(T)) \in \mathcal{L}(\mathcal{M})$ is invertible. Since $\pi \circ \pi_0 : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{M})$ is a unital $*$ -representation and T is a m -isometry, $\pi(\pi_0(T))$ is an m -isometry. But m is even and so by Proposition 1.23, $\Delta_{\pi(\pi_0(T))} = 0$. Since

$$\pi(\pi_0(\Delta_T)) = \Delta_{\pi(\pi_0(T))} = 0$$

and π is injective, $\pi_0(\Delta_T) = 0$ and so Δ_T is compact. This completes the proof of Proposition 10.6.

When Does an m -Isometry Possess a DTO of a Given Form?

Let $T \in \mathcal{L}(\mathcal{H})$ be an n -cyclic m -isometry. Are there purely operator theoretic characterizations for the existence of an n -cyclic tuple γ for T such that $(T, \gamma)^\wedge$

- (a) is smooth?
- (b) is regular?
- (c) is in Wiener-Hopf form?
- (d) is in Wiener-Hopf form with coefficients in $(L^\infty)^{n,n}$?

Recall that “smooth”, “regular”, and “ DTO in W-H form” were defined in Section 4 where it was shown in Theorem 4.18 that (b) implies (c). The above four questions are all open even in the case where $n = 1$ and $m = 2$. Though it seems entirely possible that the four notions represented in questions (a) - (d) do not correspond to any natural operator theoretic notions perhaps some modification of them does. Also observe that Proposition 6.18 implies that if (d) holds, then T is similar to the Dirichlet shift. Is the converse true?

What Operators Play the Role for m -Isometries that Brownian Unitaries Play for 2-Isometries?

We recall from Section 5 that the Brownian shifts were characterized in three distinct ways: via statistics, via operator theory, and via distribution theory. First, there was the characterization of $B_{\sigma, e^{i\theta}}$ as the time shift on the L^2 space of a scaled Brownian motion (Proposition 5.2). Second, there was the purely operator theoretic characterization as the pure 2-isometries with rank $\Delta_T = 1$ (Proposition 5.6). Third, there was the characterization on the level of DTO as those cyclic 2-isometries with the leading coefficient of $(T, \gamma)^\wedge$ a point mass (Proposition 5.10). Now, the Brownian unitaries, which were introduced as the

direct integrals in θ of Brownian shifts of constant covariance, themselves had two further characterizations one spatial (Proposition 5.12) and one C^* -algebraic (Theorem 5.20). In addition, if one regards the Brownian unitaries as given, then yet a fourth characterization of Brownian shifts is that they are the irreducible Brownian unitaries. Finally, one has via Theorem 5.99 that the Brownian unitaries of covariance σ are precisely the smallest collection of the 2-isometries of covariance $\leq \sigma$ that is closed with respect to restrictions to reducing subspaces and in addition has the property that every 2-isometry of covariance $\leq \sigma$ has an extension to an element of the subcollection. (i.e., Theorem 5.80 is true). Summarizing, we see that the Brownian shifts are a natural class of 2-isometries from four distinct points of view.

It would certainly be of interest to work out any aspect of the above notions to the cases when $m > 2$. In the invertible case (which can only occur when m is odd) one has that the Jordan operators $J = U + Q$ (i.e., U unitary, $Q^{\frac{m+1}{2}} = 0$, $UQ = QU$) play a significant role. Scott McCullough in his thesis [M1] made a deep and serious study of the lifting structure of 3-isometries. Roughly, his result can be described as follows.

Let $\mathcal{F}_\sigma$ denote the collection of 3-isometries that satisfy a certain normalization depending on a parameter σ (this is analogous to considering the 2-isometries of covariance $\leq \sigma$). In attempting to develop an operator theoretic model theory for $\mathcal{F}_\sigma$ one is naturally led to consider subcollections $\mathcal{B} \subseteq \mathcal{F}_\sigma$ such that every element $T \in \mathcal{F}_\sigma$ has an extension to an element of $\mathcal{B}$ and in addition satisfies (5.89). We refer to such subcollections $\mathcal{B}$ as boundaries for $\mathcal{F}_\sigma$ [Ag3]. One way to measure the complexity of a given boundary would be through the complexity of the irreducible elements in that boundary. For example, Theorem 5.80 asserts that the Brownian unitaries of covariance σ are a boundary for the 2-isometries of covariance $\leq \sigma$ and the irreducible element of this boundary are the rank one unitaries 1 parameter family of $[e^{i\theta}]$ and the 1 parameter family of Brownian shifts $B_{\sigma, e^{i\theta}}$. McCullough showed that such simplicity can never be achieved for any boundary for $\mathcal{F}_\sigma$. Specifically, he showed that if $\mathcal{B}$ is a boundary for $\mathcal{F}_\sigma$, then there is a distinct irreducible element $B_\varphi \in \mathcal{B}$ for each smooth automorphism φ of the circle.

How does Disconjugacy Theory Work for $m > 3$?

One way to attack the questions raised in the previous question would be to generalize the results of Section 6 to Toeplitz operators of order > 1 and then to employ the ideas in Section 7. Unfortunately, the immediate attempt to do this fails as the following example illustrates.

Let A_w be defined by

$$\begin{aligned} \text{dom}(A_w) &= \{f \in H^2 : f'' \in H^2\} \\ A_w f &= D^2 f \quad . \end{aligned}$$

Then $A_w \geq 0$ and $\dim \ker A_w = 2$. Thus, Lemma 6.7, Lemma 6.16, and Proposition 6.17 are all false for $m > 1$.

What is the Structure of the Minimal Brownian Isometric Extension of a given 2-Isometry?

In Section 9 we introduced the notion of a minimal Brownian isometric extension of covariance σ for a given 2-isometry T of covariance σ . Such extensions were required to be minimal with respect to **reducing** subspaces. While this is the most obvious way to define minimality in this context it turns out that the following definition though less obvious is more natural.

DEFINITION 10.7. Let $T \in \mathcal{L}(\mathcal{H})$ be a 2-isometry of covariance σ , let $\mathcal{K} \supseteq \mathcal{H}$, and let $B \in \mathcal{L}(\mathcal{K})$ be a Brownian isometric extension of covariance σ for T . B is a *minimal Brownian isometric extension of covariance σ for T with respect to invariant subspaces* if $\mathcal{K}$ is the smallest invariant subspace for B containing $\mathcal{H}$ such that $B|_{\mathcal{K}}$ is a Brownian isometry.

The reason that we say Definition 10.7 is less obvious than the definition of minimality considered in Section 9 is based on the observation that if $B \in \mathcal{L}(\mathcal{H})$, B is a Brownian isometry, and $\mathcal{M} \subseteq \mathcal{H}$ is invariant for B , then $B|_{\mathcal{M}}$ is not necessarily a Brownian isometry. Thus, while Corollary 5.57 gives an immediate proof via Zorn's Lemma that minimal Brownian isometric extensions with respect to **reducing** subspaces always exist, the existence of minimal Brownian isometric extensions with respect to **invariant** subspaces is not immediately clear (cf., the sentence preceding Lemma 10.8 below).

Two facts are, however, immediately clear. Firstly, Lemma 5.90 guarantees that the notion of minimal Brownian **unitary** extension of covariance σ remains unchanged if one replaces reducing subspaces with subspaces assumed only to be invariant (and on which the extension is a Brownian unitary). Secondly, Theorem 9.5 guarantees that the two notions of minimal Brownian isometric extension of covariance σ for T agree in the case when T is cyclic.

In general, however, the two notions do not agree and, indeed, the notion involving reducing subspaces is somewhat unnatural as can be seen by the following multicyclic example. Fix $\theta \in \mathbb{R}$ and positive constants $\sigma_0 \leq \sigma_1$. Let

$$B = \begin{bmatrix} B_{\sigma_1, e^{i\theta}} & 0 & 0 \\ 0 & S & \sigma_0 \otimes 1 \\ 0 & 0 & e^{i\theta} \end{bmatrix}$$

acting on $(H^2 \oplus \mathbb{C}) \oplus H^2 \oplus \mathbb{C}$ where $B_{\sigma_1, e^{i\theta}}$ is a Brownian shift, S is the shift operator acting on H^2 and $\sigma_0 \otimes 1 : \mathbb{C} \rightarrow H^2$ is defined by $(\sigma_0 \otimes 1)(\lambda)(z) = \sigma_0 \lambda$, $z \in \mathbb{D}$. Note first that $B|_{(\{0\} \oplus H^2 \oplus \mathbb{C})} = B_{\sigma_0, e^{i\theta}}$ and hence B is a Brownian isometry of covariance σ_1 . Hence if we set $T = B|_{(H^2 \oplus \mathbb{C}) \oplus H^2 \oplus \{0\}}$, then T is a 2-isometry of covariance σ_1 and B is a Brownian isometric extension of covariance σ_1 for T . Since $T \cong B_{\sigma_1, e^{i\theta}} \oplus S$, T is itself a Brownian isometry of covariance σ_1 and hence B cannot possibly be a minimal extension

with respect to invariant subspaces. On the other hand, since $(H^2 \oplus \mathbb{C}) \oplus H^2 \oplus \{0\}$ has codimension 1 and σ_0 is assumed nonzero, there are no reducing subspaces for B containing $(H^2 \oplus \mathbb{C}) \oplus H^2 \oplus \{0\}$, so that B is minimal with respect to reducing subspaces. Hence the two notions of minimal extension do not agree in the multicyclic case. In addition, the unnaturalness of the notion of minimal extension with respect to **reducing** subspaces is revealed by our example by observing that T is a Brownian isometry and yet possesses the **proper** minimal Brownian isometric extension B !

The existence of minimal Brownian isometric extensions of covariance σ with respect to invariant subspaces for a general 2-isometry of covariance σ follows by a simple application of Zorn's Lemma, Theorem 5.80 and the following lemma.

Lemma 10.8. *Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{L}(\mathcal{H})$ be a Brownian isometry. If A is a totally ordered set and $\{\mathcal{H}_\alpha : \alpha \in A\}$ is a chain of invariant subspaces for T ordered by reverse inclusion (i.e., $\alpha \leq \beta \implies \mathcal{H}_\alpha \supseteq \mathcal{H}_\beta$) such that $T|_{\mathcal{H}_\alpha}$ is a Brownian isometry for each α , then $T|_{\bigcap_{\alpha \in A} \mathcal{H}_\alpha}$ is a Brownian isometry.*

PROOF. Let $\mathcal{H}, T, A$, and $\mathcal{H}_\alpha$ be as in the statement of the lemma. Set $T_\alpha = T|_{\mathcal{H}_\alpha}$, $\mathcal{H}_\infty = \bigcap_{\alpha \in A} \mathcal{H}_\alpha$, and $T_\infty = T|_{\mathcal{H}_\infty}$. We need to show that T_∞ is a Brownian isometry.

Let $P_\alpha \in \mathcal{L}(\mathcal{H})$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_\alpha$ and let P_∞ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_\infty$. Since $\mathcal{H}_\infty = \bigcap_{\alpha \in A} \mathcal{H}_\alpha$ and $\alpha \leq \beta \implies \mathcal{H}_\alpha \supseteq \mathcal{H}_\beta$, it is clear that $P_\alpha \longrightarrow P_\infty$ in the strong operator topology.

Set $\mathcal{M}_\alpha = \ker \Delta_{T_\alpha}$ and $\mathcal{M}_\infty = \ker \Delta_{T_\infty}$. Evidently, since $\mathcal{H}_\alpha$ is invariant for T ,

$$\begin{aligned} \mathcal{M}_\alpha &= \ker \Delta_{T_\alpha} \\ &= \{x \in \mathcal{H}_\alpha : \|T_\alpha x\| = \|x\|\} \\ &= \{x \in \mathcal{H}_\alpha : \|Tx\| = \|x\|\} \\ &= \mathcal{H}_\alpha \cap \ker \Delta_T \end{aligned}$$

Similarly, $\mathcal{M}_\infty = \mathcal{H}_\infty \cap \ker \Delta_T$. Thus,

$$\begin{aligned} \mathcal{M}_\infty &= \mathcal{H}_\infty \cap \ker \Delta_T \\ &= \left(\bigcap_{\alpha \in A} \mathcal{H}_\alpha \right) \cap \ker \Delta_T \\ &= \bigcap_{\alpha \in A} (\mathcal{H}_\alpha \cap \ker \Delta_T) \\ &= \bigcap_{\alpha \in A} \mathcal{M}_\alpha \quad , \end{aligned}$$

so that if Q_α denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}_\alpha$ and Q_∞ denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}_\infty$, then $Q_\alpha \longrightarrow Q_\infty$ in the strong operator topology.

Now, T_α is a Brownian isometry for each $\alpha \in \mathcal{A}$. Hence, by Theorem 5.48,

$$(10.9) \quad Q_\alpha (TP_\alpha T^* - P_\alpha) Q_\alpha = 0$$

for all $\alpha \in \mathcal{A}$. Since $\{P_\alpha\}$ and $\{Q_\alpha\}$ are bounded, and $P_\alpha \longrightarrow P_\infty$ and $Q_\alpha \longrightarrow Q_\infty$ in the strong operator topology we deduce from (10.9) that

$$Q_\infty (TP_\infty T^* - P_\infty) Q_\infty = 0 \quad .$$

Hence by another application of theorem 5.48, T_∞ is a Brownian isometry and the proof of Lemma 10.8 is complete.

m-symmetries, *m*-isometries and the Cayley Transform

Since there is a well understood correspondence between symmetric unbounded operators and partial isometries, we now explore a correspondence between *m*-symmetries, which we shall define below, and *m*-isometries. We will not explore (and encourage others to explore) the notion of partial *m*-isometries.

We shall use the notation introduced in [C] pertaining to unbounded operators.

If T is an unbounded operator from $\text{dom}(T) \subseteq \mathcal{H}$ into $\mathcal{H}$ and $n \geq 1$, then we shall say that T is *n*-symmetric if

$$(10.10) \quad \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \langle T^j h, T^{n-j} k \rangle = 0$$

for every $h, k \in \text{dom}(T^n)$ and that T is an *n*-isometry if

$$(10.11) \quad \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \langle T^j h, T^j k \rangle = 0$$

for every $h, k \in \text{dom}(T^n)$.

The following lemma describes the domain of a power of a linear transformation of a possibly unbounded operator.

Lemma 10.12. *Let $\mathcal{H}$ be a Hilbert space, $a \in \mathbb{C}$, $b \in \mathbb{C}$ and T a possibly unbounded operator on $\mathcal{H}$. Suppose that $a \neq b$, $\ker(T + a) = \{0\}$ and $\ker(T + b) = \{0\}$. If R is the possibly unbounded operator on $\mathcal{H}$ whose domain is $\text{ran}(T + a)$ and such that*

$$R(T + a)h = (T + b)h \in \text{dom}(T),$$

then the domain of R^n is $\text{ran}(T + a)^n$.

PROOF. Since $\ker(T + a) = \{0\}$, R is well-defined.

We first show that for each $n \geq 1$, $\text{ran}(T + b)^n \cap \text{ran}(T + a) = \text{ran}(T + b)^n(T + a)$.

Clearly, $\text{ran}(T+b)^n(T+a) \subseteq \text{ran}(T+b)^n \cap \text{ran}(T+a)$. Let $h \in \text{ran}(T+b)^n \cap \text{ran}(T+a)$. There exist $k \in \text{dom}(T^n)$ and $l \in \text{dom}(T)$ such that $h = (T+b)^n k = (T+a)l$. Therefore,

$$(10.13) \quad l = \frac{1}{a-b} ((T+b)^n k - (T+b)l).$$

Now, (10.13) implies that both $l \in \text{ran}(T+b)$ and that if $l \in \text{ran}(T+b)^j$ and $j < n$, then $l \in \text{ran}(T+b)^{j+1}$. Therefore, $l \in \text{ran}(T+b)^n$ and $h = (T+a)l \in \text{ran}(T+b)^n(T+a)$. This completes the proof of our claim that $\text{ran}(T+b)^n \cap \text{ran}(T+a) = \text{ran}(T+b)^n(T+a)$.

Now, clearly, $\text{ran}(T+a)^n \subseteq \text{dom}(R^n)$ for each $n \geq 1$ and equality holds for $n = 1$ by hypothesis. We now show that equality holds for all $n \geq 1$ by induction. Toward that end, suppose that $\text{dom}(R^n) = \text{ran}(T+a)^n$ and $h \in \text{dom}(R^{n+1})$. Since $h \in \text{dom}(R^{n+1}) \subseteq \text{dom}(R^n) = \text{ran}(T+a)^n$, there exists $k \in \text{dom}(T^n)$ such that $h = (T+a)^n k$. Therefore, $(T+b)^n k = R^n k \in \text{dom}(R) = \text{ran}(T+a)$. Therefore, $(T+b)^n k \in \text{ran}(T+b)^n \cap \text{ran}(T+a) = \text{ran}(T+b)^n(T+a)$. Let $l \in \text{dom}(T^{n+1})$ be such that $(T+b)^n k = (T+b)^n(T+a)l$. Since $\ker(T+b) = \{0\}$, we see that $k = (T+a)l$ and so $h = (T+a)^n k = (T+a)^{n+1}l$. Therefore, $\text{dom}(R^{n+1}) \subseteq \text{ran}(T+a)^{n+1}$. This completes the proof of Lemma 10.12.

The following lemma will allow us to quickly compute formulas involving the Cayley transform.

Lemma 10.14. *Let T is a possibly unbounded operator on $\mathcal{H}$, $n \geq 1$, $h \in \text{dom}(T^n)$ and $k \in \text{dom}(T^n)$. If $N_1 > 0$, $N_2 > 0$, $p_1, \dots, p_{N_1}, r_1, \dots, r_{N_2} \in \mathbf{C}[x]$, $q_1, \dots, q_{N_1}, s_1, \dots, s_{N_2} \in \mathbf{C}[y]$ and*

$$(10.15) \quad \sum_{j=1}^{N_1} p_j(x) \overline{q_j}(y) = \sum_{j=1}^{N_2} r_j(x) \overline{s_j}(y)$$

where for each $q \in \mathbf{C}[y]$, $\overline{q}$ is a member of $\mathbf{C}[y]$ and is given by the formula $\overline{q}(y) = \overline{q(\overline{y})}$, then

$$(10.16) \quad \sum_{j=1}^{N_1} \langle p_j(T)h, q_j(T)k \rangle = \sum_{j=1}^{N_2} \langle r_j(T)h, s_j(T)k \rangle.$$

PROOF. Let V_1 be the complex vector space of polynomials in x of degree $\leq n$ and V_2 be the complex vector space of polynomials in y of degree $\leq n$. Since the map $f : V_1 \times V_2 \rightarrow \mathbf{C}$ given by

$$f(p, \overline{q}) = \langle p(T)h, q(T)k \rangle$$

(where, of course, $p(T)$ and $q(T)$ are as described by the Reisz Functional Calculus) is bilinear, there exists a well-defined map $F : V_1 \otimes_{\mathbf{C}} V_2 \rightarrow \mathbf{C}$ such that $F(p \otimes \overline{q}) = f(p, \overline{q})$.

Since $V_1 \otimes_{\mathbb{C}} V_2$ is naturally isomorphic to $\text{span}\{x^r y^s : r \leq n \text{ and } s \leq n\}$, the fact that (10.15) implies (10.16) follows from the fact that F is well-defined. This completes the proof of Lemma 10.14.

The following proposition gives the correspondence between m -isometries and m -symmetries.

Proposition 10.17. *Let $n \geq 1$, $\mathcal{H}$ be a Hilbert space and both T and R be possibly unbounded operators on $\mathcal{H}$.*

(a) *If $\ker(T + i) = \{0\}$, $\ker(T - i) = \{0\}$, $\text{dom}(R) = \text{ran}(T + i)$ and R is defined by*

$$(10.18) \quad R(T + i)h = (T - i)h \quad h \in \text{dom}(T),$$

then for all $h, k \in \text{dom}(T^n)$,

$$(10.19) \quad \begin{aligned} & \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \langle R^j (T + i)^n h, R^j (T + i)^n k \rangle \\ &= (2i)^n \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \langle T^j h, T^{n-j} k \rangle. \end{aligned}$$

In particular, if T is n -symmetric and R is defined as in (10.18), then R is an m -isometry.

(b) *If $\ker(R - 1) = \{0\}$, $\ker(R + 1) = \{0\}$, $\text{dom}(T) = \text{ran}(1 - R)$ and T is defined by*

$$(10.20) \quad T(1 - R)h = i(1 + R)h \quad h \in \text{dom}(R),$$

then for all $h, k \in \text{dom}(R^n)$,

$$(10.21) \quad \begin{aligned} & \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \langle T^j (1 - R)^n h, T^{n-j} (1 - R)^n k \rangle \\ &= (-2i)^n \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \langle R^j h, R^j k \rangle. \end{aligned}$$

In particular, if R is an n -isometry and neither 1 nor -1 are eigenvalues of R , then T is n -symmetric.

(c) *If T is n -symmetric and R is defined by (10.18), then R and T satisfy (10.20). If T is an n -isometry, neither 1 nor -1 are eigenvalues of R and T is defined by (10.20), then R and T satisfy (10.18).*

PROOF.

(a) Suppose that T and R are as given in the hypothesis to (a). If $h, k \in \text{dom}(T^n)$, then

$$\langle R^j (T + i)^n h, R^j (T + i)^n k \rangle = \langle (T - i)^j (T + i)^{n-j} h, (T - i)^j (T + i)^{n-j} k \rangle.$$

Now,

$$\begin{aligned}
& \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} (x-i)^j (x+i)^{n-j} (y+i)^j (y-i)^{n-j} \\
&= [(x-i)(y+i) - (x+i)(y-i)]^n \\
&= (2i(x-y))^n = (2i)^n \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} x^j y^{n-j}
\end{aligned}$$

and so by Lemma 10.14, (10.19) holds.

Now, if T is m -symmetric, then $\ker(T+i) = \{0\}$, $\ker(T-i) = \{0\}$ and the right hand side of (10.19) is 0 for any $h, k \in \text{dom}(T^n)$. Therefore, R is n -symmetric.

(b) Suppose that T and R are given in the the hypothesis to (b). If $h, k \in \text{dom}(R^n)$, then

$$\langle T^j(1-R)^n h, T^{n-j}(1-R)^n k \rangle = \langle i^j(1+R)^j(1-R)^{n-j} h, i^{n-j}(1+R)^{n-j}(1-R)^j k \rangle.$$

Now,

$$\begin{aligned}
& \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} i^j (-i)^{n-j} (1+x)^j (1-x)^{n-j} (1+y)^{n-j} (1-y)^j \\
&= i^n \sum_{j=0}^n \binom{n}{j} (1+x)^j (1-y)^j (1-x)^{n-j} (1+y)^{n-j} \\
&= i^n ((1+x)(1-y) + (1-x)(1+y))^n \\
&= i^n (2 - 2xy)^n \\
&= (-2i)^n \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} x^j y^j
\end{aligned}$$

and so by Lemma 10.14, (10.20) holds.

If R is an n -isometry and neither 1 nor -1 are eigenvalues of R , then the right hand side of (10.21) is zero for all $h, k \in \text{dom}(R^n)$ and so T is n -symmetric.

(c) The proof of (c) is the same as that of Theorem 3.1 (c) of [C, X.3]. This completes the proof of Proposition 10.17.

We now turn to a collection of concrete examples of unbounded 2-symmetric operators and show that the Cayley transforms of certain 2-symmetrics are Brownian shifts.

Lemma 10.22. *If A is a densely defined symmetric operator on $\mathcal{H}$, $f \in \mathcal{H}$, $\|f\| = 1$, $t \in \mathbf{R}$, $f \in \ker(A^* - \lambda)$ and $\text{dom}(T^2) \not\subseteq (\mathbb{C}f)^\perp$, then $A + itf \otimes f$ is 2-symmetric if and only if $\text{Im}(\lambda) = t$. Furthermore, if the spectrum of A is the upper half plane, $\text{Im}(\lambda) = t$ and R is defined as in (10.18), then R is a bounded 2-isometry and*

$$(10.23) \quad R^* R - 1 = \frac{-4t}{|\lambda - i(t+1)|^2} f \otimes f.$$

PROOF. Let $T = A + itf \otimes f$. Since $A \subseteq A^*$, $\text{dom}(T^*) = \text{dom}(A^*)$, $T^* = A^* - itf \otimes f$ and $T^*h = Th - 2it\langle h, f \rangle f$ for every $h \in \text{dom}(T)$. Now for $h, k \in \text{dom}(T^2)$,

$$\begin{aligned}\langle T^2h, k \rangle &= \langle Th, T^*k \rangle \\ &= \langle Th, Tk - 2it\langle k, f \rangle f \rangle \\ &= \langle Th, Tk \rangle + 2it\langle f, k \rangle \langle Th, f \rangle \\ &= \langle Th, Tk \rangle + 2it\langle f, k \rangle \langle h, T^*f \rangle \\ &= \langle Th, Tk \rangle + 2it\langle f, k \rangle \langle h, (\lambda - it)f \rangle \\ &= \langle Th, Tk \rangle + 2it(\bar{\lambda} + it)\langle f, k \rangle \langle h, f \rangle\end{aligned}$$

and

$$\langle h, T^2k \rangle = \langle Th, Tk \rangle - 2it(\lambda - it)\langle f, k \rangle \langle h, f \rangle.$$

Therefore,

$$\begin{aligned}\langle T^2h, k \rangle - 2\langle Th, Tk \rangle + \langle h, T^2k \rangle \\ = 2it\langle f, k \rangle \langle h, f \rangle (\bar{\lambda} + it - \lambda + it).\end{aligned}$$

Since $\text{dom}(T^2) \not\subseteq (\mathbf{C}f)^\perp$, T is 2-symmetric if and only if $\text{Im}(\lambda) = t$.

Now, suppose that the spectrum of A is the closed upper half plane, $\text{Im}(\lambda) = t$ and R is defined as in (10.18). We first show that $\text{ran}(T - i) = \mathcal{H}$. Since $\text{ran}(A + i) = \mathcal{H}$ and $T = A + itf \otimes f$, it suffices to show that $f \in \text{ran}(T + i)$. Let $g \in \mathcal{H}$ be such that $(A + i)g = f$. Now,

$$\begin{aligned}(\bar{\lambda} + i)\langle g, f \rangle &= \langle g, (\lambda - i)f \rangle \\ &= \langle g, (A^* - i)f \rangle \\ &= \langle (A + i)g, f \rangle \\ &= \|f\|^2 \\ &= 1.\end{aligned}$$

Therefore, $\langle g, f \rangle = (\bar{\lambda} + i)^{-1}$. Now,

$$\begin{aligned}(T + i)g &= (A + i)g + (itf \otimes f)g \\ &= f + it\langle g, f \rangle f \\ &= (1 + it(\bar{\lambda} + i)^{-1})f \\ &= (\bar{\lambda} + i)^{-1}(\bar{\lambda} + i + it)f.\end{aligned}$$

Since $\text{Im}(\lambda) = t$, $(T + i)g \neq 0$ and so $f \in \text{ran}(T + i)$. Thus $\text{ran}(T - i) = \mathcal{H}$.

Since A is closed, T is closed. Now to see that the graph of R is closed, note that if $(T + i)h_n \rightarrow h$ and $(T - i)h_n \rightarrow k$, then

$$\begin{aligned}h_n &\longrightarrow \frac{1}{2i}(h - k) \\ Th_n &\longrightarrow \frac{h + k}{2}.\end{aligned}$$

Since T is closed,

$$\frac{1}{2i}T(h-k) = \frac{h+k}{2}.$$

But then

$$(T+i)\left(\frac{1}{2i}(h-k)\right) = h$$

and

$$(T-i)\left(\frac{1}{2i}(h-k)\right) = k.$$

Therefore, the graph of R is closed.

Since R is defined on all of $\mathcal{H}$ and R is closed, R is a bounded operator. Since T is 2-symmetric, Proposition 10.17 implies that R is a 2-isometry. To compute $R^*R - 1$, first note that

$$\begin{aligned} (T^* - i)f &= A^*f - itf - if \\ &= (\lambda - i(t+1))f. \end{aligned}$$

Since $\text{Im}(\lambda) = t$, $\lambda - i(t+1) \neq 0$. Let $c = \lambda - i(t+1)$. Now for $h \in \text{dom}(T)$, $Th = Ah + it\langle h, f \rangle f$ and so by Proposition 10.17,

$$\begin{aligned} \|R(T+i)h\|^2 - \|(T+i)h\|^2 &= (2i)(\langle Th, h \rangle - \langle h, Th \rangle) \\ &= (-4t)|\langle h, f \rangle|^2 \\ &= (-4t) \left| \left\langle h, (T^* - i)\frac{f}{c} \right\rangle \right|^2 \\ &= \frac{-4t}{|c|^2} |\langle (T+i)h, f \rangle|^2 \\ &= \left\langle \left(\frac{-4t}{|c|^2} f \otimes f \right) (T+i)h, (T+i)h \right\rangle. \end{aligned}$$

Therefore, (10.23) holds. This completes the proof of Lemma 10.22.

Now, if one picks A and f as in Lemma 10.22 such that A cannot be expressed as the direct sum of symmetric operators, then R will be a Brownian shift. In particular, if $Ag = ig'$ for f in $\{g \in L^2(0, \infty) : \text{for all } c > 0, g \text{ is absolutely continuous on } [0, c], g(0) = 0 \text{ and } g' \in L^2(0, \infty)\}[\mathbf{C}]$, then R will be a rank one perturbation of the unilateral shift,

$$u \rightarrow u - 2e^{-x} \int_0^\infty e^s u(s) ds.$$

Where is the function theory?

There is a rich and remarkable interplay between certain basic operator theoretic questions about the unilateral shift and the function theory on H^2 . This structure was

originally discovered by Beurling [B] in two stunningly beautiful theorems that characterized both the cyclic vectors and the invariant subspace lattice for the unilateral shift. Excepting the spectral theorem and the Sz.-Nagy Dilation theorem, this seminal work of Beurling has arguably led to more mathematics than any other basic discoveries from the infancy of operator theory.

In the case of 2-isometries, it could be imagined that to solve questions pertaining to cyclicity and lattices that one could avail oneself of function theory. Here, however, it must be observed that if T is a pure cyclic 2-isometry, then except the case when T is finitely atomic (i.e., T is unitarily equivalent to M_z on $H^2_{\mu,\sigma}$ with μ supported on a finite number of points; equivalently, $T = B_{\sigma,U}$ where $\sigma(U)$ is finite; equivalently, T is a finite rank perturbation of the unilateral shift), there is no known case where function theoretic models for T solve **any** operator theoretic question about T . In particular, in the most studied case, $T = \text{Dirichlet shift}$ ($\mu = \frac{d\theta}{2\pi}$) the original questions of Beurling remain famous unsolved problems. Sarason showed how to analyze the lattice in the finitely atomic case [Sa]. Richter has the best results for the Dirichlet shift [R1].

One concrete operator theoretic question for which a pseudo-answer can be derived is to describe the commutant of a pure cyclic 2-isometry. Here, a routine argument identifies it with the space of multipliers of $H^2_{\mu,\sigma}$. For these spaces of multipliers, characterizations of 0-sets and more generally z -invariant subspaces (wk-* closed or not) would be of operator theoretic interest. One of the difficulties here is that unlike in the case of the unilateral unweighted shift where the space of multipliers is H^∞ and has a simple concrete function theoretic characterization, the function theoretic characterization of the multipliers of $H^2_{\mu,\sigma}$ is in large part unstudied. A deep result here is Stegenga's characterization of the multipliers of the Dirichlet space using a capacity condition [Steg].

The Prediction Problem for 2-Stationary Processes

If $X = \{X_n : n \geq 0\}$ is a discrete unilateral stationary Gaussian process, then one form of the prediction problem is to give the best possible estimate of the present (i.e., X_0) given the future (i.e., $\{X_1, X_2, \dots\}$). In the context of least squares estimation and Gaussian processes this is the problem of determining the orthogonal projection of X_0 on the span of $\{X_n : n \geq 1\}$ in the L^2 space of the process. As is well known the *mathematical* part of this problem is elegantly and definitively treated by the circle of ideas surrounding the Kolmogorov-Szegő-Krein Theorem and the Wiener-Hopf Factorization Theory.

Specifically, the facts that the process is stationary and the covariance is positive semidefinite imply via Bochner's Theorem (or, via many other ways as well; e.g., Herglotz) the existence of a positive measure μ on the unit circle in the complex plane with the property that the L^2 -space of the process can be identified with $H^2(\mu)$, the closure of the analytic polynomials in $L^2(\mu)$, and in this identification, X_n is identified with z^n and the time shift operator is identified with the operator M_z .

With this setup, the famous theorem of Kolmogorov, Szegő, and Krein asserts that

naturally,

$$(10.24) \quad H^2(\mu) = H^2(\mu_s) \oplus H^2\left(w \frac{d\theta}{2\pi}\right)$$

where $\mu = \mu_s + w \frac{d\theta}{2\pi}$ is the Lebesgue Decomposition of μ . Furthermore, $H^2(\mu_s) = L^2(\mu_s)$ and $H^2\left(w \frac{d\theta}{2\pi}\right) = L^2\left(w \frac{d\theta}{2\pi}\right)$ if and only if

$$\int \ln w = -\infty .$$

Finally, if

$$\int \ln w > -\infty ,$$

then $H^2\left(w \frac{d\theta}{2\pi}\right) = \frac{1}{f} H^2$ where f is the outer factorization of w (i.e., f is outer and $|f|^2 = w$ a.e. on $\partial\mathbb{D}$). In particular, $H^2\left(w \frac{d\theta}{2\pi}\right)$ consists of analytic functions on $\mathbb{D}$ and has reproducing kernel k_λ^w given by

$$k_\lambda^w(z) = \frac{1}{f(\lambda)f(z)} \frac{1}{1 - \bar{\lambda}z} .$$

Since the prediction error E is given by the formula

$$E^2 = \text{dist}_{H^2(\mu)}(1, \text{span}\{z, z^2, \dots\})^2$$

these facts are easily seen to imply that $E = 0$ if and only if $\int \ln w = -\infty$ and otherwise

$$\begin{aligned} E^2 &= \left\| P_{(\text{span}\{z, z^2, \dots\})^\perp} 1 \right\|^2 \\ &= \left\| \langle 1, k_0^w \rangle \frac{k_0^w}{\|k_0^w\|^2} \right\|^2 \\ &= \frac{1}{k_0^w(0)} \\ &= |f(0)|^2 \\ &= e^{\int_0^{2\pi} \ln w \frac{d\theta}{2\pi}} . \end{aligned}$$

Now, one can approach the prediction problem for 2-stationary processes using the circle of ideas just explicated. Our first observations that the meat of equation (10.24) is not in the fact that there is a splitting of $H^2(\mu)$; this after all is implied by the von Neumann Decomposition. Rather, it is that the decomposition arises from the Lebesgue Decomposition. Then in lieu of Lemma 9.2 the question is: if one is given a positive DTO

A of order 1, how can one concretely identify the maximal positive measure μ such that $A - \mu$ is positive and

$$H_A^2 = H_\mu^2 \oplus H_{A-\mu}^2?$$

(Here, note that necessarily, $H_\mu^2 = L^2(\mu)$).

We next remark that if T is a pure 2-isometry, T has a model as $M_{e^{i\theta}}$ on H_A^2 , and H_A^2 is a space of analytic function then as before the formula for the prediction error is given by

$$E^2 = \frac{1}{k_0(0)}$$

where k_λ is the reproducing kernel for H_A^2 . In the case where A has already been filtered into W-H form (i.e., $A = A_w, w \in L_{\mathbb{R}}^\infty$ as in (6.2)), then $k_\lambda(e^{i\theta})$ can be computed using (6.5) to yield

$$k_\lambda = A_w^{-1} \left(\frac{1}{1 - \bar{\lambda}e^{i\theta}} \right) = \Delta_w \left(\frac{1}{1 - \bar{\lambda}e^{i\theta}} \right).$$

In particular,

$$E^2 = \frac{1}{\langle \Delta_w 1, 1 \rangle}$$

and the problem of computing the error reduces to solving the nonhomogenous Toeplitz equation

$$A_w f = 1$$

There does not seem to be a highly evolved literature on the numerics of this equation.

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