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PARTITIONING GENERAL PROBABILITY MEASURES[1]

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Suppose $\mu_1, \dots, \mu_n$ are probability measures on the same measurable space $(\Omega, \mathscr{F})$. Then if all atoms of each μ_i have mass α or less, there is a measurable partition $A_1, \dots, A_n$ of Ω so that $\mu_i(A_i) \geq V_n(\alpha)$ for all $i = 1, \dots, n$, where $V_n(\cdot)$ is an explicitly given piecewise linear nonincreasing continuous function on $[0, 1]$. Moreover, the bound $V_n(\alpha)$ is attained for all n and all α. Applications are given to L_1 spaces, to statistical decision theory, and to the classical nonatomic case.

1. Introduction. The underlying space of any nonatomic probability measure may always be partitioned into n measurable subsets each having measure exactly $1/n$. More generally, if there are k nonatomic probability measures on the same space, Neyman [6] showed there is a measurable partition of the space into n subsets so that *each* probability assigns measure exactly $1/n$ to each subset, thereby solving Fisher's "Problem of the Nile" [4]. In the case of n continuous probability measures, Steinhaus, Banach and Knaster [7] gave a practical method for determining a partition into n sets with the property that the ith measure of the ith subset is *at least* $1/n$. Extensions of these results, many using Lyapounov's convexity theorem [5] ("the range of every nonatomic finite-dimensional, vector valued (finite) measure is convex (and compact)") and generalizations were obtained by Dvoretzky, Wald and Wolfowitz [2] and Dubins and Spanier [1].

In general, all of the above-mentioned results fail if the measures have atoms, and it is the purpose of this paper to determine some best possible partitioning bounds as a function of the maximum size of the atoms.

Throughout this paper $(\Omega, \mathscr{F}) = (\mathbb{R}, \text{Borels})$, but any measurable space admitting nonatomic probability measures will do; this particular choice is mainly for notational convenience since a measure μ on $(\mathbb{R}, \text{Borels})$ is nonatomic if and only if $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$.

DEFINITION. For each $\alpha \in [0, 1]$,

$$\mathscr{P}(\alpha) = \{\mu\colon \mu \text{ is a probability measure on } (\Omega, \mathscr{F}) \text{ with } \mu(\{x\}) \leq \alpha \text{ for all } x \in \Omega\}.$$

DEFINITION. $V_n\colon [0, 1] \to [0, n^{-1}]$ is the unique nonincreasing function (see Figure 1) satisfying

$$V_n(\alpha) = 1 - k(n-1)\alpha \tag{1}$$

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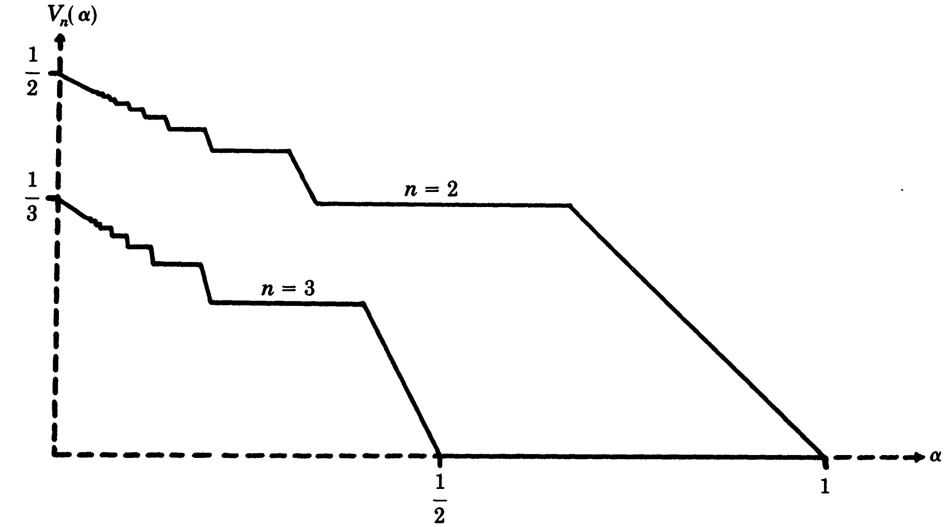

FIG. 1. *Graphs of V_2 and V_3.*

for

$$\alpha \in \left[(k+1)k^{-1}((k+1)n-1)^{-1}, (kn-1)^{-1}\right],$$

for all $k \geq 1$.

The main results of this paper are the following two closely related theorems.

THEOREM 1.1. *Let $\mu \in \mathscr{P}(\alpha)$. Then for each $n > 1$ there exists a measurable partition $\{A_i\}_{i=1}^n$ of Ω satisfying*

$$\mu(A_i) \geq V_n(\alpha), \quad \textit{for all } i = 1, \ldots, n; \tag{2}$$

moreover, V_n is the best possible bound in (2), *and is attained for all α.*

THEOREM 1.2. *Let $\mu_1, \ldots, \mu_n \in \mathscr{P}(\alpha)$. Then there exists a measurable partition $\{A_i\}_{i=1}^n$ of Ω satisfying*

$$\mu_i(A_i) \geq V_n(\alpha), \quad \textit{for all } i = 1, \ldots, n; \tag{3}$$

again, V_n is the best possible bound in (3), *and is attained for all α.*

REMARK. Theorems 1.1 and 1.2 are "dual" in the following sense: the bound (2) in Theorem 1.1 follows from (3) of Theorem 1.2 by taking $\mu_1 = \cdots = \mu_n$, whereas the *sharpness* of the bound (3) in Theorem 1.2 follows similarly from the sharpness of (2) in Theorem 1.1.

A "cake-cutting" interpretation of Theorem 1.2 based on a description by Dubins and Spanier [1] is this. Suppose a cake Ω is to be divided among n

people whose values $\{\mu_i\}_{i=1}^n$ of different portions of the cake may differ [here $\mu_i(A)$ represents the value of piece A to person i]. Then if no one values any *crumb* (indivisible portion of the cake) more than α, the cake may divided so that each person receives a piece he himself values at least $V_n(\alpha)$, and in general it is not possible to do better.

EXAMPLE 1.3. Suppose three people must divide a cake, and each agrees that no crumb is worth more than 10^{-3} the value of the whole cake. Then there is a way of cutting the cake into three pieces, and giving each person a piece, in such a way that each person values his own piece at least $V_n(\alpha) = V_3(10^{-3}) = 83/250$ and in general it is not possible to do better.

(A similar interpretation of Theorem 1.1 is also possible. Suppose a cake of total volume (or weight) one is to be cut into n pieces so that the smallest piece has as large a volume as possible. If each atom (or molecule, or crumb, or other indivisible piece) has volume α or less, then in an optimal partitioning the smallest piece has volume at least $V_n(\alpha)$, and in general this is the best possible bound.)

Intuitively, it is clear that the nonatomic case is the limit of the general case as the maximum atom size approaches zero.

COROLLARY 1.4 ([1], [2], [7]). *Suppose $\mu_1, \dots, \mu_n$ are nonatomic measures on $(\Omega, \mathscr{F})$. Then there exists a measurable partition $\{A_i\}_{i=1}^n$ of Ω so that*

$$\mu_i(A_i) \geq n^{-1}, \quad \textit{for all } i = 1, \dots, n.$$

This paper is organized as follows: Section 2 contains the proof of Theorem 1.1; Section 3 the proof of Theorem 1.2; Section 4 further observations about the upper bound function $V_n(\alpha)$; and Section 5 contains several applications to L_1 function spaces and statistical decision theory.

2. Partitioning a single probability measure.

The main objective of this section is to prove Theorem 1.1. Throughout this paper, $\Pi_{\mathscr{G}}$ will denote the collection of $\mathscr{G}$-measurable partitions of Ω, where $\mathscr{G}$ is a sub-σ-algebra of $\mathscr{F}$, and $\sigma(\mathscr{C})$ will denote the σ-algebra generated by $\mathscr{C}$.

DEFINITION 2.1. Suppose μ is a probability measure on $(\Omega, \mathscr{F})$. Then

$$U_n(\mu) = \sup\left\{ \min_{1 \leq i \leq n} \{\mu(A_i)\} : \{A_i\}_{i=1}^n \in \Pi_{\mathscr{F}} \right\}$$

and

$$U_n(\alpha) = \inf\{U_n(\mu) : \mu \in \mathscr{P}(\alpha)\}.$$

LEMMA 2.2. *Fix $\alpha \in (0,1]$. For each $\mu \in \mathscr{P}(\alpha)$ there exists a purely atomic $\hat{\mu} \in \mathscr{P}(\alpha)$ having at most $2\alpha^{-1}$ atoms, and satisfying*

$$U_n(\hat{\mu}) \leq U_n(\mu), \quad \textit{for all } n \geq 1. \tag{4}$$

PROOF. The idea of the proof is simply that collapsing mass to atoms reduces the partitioning options available, and thus reduces U_n; for completeness the first step will be given in some detail. Let $A = \{x_1, x_2, \dots\} \subset \Omega$ denote the atoms of μ and $A^c = \Omega \setminus A$. If $\mu(A^c) > 0$, let $A_1, A_2, \dots$ be a measurable partition of A^c satisfying $0 < \mu(A_i) \le \alpha$ for all i, which is possible since μ is nonatomic on A^c. For each i, fix $y_i \in A_i$, and let $\mu_1 \in \mathscr{P}(\alpha)$ be the purely atomic probability measure defined by $\mu_1(\{x_i\}) = \mu(\{x_i\})$ and $\mu_1(\{y_i\}) = \mu(A_i)$. Since μ restricted to $\sigma(\{x_1\}, A_1, \{x_2\}, A_2, \dots)$ is isomorphic to $\hat{\mu}$ restricted to $\sigma(\{x_1\}, \{y_1\}, \{x_2\}, \{y_2\}, \dots)$, and since (recall $\{x_1\}, A_1, \{x_2\}, A_2, \dots$ are disjoint) $\sigma(\{x_1\}, A_1, \{x_2\}, A_2, \dots) \subset \mathscr{F}$, it follows that $U_n(\mu_1) \le U_n(\mu)$ for all $n \ge 1$.

The next step is to replace μ_1 by a purely atomic measure with each atom having mass at least $\alpha 2^{-1}$ (and hence having at most $2\alpha^{-1}$ atoms). This is done by first combining the tail $\{x_N\}, \{y_N\}, \{x_{N+1}\}, \{y_{N+1}\}, \dots$ into one atom (where $\sum_{i=N}^{\infty}[\mu_1(\{x_i\}) + \mu_1(\{y_i\})] \le \alpha$) to reduce to a finite number of atoms, and then by repeatedly combining any two atoms with mass $\le \alpha/2$. □

LEMMA 2.3. *For each $\alpha \in [0,1]$ and $n \ge 1$, there exists a $\mu \in \mathscr{P}(\alpha)$ and a partition $\{A_i\}_{i=1}^n \in \Pi_{\mathscr{F}}$ satisfying*

$$U_n(\alpha) = \mu(A_1) \le \mu(A_2) \le \cdots \le \mu(A_n). \tag{5}$$

PROOF. For $\alpha = 0$ (which will not be needed in this paper) the result is an easy consequence (taking $\mu_1 = \cdots = \mu_n$) of Lyapounov's convexity theorem [5].

Fix $\alpha \in (0,1]$ and $k > \max\{n, 2\alpha^{-1}\}$, and choose k distinct points $x_1, \dots, x_k$ in Ω. By the definition of $U_n(\alpha)$ and Lemma 2.2, $U_n(\alpha) = \inf\{U_n(\mu): \mu \in \mathscr{P}(\alpha, k)\}$, where $\mathscr{P}(\alpha, k) = \{\mu \in \mathscr{P}(\alpha): \sum_{i=1}^k \mu(\{x_i\}) = 1\}$. Since $\mathscr{P}(\alpha, k)$ is compact, and since U_n is a continuous function of $\mu \in \mathscr{P}(\alpha, k)$, $\inf\{U_n(\mu): \mu \in \mathscr{P}(\alpha, k)\}$ is attained by some $\hat{\mu} \in \mathscr{P}(\alpha, k)$. Since the support of $\hat{\mu}$ is a *finite* set (subset of $\{x_1, \dots, x_k\}$), it is clear that there is a partition $\{A_i\}_{i=1}^n \in \Pi_{\mathscr{F}}$ satisfying (5) with $\hat{\mu}$ in place of μ. □

PROOF OF THEOREM 1.1. Fix $n > 1$ and $k \ge 1$ and let $\alpha \in I(n, k)$, where $I(n, k) = [(k+1)k^{-1}((k+1)n - 1)^{-1}, (kn-1)^{-1}] \subset (0,1)$.

It first will be shown that on $I(n, k)$, $V_n = U_n$. By Lemmas 2.3 and 2.2 there exists a purely atomic measure $\mu \in \mathscr{P}(\alpha)$ with at most $2\alpha^{-1}$ atoms, and a partition $\{A_i\}_{i=1}^n \in \Pi_{\mathscr{F}}$ satisfying (5).

Suppose, by way of contradiction, that $\mu(A_1) < 1 - k(n-1)\alpha$. Since μ is a probability measure, $\mu(\bigcup_{i=2}^n A_i) > k(n-1)\alpha$, and since the $\{A_i\}$ are disjoint, this implies that for some $j \in \{2, 3, \dots, n\}$, $\mu(A_j) > k\alpha$. Since μ is purely atomic and in $\mathscr{P}(\alpha)$, A_j must contain at least $k+1$ μ-atoms. Let $\{x_j\} \in A_j$ be the smallest atom in A_j (which exists since μ has only a finite number of atoms) and observe that

$$\mu(A_1 \cup \{x_j\}) > \mu(A_1). \tag{6}$$

Since $\{x_j\}$ is the smallest atom in A_j, and there are at least $k+1$ atoms in A_j, this implies

(7) $$\mu(A_j \setminus \{x_j\}) \geq k(k+1)^{-1}\mu(A_j) > k^2(k+1)^{-1}\alpha \geq 1 - k(n-1)\alpha,$$

where the last inequality in (7) follows since $\alpha \geq (k+1)k^{-1}[(k+1)n-1]^{-1}$.

If $\mu(A_2) > \mu(A_1)$, then together (6) and (7) contradict the assumed optimality (5) of μ and the partition $\{A_i\}_{i=1}^n$; otherwise [i.e., if $\mu(A_2) = \mu(A_1)$], repeat the procedure with A_2, etc. Since there are only a finite number of sets in the partition, eventually such a contradiction is reached. This implies that $U_n(\mu) \geq V_n(\alpha)$, and hence that $U_n \geq V_n$ on $I(n, k)$.

To show $U_n(\alpha) \leq V_n(\alpha)$, let $\hat{\mu} \in \mathscr{P}(\alpha)$ be a purely atomic measure with $kn-1$ atoms of mass α, and one atom of mass $1-\alpha(kn-1)$. [Since $\alpha \in I(n, k)$, it follows that $0 \leq 1 - \alpha(kn-1) \leq \alpha$.] Clearly an optimal partition for $\hat{\mu}$ has

$$\begin{aligned}\hat{\mu}(A_1) &= (k-1)\alpha + 1 - \alpha(kn-1)\\ &= 1 - k(n-1)\alpha \leq k\alpha\\ &= \hat{\mu}(A_2) = \cdots = \hat{\mu}(A_n),\end{aligned}$$

which shows that $U_n = V_n$ on $I(n, k)$, and in fact that $V_n(\alpha)$ is attained (by $\hat{\mu}$).

To complete the proof, observe that the value of V_n at the left endpoint of $I(n, k)$ is the same as the value of V_n at the right endpoint of $I(n, k+1)$, that is, $1 - k(n-1)x = 1 - (k+1)(n-1)y$ for $x = (k+1)k^{-1}[(k+1)n-1]^{-1}$ and $y = ((k+1)n-1)^{-1}$. Then since V_n was defined to be nonincreasing, it must be constant on $[0,1] \setminus \bigcup_{k=1}^{\infty} I(n, k)$.

That $V_n(0) = n^{-1}$ and $V_n(1) = 0$ are also attained is easy. □

3. Partitioning several probability measures. The main objective of this section is to prove Theorem 1.2; the first two results (Lemma 3.2 and Proposition 3.3) concern stochastic matrices and are purely combinatorial in nature.

Throughout this section, the following notation is used:

$\mathscr{S}_{n,k}$ is the set of $n \times k$ stochastic matrices;
Π_k is the collection of partitions of the set $\{1, 2, \ldots, k\}$; and
P_k is the set of permutations of $\{1, 2, \ldots, k\}$.

DEFINITION 3.1. Suppose $A = (a_{i,j}) \in \mathscr{S}_{n,k}$. Then

$$W_n(A) = \max\left\{\min_{1\leq i\leq n}\left\{\sum_{j\in J_i} a_{i,j}\right\}: \{J_i\}_{i=1}^n \in \Pi_k\right\}.$$

LEMMA 3.2. *For each* $A = (a_{i,j}) \in \mathscr{S}_{n,n}$ *there exist* $\pi \in P_n$ *and* $j \in \{1, \ldots, n\}$ *satisfying both*

(8) $$W_n(A) = \min_{1\leq i\leq n}\{a_{i,\pi(i)}\}$$

and

(9) $$a_{j,\pi(j)} = \max_{1\leq k\leq n}\{a_{k,\pi(j)}\}.$$

Proof. Since $A \in \mathscr{S}_{n,n}$, it is easy to see that

$$W_n(A) = \max\left\{ \min_{1 \le i \le n} \{a_{i,\pi(i)}\} : \pi \in P_n \right\}. \bullet$$

Let $\pi^* \in P_n$ satisfy (10) and (11),

$$W_n(A) = \min_{1 \le i \le n} \{a_{i,\pi^*(i)}\}, \tag{10}$$

$$\sum_{i=1}^{n} a_{i,\pi^*(i)} = \max\left\{ \sum_{i=1}^{n} a_{i,\pi(i)} : \pi \in P_n, \min_{1 \le i \le n} \{a_{i,\pi(i)}\} = W_n(A) \right\}. \tag{11}$$

Renumbering if necessary, assume $\pi^*(i) = i$ for all $i = 1, \dots, n$, and $W_n(A) = a_{1,1} \le a_{2,2} \le \cdots \le a_{n,n}$. It will now be shown that

$$a_{jj} = \max_{1 \le k \le n} a_{k,j}, \quad \text{for some } j \in \{1, \dots, n\}, \tag{12}$$

which, with (10), will complete the proof.

To establish (12), suppose by way of contradiction that for each $j \in \{1, \dots, n\}$, $a_{jj} < \max_{1 \le k \le n}\{a_{k,j}\}$. Then there exist $i_1, i_2, \dots, i_n \in \{1, \dots, n\}$ satisfying (with $i_0 := 1$)

$$\begin{aligned} W_n(A) = a_{i_0,i_0} &< a_{i_1,i_0}, \\ a_{i_1,i_1} &< a_{i_2,i_1}, \\ &\vdots \\ a_{i_{n-1},i_{n-1}} &< a_{i_n,i_{n-1}}. \end{aligned} \tag{13}$$

Since $i_0, \dots, i_n \in \{1, \dots, n\}$, the ordered $(n+1)$-tuple $(i_0, i_1, \dots, i_n)$ contains a primitive cycle, that is, there exists $j \in \{0, \dots, n-1\}$ and $k \in \{0, \dots, n-j\}$ such that $i_j, i_{j+1}, \dots, i_{j+k}$ are distinct and $i_j = i_{j+k+1}$.

Next consider the permutation $\tilde{\pi} \in P_n$ defined by $\tilde{\pi}(i_{j+m+1}) = i_{j+m}$ for $m = 0, 1, \dots, k$, and $= \pi^*$ otherwise. By (13),

$$a_{i_{j+m+1},\tilde{\pi}(i_{j+m+1})} = a_{i_{j+m+1},i_{j+m}} > a_{i_{j+m},i_{j+m}} \ge a_{1,1} = W_n(A),$$

for $m = 0, 1, \dots, k$, so the definition of $\tilde{\pi}$ implies that $W_n(A) \le \min_{1 \le i \le n} a_{i,\tilde{\pi}(i)}$, and hence by the definition of $W_n(A)$ that

$$W_n(A) = \min_{1 \le i \le n} a_{i,\tilde{\pi}(i)}. \tag{14}$$

But (13) and the definition of $\tilde{\pi}$ also imply that $\sum_{i=1}^{n} a_{i,\tilde{\pi}(i)} > \sum_{i=1}^{n} a_{i,\pi^*(i)}$ which, with (14), contradicts (11). This completes the proof of (12), and the lemma. □

The next proposition states that there is always an optimal partitioning of a stochastic matrix in which the "cooperative value," that is, the sum of the partition-assignment values, is at least one.

Proposition 3.3. *For each $A = (a_{i,j}) \in S_{n,m}$ there is a partition $\{J_i\}_{i=1}^{n} \in \Pi_m$ satisfying both*

$$W_n(A) = \min_{1 \le i \le n} \left\{ \sum_{j \in J_i} a_{i,j} \right\} \tag{15}$$

and

$$\sum_{i=1}^{n} \sum_{j \in J_i} a_{i,j} \geq 1. \tag{16}$$

PROOF. Fix $A = (a_{i,j}) \in S_{n,m}$. By the definition of W_n, there exists a partition $\{J_i\}_{i=1}^n$ of Π_m satisfying (15). If $m > n$, let $\hat{A} = (\hat{a}_{i,j}) \in S_{n,n}$ be the matrix defined by $\hat{a}_{i,j} = \sum_{k \in J_j} a_{i,k}$, and observe that both

$$W_n(A) = W_n(\hat{A}) = \min_{1 \leq i \leq n} \hat{a}_{i,i} \tag{17}$$

and

$$\sum_{i=1}^{n} \hat{a}_{i,i} = \sum_{i=1}^{n} \sum_{j \in J_i} a_{i,j}. \tag{18}$$

By (17) and (18), it is enough to establish the proposition for $n \times n$ stochastic matrices A (if $m < n$, simply add $n - m$ columns of zeros to A). The proof will proceed by induction on n; for $n = 1$ the conclusion is trivial, so assume it holds for $1, 2, \ldots, n - 1$ and let $A = (a_{i,j}) \in S_{n,n}$.

By Lemma 3.2 there exists $\pi \in P_n$ and $j \in \{1, \ldots, n\}$ satisfying (8) and (9). Reordering if necessary, assume $j = n = \pi(n)$, and observe that by (9) the $(n - 1) \times (n - 1)$ matrix $\tilde{A}$ obtained from A by deleting the nth row and column is substochastic with row sums $\sum_{j=1}^{n-1} a_{i,j} \geq 1 - a_{n,n}$ for all $i = 1, \ldots, n - 1$. It follows easily from the induction hypothesis that there exists $\tilde{\pi} \in P_{n-1}$ satisfying both

$$W_{n-1}(\tilde{A}) = \min_{1 \leq i \leq n-1} \{a_{i,\tilde{\pi}(i)}\} \tag{19}$$

and

$$\sum_{i=1}^{n-1} a_{i,\tilde{\pi}(i)} \geq 1 - a_{n,n}. \tag{20}$$

Defining $\hat{\pi} \in P_n$ by $\hat{\pi}(i) = \tilde{\pi}(i)$ for $i < n$ and $\hat{\pi}(n) = \pi(n) = n$, (8) and (19) together imply that

$$W_n(A) = \min\{W_{n-1}(\tilde{A}), a_{n,n}\} = \min_{1 \leq i \leq n} \{a_{i,\hat{\pi}(i)}\}, \tag{21}$$

and (20) and the definition of $\hat{\pi}$ imply that

$$\sum_{i=1}^{n} a_{i,\hat{\pi}(i)} \geq 1. \tag{22}$$

The induction conclusion then follows from (21) and (22) by taking $J_i = \{\hat{\pi}(i)\}$ for $i = 1, \ldots, n$. □

Not all optimal partitions [partitions achieving $W_n(A)$] satisfy (16).

EXAMPLE 3.4. Let

$$A = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}.$$

The partition $\{J_i\} = \{i\}$, $i = 1,2,3$, satisfies

$$W_3(A) = 0.3 = \min_{1 \le i \le 3} \left\{ \sum_{j \in J_i} a_{i,j} \right\}$$
$$= \min\{a_{1,1}, a_{2,2}, a_{3,3}\},$$

but $\sum_{i=1}^{n} \sum_{j \in J_i} a_{i,j} = 0.9 < 1$.

PROOF OF THEOREM 1.2. That $V_n(\alpha)$ is attained for all α follows from Theorem 1.1 by taking $\mu_1 = \mu_2 = \cdots = \mu_n$.

Fix $\alpha \in (0,1]$ and $\mu_1, \dots, \mu_n \in \mathscr{P}(\alpha)$. By an argument directly analogous to that in the proof of Lemma 2.2, it may be assumed without loss of generality that $\{\mu_i\}_{i=1}^{n}$ are purely atomic each with at most $m \le 2n\alpha^{-1}$ atoms. In other words, it suffices to show that if

$$A = (a_{i,j}) \in S_{n,m} \quad \text{and} \quad a_{i,j} \le \alpha \tag{23}$$
$$\text{for all } i = 1, \dots, n \quad \text{and} \quad j = 1, \dots, m,$$

then

$$W_n(A) \ge V_n(\alpha). \tag{24}$$

Fix A satisfying (23). By Proposition 3.3 there exists a partition $\{J_i\}_{i=1}^{n} \in \Pi_m$ satisfying (15) and (16). To prove (24), fix $n > 1$, $k \ge 1$, and

$$\alpha \in I(n,k) = \left[(k+1)k^{-1}((k+1)n - 1)^{-1}, (kn-1)^{-1}\right].$$

Suppose, by way of contradiction, that $\sum_{j \in J_1} a_{i,j} < 1 - k(n-1)\alpha$. By (16), $\sum_{i=2}^{n} \sum_{j \in J_i} a_{i,j} > k(n-1)\alpha$, so for some $i \in \{2, \dots, n\}$, $\sum_{j \in J_i} a_{i,j} > k\alpha$. The argument now proceeds as in the proof of Theorem 1.1, the key difference having been the use of Proposition 3.3 (which is trivial for the $\mu_1 = \mu_2 = \cdots = \mu_n$ context of Theorem 1.1). □

4. Several remarks concerning $V_n(\alpha)$. The following proposition is an easy consequence of the definition of $V_n(\alpha)$.

PROPOSITION 4.1. *For each $n \ge 1$, $V_n(\cdot)$ is continuous and nonincreasing on* $[0,1]$, *piecewise linear on* $(0,1]$, *and satisfies*

(i) $$V_n(0) = n^{-1}, \qquad V_n(1) = 0;$$

(ii) $$V_{n+1}(\alpha) < V_n(\alpha), \quad \textit{if } V_n(\alpha) > 0, \quad \textit{and}$$
$$V_{n+1}(\alpha) = V_n(\alpha), \quad \textit{if } V_n(\alpha) = 0;$$

and

(iii) $$V_n(\alpha) \ge n^{-1} - (n-1)n^{-1}\alpha.$$

The critical points at the left-hand endpoints of the intervals where V_n is constant are local minima. For example, V_2 has local minima at $1/3$, $1/5$, $1/7, \dots$; and for the first of these, one interpretation is that in the case of bisection ($n = 2$), atoms of mass exactly $1/3$ are locally the worst—in general atoms slightly less than *or* slightly greater than $1/3$ allow better partitions.

5. Applications to L_1 spaces and statistical decision theory It is easy to translate the settings of Theorems 1.1 and 1.2 to the theory of L_1 spaces; the next theorem is the analog of Theorem 1.2.

THEOREM 5.1. *Suppose λ is a Borel measure on $\mathbb{R}$. If $f_1, f_2, \ldots, f_n \in L_1(\lambda)$ satisfy*

(i) $f_i \geq 0$, $i = 1, \ldots, n$;
(ii) $\int f_i \, d\lambda = 1$, $i = 1, \ldots, n$; *and*
(iii) $\lambda(\{x\}) f_i(x) \leq \alpha$, *for all* $x \in \mathbb{R}$,

then there exists a measurable partition $\{A_i\}_{i=1}^n$ of $\mathbb{R}$ satisfying

$$\int_{A_i} f_i \, d\lambda \geq V_n(\alpha), \quad \text{for all } i = 1, \ldots, n.$$

Moreover, this bound is best possible, and is attained for all α and n.

The final theorem is an application of Theorem 1.2 to statistical decision theory which is related to similar applications of partitioning inequalities in [2] and [3].

Suppose there is an Ω-valued random variable X which has one of the known distributions $\mu_1, \ldots, \mu_n$ (but it is not known which one). A single observation $X(\omega)$ of X is made, and then it is to be guessed from which of the distributions $\mu_1, \ldots, \mu_n$ the observation came. A *decision rule* is simply a (measurable) partition $\{A_i\}_{i=1}^n$ of Ω ("if $X(\omega) \in A_i$, then guess distribution μ_i"). A *minimax* decision rule is a partition which attains the "minimax risk" R given by

$$(25) \quad R(\mu_1, \ldots, \mu_n) = \inf\left\{ \max_{1 \leq i \leq n} P(X \notin A_i | \text{dist}(X) = \mu_i) : \{A_i\}_{i=1}^n \in \Pi_{\mathscr{F}} \right\}.$$

Since

$$R(\mu_1, \ldots, \mu_n) = \inf\left\{ \max_{1 \leq i \leq n} \{1 - \mu_i(A_i)\} : \{A_i\}_{i=1}^n \in \Pi_{\mathscr{F}} \right\}$$
$$= 1 - \sup\left\{ \min_{1 \leq i \leq n} \{\mu_i(A_i)\} : \{A_i\}_{i=1}^n \in \Pi_{\mathscr{F}} \right\},$$

Theorem 1.2 has the following immediate consequence.

THEOREM 5.2. *Let $\mu_1, \ldots, \mu_n \in \mathscr{P}(\alpha)$. Then*

$$R(\mu_1, \ldots, \mu_n) \leq 1 - V_n(\alpha),$$

and this bound is attained for all α and all n.

A similar application (see [2]) can also be made to the theory of zero-sum two-person games.

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